B. N. PRASAD (*)

On areal metrics on complex manifolds (**)

Abstract : The 4-index areal metric tensor has been defined and its properties have been studied by Rund [3], [4]. The geometry of areal spaces in these works are based on real coordinate system. The purpose of this paper is to investigate a certain class of 4-index metric tensor on complex manifolds. After giving the outline of complex manifolds in §1, we introduce this 4-index metric tensor in § 2. The section 3 is devoted to a fairly detailed description of the many identities which result from the homogeneity condition of the metric function. In § 4 these preliminary results, all of which are basically concerned with a 4-index metric tensor, are used in the construction of suitable connection coefficients, which are explicily derivable from the metric tensor and its various derivatives. The last section is devoted in construction of the covariant partial derivatives of a given vector field with the help of these connection coefficients.
Our considerations are purely local in character. In some places the detailed calculations have been suppressed for the sake of brevity. With regard to such instances reference is made to the report [5].
Throughout this paper the Latin indices i, j, h, k, l, m, p, q run over 1 to n while Greek indices α, β, γ, λ run over 1 to m.

1. - Introduction

We consider a 2n-dimensional real manifold X_{2n} (of class C^∞) referred to local coodinates (x^j, y^j). Corresponding to each point P of X_{2n} we introduce complex numbers z^j,

$$z^j = x^j + i\, y^j \qquad (i^2 = -1) \tag{1.1}$$

which may be regarded as the complex coordinate of P (with respect to the given coördinate system). If there exist complex coordinate neighbourhoods $U(z^j)$, $U(\bar z^j)$ (where $\bar z^j$ refer to another local coordinate system) such that in the intersection of these neighbourhoods, we have

$$\bar z^j = \bar z^j (z^h) \qquad \det\left[\frac{\partial \bar z^j}{\partial z^h}\right] \neq 0 \tag{1.2}$$

where $\bar z^j (z^h)$ are holomorphic functions of z^h, then space X_{2n} is said to admit a complex structure. Under these circumstances X_{2n} is called a complex space of complex dimension n and is denoted by C_n.

(*) Department of Mathematics, St. Andrews College, Gorakhpur, (U.P.) India.
(**) Memoria presentata dall'Accademico dei XL E. Bompiani il 21-9-1983.

With (1.1) we may associate the conjugate complex

(1.3) $$\overline{z}^{j*} = x^j - i\, y^j$$

so that (1.2) carries with it the corresponding conjugate complex transformation,

(1.4) $$\overline{z}^{j*} = \overline{z}^{j*}(z^{h*}) .$$

An analytic m-dimensional subspace C_m of C_n $(m < n)$ is represented parametrically by equations ([1] page 104)

(1.5) $$z^j = z^j(u^\alpha) \quad , \quad z^{j*} = z^{j*}(u^{\alpha*}) \qquad .$$

in which the z^j, z^{j*} are holomorphic functions of the complex variables u^α, $u^{\alpha*}$ respe ctively. Thus the derivatives $\dot{z}^j_\alpha = \dfrac{\partial z^j}{\partial u^\alpha}$ and their complex conjugate $\dot{z}^{j*}_{\alpha*} = \dfrac{\partial z^{j*}}{\partial u^{\alpha*}}$ are defined, each of which are the elements of an $n \times m$ matrix which is always supposed to be of rank m.

2. - Fundamental formulae

We consider real Lagrange function L of the form,

(2.1) $$L = L(z^j, z^{j*}, \dot{z}^j_\alpha, \dot{z}^{j*}_{\alpha*})$$

satisfying the conditions

(A) The Lagrangian L is of class C^4 in all its arguments and it is scalar with respect to transfor mations (1.2) and (1.4).

(B) The Lagrangian L is positive for all independent sets of arguments $\dot{z}^j_\alpha$, $\dot{z}^{j*}_{\alpha*}$.

(C) The integral

(2.2) $$I = \int_G L\, du^1 \wedge \ldots\ldots \wedge du^m \wedge du^{1*} \wedge \ldots\ldots \wedge du^{m*}$$

over a fixed region G of C_m is invariant under the holomorphic transformations of the complex parameters

(2.3) $$\overline{u}^\alpha = \overline{u}^\alpha(u^\beta) \quad , \quad \overline{u}^{\alpha*} = \overline{u}^{\alpha*}(u^{\beta*})$$

(D) The $n\,m \times n\,m$ determinant

$$D = \det \left[\frac{2\,m}{2} \frac{\partial^2 L^{\frac{2}{2m}}}{\partial \dot{z}^h_\beta\, \partial \dot{z}^{j*}_{\alpha*}} \right]$$

is non vanishing for linearly independent $\dot{z}^h_\beta$, $\dot{z}^{j*}_{\alpha*}$.

The condition C is equivalent to the relations [2]

(2.4) (a) $\frac{\partial L}{\partial \dot z^i_\alpha} \dot z^i_\beta = \delta^\alpha_\beta L$ (b) $\frac{\partial L}{\partial \dot z^{i*}_{\alpha*}} \dot z^{i*}_{\beta*} = \delta^{\alpha*}_{\beta*} L$.

In particular if we put $\alpha = \beta$ in (2.4) we get

(2.5) (a) $\frac{\partial L}{\partial \dot z^i_\alpha} \dot z^i_\alpha = m L$; (b) $\frac{\partial L}{\partial \dot z^{i*}_{\alpha*}} \dot z^{i*}_{\alpha*} = m L$.

Differentiating both the equations of (2.5) with respect to $\dot z^h_\beta$ and adding we get

(2.6) $$\frac{\partial^2 L}{\partial \dot z^h_\beta \partial \dot z^i_\alpha} \dot z^i_\alpha + \frac{\partial^2 L}{\partial \dot z^h_\beta \partial \dot z^{i*}_{\alpha*}} \dot z^{i*}_{\alpha*} = (2m-1) \frac{\partial L}{\partial \dot z^h_\beta}.$$

From (2.5) and (2.6 one can find

(2.7) $$\frac{2m}{2}\left[\frac{\partial^2 L^{\frac{2}{2m}}}{\partial \dot z^h_\beta \partial \dot z^i_\alpha} \dot z^i_\alpha + \frac{\partial^2 L^{\frac{2}{2m}}}{\partial \dot z^h_\beta \partial \dot z^{i*}_{\alpha*}} \dot z^{i*}_{\alpha*}\right] = L^{\frac{2}{2m}-1} \frac{\partial L}{\partial \dot z^h_\beta}$$

alongwith

(2.8) $$\frac{2m}{2}\left[\frac{\partial^2 L^{\frac{2}{2m}}}{\partial \dot z^{h*}_{\beta*} \partial \dot z^i_\alpha} \dot z^i_\alpha + \frac{\partial^2 L^{\frac{2}{2m}}}{\partial \dot z^{h*}_{\beta*} \partial \dot z^{i*}_{\alpha*}} \dot z^{i*}_{\alpha*}\right] = L^{\frac{2}{2m}-1} \frac{\partial L}{\partial \dot z^{h*}_{\beta*}}.$$

Multiplying (2.7) by $\dot z^h_\beta$ and (2.8) by $\dot z^{h*}_{\beta*}$ and adding we get (in view of (2.5))

(2.9) $$2m L^{\frac{2}{2m}} = g^{\beta\alpha}_{hj} \dot z^h_\beta \dot z^j_\alpha + 2 g^{\beta\alpha*}_{hj*} \dot z^h_\beta \dot z^{j*}_{\alpha*} + g^{\beta*\alpha*}_{h*j*} \dot z^{h*}_{\beta*} \dot z^{j*}_{\alpha*}$$

where

(2.10) $$g^{\beta\alpha}_{hj}(z^i, z^{i*}, \dot z^i_\lambda, \dot z^{i*}_{\lambda*}) = \frac{2m}{2} \frac{\partial^2 L^{\frac{2}{2m}}}{\partial \dot z^h_\beta \partial \dot z^j_\alpha},$$

(2.11) $$g^{\beta\alpha*}_{hj*}(z^i, z^{i*}, \dot z^i_\lambda, \dot z^{i*}_{\lambda*}) = \frac{2m}{2} \frac{\partial^2 L^{\frac{2}{2m}}}{\partial \dot z^h_\beta \partial \dot z^{j*}_{\alpha*}},$$

(2.12) $$g^{\beta*\alpha*}_{h*j*}(z^i, z^{i*}, \dot z^i_\lambda, \dot z^{i*}_{\lambda*}) = \frac{2m}{2} \frac{\partial^2 L^{\frac{2}{2m}}}{\partial \dot z^{h*}_{\beta*} \partial \dot z^{j*}_{\alpha*}}.$$

From (2.9) it is evident that if L is interpreted as measure of the area d A of an m-dimensional complex subspace (2m-dimensional real subspace) spanned by $\dot z^i_\alpha$ $\dot z^{i*}_{\alpha*}$ at the points z^i, z^{i*} of C_n in the sense that

(2.13) $$dA = L(z^i, z^{i*}, \dot z^i_\alpha, \dot z^{i*}_{\alpha*})\, du^1 \wedge \ldots\ldots \wedge du^m \wedge du^{1*} \wedge \ldots\ldots \wedge du^{m*}$$

then the tensors (2.10), (2.11) and (2.12) can be regarded as a suitable areal metric tensor ([3] page 289).

It is to be noted that $g^{\beta\alpha}_{hj}$ is symmetric in pairs of indices such as (β, h), (α, j). The similar symmetries exist for the tensors $g^{\beta\alpha^*}_{hj^*}$ and $g^{\beta^*\alpha^*}_{h^*j^*}$. Furthermore $g^{\beta\alpha}_{hj} \neq g^{\beta\alpha}_{jh}$.

3. - Identities resulting from homogeneity condition (2.4)

Differentiating (2.5) with respect to $\dot{z}^h_\beta$ and $\dot{z}^{h^*}_{\beta^*}$ respectively we get after some simplifications

$$(3.1) \quad (a) \quad \frac{\partial^2 L}{\partial \dot{z}^h_\beta \, \partial \dot{z}^j_\alpha} \dot{z}^j_\alpha = (m-1) \frac{\partial L}{\partial \dot{z}^h_\beta} \qquad (b) \quad \frac{\partial^2 L}{\partial \dot{z}^h_\beta \, \partial \dot{z}^{j^*}_{\alpha^*}} \dot{z}^{j^*}_{\alpha^*} = m \frac{\partial L}{\partial \dot{z}^h_\beta}$$

$$(3.2) \quad (a) \quad \frac{\partial^2 L}{\partial \dot{z}^{h^*}_{\beta^*} \, \partial \dot{z}^j_\alpha} \dot{z}^j_\alpha = m \frac{\partial L}{\partial \dot{z}^{h^*}_{\beta^*}} \qquad (b) \quad \frac{\partial^2 L}{\partial \dot{z}^{h^*}_{\beta^*} \, \partial \dot{z}^{j^*}_{\alpha^*}} \dot{z}^{j^*}_{\alpha^*} = (m-1) \frac{\partial L}{\partial \dot{z}^{h^*}_{\beta^*}}.$$

But we have from (2.10) and (2.11)

$$(3.3) \qquad g^{\beta\alpha}_{hj} = \left(\frac{2}{2m} - 1\right) L^{\frac{2}{2m}-2} \frac{\partial L}{\partial \dot{z}^h_\beta} \frac{\partial L}{\partial \dot{z}^j_\alpha} + L^{\frac{2}{2m}-1} \frac{\partial^2 L}{\partial \dot{z}^h_\beta \, \partial \dot{z}^j_\alpha}$$

$$(3.4) \qquad g^{\beta\alpha^*}_{hj^*} = \left(\frac{2}{2m} - 1\right) L^{\frac{2}{2m}-2} \frac{\partial L}{\partial \dot{z}^h_\beta} \frac{\partial L}{\partial \dot{z}^{j^*}_{\alpha^*}} + L^{\frac{2}{2m}-1} \frac{\partial^2 L}{\partial \dot{z}^h_\beta \, \partial \dot{z}^{j^*}_{\alpha^*}}.$$

Differentiating (3.4) with respecto $\dot{z}^k_\gamma$ we get,

$$(3.5) \qquad \frac{\partial g^{\beta\alpha^*}_{hj^*}}{\partial \dot{z}^k_\gamma} = \left(\frac{2}{2m} - 1\right) \left\{ \left(\frac{2}{2m} - 2\right) L^{\frac{2}{2m}-3} \frac{\partial L}{\partial \dot{z}^k_\gamma} \frac{\partial L}{\partial \dot{z}^h_\beta} \frac{\partial L}{\partial \dot{z}^{j^*}_{\alpha^*}} \right.$$
$$+ L^{\frac{2}{2m}-2} \left(\frac{\partial^2 L}{\partial \dot{z}^k_\gamma \, \partial \dot{z}^h_\beta} \frac{\partial L}{\partial \dot{z}^{j^*}_{\alpha^*}} + \frac{\partial L}{\partial \dot{z}^h_\beta} \frac{\partial^2 L}{\partial \dot{z}^k_\gamma \, \partial \dot{z}^{j^*}_{\alpha^*}} \right)$$
$$\left. + L^{\frac{2}{2m}-2} \frac{\partial L}{\partial \dot{z}^k_\gamma} \frac{\partial^2 L}{\partial \dot{z}^h_\beta \, \partial \dot{z}^{j^*}_{\alpha^*}} \right\} + L^{\frac{2}{2m}-1} \frac{\partial^3 L}{\partial \dot{z}^k_\gamma \, \partial \dot{z}^h_\beta \, \partial \dot{z}^{j^*}_{\alpha^*}}.$$

Multiplying (3.5) by $\dot{z}^k_\gamma$ and using (2.5), (3.1) a and (3.2) a we get after some simplifications,

$$(3.6) \qquad \frac{\partial g^{\beta\alpha^*}_{hj^*}}{\partial \dot{z}^k_\gamma} \dot{z}^k_\gamma = L^{\frac{2}{2m}-1} \left\{ (1-m) \frac{\partial^2 L}{\partial \dot{z}^h_\beta \, \partial \dot{z}^{j^*}_{\alpha^*}} + \frac{\partial^3 L}{\partial \dot{z}^k_\gamma \, \partial \dot{z}^h_\beta \, \partial \dot{z}^{j^*}_{\alpha^*}} \dot{z}^k_\gamma \right\}.$$

However differentiation of (3.1) a with respect to $\dot{z}^{k^*}_{\gamma^*}$ gives

$$(3.7) \qquad \frac{\partial^3 L}{\partial \dot{z}^{k^*}_{\gamma^*} \, \partial \dot{z}^h_\beta \, \partial \dot{z}^j_\alpha} \dot{z}^j_\alpha = (m-1) \frac{\partial^2 L}{\partial \dot{z}^{k^*}_{\gamma^*} \, \partial \dot{z}^h_\beta}$$

and this shows that (3.6) reduces to

$$\frac{\partial g^{\beta\alpha^*}_{hj^*}}{\partial \dot z^k_\gamma}\, \dot z^k_\gamma = 0 . \tag{3.8}$$

Since $\frac{\partial g^{\beta\alpha^*}_{hj^*}}{\partial \dot z^k_\gamma}$ is symmetric in the pairs of indices (β, h) ; (γ, k) it also follows from (3.8) that

$$\frac{\partial g^{\gamma\alpha^*}_{kj^*}}{\partial \dot z^h_\beta}\, \dot z^k_\gamma = 0 . \tag{3.9}$$

Similarly we can show that

$$\frac{\partial g^{\beta^*\alpha}_{h^*j}}{\partial \dot z^{k^*}_{\gamma^*}}\, \dot z^{k^*}_{\gamma^*} = 0 = \frac{\partial g^{\gamma^*\alpha}_{k^*j}}{\partial \dot z^{h^*}_{\beta^*}}\, \dot z^{k^*}_{\gamma^*} . \tag{3.10}$$

On the other hand multiplying (3.5) by $\dot z^{j^*}_{\alpha^*}$ and using (2.5) and (3.1) b we obtain

$$\frac{\partial g^{\beta\alpha^*}_{hj^*}}{\partial \dot z^k_\gamma}\, \dot z^{j^*}_{\alpha^*} = \left(\frac{2}{2\,m} - 1\right) L^{\frac{2}{2m}-2}\, \frac{\partial L}{\partial \dot z^k_\gamma}\, \frac{\partial L}{\partial \dot z^h_\beta} + \tag{3.11}$$

$$+ L^{\frac{2}{2m}-1} \left\{ \frac{\partial^2 L}{\partial \dot z^k_\gamma\, \partial \dot z^h_\beta}\,(1 - m) + \frac{\partial^3 L}{\partial \dot z^k_\gamma\, \partial \dot z^h_\beta\, \partial \dot z^{j^*}_{\alpha^*}}\, \dot z^{j^*}_{\alpha^*} \right\} .$$

Differentiation of (3.1) b with respect to $\dot z^k_\gamma$ yields

$$\frac{\partial^3 L}{\partial \dot z^k_\gamma\, \partial \dot z^h_\beta\, \partial \dot z^{j^*}_{\alpha^*}}\, \dot z^{j^*}_{\alpha^*} = m\, \frac{\partial^2 L}{\partial \dot z^k_\gamma\, \partial \dot z^h_\beta} . \tag{3.12}$$

Applying (3.12) to (3.11) we find that

$$\frac{\partial g^{\beta\alpha^*}_{hj^*}}{\partial \dot z^k_\gamma}\, \dot z^{j^*}_{\alpha^*} = \left(\frac{2}{2\,m} - 1\right) L^{\frac{2}{2m}-2}\, \frac{\partial L}{\partial \dot z^k_\gamma}\, \frac{\partial L}{\partial \dot z^h_\beta} + L^{\frac{2}{2m}-1}\, \frac{\partial^2 L}{\partial \dot z^k_\gamma\, \partial \dot z^h_\beta}$$

which in view of (3.3) yields

$$\frac{\partial g^{\beta\alpha^*}_{hj^*}}{\partial \dot z^k_\gamma}\, \dot z^{j^*}_{\alpha^*} = g^{\beta\gamma}_{hk} . \tag{3.13}$$

Similarly we can show that

$$\frac{\partial g^{\beta^*\alpha}_{h^*j}}{\partial \dot z^{k^*}_{\gamma^*}}\, \dot z^j_\alpha = g^{\beta^*\gamma^*}_{h^*k^*} . \tag{3.14}$$

Multiplying (3.13) by $\dot z^h_\beta$ and using (3.9) we find

$$g^{\beta\gamma}_{hk}\, \dot z^h_\beta = 0 \tag{3.15}$$

and similarly

$$g^{\beta^*\gamma^*}_{h^*k^*}\, \dot z^{h^*}_{\beta^*} = 0 . \tag{3.16}$$

Furthermore from (3.3), (3.1) a and (2.5) it follows that

$$(3.17)\qquad \frac{\partial g_{hj}^{\beta\alpha}}{\partial \dot{z}_{\gamma}^{k}} \dot{z}_{\gamma}^{k} = -\left(\frac{2}{2\,m} - 1\right) L^{\frac{2}{2m}-2} \frac{\partial L}{\partial \dot{z}_{\beta}^{h}} \frac{\partial L}{\partial \dot{z}_{\alpha}^{j}} +$$

$$+ L^{\frac{2}{2m}-1} \left\{ (1-m) \frac{\partial^2 L}{\partial \dot{z}_{\beta}^{h} \partial \dot{z}_{\alpha}^{j}} + \frac{\partial^3 L}{\partial \dot{z}_{\gamma}^{k} \partial \dot{z}_{\beta}^{h} \partial \dot{z}_{\alpha}^{j}} \dot{z}_{\gamma}^{k} \right\}$$

while differentiation of (3.1) a gives

$$(3.18)\qquad \frac{\partial^3 L}{\partial \dot{z}_{\gamma}^{k} \partial \dot{z}_{\beta}^{h} \partial \dot{z}_{\alpha}^{j}} \dot{z}_{\gamma}^{k} = (m-2) \frac{\partial^2 L}{\partial \dot{z}_{\beta}^{h} \partial \dot{z}_{\alpha}^{j}} .$$

Substituting (3.18) in (3.17) and using (3.3) once more, we obtain,

$$(3.19)\qquad \frac{\partial g_{hj}^{\beta\alpha}}{\partial \dot{z}_{\gamma}^{k}} \dot{z}_{\gamma}^{k} = - g_{hj}^{\beta\alpha}$$

and by symmetry in the pairs of indices (β, h), (γ, k) we get

$$(3.20)\qquad \frac{\partial g_{kj}^{\gamma\alpha}}{\partial \dot{z}_{\beta}^{h}} \dot{z}_{\gamma}^{k} = - g_{hj}^{\beta\alpha} .$$

Similarly it can be shown that

$$(3.21)\qquad \frac{\partial g_{h^*j^*}^{\beta^*\alpha^*}}{\partial \dot{z}_{\gamma^*}^{k^*}} \dot{z}_{\gamma^*}^{k^*} = \frac{\partial g_{k^*j^*}^{\gamma^*\alpha^*}}{\partial \dot{z}_{\beta^*}^{h^*}} \dot{z}_{\gamma^*}^{k^*} = - g_{h^*j^*}^{\beta^*\alpha^*} .$$

A direct application of relations (3.15), (3.16), (3.19) and (3.21) will yield,

$$(3.22)\qquad \frac{\partial g_{hj}^{\beta\alpha}}{\partial \dot{z}_{\gamma}^{k}} \dot{z}_{\gamma}^{k} \dot{z}_{\beta}^{h} = 0 \quad ; \quad \frac{\partial g_{h^*j^*}^{\beta^*\alpha^*}}{\partial \dot{z}_{\gamma^*}^{k^*}} \dot{z}_{\gamma^*}^{k^*} \dot{z}_{\beta^*}^{h^*} = 0 .$$

Also in consequence of (3.15) and (3.16) the relation (2.9) reduces to

$$(3.23)\qquad m\, L^{\frac{1}{m}} = g_{hj^*}^{\beta\alpha^*} \dot{z}_{\beta}^{h} \dot{z}_{\alpha^*}^{j^*} .$$

4. - The connection coefficients

Let us suppose that we are given an m-dimensional complex subspace C_m. Under the holomorphic transformations of the form (1.2) and (1.4) of the local coordinates of C_n, the quantities $\dot{z}_{\alpha}^{j}$ and $\dot{z}_{\alpha^*}^{j^*}$ transform as follows,

$$(4.1)\qquad \dot{z}_{\alpha}^{j} = B_{h}^{j}(\overline{z}^{l})\, \dot{\overline{z}}_{\alpha}^{h} \quad ; \quad \dot{z}_{\alpha^*}^{j^*} = B_{h^*}^{j^*}(\overline{z}^{l^*})\, \dot{\overline{z}}_{\alpha^*}^{h^*}$$

where we have written

$$(4.2)\qquad B_{h}^{j}(\overline{z}^{l}) = \frac{\partial z^{j}}{\partial \overline{z}^{h}} \quad ; \quad B_{h^*}^{j^*}(\overline{z}^{l^*}) = \frac{\partial z^{j^*}}{\partial \overline{z}^{h^*}} .$$

We shall also put

$$(4.3)\qquad B^j_{hl} = \frac{\partial^2 z^j}{\partial \bar{z}^h \partial \bar{z}^l} = B^j_{lh} \quad , \quad B^{j^*}_{h^* l^*} = \frac{\partial^2 z^{j^*}}{\partial \bar{z}^{h^*} \partial \bar{z}^{l^*}} = B^{j^*}_{l^* h^*} \,.$$

Since the functions (1.2) and (1.4) are assumed to be holomorphine, we have,

$$(4.4)\qquad \frac{\partial z^j}{\partial \bar{z}^{h^*}} = 0 \quad ; \quad \frac{\partial z^{j^*}}{\partial \bar{z}^h} = 0$$

and

$$(4.5)\qquad \frac{\partial B^j_h}{\partial \bar{z}^{k^*}} = 0 \quad ; \quad \frac{\partial B^{j^*}_{h^*}}{\partial \bar{z}^k} = 0 \,.$$

The equations (4.2), (4.3), (4.4) and (4.5) have been frequently used in the remaining part of this paper.

For future reference we note that

$$(4.6)\qquad \frac{\partial \dot{z}^j_\alpha}{\partial u^\beta} = \dot{z}^j_{\alpha\beta} = B^j_h(\bar{z}^l)\, \dot{\bar{z}}^h_{\alpha\beta} + B^j_{kh}\, \dot{\bar{z}}^k_\beta\, \dot{\bar{z}}^h_\alpha \,,$$

$$(4.7)\qquad \frac{\partial \dot{z}^{j^*}_{\alpha^*}}{\partial u^{\beta^*}} = \dot{z}^{j^*}_{\alpha^*\beta^*} = B^{j^*}_{h^*}(\bar{z}^{l^*})\, \dot{\bar{z}}^{h^*}_{\alpha^*\beta^*} + B^{j^*}_{k^* h^*}\, \dot{\bar{z}}^{k^*}_{\beta^*}\, \dot{\bar{z}}^{h^*}_{\alpha^*} \,.$$

Clearly, the quantities $\dot{z}^j_{\alpha\beta}$ and $\dot{z}^{j^*}_{\alpha^*\beta^*}$ do not represent components of a tensor field and consequently we must construct the corresponding covariant derivatives.

Since the given Lagrangian L is supposed to be scalar with respect to the transformations (1.2) and (1.4) it follows directly from (4.1) that the quantities (2.10), (2.11) and (2.12) represent componentes of a covariant tensor of rank two. We, therefore, have,

$$(4.8)\qquad \bar{g}^{\beta\alpha^*}_{hj^*}(\bar{z}^l, \bar{z}^{l^*}, \dot{\bar{z}}^l_\lambda, \dot{\bar{z}}^{l^*}_{\lambda^*}) = g^{\beta\alpha^*}_{km^*}(z^t, z^{t^*}, \dot{z}^t_\gamma, \dot{z}^{t^*}_{\gamma^*})\, B^k_h\, B^{m^*}_{j^*} \,.$$

Differentiating (4.8) with respect to $\bar{z}^{l^*}$ we get

$$(4.9)\qquad \frac{\partial \bar{g}^{\beta\alpha^*}_{hj^*}}{\partial \bar{z}^{l^*}} = \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial z^{t^*}}\, B^{t^*}_{l^*}\, B^k_h\, B^{m^*}_{j^*} + $$

$$+ \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial \dot{z}^{t^*}_{\gamma^*}}\, B^{t^*}_{r^* l^*}\, \dot{\bar{z}}^{r^*}_{\gamma^*}\, B^k_h\, B^{m^*}_{j^*} + g^{\beta\alpha^*}_{km^*}\, B^k_h\, B^{m^*}_{j^* l^*} \,.$$

Multiplying (4.9) by $\dot{\bar{z}}^{j^*}_{\alpha^*}$ and using (4.1), (3.10) we find,

$$(4.10)\qquad \frac{\partial \bar{g}^{\beta\alpha^*}_{hj^*}}{\partial \bar{z}^{l^*}}\, \dot{\bar{z}}^{j^*}_{\alpha^*} = \left\{ \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial z^{t^*}}\, B^{t^*}_{l^*}\, \dot{z}^{m^*}_{\alpha^*} + g^{\beta\alpha^*}_{km^*}\, B^{m^*}_{j^* l^*}\, \dot{\bar{z}}^{j^*}_{\alpha^*} \right\} B^k_h \,.$$

If, in addition, this is multiplied by $\dot{\bar{z}}^{l^*}_{\gamma^*}$, we find

$$(4.11)\qquad \frac{\partial \bar{g}^{\beta\alpha^*}_{hj^*}}{\partial \bar{z}^{l^*}}\, \dot{\bar{z}}^{j^*}_{\alpha^*}\, \dot{\bar{z}}^{l^*}_{\gamma^*} = B^k_h \left\{ \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial z^{t^*}}\, \dot{z}^{t^*}_{\gamma^*}\, \dot{z}^{m^*}_{\alpha^*} + g^{\beta\alpha^*}_{km^*}\, B^{m^*}_{j^* l^*}\, \dot{\bar{z}}^{j^*}_{\alpha^*}\, \dot{\bar{z}}^{l^*}_{\gamma^*} \right\} .$$

Now multiplying (4.9) by $\dot{\bar{z}}^h_\beta$ and using (4.1), (3.14) we get

$$(4.12)\qquad \frac{\partial \bar{g}^{\beta\alpha^*}_{hj^*}}{\partial \bar{z}^{l^*}}\, \dot{\bar{z}}^h_\beta = \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial z^{t^*}}\, B^{t^*}_{l^*}\, B^{m^*}_{j^*}\, \dot{z}^k_\beta + g^{\beta^*\alpha^*}_{t^*m^*}\, B^{t^*}_{r^*l^*}\, \dot{\bar{z}}^{r^*}_{\beta^*}\, B^{m^*}_{j^*}$$

$$+ g^{\beta\alpha^*}_{km^*}\, \dot{z}^k_\beta\, B^{m^*}_{j^*l^*}\,.$$

Again differentiating (4.8) with respect to $\bar{z}^l$ we get

$$(4.13)\qquad \frac{\partial \bar{g}^{\beta\alpha^*}_{hj^*}}{\partial \bar{z}^l} = \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial z^t}\, B^t_l\, B^k_h\, B^{m^*}_{j^*} + \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial \dot{z}^t_\lambda}\, B^t_{rl}\, \dot{\bar{z}}^r_\lambda\, B^k_h\, B^{m^*}_{j^*}$$

$$+ g^{\beta\alpha^*}_{km^*}\, B^k_{hl}\, B^{m^*}_{j^*}\,.$$

Multiplication of (4.13) by $\dot{\bar{z}}^{j^*}_{\alpha^*}$ yields (in view of (4.1) and (3.13)),

$$(4.14)\qquad \frac{\partial \bar{g}^{\beta\alpha^*}_{hj^*}}{\partial \bar{z}^l}\, \dot{\bar{z}}^{j^*}_{\alpha^*} = \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial z^t}\, B^t_l\, B^k_h\, \dot{z}^{m^*}_{\alpha^*} + g^{\beta\lambda}_{kt}\, B^t_{rl}\, \dot{\bar{z}}^r_\lambda\, B^k_h$$

$$+ g^{\beta\alpha^*}_{km^*}\, \dot{z}^{m^*}_{\alpha^*}\, B^k_{hl}\,.$$

Further multiplication of (4.14) by $\dot{\bar{z}}^l_\gamma$ gives

$$(4.15)\qquad \frac{\partial \bar{g}^{\beta\alpha^*}_{hj^*}}{\partial \bar{z}^l}\, \dot{\bar{z}}^{j^*}_{\alpha^*}\, \dot{\bar{z}}^l_\gamma = \left\{ \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial z^t}\, \dot{z}^t_\gamma\, \dot{z}^{m^*}_{\alpha^*} + g^{\beta\lambda}_{kt}\, B^t_{rl}\, \dot{\bar{z}}^r_\lambda\, \dot{\bar{z}}^l_\gamma \right\} B^k_h$$

$$+ g^{\beta\alpha^*}_{km^*}\, \dot{z}^{m^*}_{\alpha^*}\, B^k_{hl}\, \dot{\bar{z}}^l_\lambda\,.$$

Now let us interchange l and h in (4.13) and subtract the result from (4.13). We thus obtain,

$$(4.16)\qquad \frac{\partial \bar{g}^{\beta\alpha^*}_{hj^*}}{\partial \bar{z}^l} - \frac{\partial \bar{g}^{\beta\alpha^*}_{lj^*}}{\partial \bar{z}^h} = \left\{ \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial z^t} - \frac{\partial g^{\beta\alpha^*}_{tm^*}}{\partial z^k} \right\} B^{m^*}_{j^*}\, B^t_l\, B^k_h$$

$$+ \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial \dot{z}^t_\lambda} \left\{ B^t_{rl}\, B^k_h - B^t_{rh}\, B^k_l \right\} \dot{\bar{z}}^r_\lambda\, B^{m^*}_{j^*}\,.$$

Transvecting (4.16) by $\dot{\bar{z}}^{j^*}_{\alpha^*}$ and using (4.1) and (3.13) once more we find,

$$(4.17)\qquad \left\{ \frac{\partial \bar{g}^{\beta\alpha^*}_{hj^*}}{\partial \bar{z}^l} - \frac{\partial \bar{g}^{\beta\alpha^*}_{lj^*}}{\partial \bar{z}^h} \right\} \dot{\bar{z}}^{j^*}_{\alpha^*} = \left\{ \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial z^t} - \frac{\partial g^{\beta\alpha^*}_{tm^*}}{\partial z^k} \right\} \dot{z}^{m^*}_{\alpha^*}\, B^t_l\, B^k_h$$

$$+ g^{\beta\lambda}_{kt} \left\{ B^t_{rl}\, B^k_h - B^t_{rh}\, B^k_l \right\} \dot{\bar{z}}^r_\lambda$$

while further multiplication by $\dot{\bar{z}}^l_\beta$ gives in view of (3.15),

$$(4.18)\qquad \left\{ \frac{\partial \bar{g}^{\beta\alpha^*}_{hj^*}}{\partial \bar{z}^l} - \frac{\partial \bar{g}^{\beta\alpha^*}_{lj^*}}{\partial \bar{z}^h} \right\} \dot{\bar{z}}^{j^*}_{\alpha^*}\, \dot{\bar{z}}^l_\beta = B^k_h \left\{ \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial z^t} - \frac{\partial g^{\beta\alpha^*}_{tm^*}}{\partial z^k} \right\} \dot{z}^{m^*}_{\alpha^*}\, \dot{z}^t_\beta$$

$$+ g^{\beta\lambda}_{kt}\, B^t_{rl}\, \dot{\bar{z}}^r_\lambda\, \dot{\bar{z}}^l_\beta\,.$$

The relations (4.11) and (4.18) provide some indication as to how the covariant derivatives can be forme d. In connection of first of these we have from (4.7),

$$(4.19) \qquad g^{\beta\alpha^*}_{km^*}\, \dot{z}^{m^*}_{\alpha^*\gamma^*}\, B^k_h = \left\{ g^{\beta\alpha^*}_{km^*}\, B^{m^*}_{\cdot *}\, \dot{\bar{z}}^{t^*}_{\alpha^*\gamma^*} + g^{\beta\alpha^*}_{km^*}\, B^{m^*}_{j^*t^*}\, \dot{\bar{z}}^{j^*}_{\alpha^*}\, \dot{\bar{z}}^{t^*}_{\gamma^*} \right\} B^k_h .$$

To eliminate $B^{m^*}_{j^*t^*}$ from (4.19) and (4.11) we subtract the latter from former. Thus we get with the use of (4.8),

$$(4.20) \qquad B^k_h \left\{ g^{\beta\alpha^*}_{km^*}\, \dot{z}^{m^*}_{\alpha^*\gamma^*} + \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial z^{t^*}}\, \dot{z}^{m^*}_{\alpha^*}\, \dot{z}^{t^*}_{\gamma^*} \right\}$$

$$= \bar{g}^{\beta\alpha^*}_{hl^*}\, \dot{\bar{z}}^{l^*}_{\alpha^*\gamma^*} + \frac{\partial \bar{g}^{\beta\alpha^*}_{hj^*}}{\partial \bar{z}^{l^*}}\, \dot{\bar{z}}^{j^*}_{\alpha^*}\, \dot{\bar{z}}^{l^*}_{\gamma^*} .$$

This relation shows that the quantities defined by

$$(4.21) \qquad Z^{\beta}_{k,\cdot,*} = g^{\beta\alpha^*}_{km^*}\, \dot{z}^{m^*}_{\alpha^*\gamma^*} + \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial z^{t^*}}\, \dot{z}^{m^*}_{\alpha^*}\, \dot{z}^{t^*}_{\gamma^*}$$

are the components of a covariant vector.

Similarly it can be verified that the quantities defined by

$$(4.22) \qquad Z^{\beta^*}_{k^*,\gamma} = g^{\beta^*\alpha}_{k^*m}\, \dot{z}^{m}_{\alpha\gamma} + \frac{\partial g^{\beta^*\alpha}_{k^*m}}{\partial z^{t}}\, \dot{z}^{m}_{\alpha}\, \dot{z}^{t}_{\gamma}$$

satisfies the transformation law

$$(4.23) \qquad \bar{Z}^{\beta^*}_{k^*,\gamma} = B^{l^*}_{k^*}\, Z^{\beta^*}_{l^*,\gamma}$$

so that (4.21) is to be taken in conjugation with (4.22).

These are the required relations resulting from (4.11).

On the other hand from (4.6) and (4.8) we have

$$(4.24) \qquad g^{\beta\alpha}_{km}\, B^m_{jl}\, \dot{\bar{z}}^j_{\alpha}\, \dot{\bar{z}}^l_{\beta}\, B^k_h = g^{\beta\alpha}_{km}\, \dot{z}^m_{\alpha\beta}\, B^k_h - \bar{g}^{\beta\alpha}_{hl}\, \dot{\bar{z}}^l_{\alpha\beta} .$$

This is substituted in the right hand side of (4.18).

In this manner we find that

$$(4.25) \qquad \bar{X}_h = B^k_h\, X_k$$

where

$$(4.26) \qquad X_k = g^{\beta\alpha}_{km}\, \dot{z}^m_{\alpha\beta} + \left\{ \frac{\partial g^{\beta\alpha^*}_{km^*}}{\partial z^t} - \frac{\partial g^{\beta\alpha^*}_{tm^*}}{\partial z^k} \right\} \dot{z}^{m^*}_{\alpha^*}\, \dot{z}^t_{\beta} .$$

It follows that the quantities thus defined also form the components of a covariant vector.

Similarly we can easily verify that the quantities defined by

$$(4.27) \qquad X_{k^*} = g^{\beta^*\alpha^*}_{k^*m^*}\, \dot{z}^{m^*}_{\alpha^*\beta^*} + \left\{ \frac{\partial g^{\beta^*\alpha}_{k^*m}}{\partial z^{t^*}} - \frac{\partial g^{\beta^*\alpha}_{t^*m}}{\partial z^{k^*}} \right\} \dot{z}^m_{\alpha}\, \dot{z}^{t^*}_{\beta^*}$$

satisfy the transformation law

$$\bar{X}_{h^*} = B_{h^*}^{k^*} X_{k^*} . \tag{4.28}$$

In the above we have found a way of constructing covariant vectors in terms of derivatives of $\dot{z}_\alpha^j$ and $\dot{z}_{\alpha^*}^{j^*}$ which indicates that there exist certain connection coefficients which are in the construction of these vectors. In order to obtain the explicit form of these coefficients we suppose that there exist quantities $g_{\gamma\alpha^*}^{kj^*}$ such that

$$g_{hj^*}^{\beta\alpha^*} \, g_{\gamma\alpha^*}^{kj^*} = \delta_h^k \, \delta_\gamma^\beta . \tag{4.29}$$

The existence of (4.29) follows from the condition (D) of section 2. From (4.29) it also follows that

$$g_{h^*j}^{\beta^*\alpha} \, g_{\gamma^*\alpha}^{k^*j} = \delta_{h^*}^{k^*} \, \delta_{\gamma^*}^{\beta^*} . \tag{4.30}$$

Also if in analogy to (4.2) we write

$$A_h^j = \frac{\partial \bar{z}^j}{\partial z^h} \quad , \quad A_{h^*}^{j^*} = \frac{\partial \bar{z}^{j^*}}{\partial z^{h^*}} \tag{4.31}$$

and thereby obtain

$$\bar{g}_{\gamma\alpha^*}^{tp^*} \, A_{p^*}^{j^*} = \bar{g}_{\gamma\alpha^*}^{sj^*} \, B_s^t . \tag{4.32}$$

Multiplying (4.13) by $\dot{\bar{z}}_\beta^h$, noting (4.1 and (3.9)we get

$$\frac{\partial \bar{g}_{hj^*}^{\beta\alpha^*}}{\partial \bar{z}^l} \, \dot{\bar{z}}_\beta^h = \frac{\partial g_{km^*}^{\beta\alpha^*}}{\partial z^t} \, \dot{z}_\beta^k \, B_{j^*}^{m^*} \, B_l^t + g_{km^*}^{\beta\alpha^*} \, B_{hl}^k \, B_{j^*}^{m^*} \, \dot{\bar{z}}_\beta^h . \tag{4.33}$$

Transvecting this relation by $g_{\gamma\alpha^*}^{tp^*} \, A_{p^*}^{j^*}$ and using (4.32) we get after some simplification

$$B_{hl}^t \, \dot{\bar{z}}_\gamma^h = \bar{g}_{\gamma\alpha^*}^{sj^*} \, \frac{\partial \bar{g}_{hj^*}^{\beta\alpha^*}}{\partial \bar{z}^l} \, \dot{\bar{z}}_\beta^h \, B_s^t - g_{\gamma\alpha^*}^{tm^*} \, \frac{\partial g_{km^*}^{\beta\alpha^*}}{\partial z^p} \, \dot{z}_\beta^k \, B_l^p . \tag{4.34}$$

This relation shows that

$$B_{hl}^t \, \dot{\bar{z}}_\gamma^h = B_s^t \, \bar{G}_{l,\gamma}^s - G_{p,\gamma}^t \, B_l^p \tag{4.35}$$

where

$$G_{p,\gamma}^t = g_{\gamma\alpha^*}^{tm^*} \, \frac{\partial g_{km^*}^{\beta\alpha^*}}{\partial z^p} \, \dot{z}_\beta^k \tag{4.36}$$

and a corresponding relation on $\bar{G}_{p,\gamma}^t$ in the barred system. From (4.36) and (4.30) we can deduce that

$$g_{k^*m}^{\beta^*\alpha} \, G_{p,\alpha}^m = \frac{\partial g_{k^*m}^{\beta^*\alpha}}{\partial z^p} \, \dot{z}_\alpha^m \tag{4.37}$$

so that the covariant vector (4.22) can be expressed in the form

(4.38) $$Z^{\beta *}_{k *, \gamma} = g^{\beta * \alpha}_{k * m} \left(\dot{z}^{m}_{\alpha \gamma} + G^{m}_{p . \alpha} \, \dot{z}^{p}_{\gamma} \right) .$$

Now let us put

(4.39) $$G^{t \beta}_{kh , \gamma} = \frac{\partial G^{t}_{k . \gamma}}{\partial \dot{z}^{h}_{\beta}} .$$

From (4.1) we can deduce that

(4.40) $$\frac{\partial \dot{z}^{k}_{\gamma}}{\partial \dot{z}^{h}_{\beta}} = \delta^{k}_{h} \, \delta^{\beta}_{\gamma} .$$

Differentiating (4.35) with respect to $\dot{z}^{k}_{\beta}$, using (4.39) and (4.40) we get

(4.41) $$B^{t}_{kl} \, \delta^{\beta}_{\gamma} = B^{t}_{s} \, \bar{G}^{s \beta}_{lk . \gamma} - G^{t \beta}_{ph . \gamma} \, B^{h}_{k} \, B^{p}_{l} .$$

In particular if we put $\beta = \gamma$ in (4.41) we get

(4.42) $$B^{t}_{kl} = B^{t}_{s} \, \bar{\Gamma}^{s}_{lk} - \Gamma^{t}_{ph} \, B^{p}_{l} \, B^{h}_{k}$$

where

(4.43) $$\Gamma^{t}_{ph} = \frac{1}{m} \, G^{t \beta}_{ph , \beta} .$$

The relation (4.42) shows that Γ^{t}_{ph} obey the transformation law of the connection coefficients. Accordingly we shall regard the quantities (4.43) as the connection coefficients of our manifold C_n.

Similarly we can show that the quantities defined by

(4.44) $$\Gamma^{t *}_{p * h *} = \frac{1}{m} \, G^{t * \beta *}_{p * h * , \beta *}$$

where

(4.45) $$G^{t * \beta *}_{p * h * , \gamma *} = \frac{\partial G^{t *}_{p * , \gamma *}}{\partial \dot{z}^{h *}_{\beta *}}$$

and

(4.46) $$G^{t *}_{p * , \gamma *} = g^{t * m}_{\gamma * \alpha} \, \frac{\partial g^{\beta * \alpha}_{k * m}}{\partial z^{p *}} \, \dot{z}^{k *}_{\beta *}$$

satisfy the transformation law

(4.47) $$B^{t *}_{k * l *} = B^{t *}_{s *} \, \bar{\Gamma}^{s *}_{l * k *} - \Gamma^{t *}_{p * h *} \, B^{p *}_{l *} \, B^{h *}_{k *} .$$

From (4.36), (4.39) and (4.43) it follows that

(4.48) $$m \, \Gamma^{t}_{ph} = \frac{\partial g^{tm *}_{\beta \alpha *}}{\partial \dot{z}^{h}_{\beta}} \, \frac{\partial g^{\gamma \alpha *}_{km *}}{\partial z^{p}} \, \dot{z}^{k}_{\gamma} + g^{tm *}_{\beta \alpha *} \, \frac{\partial g^{\beta \alpha *}_{hm *}}{\partial z^{p}}$$

which shows that the connection coefficient Γ^{l}_{ph} is explicitly derivable from the metric tensor and its various derivatives. Similarly $\Gamma^{l^*}_{p^*h^*}$ is derivable from the metric tensor and its derivatives.

5. - Partial covariant derivatives

We shall now use the connection coefficients defined by (4.43) and (4.44) to construct partial covariant derivatives of a given vector field $V^{j}_{\in}(z^{l}, z^{l^*}, \dot{z}^{l}_{\lambda}, \dot{z}^{l^*}_{\lambda^*})$ which is such that

$$V^{j}_{\in}(z^{l}, z^{l^*}, \dot{z}^{l}_{\lambda}, \dot{z}^{l^*}_{\lambda^*}) = B^{j}_{h}\, \overline{V}^{h}_{\in}(\overline{z}^{l}, \overline{z}^{l^*}\, \dot{\overline{z}}^{l}_{\lambda}, \dot{\overline{z}}^{l^*}_{\lambda^*})\,. \tag{5.1}$$

Differentiation of this with respect to $\dot{\overline{z}}^{l}_{\lambda}$, and $\dot{z}^{l^*}_{\lambda^*}$ respectively gives

$$\frac{\partial V^{j}_{\in}}{\partial \dot{z}^{l}_{\lambda}} = B^{j}_{h} \frac{\partial \overline{V}^{h}_{\in}}{\partial \dot{\overline{z}}^{l}_{\lambda}} A^{l}_{i}\,, \tag{5.2}$$

$$\frac{\partial V^{j}_{\in}}{\partial \dot{z}^{l^*}_{\lambda^*}} = B^{j}_{h} \frac{\partial \overline{V}^{h}_{\in}}{\partial \dot{\overline{z}}^{l^*}_{\lambda^*}} A^{l^*}_{i^*}\,. \tag{5.3}$$

From (4.1) and (4.31) it also follow that

$$\frac{\partial \dot{\overline{z}}^{l}_{\lambda}}{\partial z^{j}} = A^{l}_{kj}\, \dot{z}^{k}_{\lambda} \quad , \quad \frac{\partial \dot{\overline{z}}^{l^*}_{\lambda^*}}{\partial z^{j^*}} = A^{l^*}_{k^*j^*}\, \dot{z}^{k^*}_{\lambda^*} \tag{5.4}$$

where

$$A^{l}_{kj} = \frac{\partial^2 \overline{z}^{l}}{\partial z^{k}\, \partial z^{j}} \quad , \quad A^{l^*}_{k^*j^*} = \frac{\partial^2 \overline{z}^{l^*}}{\partial z^{k^*}\, \partial z^{j^*}} \tag{5.5}$$

and since (4.31) is the inverse of (4.2) ie.

$$B^{j}_{h}\, A^{h}_{i} = \delta^{j}_{i} \quad , \quad B^{j^*}_{h^*}\, A^{h^*}_{i^*} = \delta^{j^*}_{i^*} \tag{5.6}$$

we have

$$A^{l}_{kj} = -\, A^{l}_{m}\, B^{m}_{pq}\, A^{p}_{k}\, A^{q}_{j} \quad ; \quad A^{l^*}_{k^*j^*} = -\, A^{l^*}_{m^*}\, B^{m^*}_{p^*q^*}\, A^{p^*}_{k^*}\, A^{q^*}_{j^*}\,. \tag{5.7}$$

Substituting (5.7) in (5.4) it follows that

$$\frac{\partial \dot{\overline{z}}^{l}_{\lambda}}{\partial z^{j}} = -\, B^{m}_{pq}\, \dot{\overline{z}}^{p}_{\lambda}\, A^{q}_{j}\, A^{l}_{m} \quad ; \quad \frac{\partial \dot{\overline{z}}^{l^*}_{\lambda^*}}{\partial z^{j^*}} = -\, B^{m^*}_{p^*q^*}\, \dot{\overline{z}}^{p^*}_{\lambda^*}\, A^{q^*}_{j^*}\, A^{l^*}_{m^*}\,. \tag{5.8}$$

Now differentiating (5.1) with respect to z^{k}, we obtain

$$\frac{\partial V^{j}_{\in}}{\partial z^{k}} = B^{j}_{ih}\, A^{i}_{k}\, \overline{V}^{h}_{\in} + B^{j}_{h} \left\{ \frac{\partial \overline{V}^{h}_{\in}}{\partial \overline{z}^{i}} A^{i}_{k} + \frac{\partial \overline{V}^{h}_{\in}}{\partial \dot{\overline{z}}^{l}_{\lambda}} \frac{\partial \dot{\overline{z}}^{l}_{\lambda}}{\partial z^{k}} \right\}. \tag{5.9}$$

In the last term of the right hand side of this equation we substitute the value of $\partial \dot{z}^l_\lambda / \partial z^k$ from (5.8) and then B^m_{pq} are eliminated with the help of (4.42). This process gives the following expression

$$- B^j_h \frac{\partial \overline{V}^h_\in}{\partial \dot{\overline{z}}^l_\lambda} A^l_m A^q_k (B^m_t \overline{\Gamma}^t_{pq} - \Gamma^m_{ri} B^r_p B^i_q) \dot{z}^p_\lambda$$

and by means of (5.2) we can easily reduce it to

$$- B^j_h A^q_k \frac{\partial \overline{V}^h_\in}{\partial \dot{\overline{z}}^t_\lambda} \overline{\Gamma}^t_{pq} \dot{\overline{z}}^p_\lambda + \frac{\partial V^j_\in}{\partial \dot{z}^t_\lambda} \Gamma^t_{rk} \dot{z}^r_\lambda .$$

Thus (5.9) can be written in the form

$$\frac{\partial V^j_\in}{\partial z^k} - \frac{\partial V^j_\in}{\partial \dot{z}^l_\lambda} \Gamma^l_{ik} \dot{z}^i_\lambda = B^j_h A^l_k \overline{V}^h_\in \tag{5.10}$$

$$+ B^j_h A^q_k \left\{ \frac{\partial \overline{V}^h_\in}{\partial \overline{z}^q} - \frac{\partial \overline{V}^h_\in}{\partial \dot{\overline{z}}^l_\lambda} \overline{\Gamma}^l_{pq} \dot{\overline{z}}^p_\lambda \right\} .$$

By means of (4.42) the first term on the right hand side of this relation can be expressed as

$$A^h_k B^j_i \overline{\Gamma}^i_{hl} \overline{V}^l_\in - \Gamma^j_{kp} V^p_\in .$$

Substituting this in (5.10) we get,

$$V^j_{\in|k} = B^j_h A^q_k \overline{V}^h_{\in|q} \tag{5.11}$$

where

$$V^j_{\in|k} = \frac{\partial V^j_\in}{\partial z^k} - \frac{\partial V^j_\in}{\partial \dot{z}^l_\lambda} \Gamma^l_{mk} \dot{z}^m_\lambda + \Gamma^j_{kl} V^l_\in . \tag{5.12}$$

The relation (5.11) shows that the quantities $V^j_{\in|k}$ defined by (5.12) represent components of a mixed tensor. Thus $V^j_{\in|k}$ represents a covariant partial derivative of the quantities $V^j_\in (z^i, z^{i^*}, \dot{z}^i_\lambda, \dot{z}^{i^*}_{\lambda^*})$ with respect to z^k .

Now we shall find the covariant partial derivative of $V^j_\in$ with respect to z^{k^*}. For this purpose let us differentiate (5.1) with respect to z^{k^*}. Thus

$$\frac{\partial V^j_\in}{\partial z^{k^*}} = B^j_h \left\{ \frac{\partial \overline{V}^h_\in}{\partial \overline{z}^{i^*}} A^{i^*}_{k^*} + \frac{\partial \overline{V}^h_\in}{\partial \dot{\overline{z}}^{i^*}_{\lambda^*}} \frac{\partial \dot{\overline{z}}^{i^*}_{\lambda^*}}{\partial z^{k^*}} \right\} \tag{5.13}$$

Substituting the value of $\partial \dot{z}^{l^*}_{\lambda^*} / \partial z^{k^*}$ from (5.8) and then eliminating $B^{m^*}_{p^* q^*}$ with the help of (4.47) we get (after using (5.3))

$$\frac{\partial V^j_\in}{\partial z^{k^*}} - \frac{\partial V^j_\in}{\partial \dot{z}^{l^*}_{\lambda^*}} \Gamma^{l^*}_{p^* k^*} \dot{z}^{p^*}_{\lambda^*} \tag{5.14}$$

$$= B^j_h A^{q^*}_{k^*} \left\{ \frac{\partial \overline{V}^h_\in}{\partial \overline{z}^{q^*}} - \frac{\partial \overline{V}^h_\in}{\partial \dot{\overline{z}}^{l^*}_{\lambda^*}} \overline{\Gamma}^{l^*}_{p^* q^*} \dot{\overline{z}}^{p^*}_{\lambda^*} \right\} .$$

This relation shows that the quantities defined by

$$V^{j}_{\in|k^*} = \frac{\partial V^{j}_{\in}}{\partial z^{k^*}} - \frac{\partial V^{j}_{\in}}{\partial \dot{z}^{l^*}_{\lambda^*}} \Gamma^{l^*}_{p^*k^*} \dot{z}^{p^*}_{\lambda^*} \tag{5.15}$$

represent a component of a mixed tensor. Thus $V^{j}_{\in|k^*}$ represent a covariant partial derivative of $V^{j}_{\in}$ with respect to z^{k^*}.

In the similar manner we can construct the covariant partial derivative of a vector field $V^{j^*}_{\in^*}(z^l, z^{l^*}, \dot{z}^l_\lambda, \dot{z}^{l^*}_{\lambda^*})$ with respect to z^k and z^{k^*} respectively. These are given by

$$V^{j^*}_{\in^*|k} = \frac{\partial V^{j^*}_{\in^*}}{\partial z^{k}} - \frac{\partial V^{j^*}_{\in^*}}{\partial \dot{z}^{l}_{\lambda}} \Gamma^{l}_{pk} \dot{z}^{p}_{\lambda}, \tag{5.16}$$

$$V^{j^*}_{\in^*|k^*} = \frac{\partial V^{j^*}_{\in^*}}{\partial z^{k^*}} - \frac{\partial V^{j^*}_{\in^*}}{\partial \dot{z}^{l^*}_{\lambda^*}} \Gamma^{l^*}_{p^*k^*} \dot{z}^{p^*}_{\lambda^*} + \Gamma^{j^*}_{k^*l^*} V^{l^*}_{\in^*}. \tag{5.17}$$

The existence of connection coefficients and covariant partial derivatives of a vector field suggests the existence of curvature tensors and a corresponding theory of curvature. We hope to deal the brief discussion of these concepts in a separete pubblication.

The author is tankful to Dr. U.P. Singh for his valuable suggestions in preparation of this paper.

REFERENCES

[1] K. YANO: Differential geometry on complex and almost complex spaces. Oxford (1965).

[2] H. RUND: Generalized metrics on complex manifolds Math. Nachr. 34, (1967) 55-77.

[3] H. RUND: The Hamilton-Jacobi theory in the calculus of variations: London and New York (1966).

[4] H. RUND: A geometrical theory of multiple integral problems in the calculus of variations. Con. J. Math. 20, (1968), 639-657.

[5] H. RUND: Calculus of variations on complex manifolds. Third colloquium on the calculus of variations. University of South Africa. Pretoria (1965), 10-85.