Wave solutions of the Einstein-Maxwell equations in a generalized Peres space-time

1. Introduction

The Einstein-Maxwell equations in the absence of source are

\begin{align}
R^i_k - (1/2) R \delta^i_k + F^j_i F_{jk} + \ast F^i \ast F_{ik} &= 0, \\
F_{ij} &= 0 \quad \text{and} \quad \ast F_{ij} = 0,
\end{align}

where $R^i_k$ is the Ricci tensor, $F_{ij} = - \ast F_{ij}$ the electromagnetic field tensor and $\ast F_{ij} = (1/2) (-g)^{1/2} \epsilon_{ijk} F^{kl}$. The covariant differentiation with respect to the metric tensor $g_{ij}$ is denoted by a semi-colon.


\begin{equation}
ds^2 = -A dx^2 - B dy^2 - (1 - E) dz^2 - 2 E dz dt + (1 + E) dt^2
\end{equation}

where $A$ and $B$ are any functions of $z$, $t$ and $E$ is any function of $x$, $y$, $z$, $t$; and have also obtained the wave-like solutions of the field equations of general relativity. In this paper we propose to obtain the wave-like solutions (in the sense of Takeno [5]) of Einstein-Maxwell equations (1.1) and (1.2) in the generalized Peres space-time (1.3), which is non-flat in general.

The values of $g^{ij}$, $R_{ij}$ etc. have been calculated for the metric (1.3) in [1], however it has been preferred to give below the values of $g^{ij}$, $R_{ij}$ and $R$ only as:

\begin{align}
g^{11} &= -1/A, \quad g^{22} = -1/B, \quad g^{33} = -(1 + E), \quad g^{44} = -E = g^{42}, \quad g^{44} = (1 - E), \\
\text{and other } g^{ij} &= 0,
\end{align}

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\[
\begin{align*}
R_{11} = & \left( A_{33} - A_{44} \right)/2 + E \left( A_{33} + 2 A_{44} + A_{11} \right)/2 - \left( A_{33} - A_{44} \right)/4 A + \\
& + W \left( A_{33} + A_{44} \right)/2 + \left( A_{33} - A_{44} \right)/4 B - E \left( A_{33} + A_{44} \right) P/4, \\
R_{13} = & - R_{14} = (1/2) E_{13} + E_{14} - (1/4) E_1 P, \\
R_{22} = & \left( B_{33} - B_{44} \right)/2 + E \left( B_{33} + 2 B_{44} + B_{11} \right)/2 - \left( B_{33} - B_{44} \right)/4 B + \\
& + W \left( B_{33} + B_{44} \right)/2 + \left( A_{33} - A_{44} \right)/4 A + E \left( B_{33} + B_{44} \right) P/4, \\
R_{23} = & - R_{24} = (1/2) E_{23} + E_{24} + (1/4) E_2 P, \\
R_{33} = & A_{33}^2 A + B_{33}^2 B - U + V/2 - \left( A_{33}^2 A^2 + B_{33}^2 B^2 \right)/4 + E_3 S + (W + E_3) T, \\
R_{34} = & A_{44}^2 A + B_{44}^2 B + U - \left( A_{44}^2 A^2 + B_{44}^2 B^2 \right)/4 - E_4 T + E_3 S, \\
R_{44} = & A_{11}^2 A + B_{11}^2 B - U - V/2 - \left( A_{11}^2 A^2 + B_{11}^2 B^2 \right)/4 - (W + E_4) S - E_4 T, \\
\text{and other } R_{ii} = 0, \\
\end{align*}
\]

\begin{align*}
(1.5) \quad R = & \left( A_3^2 - A_4^2 \right)/2 A^2 + \left( B_3^2 - B_4^2 \right)/2 B^2 - E \left( A_3^2 + A_4^2 \right) \left( B_3^2 + B_4^2 \right)/2 A B - \\
& - \left( A_3 B_3 - A_4 B_4 \right)/2 A B - WQ - V - \left( A_{33} - A_{44} \right)/A - \left( B_{33} - B_{44} \right)/B - \\
& - E \left( A_{33} + 2 A_{44} + A_{11} \right)/A + \left( B_{33} + 2 B_{44} + B_{11} \right)/B, \\
\end{align*}

where \( P = (A_3 + A_4)/A - (B_3 + B_4)/B, \) \( Q = (A_3 + A_4)/A + (B_3 + B_4)/B, \) \( V = E_{23} + 2 E_{45} + E_{44}, \) \( S = A_{33}/A + B_{33}/B, \) \( T = A_{44}/A + B_{44}/B, \) \( W = E_3 + E_4, \) \( U = (E_{11}/A + E_{33}/B)/2 + E V/2 - E W Q/4 \) and the suffixes 1, 2, 3, 4 denote partial derivatives of \( A, B, E \) with respect to \( x, y, z, t \) respectively.

The values of \( R_{ki} = g^{ij} R_{kij} \) can easily be calculated with the help of (1.4) and (1.5).

The null electromagnetic field \( F_{ij} \) (in the sense of Synge \cite{6}) and its contravariant components \( F^\alpha \) considered in \cite{1} are

\[
\begin{align*}
(1.7) \quad \text{a) } F_{ij} = & \begin{vmatrix}
0 & 0 & -\sigma & \sigma \\
0 & \rho & \rho & -\rho \\
\sigma & -\rho & 0 & 0 \\
-\sigma & \rho & 0 & 0
\end{vmatrix}
\quad \text{and b) } F^\alpha = \begin{vmatrix}
0 & 0 & -\sigma/A & -\sigma/A \\
0 & \sigma/B & \rho/B & \rho/B \\
\sigma/A & -\sigma/B & 0 & 0 \\
-\sigma/A & -\sigma/B & 0 & 0
\end{vmatrix}
\end{align*}
\]

and their dual tensors are

\[
(1.8) \quad \text{a) } F^*_ij = \begin{vmatrix}
0 & 0 & -\rho \sqrt{A/B} & \rho \sqrt{A/B} \\
0 & 0 & -\sigma \sqrt{B/A} & \sigma \sqrt{B/A} \\
\rho \sqrt{A/B} & \sigma \sqrt{B/A} & 0 & 0 \\
-\rho \sqrt{A/B} & -\sigma \sqrt{B/A} & 0 & 0
\end{vmatrix}
\]
and b) \[ F_\sigma = \begin{vmatrix} 0 & 0 & -\rho/\sqrt{AB} & -\rho/\sqrt{AB} \\ 0 & 0 & -\sigma/\sqrt{AB} & -\sigma/\sqrt{AB} \\ \rho/\sqrt{AB} & \sigma/\sqrt{AB} & 0 & 0 \\ \rho/\sqrt{AB} & \sigma/\sqrt{AB} & 0 & 0 \end{vmatrix} \]

where \( \rho \) and \( \sigma \) are functions of \( x, y, z \rightarrow t \) satisfying

(1.9) \[ \lambda_1 \rho + \lambda_2 \sigma = 0 \ , \quad (\lambda_1 \sim \lambda_2 x \text{ etc.}) \]

2. Solution of Equation (1.1) and (1.2)

On substituting the relevant quantities in (1.2) a) we find

(2.1) \[ a) \quad P = 0 \quad \text{and} \quad b) \quad A \lambda_2 \rho - B \lambda_1 \sigma = 0 \]

for \( i = 1, 2 \) and \( i = 3, 4 \) respectively, and (1.2) b) is identically satisfied with the aid of (1.9).

Again the non-vanishing components of \( F_\sigma F_{14} \) and \( F_\sigma \ast F_{14} \) are

(2.2) \[ \begin{cases} F_\sigma F_{14} = -F_\sigma F_{14} = -F_\sigma F_{14} = F_\sigma F_{14} = \lambda \\ \ast F_\sigma \ast F_{14} = -\ast F_\sigma \ast F_{14} = -\ast F_\sigma \ast F_{14} = \ast F_\sigma \ast F_{14} = \lambda \end{cases} \]

where \( \lambda = \rho^2/B + \sigma^2/A \). Since \( \delta_j \) are defined as

(2.3) \[ \delta_j = 1 \quad \text{if} \quad i = j , \quad \delta_i = 0 \quad \text{if} \quad i \neq j \]

the equation (1.1), with the help of (2.2), reduces to

(2.4) \[ \begin{cases} a) \quad R_{13}/A = (1/2) (R_{23} - R_{44}) = R_{25} / B = -R/2 \\ b) \quad R_{23} + 2 R_{24} + R_{44} = 0 \\ c) \quad E_\sigma R / 2 - R_{44} + \mu = 0 , \quad (\mu = 2 \lambda) \\ d) \quad (E_{13} + E_{14})/E_4 = -(E_{23} + E_{24})/E_5 = P/2 \end{cases} \]

Now we shall find the solutions of (2.4) using (2.1) a).

Case I. Since \( P = 0 \), let the quantities on either side vanish separately i.e.,

(2.5) \[ (A_3 + A_4)/A = (B_3 + B_4)/B = 0 \ , \]

which gives on integration

a) \( A = A (z - t) \) \quad \text{and} \quad b) \( B = B (z - t) \) .
This helps the simplification of (2.4) to

\[
\begin{align*}
\text{a) } & \quad \mathbf{V} = 0 \\
\text{b) } & \quad (E_{11} + E_{22} + E_{33})/2 - (A^2 + B^2)/4 + W(A/A + B/B) - \frac{W(A + B)}{2} = 2 (\mathbf{g}^2 + \mathbf{\sigma}^2) \\
\text{c) } & \quad (E_{11} + E_{44})/E_1 = -(E_{33} + E_{44})/E_3 = 0 , \quad (\tau_{-}) = \chi(z - t) .
\end{align*}
\]

Integrating (2.6) a) and (2.6) c) we find that

\[
(2.7) \quad \mathbf{E} = f(x, y, z - t) + z g(z - t) .
\]

Thus, we have:

A necessary and sufficient condition that \(g_i\) and \(F_i\) constitute a solution of (1.1) and (1.2) is that \(A, B, E, \mathbf{g}, \mathbf{\sigma}\) satisfying (2.5) a), (2.5) b), (2.7), (1.9); satisfy (2.1) and (2.6) b).

Case II. Since \(P = 0\), let the quantities on either side are non-vanishing ie.,

\[
(2.8) \quad (A_3 + A_4)/A = (B_3 + B_4)/B \neq 0 .
\]

It follows easily from (2.8) that

\[
(2.9) \quad (A_{33} + 2A_{44})/A = (B_{33} + 2B_{44})/B
\]

and

\[
(2.10) \quad (A_{33} - A_{44})/A = (B_{33} - B_{44})/B = v \left( (A_3 - A_4)/A - (B_3 - B_4)/B \right) ,
\]

where \(v = (A_3 + A_4)/A\).

Again, using (2.8) and (2.9) in \(R_{11}/A = R_{22}/B\) of (2.4) a) we find that

\[
(2.11) \quad (A_{33} - A_{44})/A - (B_{33} - B_{44})/B = (v/2) \left( (A_3 - A_4)/A - (B_3 - B_4)/B \right).
\]

Hence, it follows from (2.10) and (2.11) that

\[
(2.12) \quad (A_3 - A_4)/A = (B_3 - B_4)/B .
\]

(2.8) and (2.12) give \(A/B = \text{constant}\) which, by certain transformations, can be reduced to

\[
(2.13) \quad A = B .
\]

Therefore, (2.4) b) can be written as

\[
\lambda_3 v + \lambda_4 v + (1/2) v^2 = 0
\]

which gives on integration

\[
(2.14) \quad A \psi (z - t) = \{z - \Phi(z - t)\}^2 .
\]
The equation (2.4) d) and (2.4) a) give on simplification

\[(2.15) \quad E = (x \ y)^{1/2} f (z - t) + g (z, t)\]

and

\[(2.16) \quad \nu (3 \ E \ \nu/2 + W) = (A_{14} - A_{23})/A\]

respectively. Therefore we have:

A necessary and sufficient condition that \(g_{00}\) and \(F_{ij}\) constitute a solution of (1.1) and (1.2) is that \(A, E, \rho, \sigma\) satisfying (2.14), (2.15), (2.16), (1.9) ; satisfy (2.1) and (2.4) e).

3. Wave-like solutions in Peres space-time

The metric (1.3) reduces to Peres metric [2] if we take \(A = B = 1\) and \(E = E (x, y, z - t)\). Therefore (2.1) and (2.4) reduces to

\[(3.1) \quad \lambda_2 \rho - \lambda_1 \sigma = 0\]

and

\[(3.2) \quad (\lambda_{11} + \lambda_{22}) E = 4 (\rho^2 + \sigma^2)\]

respectively. Hence, the equations (1.2) are equivalently to the following Cauchy-Riemann type equations

\[(3.3) \quad \lambda_1 \rho + \lambda_2 \sigma = 0, \quad \lambda_2 \rho - \lambda_1 \sigma = 0 .\]

Thus, a necessary and sufficient condition that \(g_{00}\) given by

\[ds^2 = - dx^2 - dy^2 - (1 - E) dz^2 - 2 Edzdt + (1 + E) dt^2, E = E (x, y, z - t)\]

and \(F_{ij}\) given by (1.7) a) where \(E, \rho, \sigma\) are functions of \(x, y, z - t\), satisfy the field equations (1.1) and (1.2) is that \(E, \rho, \sigma\) satisfy (3.2) and (3.3).

Again it should be noted that the conditions (3.2) and (3.3) give no restriction to the dependence of \(E, \rho, \sigma\) on \(z - t\). When \(\rho, \sigma\) vanish, the above result is reduced to Peres [2] result concerning in an empty region. Likewise the Einstein-Maxwell equation is reduced to Einstein tensor

\[R^t_t = (1/2) R \ \delta^t_t = 0\]

which, subsequently, gives \(R_{00} = 0\), Einstein's field equation in empty region.

As the space-time is non-flat and the metric tensor is not only the function of \(z - t\), the solutions have been called wave-like in the sense of Takeno [5].
REFERENCES


