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On projective recurrent Finsler spaces of the first order (**)

1. Introduction — In one of the recent papers R. B. Misra [1] has given a comparative study of various types of recurrent Finsler spaces. In the present paper the matter has been decomposed in three sections. The first one is introductory and the second, and third sections deal with W — recurrent and W* — recurrent Finsler spaces with Cartan's first and second co-variant derivatives respectively.

We shall consider an n-dimensional Finsler space F_n, [3] with homogeneous metric function $F(x, \dot{x})$ of degree one in $\dot{x}^i$'s which is defined by $g_{ij}(x, \dot{x}) = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x})$. The tensor $C_{ijk}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2}\dot{\partial}_k g_{ij}(x, \dot{x})$ satisfies the identity:

$$C_{ijk}(x, \dot{x})\,\dot{x}^i = C_{ijk}(x, \dot{x})\,\dot{x}^j = C_{ijk}(x, \dot{x})\,\dot{x}^k = 0\,. \tag{1.1}$$

Cartan. ([3] ch. II-III) defined two types of covariant derivatives; for instance the two covariant derivatives for a mixed tensor $T^i_j(x, \dot{x})$ are given by

$$T^i_{j|k} = \partial_k T^i_j - \dot{\partial}_l T^i_j \partial_k G^l + T^h_j \Gamma^{*i}_{hk} - T^i_h \Gamma^{*h}_{jk} \tag{1.2}$$

and

$$T^i_j\big|_k = F\dot{\partial}_k T^i_j + T^m_j A^i_{mk} - T^i_m A^m_{jk}, \tag{1.3}$$

where

$$A^m_{jk} \overset{\text{def}}{=} F\,C^m_{jk}$$

The commutation formulae are as follows:

$$2\,T^i_{j|[hk]} = K^i_{mhk} T^m_j - K^m_{jhk} T^i_m - \dot{\partial}_m T^i_j K^m_{\nu hk}\dot{x}^\nu, \tag{1.4}$$

$$2\,T^i_j\big|_{[hk]} = F_{\dot{x}k} T^i_j\big|_h + F_{\dot{x}h} T^i_j\big|_k + S^i_{mkh} T^m_j - S^m_{jkh} T^i_m, \tag{1.5}$$

$$(\dot{\partial}_h T^i_j)_{|k} - (\dot{\partial}_h T^i_{j|k}) = \dot{\partial}_l T^i_j A^l_{hk|\nu} l^\nu - \dot{\partial}_h \Gamma^{*i}_{\nu k} T^\nu_j + \dot{\partial}_h \Gamma^{*\nu}_{jk} T^i_\nu \tag{1.6}$$

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([1]) The notations ∂_i and $\dot{\partial}_i$ denote the operators $\partial/\partial x^i$ and $\partial/\partial \dot{x}^i$ respectively.

and

$$(1.7) \qquad (\dot{\partial}_h T^i_j)|_k - \dot{\partial}_h T^i_j|_k = -\{F \dot{x}^h \dot{\partial}_k T^i_j + \dot{\partial}_h A^i_{mk} T^m_j - \dot{\partial}_h A^m_{jk} T^i_m + A^m_{hk} \dot{\partial}_m T^i_j\} .$$

The projective curvature tensor is defined by

$$(1.8) \qquad \text{(a)} \quad W^i_{jhk}(x, \dot{x}) = \dot{\partial}_j W^i_{hk} = \frac{2}{3} \dot{\partial}_j \dot{\partial}_{[h} W^i_{k]}$$

$$\text{(b)} \quad W^i_{hk}(x, \dot{x}) = \frac{2}{3} \dot{\partial}_{[h} W^i_{k]} \quad , \quad \text{(c)} \quad W^i_{hk} = -W^i_{kh} .$$

Here the square brackets denote the skew symmetric part with respect to the indices enclosed with in them. Noting that W^i_j, is homogeneous of degree two in its directional argument, we have the following identities

$$(1.9) \qquad \text{(a)} \quad W^i_{hk} \dot{x}^h = W^i_k \quad , \quad \text{(b)} \quad W^i_{jhk} \dot{x}^j \dot{x}^h = W^i_k \quad ,$$

$$\text{(c)} \quad W^i_{jhk} \dot{x}^j = W^i_{hk}$$

$$(1.10) \quad \text{(a)} \quad W^i_k \dot{x}^k = 0 \quad , \quad \text{(b)} \quad \dot{\partial}_h W^i_k \dot{x}^k = -W^i_h \quad , \quad \text{(c)} \quad \dot{\partial}_i W^i_k = 0 .$$

2. PROJECTIVE RECURRENT FINSLER SPACE WITH CARTAN'S FIRST CO-VARIANT DERIVATIVE

Definition (2.1): In an n-dimensional Finsler space F_n the projective curvature tensor is called W — recurrent F_n if it satisfies the relation

$$(2.1) \qquad W^i_{jhk|l} = \lambda_l W^i_{jhk} \quad , \quad (\lambda_l \neq 0)$$

Transvecting (2.1) by $\dot{x}^j$ and noting (1.9b), we find

$$(2.2) \qquad W^i_{hk|l} = \lambda_l W^i_{hk} .$$

Hence the tensor field W^i_{hk} is also recurrent in an W — recurrent F_n. Again transvecting the equation (2.2) by $\dot{x}^h$ and using the equation (1.9a), we get

$$(2.3) \qquad W^i_{k|l} = \lambda_l W^i_k .$$

So that W^i_k is also recurrent in W — recurrent F_n.

Theorem (2.1): An W^i_{jk} — recurrent F_n will be W — recurrent F_n if and only if the recurrence vector λ_l satisfies

$$(2.4) \qquad (\dot{\partial}_j \lambda_l) W^i_{hk} = (\dot{\partial}_j \Gamma^{*i}_{vl}) W^v_{hk} - (\dot{\partial}_m W^i_{hk}) A^m_{jl|v} l^v + 2 \dot{\partial}_j \Gamma^{*v}_{l[h} W^i_{|v|k]} .$$

Proof: Let us suppose that a F_n be W^i_{jk} — recurrent space.

Differentiating (2.2) with respect to $\dot{x}^l$ and applying the commutation formula (1.6), we get

$$(2.5) \qquad W^i_{lhk|l} - \lambda_l W^i_{lhk} = (\dot{\partial}_l \lambda_l) W^i_{hk} + \dot{\partial}_m W^i_{hk} A^m_{il|v} l^v$$
$$- \dot{\partial}_l \Gamma^{*i}_{vl} W^v_{hk} + \dot{\partial}_l \Gamma^{*v}_{hl} W^i_{vk} + \dot{\partial}_l \Gamma^{*v}_{kl} W^i_{hv}$$

Hence for W — recurrent space the first member of the equation (2.5) vanishes and after rearranging the terms, it gives the theorem.

Theorem (2.2): In an W^i_{jk} — recurrent F_n the relation

$$(2.6) \qquad (\dot{\partial}_l \lambda_l) W^i_{hk} \dot{x}^l = \dot{\partial}_l \Gamma^{*i}_{vl} W^v_{hk} \dot{x}^l + 2 \dot{\partial}_l \Gamma^{*v}_{l[h} W^i_{\boxed{v} k]} \dot{x}^l$$

holds good.

Proof: Transvecting (2.5) by $\dot{x}^l$ and using (1.8c) and (1.9c), we get

$$(2.7) \qquad W^i_{hk|l} - \lambda_l W^i_{hk} = (\dot{\partial}_l \lambda_l) W^i_{hk} \dot{x}^l - \dot{x}^l \dot{\partial}_l \Gamma^{*i}_{vl} W^v_{hk}$$
$$+ \dot{x}^l \dot{\partial}_l \Gamma^{*v}_{hl} W^i_{vk} + \dot{x}^l \dot{\partial}_l \Gamma^{*v}_{kl} W^i_{hv} .$$

Using the relation (2.2) for W^i_{jk} — recurrent Finsler space, we get (2.6).

Theorem (2.3): The necessary and sufficient condition that an W^i_j — recurrent F_n will be an W^i_{jk} — recurrent F_n is,

$$(2.8) \qquad W^i_{[j} \dot{\partial}_{h]} \lambda_l + \dot{\partial}_m W^i_{[j} A^m_{h]l|v} l^v - W^v_{[j} \dot{\partial}_{h]} \Gamma^{*i}_{vl} + \dot{\partial}_{[h} \Gamma^{*v}_{j]l} W^i_v = 0 .$$

Proof: An W^i_j — recurrent F_n is characterised by the relation (2.3). Differentiating (2.3) with respect to $\dot{x}^h$ and applying the commutation formula (1.6), we get

$$(2.9) \qquad (\dot{\partial}_h W^i_j)_{|l} - (\dot{\partial}_h \lambda_l) W^i_j - \lambda_l \dot{\partial}_h W^i_j = - \dot{\partial}_h \Gamma^{*i}_{vl} W^v_j$$
$$+ \dot{\partial}_m W^i_j A^m_{hl|v} l^v + \dot{\partial}_h \Gamma^{*v}_{jl} W^i_v .$$

Interchanging the indices h and j in (2.9) and substracting it from (2.9) and using the relation (1.8b), we get

$$(2.10) \qquad W^i_{hj|l} - \lambda_l W^i_{hj} = \frac{2}{3} \{ W^i_{[j} \dot{\partial}_{h]} \lambda_l + \dot{\partial}_m W^i_{[j} A^m_{h]l|v} l^v$$
$$- W^v_{[j} \dot{\partial}_{h]} \Gamma^{*i}_{vl} + W^i_v \dot{\partial}_{[h} \Gamma^{*v}_{j]l} .$$

From equation (2.2) and (2.10) we obtain the result (2.8).

Theorem (2.4): In an W^i_j — recurrent F_n the following relation is true:

$$(2.11) \qquad \dot{x}^h W^i_j \dot{\partial}_h \lambda_l - \dot{x}^h W^v_j \dot{\partial}_h \Gamma^{*i}_{vl} + W^i_m A^m_{jl|v} l^v$$
$$+ 2 W^i_v \dot{\partial}_{[h} \Gamma^{*v}_{j]l} \dot{x}^h = 0 .$$

Proof : Multiplying (2.10) by $\dot{x}^h$ and using (1.10a) and $A^i_{hj|v}\,\dot{x}^h = 0$, we get (2.11) in view of W^i_j — recurrent F_n.

Theorem (2.5) : The recurrence vector λ in W^i_j — recurrent F_n, satisfies the relation

$$2\,\lambda_{[l|m]}\,W^i_j = W^p_j\,k^i_{plm} - W^i_p\,K^i_{jlm} - \dot{\partial}_p\,W^i_j\,K^p_{vlm}\,\dot{x}^v . \tag{2.12}$$

Proof : Differentiating (2.3) co-variantly with respect to x^m, we get

$$W^i_{j|lm} = (\lambda_{l|m} + \lambda_l\,\lambda_m)\,W^i_j . \tag{2.13}$$

Interchanging the indices l and m in (2.13) and substracting it from (2.13) and applying the commutation formula (1.4), we get the required result.

3. PROJECTIVE RECURRENT FINSLER SPACE WITH CARTAN'S SECOND CO-VARIANT DERIVATIVE

Definition (3.1) : An n-dimensional Finsler space F_n is said to be an W^* — recurrent F_n if the Cartan's second co-variant derivative of the projective curvature tensor satisfies the relation :

$$W^i_{ihk}\big|_l = V_l\,W^i_{ihk} \quad , \quad (V_l \neq 0) \tag{3.1}$$

Transvecting the relation (3.1) by $\dot{x}^i$ and $\dot{x}^h$, we get

$$W^i_{hk}\big|_l = V_l\,W^i_{hk} \quad , \quad \text{if } i \neq l \tag{3.2}$$

and

$$W^i_k\big|_l = V_l\,W^i_k \quad , \quad \text{if } h \neq l \tag{3.3}$$

Hence the tensor fields W^i_{jk} and W^i_k are also recurrent in an W^* — recurrent F_n and known as W^{*i}_{jk} — recurrent and W^{*i}_k — recurrent Finsler space respectively.

Theorem (3.1) : An W^{*i}_{jk} — recurrent F_n will be an W^* — recurrent F_n if and only if it satisfies the relation :

$$(\dot{\partial}_i V_l)\,W^i_{hk} = F_{\dot{x}^l}\,W^i_{ihk} + \dot{\partial}_i\,A^i_{ml}\,W^m_{hk} - 2\,\dot{\partial}_i\,A^m_{l[h}\,W^{\,i}_{\boxed{m}\,k]} + A^m_{il}\,W^i_{mhk} . \tag{3.4}$$

Proof : Differentiating (3.2) with respect to $\dot{x}^i$ and applying the commutation formula (1.7), we get

$$W^i_{ihk}\big|_l - V_l\,W^i_{ihk} = (\dot{\partial}_i V_l)\,W^i_{hk} - \{\,F_{\dot{x}^l}\,W^i_{ihk} + \dot{\partial}_i\,A^i_{ml}\,W^m_{hk} - \dot{\partial}_i\,A^m_{hl}\,W^i_{mk} - \dot{\partial}_i\,A^m_{kl}\,W^i_{hm} + A^m_{il}\,\dot{\partial}_m\,W^i_{hk}\,\} . \tag{3.5}$$

For W^* — recurrent space we get the theorem from (3.5) by using the relation (3.1).

Theorem (3.2): In an W^{*i}_{jk} — recurrent F_n the following condition holds good:

$$(3.6)\qquad (\dot\partial_l V_l)\, W^i_{hk}\, \dot x^l = F\, W^i_{[hk} + \dot\partial_l A^i_{ml}\, W^m_{hk}\, \dot x^l - 2\, \dot\partial_l A^m_{l[h}\, W^{\ i}_{\boxed{m}\ k]}\, \dot x^l .$$

Proof: Multiplying (3.5) by $\dot x^h$ and using relations (1.9c) and (1.1) we get the result (3.6) in view of the equation (3.2).

Theorem (3.3): An W^{*i}_k — recurrent F_n becomes W^{*i}_{jk} — recurrent F_n if it satisfies the following relation

$$(3.7)\qquad W^i_{[k}\, \dot\partial_{h]}\, V_l = \dot\partial_l\, W^i_{[k}\, \dot\partial_{h]}\, F + W^m_{[k}\, \dot\partial_{h]}\, A^i_{ml} - \dot\partial_{[h}\, A^m_{k]l}\, W^i_m$$
$$+ \dot\partial_m\, W^i_{[k}\, A^m_{h]l} .$$

Proof: An W^{*i}_j — recurrent F_n is characterised by (3.3).

Differentiating (3.3) with respect to $\dot x^h$ and applying the commutation formula (1.7), we get

$$(3.8)\qquad (\dot\partial_h W^i_k)|_l - (\dot\partial_h W^i_k)\, V_l - (\dot\partial_h V_l)\, W^i_k = -\{\dot\partial_h F\, \dot\partial_l W^i_k$$
$$+ \dot\partial_h A^i_{ml}\, W^m_k - \dot\partial_h A^m_{kl}\, W^i_m + A^m_{hl}\, \dot\partial_m W^i_k\} .$$

Now interchanging the indices h and k in (3.8) and substracting it from (3.8) and applying the relation (1.8b), we obtain

$$(3.9)\qquad W^i_{hk}|_l - V_l\, W^i_{hk} = \frac{2}{3}\, W^i_{[k}\, \dot\partial_{h]}\, V_l - \frac{2}{3}\, \{\dot\partial_l\, W^i_{[k}\, \dot\partial_{h]}\, F$$
$$+ \dot\partial_{[h}\, A^{\ i}_{\boxed{ml}}\, W^m_{k]} - \dot\partial_{[h}\, A^m_{k]l}\, W^i_m + \dot\partial_m\, W^i_{[k}\, A^m_{h]l}\} .$$

But for W^{*i}_{jk} — recurrent F_n the left hand side of (3.9) vanishes and then we get (3.7).

Theorem (3.4): In an W^{*i}_j — recurrent Finsler space, we have

$$(3.10)\qquad W^i_k\, \dot x^h\, (\dot\partial_h V_l) = F\, \dot\partial_l\, W^i_k + F_{\dot x^k}\, W^i_l + W^m_k\, \dot\partial_h\, A^i_{ml}\, \dot x^h$$
$$- 2\, \dot\partial_{[h}\, A^m_{k]l}\, W^i_m\, \dot x^h + W^i_m\, A^m_{kl} .$$

Proof: Multiplying (3.9) by $\dot x^h$ and using equations (1.10a), (1.10b) and (1.1) we get (3.10) in view of the relation (3.3).

Theorem (3.5): The recurrence vector of W^{*i}_j — recurrent F_n satisfies the relation

$$(3.11)\qquad 2\, V_{[l}|_{m]}\, W^i_j = F_{\dot x^m}\, W^i_j|_l + F_{\dot x^l}\, W^i_j|_m + S^i_{\nu ml}\, W^\nu_j - S^\nu_{jml}\, W^i_\nu .$$

Proof: Differentiating (3.3) co-variantly with respect to x^m, we get

$$(3.12)\qquad W^i_j|_{lm} = (V_l|_m + V_l\, V_m)\, W^i_j .$$

Interchanging l and m in (3.12) and substracting it from (3.12) and applying the commutation formula (1.5) we easily get the result (3.11).

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