S. N. PANDEY (*)

Wave solutions of the Einstein-Maxwell equations in a generalized Peres space-time (**)

1. Introduction

The Einstein-Maxwell equations in the absence of source are

$$R^i_k - (1/2)\, R\, \delta^i_k + F^{ij} F_{jk} + {}^*F^{ij}\, {}^*F_{jk} = 0 \,, \tag{1.1}$$

$$\text{a)}\quad F^{ij}{}_{;j} = 0 \qquad \text{and} \qquad \text{b)}\quad {}^*F^{ij}{}_{;j} = 0 \,, \tag{1.2}$$

where R^i_k is the Ricci tensor, $F_{ij} = -F_{ji}$ the electromagnetic field tensor and ${}^*F_{ij} = (1/2)\,(-g)^{1/2}\, \epsilon_{ijkl}\, F^{kl}$ its dual. The covariant differentiation with respect to the metric tensor g_{ij} is denoted by a semi-colon.

Lal and Pandey [1] have considered a generalized Peres metric [2] in view of the works done by Patel and Vaidya [3], Lal and Ali [4] and Takeno [5], given by

$$ds^2 = -A dx^2 - B dy^2 - (1 - E)\, dz^2 - 2\, E dz dt + (1 + E)\, dt^2 \tag{1.3}$$

where A and B are any functions of z, t and E is any function of x, y, z, t; and have also obtained the wave-like solutions of the field equations of general relativity. In this paper we propose to obtain the wave-like solutions (in the sense of Takeno [5]) of Einstein-Maxwell equations (1.1) and (1.2) in the generalized Peres space-time (1.3), which is non-flat in general.

The values of g^{ij}, R_{ij} etc. have been calulated for the metric (1.3) in [1], however it has been preferred to give below the values of g^{ij}, R_{ij} and R only as:

$$g^{11} = -1/A,\ g^{22} = -1/B,\ g^{33} = -(1 + E),\ g^{34} = -E = g^{43},\ g^{44} = (1 - E), \tag{1.4}$$
$$\text{and other } g^{ij} = 0 \,,$$

(*) Department of Applied Sciences M.M.M. Engineering College, Gorakhpur, India.

(**) Memoria presentata dall'Accademico dei XL Enrico Bompiani il 21-1-1974.

$$
(1.5)\quad \left\{
\begin{aligned}
R_{11} &= (A_{33} - A_{44})/2 + E\,(A_{33} + 2\,A_{34} + A_{44})/2 - (A^2_3 - A^2_4)/4\,A + \\
&\qquad + W\,(A_3 + A_4)/2 + (A_3\,B_3 - A_4\,B_4)/4\,B - E\,(A_3 + A_4)\,P/4\ , \\
R_{13} &= -\,R_{14} = (1/2)\,(E_{13} + E_{14}) - (1/4)\,E_1\,P\ , \\
R_{22} &= (B_{33} - B_{44})/2 + E\,(B_{33} + 2\,B_{34} + B_{44})/2 - (B^2_3 - B^2_4)/4\,B + \\
&\qquad + W\,(B_3 + B_4)/2 + (A_3\,B_3 - A_4\,B_4)/4\,A + E\,(B_3 + B_4)\,P/4\ , \\
R_{23} &= -\,R_{24} = (1/2)\,(E_{23} + E_{24}) + (1/4)\,E_2\,P\ , \\
R_{33} &= A_{33}/2\,A + B_{33}/2\,B - U + V/2 - (A^2_3/A^2 + B^2_3/B^2)/4 + E_3\,S + (W + E_3)\,T, \\
R_{34} &= A_{34}/2\,A + B_{34}/2\,B + U - (A_3 A_4/A^2 + B_3\,B_4/B^2)/4 - E_3\,T + E_4\,S \\
R_{44} &= A_{44}/2\,A + B_{44}/2\,B - U - V/2 - (A^2_4/A^2 + B^2_4/B^2)/4 - (W + E_4)\,S - E_4\,T \\
&\text{and other } R_{ij} = 0\ ,
\end{aligned}
\right.
$$

$$
\begin{aligned}
(1.6)\quad R &= (A^2_3 - A^2_4)/2\,A^2 + (B^2_3 - B^2_4)/2\,B^2 - E\,(A_3 + A_4)\,(B_3 + B_4)/2\,A\,B - \\
&- (A_3\,B_3 - A_4\,B_4)/2AB - WQ - V - (A_{33} - A_{44})/A - (B_{33} - B_{44})/B - \\
&\qquad - E\,\{(A_{33} + 2A_{34} + A_{44})/A + (B_{33} + 2B_{34} + B_{44})/B\}\ ,
\end{aligned}
$$

where $P = (A_3 + A_4)/A - (B_3 + B_4)/B$, $Q = (A_3 + A_4)/A + (B_3 + B_4)/B$, $V = E_{33} + 2E_{34} + E_{44}$, $S = A_3/4A + B_3/4B$, $T = A_4/4A + B_4/4B$, $W = E_3 + E_4$, $U = (E_{11}/A + E_{22}/B)/2 + EV/2 - EWQ/4$ and the suffixes 1, 2, 3, 4 denote partial derivatives of A, B, E with respect to x, y, z, t respectively.

The values of $R^i{}_k = g^{ij}\,R_{kj}$ can easily be calculated with the help of (1.4) and (1.5).

The null electromagnetic field F_{ij} (in the sense of Synge [6]) and its contravariant components F^{ij} considered in [1] are

$$
(1.7)\quad \text{a)}\ F_{ij} = \left|\begin{matrix} 0 & 0 & -\sigma & \sigma \\ 0 & 0 & \rho & -\rho \\ \sigma & -\rho & 0 & 0 \\ -\sigma & \rho & 0 & 0 \end{matrix}\right| \quad \text{and} \quad \text{b)}\ F^{ij} = \left|\begin{matrix} 0 & 0 & -\sigma/A & -\sigma/A \\ 0 & 0 & \rho/B & \rho/B \\ \sigma/A & -\rho/B & 0 & 0 \\ \sigma/A & -\rho/B & 0 & 0 \end{matrix}\right|
$$

and their dual tensors are

$$
(1.8)\quad \text{a)}\ {}^*F_{ij} = \left|\begin{matrix} 0 & 0 & -\rho\sqrt{A/B} & \rho\sqrt{A/B} \\ 0 & 0 & -\sigma\sqrt{B/A} & \sigma\sqrt{B/A} \\ \rho\sqrt{A/B} & \sigma\sqrt{B/A} & 0 & 0 \\ -\rho\sqrt{A/B} & -\sigma\sqrt{B/A} & 0 & 0 \end{matrix}\right|
$$

and b) $$* F^{ij} = \begin{vmatrix} 0 & 0 & -\rho/\sqrt{AB} & -\rho/\sqrt{AB} \\ 0 & 0 & -\sigma/\sqrt{AB} & -\sigma/\sqrt{AB} \\ \rho/\sqrt{AB} & \sigma/\sqrt{AB} & 0 & 0 \\ \rho/\sqrt{AB} & \sigma/\sqrt{AB} & 0 & 0 \end{vmatrix}$$

where ρ and σ are functions of x, y, $z - t$ satisfying

(1.9) $$\partial_1 \rho + \partial_2 \sigma = 0\,, \qquad (\partial_1 = \partial/\partial x \text{ etc.}).$$

2. Solution of equation (1.1) and (1.2)

On substituing the relevant quantities in (1.2) a) we find

(2.1) $$\text{a)}\quad P = 0 \qquad \text{and} \qquad \text{b)}\quad A\,\partial_2 \rho - B\,\partial_1 \sigma = 0$$

for $i = 1, 2$ and $i = 3, 4$ respectively, and (1.2) b) is identically satisfied with the aid of (1.9).

Again the non-vanishing components of $F^{ij} F_{jk}$ and $* F^{ij} * F_{jk}$ are

(2.2) $$\begin{cases} F^{3j} F_{j4} = - F^{3j} F_{j3} = - F^{4j} F_{j3} = F^{4j} F_{j4} = \lambda \\ * F^{3j} * F_{j4} = - * F^{3j} * F_{j3} = - * F^{4j} * F_{j3} = * F^{4j} * F_{j4} = \lambda \end{cases}$$

where $\lambda = \rho^2/B + \sigma^2/A$. Since δ^i_j are defined as

(2.3) $$\delta^i_j = 1 \quad \text{if} \quad i = j\,, \qquad \delta^i_j = 0 \quad \text{if} \quad i \neq j$$

the equation (1.1), with the help of (2.2), reduces to

(2.4) $$\begin{cases} \text{a)} & R_{11}/A = (1/2)\,(R_{33} - R_{44}) = R_{22}/B = - R/2 \\ \text{b)} & R_{33} + 2\,R_{34} + R_{44} = 0 \\ \text{c)} & ER/2 - R_{34} + \mu = 0\,, \qquad (\mu = 2\,\lambda) \\ \text{d)} & (E_{13} + E_{14})/E_1 = - (E_{23} + E_{24})/E_2 = P/2 \end{cases}$$

Now we shall find the solutions of (2.4) using (2.1) a).

Case I. Since $P = 0$, let the quantities on either side vanish seperately i.e.,

$$(A_3 + A_4)/A = (B_3 + B_4)/B = 0\,,$$

which gives on integration

(2.5) $$\text{a)}\quad A = A\,(z - t) \qquad \text{and} \qquad \text{b)}\quad B = B\,(z - t)\,.$$

This helps the simplification of (2.4) to

$$(2.6)\quad \begin{cases} \text{a)}\ V = 0 \\ \text{b)}\ (E_{11}/A + E_{22}/B)/2 - (\bar{A}^2/A^2 + \bar{B}^2/B^2)/4 + W(\bar{A}/4A + \bar{B}/4B) - \\ \qquad\qquad - (\bar{\bar{A}}/A + \bar{\bar{B}}/B) = 2(\rho^2/B + \sigma^2/A) \\ \text{c)}\ (E_{13} + E_{14})/E_1 = -(E_{23} + E_{24})/E_2 = 0\ ,\ ({}^{\prime}\!-\!{}^{\prime} \equiv \partial/\partial(z-t))\ . \end{cases}$$

Integrating (2.6) a) and (2.6) c) we find that

$$(2.7)\qquad E = f(x, y, z-t) + zg(z-t)\ .$$

Thus, we have :

A necessary and sufficient condition that g_{ij} and F_{ij} constitute a solution of (1.1) and (1.2) is that A, B, E, ρ, σ satisfying (2.5) a), (2.5) b), (2.7), (1.9); satisfy (2.1) and (2.6) b).

Case II. Since $P = 0$, let the quantities on either side are non-vanishing ie.,

$$(2.8)\qquad (A_3 + A_4)/A = (B_3 + B_4)/B \neq 0\ .$$

It follows easily from (2.8) that

$$(2.9)\qquad (A_{33} + 2A_{34} + A_{44})/A = (B_{33} + 2B_{34} + B_{44})/B$$

and

$$(2.10)\quad (A_{33} - A_{44})/A - (B_{33} - B_{44})/B = \nu\,\{(A_3 - A_4)/A - (B_3 - B_4)/B\}\ ,$$

where $\nu = (A_3 + A_4)/A$.

Again, using (2.8) and (2.9) in $R_{11}/A = R_{22}/B$ of (2.4) a) we find that

$$(2.11)\quad (A_{33} - A_{44})/A - (B_{33} - B_{44})/B = (\nu/2)\,\{(A_3 - A_4)/A - (B_3 - B_4)/B\}\ .$$

Hence, it follows from (2.10) and (2.11) that

$$(2.12)\qquad (A_3 - A_4)/A = (B_3 - B_4)/B\ .$$

(2.8) and (2.12) give A/B = constant which, by certain transformations, can be reduced to

$$(2.13)\qquad A = B\ .$$

Therefore, (2.4) b) can be written as

$$\partial_3 \nu + \partial_4 \upsilon + (1/2)\nu^2 = 0$$

which gives on integration

$$(2.14)\qquad A\,\psi(z-t) = \{z - \Phi(z-t)\}^2\ .$$

The equation (2.4) d) and (2.4) a) give on simplification

$$E = (x\,y)^{1/2} f(z - t) + g(z, t) \tag{2.15}$$

and

$$\nu\,(3\,E\,\nu/2 + W) = (A_{44} - A_{33})/A \tag{2.16}$$

respectively. Therefore we have:

A necessary and sufficient condition that g_{ij} and F_{ij} constitute a solution of (1.1) and (1.2) is that A, E, ρ, σ satisfying (2.14), (2.15), (2.16), (1.9); satisfy (2.1) and (2.4) c).

3. Wave-like solutions in Peres space-time

The metric (1.3) reduces to Peres metric [2] if we take $A = B = 1$ and $E = E(x, y, z - t)$. Therefore (2.1) and (2.4) reduces to

$$\partial_2 \rho - \partial_1 \sigma = 0 \tag{3.1}$$

and

$$(\partial_{11} + \partial_{22})\,E = 4\,(\rho^2 + \sigma^2) \tag{3.2}$$

respectively. Hence, the equations (1.2) are equivalently to the following Cauchy-Riemann type equations

$$\partial_1 \rho + \partial_2 \sigma = 0\,, \quad \partial_2 \rho - \partial_1 \sigma = 0\,. \tag{3.3}$$

Thus, a necessary and sufficient condition that g_{ij} given by

$$ds^2 = -dx^2 - dy^2 - (1 - E)\,dz^2 - 2\,E dz dt + (1 + E)\,dt^2,\ E = E(x, y, z - t)$$

and F_{ij} given by (1.7) a) where E, ρ, σ, are functions of x, y, $z - t$, satisfy the field equations (1.1) and (1.2) is that E, ρ, σ satisfy (3.2) and (3.3).

Again it should be noted that the conditions (3.2) and (3.3) give no restriction to the dependence of E, ρ, σ on $z - t$. When ρ, σ vanish, the above result is reduced to Peres [2] result concerning in an empty region. Likewise the Einstein-Maxwell equation is reduced to Einstein tensor

$$R^i_{\ k} - (1/2)\,R\,\delta^i_k = 0$$

which, subsequently, gives $R_{ij} = 0$, Einstein's field equation in empty region.

As the space-time is non-flat and the metric tensor is not only the function of $z - t$, the solutions have been called wave-like in the sense of Takeno [5].

REFERENCES

[1] K. B. Lal and S. N. Pandey : Wave solutions of the field equations of general relativity in a generalized Peres space-time. Tensor, N.S., *28* (1974) (in Press).

[2] A. Peres : Some gravitational waves, Phys Rev. Let. *3* (1959), 571-572.

[3] L. K. Patel and P. C. Vaidya : A generalized plane wave metric, Ann. Inst. Henri Poincaré, XV, n° 2 (1971), 169-175.

[4] K. B. Lal and N. Ali : Thewave solutions of field equations of general relativity in a generalized Takeno space-time, Tensor, N. S. *21* (1970), 134-137.

[5] H. Takeno : The mathematical theory of plane gravitational waves in general relativity, Sci-Rep. Res. Inst. Theo. Phys. Hiroshima Univ. *1* (1961).

[6] J. L. Synge : Relativity ; The special theory, Amsterdam, (1956), Chap. I.