On projective recurrent Finsler spaces of the first order

1. Introduction — In one of the recent papers B. B. Misra [1] has given a comparative study of various types of recurrent Finsler spaces. In the present paper the matter has been decomposed in three sections. The first one is introductory and the second, and third sections deal with $W$ — recurrent and $W^*$ — recurrent Finsler spaces with Cartan's first and second co-variant derivatives respectively.

We shall consider an $n$-dimensional Finsler space $F_m$ [3] with homogeneous metric function $F(x, \dot{x})$ of degree one in $x^i$, $\dot{x}^i$ which is defined by $g_{ij}(x, \dot{x}) = \frac{1}{2} \dot{x}^i \dot{x}^j F^2(x, \dot{x})$. The tensor $C_{i j k}(x, \dot{x})$ defined by $g_{ij}(x, \dot{x})$ satisfies the identity:

$$C_{i j k}(x, \dot{x}) \dot{x}^i = C_{j i k}(x, \dot{x}) \dot{x}^j = C_{i j k}(x, \dot{x}) \dot{x}^k = 0.$$  

(1.1)

Cartan, ([3] ch. II-III) defined two types of covariant derivatives; for instance the two covariant derivatives for a mixed tensor $T^i_j(x, \dot{x})$ are given by

$$T^i_j|_k = \delta^i_k T^i_j + \delta^i_k \Gamma^m_{jk} T^m_j - T^m_j \Gamma^i_{jk}$$  

and

$$T^i_j|_k = F \delta^i_k T^i_j + T^m_j A^m_{jk} - T^m_j A^m_{jk},$$

(1.2)

(1.3)

where

$$A^m_{jk} \overset{\text{def}}{=} F C^m_{jk}.$$  

The commutation formulae are as follows:

$$2 T^i_j|_{[hk]} = K^m_{hjk} T^m_j - K^m_{jk} T^m_h - \delta^m_j T^m_k K^m_{hjk} \dot{x}^k,$$  

(1.4)

$$2 T^i_j|_{[hk]} = F \delta^i_k T^i_j + F \delta^i_k T^i_h + S^m_{hjk} T^m_j - S^m_{jk} T^m_h,$$  

(1.5)

$$\left(\delta^i_k T^i_j\right)_{[h]} - \left(\delta^i_k T^i_j\right)_{[k]} = \delta^i_k A^m_{jk} \gamma^m - \delta^i_k \Gamma^m_{jk} T^m_j + \delta^i_k \Gamma^m_{jk} T^m_k,$$  

(1.6)

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(**) Memoria presentata dall'Accademia E. Bonaparte il 6-5-1974.

(*) The notations $\delta$ and $\delta$ denote the operators $\partial/\partial x^i$ and $\partial/\partial \dot{x}^i$ respectively.
and

\( (\delta_b T^i_j)_k = -\lambda_b T^i_j|_k = -(F \cdot x^b \delta_h T^i_j + \lambda_b A^p_{mk} T^i_m - \lambda_b A^p_{mk} T^i_m + A^p_{mk} \delta_h T^i_j) \).

The projective curvature tensor is defined by

\[
\begin{align*}
(1.8) & \\
(a) & W^1_{1h} (x, x) = \lambda_b W^1_{hk} = \frac{2}{3} \lambda_h \lambda_b W^1_{k1} \\
(b) & W^1_{hk} (x, x) = \frac{2}{3} \lambda_h \lambda_k W^1_{k1} , \\
(c) & W^1_{hk} = -W^1_{kh} .
\end{align*}
\]

Here the square brackets denote the skew symmetric part with respect to the indices enclosed with in them. Noting that \( W^1_1 \) is homogeneous of degree two in its directional argument, we have the following identities

\[
(1.9) \quad \begin{align*}
(a) & W^1_{hk} x^h = W^1_k \\
(b) & W^1_{hk} x^k x^h = W^1_k , \\
(c) & W^1_{hk} x^k = W^1_{hk}.
\end{align*}
\]

\[
(1.10) \quad \begin{align*}
(a) & W^1_k x^k = 0 \\
(b) & \delta_h W^1_k x^h = -W^1_k , \\
(c) & \lambda_h W^1_k = 0 .
\end{align*}
\]

2. PROJECTIVE RECURRENT FINSLER SPACE WITH CARTAN'S FIRST CO-VARIANT DERIVATIVE

Definition (2.1): In an \( n \)-dimensional Finsler space \( F \), the projective curvature tensor is called \( W \) — recurrent \( F \) if it satisfies the relation

\[
(2.1) \quad W^1_{i12} = \lambda_i W^1_{kh} , \quad (\lambda_i \neq 0)
\]

Transvecting (2.1) by \( x^i \) and noting \((1.9b)\), we find

\[
(2.2) \quad W^1_{kh} = \lambda_i W^1_{kh} .
\]

Hence the tensor field \( W^1_{kh} \) is also recurrent in an \( W \) — recurrent \( F \). Again transvecting the equation (2.2) by \( x^h \) and using the equation (1.9a), we get

\[
(2.3) \quad W^1_{i1} = \lambda_i W^1_{ki} .
\]

So that \( W^1_k \) is also recurrent in \( W \) — recurrent \( F \).

Theorem (2.1): An \( W^1_k \) — recurrent \( F \) will be \( W \) — recurrent \( F \) if and only if the recurrence vector \( \lambda_i \) satisfies

\[
(2.4) \quad (\delta_i \lambda_i) W^1_{kh} = (\delta_i \Gamma^p_{ki}) W^1_{kh} - (\delta_m W^1_{kh}) A^m_{1i} \Gamma^p + 2 \delta_i \Gamma^p_{kj} W^1_{kj} .
\]
Proof: Let us suppose that a $F$, be $W_{jk}^j$ — recurrent space.
Differentiating (2.2) with respect to $\dot{x}^j$ and applying the commutation formula (1.6), we get

$$W_{lk}^i - \dot{\lambda}_l W_{lk}^i = (\delta_l \lambda_j) W_{kj}^i + \dot{\lambda}_m W_{lk}^i A_{ll}^{m} \Gamma^m$$

$$- \dot{\lambda}_l \Gamma_{jk}^{i} W_{lk}^i + \dot{\lambda}_i \Gamma_{lj}^{i} W_{lk}^i + \lambda_k \Gamma_{kl}^{i} W_{kl}^i$$

Hence for $W$ — recurrent space the first member of the equation (2.5) vanishes and after rearranging the terms, it gives the theorem.

Theorem (2.2): In an $W_{jk}^j$ — recurrent $F$ the relation

$$\dot{\lambda}_l W_{lk}^i \dot{x}^l = \dot{\lambda}_i \Gamma_{lj}^{i} W_{lk}^i \dot{x}^l + 2 \dot{\lambda}_i \Gamma_{lj}^{i} W_{lk}^i W_{[lj]^i k}^j \dot{x}^i$$

holds good.

Proof: Transvecting (2.5) by $\dot{x}^i$ and using (1.8c) and (1.9c), we get

$$W_{lk}^i - \dot{\lambda}_l W_{lk}^i = (\delta_l \lambda_j) W_{lk}^i \dot{x}^l - \dot{x}^l \delta_l \Gamma_{lj}^{i} W_{lk}^i$$

$$+ \dot{x}^l \dot{\lambda}_l \Gamma_{lj}^{i} W_{lk}^i + \dot{x}^l \delta_l \Gamma_{lj}^{i} W_{lk}^i$$

Using the relation (2.2) for $W_{jk}^j$ — recurrent Finsler space, we get (2.6).

Theorem (2.3): The necessary and sufficient condition that an $W_{jk}^j$ — recurrent $F$ will be an $W_{jk}^j$ — recurrent $F$ is,

$$W_{lj}^{i} \delta_{h} \lambda_{h} A_{h}^{m} \Gamma^{m} - W_{lj}^{i} \delta_{h} \Gamma_{lj}^{i} A_{h}^{m} \Gamma^{m} + \dot{\lambda}_{h} \Gamma_{lj}^{i} W_{lk}^i = 0.$$

Proof: An $W_{jk}^j$ — recurrent $F$ is characterised by the relation (2.3). Differentiating (2.3) with respect to $\dot{x}^h$ and applying the commutation formula (1.6), we get

$$\dot{\lambda}_h W_j^j - (\delta_h \lambda_j) W_j^j - \dot{\lambda}_i W_j^j = - \dot{\lambda}_i \Gamma_{lj}^{i} W_j^j$$

$$+ \dot{\lambda}_m W_j^j A_{lj}^{m} \Gamma^{m} + \dot{\lambda}_i \Gamma_{lj}^{i} W_j^i.$$

Interchanging the indices $h$ and $j$ in (2.9) and subtracting it from (2.9) and using the relation (1.8b), we get

$$W_{lj}^{i} - \lambda_{l} W_{lj}^{i} = \frac{2}{3} \{ W_{lj}^{i} \delta_{h} \lambda_{h} + \delta_{h} W_{lj}^{i} A_{lj}^{m} \Gamma^{m}$$

$$- W_{lj}^{i} \delta_{h} \Gamma_{lj}^{i} + W_{lj}^{i} \delta_{h} \Gamma_{lj}^{i} \Gamma_{lj}^{i} \}.$$  

From equation (2.2) and (2.10) we obtain the result (2.8).

Theorem (2.4): In an $W_{jk}^j$ — recurrent $F$ the following relation is true:

$$\dot{x}^h W_j^j \delta_{h} \lambda_{h} - \dot{x}^h W_j^j \delta_{h} \Gamma_{lj}^{i} + W_m A_{lj}^{m} \Gamma^{m}$$

$$+ 2 W_j^j \delta_{h} \Gamma_{lj}^{i} \dot{x}^h = 0.$$
Proof: Multiplying (2.10) by $\dot{x}^b$ and using (1.10a) and $A^i_{b1}\gamma^i x^b = 0$, we get (2.11) in view of $W_j^i$ — recurrent $F_n$.

Theorem (2.5): The recurrence vector $\lambda$ in $W_j^i$ — recurrent $F_n$, satisfies the relation

$$2 \lambda_{[i\mid m]} W_j^i = W_j^p k_{p\mid m} - W_j^i K_j^i_{\mid lm} - \dot{\lambda}_r W_j^i K^r_{\mid im} \dot{x}^r.$$  

Proof: Differentiating (2.3) co-variantly with respect to $x^m$, we get

$$W_j^i_{\mid lm} = (\lambda_{[i\mid m} + \lambda_r \lambda_m) W_j^i.$$  

Interchanging the indices $l$ and $m$ in (2.13) and subtracting it from (2.13) and applying the commutation formula (1.4), we get the required result.

3. PROJECTIVE RECURRENT FINSLER SPACE WITH CARTAN'S SECOND CO-VARIANT DERIVATIVE

Definition (3.1): An n-dimensional Finsler space $F_n$ is said to be an $W^*$ — recurrent $F_n$ if the Cartan's second co-variant derivative of the projective curvature tensor satisfies the relation:

$$W^i_{\mid i} = V_i W_{i\mid}^i, \quad (V_i \neq 0)$$

Transvecting the relation (3.1) by $\dot{x}^i$ and $\dot{x}^b$, we get

$$W^i_{\mid i} = V_i W^i_{\mid i}, \quad \text{if } i \neq 1$$

and

$$W^i_{\mid i} = V_i W^i_{\mid i}, \quad \text{if } h \neq 1$$

Hence the tensor fields $W^i_{\mid i}$ and $W^i_{\mid h}$ are also recurrent in an $W^*$ — recurrent $F_n$ and known as $W^*_i$ — recurrent and $W^*_i$ — recurrent Finsler space respectively.

Theorem (3.1): An $W^*_i$ — recurrent $F_n$ will be an $W^*$ — recurrent $F_n$ if and only if it satisfies the relation:

$$(\ddot{\gamma}_i V_i) W^i_{\mid i} = F^i_{\mid i} W^i_{\mid i} + \dot{\gamma}_i A^i_{\mid m} W^m_{\mid i} - 2 \dot{\gamma}_i A^i_{\mid m} W^m_{\mid i} - 2 \dot{\gamma}_i A^i_{\mid m} W^m_{\mid i} + A^i_{\mid m} W^m_{\mid i}.$$  

Proof: Differentiating (3.2) with respect to $\dot{x}^i$ and applying the commutation formula (1.7), we get

$$W^i_{\mid i} = - V_i W^i_{\mid i} = (\ddot{\gamma}_i V_i) W^i_{\mid i} - (F^i_{\mid i} W^i_{\mid i} + \dot{\gamma}_i A^i_{\mid m} W^m_{\mid i}$$

$$- \dot{\gamma}_i A^i_{\mid m} W^m_{\mid i} - \ddot{\gamma}_i A^i_{\mid m} W^m_{\mid i} + A^i_{\mid m} \ddot{\gamma}_i W^i_{\mid i}).$$
For $W^*$ — recurrent space we get the theorem from (3.5) by using the relation (3.1).

Theorem (3.2): In an $W^*_{ik}$ — recurrent $F_n$ the following condition holds good:

\[ (\delta_{i} V_{i}) W_{hk} = F W_{ihk}^{i} + \delta_{i} A^{i}_{m1} W_{k}^{m} - 2 \delta_{i} A^{i}_{m1} W_{k}^{m} x^{i} \]

Proof: Multiplying (3.5) by $x^{h}$ and using relations (1.9e) and (1.1) we get the result (3.6) in view of the equation (3.2).

Theorem (3.3): An $W^*_{ik}$ — recurrent $F_n$ becomes $W^*_{jk}$ — recurrent $F_n$ if it satisfies the following relation

\[ W_{jk}^{i} \delta_{h} V_{i} = \delta_{i} W_{jk}^{i} \delta_{h} F + W_{jk}^{i} \delta_{h} A^{i}_{m1} = \delta_{i} A^{i}_{m1} W_{m}^{i} + \delta_{m} W_{jk}^{i} A^{i}_{m1} \]

Proof: An $W^*_{ik}$ — recurrent $F_n$ is characterised by (3.3).

Differentiating (3.3) with respect to $x^{h}$ and applying the commutation formula (1.7), we get

\[ (\delta_{h} W_{jk}^{i})_{i} = (\delta_{h} W_{jk}^{i})_{i} - (\delta_{h} V_{i}) W_{k}^{i} = \{} \delta_{i} F \delta_{i} W_{k}^{i} + \delta_{i} A^{i}_{m1} W_{k}^{m} - \delta_{h} A^{i}_{m1} W_{m}^{i} + \delta_{m} W_{k}^{i} A^{i}_{m1} \{ \]

Now interchanging the indices $h$ and $k$ in (3.8) and subtracting it from (3.8) and applying the relation (1.8b), we obtain

\[ W_{hk}^{i} = \{} \frac{2}{3} \{ \delta_{i} W_{jk}^{i} \delta_{h} F + \delta_{h} A^{i}_{m1} W_{k}^{m} - \delta_{i} A^{i}_{m1} W_{m}^{i} + \delta_{m} W_{jk}^{i} A^{i}_{m1} \} \]

But for $W^*_{ik}$ — recurrent $F_n$ the left hand side of (3.9) vanishes and then we get (3.7).

Theorem (3.4): In an $W^*_{ik}$ — recurrent Finsler space, we have

\[ W_{k}^{i} x^{h} (\delta_{h} V_{i}) = F \delta_{i} W_{k}^{i} + F x^{h} W_{i}^{i} + W_{k}^{i} \delta_{h} A^{i}_{m1} x^{i} - 2 \delta_{i} A^{i}_{m1} W_{k}^{m} x^{i} + W_{k}^{m} A^{i}_{m1} \]

Proof: Multiplying (3.9) by $x^{h}$ and using equations (1.10a), (1.10b) and (1.1) we get (3.10) in view of the equation (3.3).

Theorem (3.5): The recurrence vector of $W^*_{ik}$ — recurrent $F_n$ satisfies the relation

\[ 2 V_{i} W_{i}^{i} = F_{i} W_{i}^{i} + F_{i} W_{i}^{i} + S_{i} W_{i}^{i} - S_{i} W_{i}^{i} \]

Proof: Differentiating (3.3) co-variantly with respect to $x^{m}$, we get

\[ W_{i}^{i} = (V_{i} W_{i}^{i} + V_{i} W_{i}^{i}) \]

Interchanging $i$ and $m$ in (3.12) and subtracting it from (3.12) and applying the commutation formula (1.5) we easily get the result (3.11).
REFERENCES


