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Semi-Symmetric connections in a Finsler space (**)

Recently Imai [5] has studies the semi-symmetric metric connections in a Riemannian space. Similar to his definition, we define the connections with respect to Berwald's connections in a Finsler space. The covariant derivative of a metric tensor with respect to these connections do not vanish (cf. cororally (2.1)). So we call these connections as semi-symmetric connections in the Finsler space. In this paper we have defined the covariant derivative of a vector field for these connections and studies its further effect. Different types of commutation formulae are reduced and at the end, the relation of R^i_{jkh} (1) and H^i_{jkh} is taken into consideration.

1. - Preliminaries

Let F_n be an n-dimensional Finsler space in which the metric function $F(x, \dot{x})$ (2) satisfies the requisite conditions. The entities are given by

$$(1.1) \qquad \text{(i)} \quad g_{ij}(x, \dot{x}) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F, \quad \text{(ii)} \quad g^{si} g_{sl} = \delta^i_l ,$$

constitute a tensor-field which is symmetric and positively homogeneous function of degree zero in the directional arguments. On transvection, it gives

$$(1.2) \qquad g_{ij} \dot{x}^j = F \dot{\partial}_i F .$$

The partial differentiation of g_{ij} with respect to $\dot{x}^h$ becomes a tensor-field C_{hij} which on transvection with g^{js} defines another tensor-field C^s_{hi}, symmetric in its indices. For these entities the covariant differentiation of a vector-field $X^i(x, \dot{x})$ with respect to $\dot{x}^j$ has been defined by ([2], p. 187)

$$(1.3) \qquad \dot{\nabla}_j X^i = \dot{\partial}_j X^i + C^i_{jk} X^k .$$

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(1) R^i_{jkh} is the curvature tensor-type quantities defined in (3.8) and H^i_{jkh} is the curvature tensorfield due to the Berwald's connections.

(2) The line-element $(x^i, \dot{x}^i)$ is briefly denoted as $(x, \dot{x})$ and $\dot{x}^i$ stands for $\frac{d x^i}{d t}$.

Berwald introduces the connection parameters $G^i_{jk}(x, \dot{x})$ which are homogeneous functions of degree zero in the directional arguments and symmetric in their indices. The covariant derivative of a vector-field $X^i(x, \dot{x})$ with respect to G^i_{jk} is given by

$$\boldsymbol{B}_j X^i = \partial_j X^i - (\dot{\partial}_s X^i)\, G^s_{jr}\, \dot{x}^r + X^s G^i_{js} \,. \tag{1.4}$$

It can be easily varified that the covariant derivatives of the element of support $\dot{x}^i$, unit vector-field $l^i(x, \dot{x})$ and the metric function $F(x, \dot{x})$ vanish whereas the metric tensor-field g_{ij} does not.

The commutation formula for the operators $\boldsymbol{B}_j$ and $\dot{\partial}_k$ is related according to ([2], eqn. (4.11 a))

$$(\boldsymbol{B}_j \dot{\partial}_k - \dot{\partial}_k \boldsymbol{B}_j)\, X^i = - X^h G^i_{hjk} \,. \tag{1.5}$$

where $G^i_{hjk} = \dot{\partial}_h G^i_{jk}$, and for $\boldsymbol{B}_j$

$$2\, \boldsymbol{B}_{[j} \boldsymbol{B}_{k]} X^i = X^h H^i_{jkh} - (\dot{\partial}_h X^i)\, H^h_{jk} \,, \tag{1.6}$$

where the tensor-field H^i_{jkh} is defined by

$$H^i_{jkh} = 2 \left\{ \partial_{[j} G^i_{k]h} - \dot{x}^r G^s_{r[j} G^i_{k]hs} + G^i_{s[j} G^s_{k]h} \right\} (^2) \,, \tag{1.7}$$

and related with H^i_{jk} by $H^i_{jkh} = H^i_{jkh}\, \dot{x}^h$.

2. - Covariant Differentiation

Similar to the definition of Imai [5] we define

$$\pi^i_{jk} = G^i_{jk} + U^i_{jk} \,, \tag{2.1}$$

as a semi-symmetric connections in F_n. We have noted here

$$U^i_{jk} = \delta^i_j\, p_k - g_{jk}\, p^i \,, \tag{2.2}$$

where p_j is a covariant vector related with the contravariant vector p^i by $p_j = g_{ji}\, p^i$. The vector-field p_j, in general, is not a homogeneous function in the directional arguments. Thus, the connection parameters π^i_{jk} are not homogeneous functions in $\dot{x}^i$'s and also not symmetric in its indices. The skew-symmetric part of (2.2) is given by

$$U^i_{[jk]} = \delta^i_{[j}\, p_{k]} \,, \tag{2.3}$$

because the metric tensor is symmetric in its indices.

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(2) The symmetric and skew-symmetric parts of a geometric object Ω_{ivj} with respect to the indices i, j have been defined by

$$\Omega_{(i|v|j)} = \tfrac{1}{2} (\Omega_{ivj} + \Omega_{jvi})$$

and

$$\Omega_{[i|v|j]} = \tfrac{1}{2} (\Omega_{ivj} - \Omega_{jvi})$$

respectively where the index v enclosed withen two vertical bars contributes nothing towards the symmetric and skew-symmetric parts of the object.

The covariant derivative with respect to π^i_{jk} is defined in the manner analogous to (1.4). Denoting by $\boldsymbol{s}_j X^i$ the corresponding covariant derivative of a vectorfield $X^i(x, \dot{x})$ we thus have

$$\boldsymbol{s}_j X^i = \partial_j X^i - (\dot{\partial}_s X^i)\, \pi^s_{jr}\, \dot{x}^r + X^s \pi^i_{js}\,, \tag{2.4}$$

which can be reduced, by the help of (1.4) and (2.1), to

$$\boldsymbol{s}_j X^i = \boldsymbol{B}_j X^i + X^s U^i_{js} - (\dot{\partial}_s X^i)\, U^s_{jr}\, \dot{x}^r\,. \tag{2.5}$$

THEOREM 2.1. — *The covariant derivatives* $\boldsymbol{s}_j$ *of* $\dot{x}^i$, *the unit vector* $\boldsymbol{l}^i(x, \dot{x})$ *and the metric function* $F(x, \dot{x})$ *vanish.*

Proof. - We have already mentioned earlier that the covariant derivatives B_j of $\dot{x}^i$, $\boldsymbol{l}^i$ and F vanish. Applyng (2.4) we can easily see that $\boldsymbol{s}_j \dot{x}^i = 0$. Considering (2.5) for $\boldsymbol{l}^i$ we obtain

$$\boldsymbol{s}_j \boldsymbol{l}^i = \boldsymbol{l}^s U^i_{js} - (\dot{\partial}_s \boldsymbol{l}^i)\, U^s_{jr}\, \dot{x}^r\,, \tag{2.6}$$

where $\boldsymbol{B}_j \boldsymbol{l}^i = 0$. Putting $\boldsymbol{l}^i = \dfrac{\dot{x}^i}{F}$ and using (2.2), the above relation is reduced to

$$\boldsymbol{s}_j \boldsymbol{l}^i = F^{-2}\, (\dot{\partial}_s F)\, \dot{x}^i \left\{ \delta^s_j\, p^m g_{rm}\, \dot{x}^r - p^s g_{jr}\, \dot{x}^r \right\}.$$

Applying (1.2) we can see $\boldsymbol{s}_j \boldsymbol{l}^i = 0$. In the same manner, with the help of (2.1), (2.2) and (1.2), we get the truth of the above theorem.

The derivative of (2.5) may be generalised to an arbitrary tensor-field. Let $T^{\cdots\cdots}_{\cdots\cdots}(x, \dot{x})$ be a tensor of rank (r, s). Its covariant derivative $\boldsymbol{s}_k$ is given by

$$\begin{aligned} \boldsymbol{s}_k T^{\cdots\cdots}_{\cdots\cdots} &= \boldsymbol{B}_k T^{\cdots\cdots}_{\cdots\cdots} + \sum_m^r T^{\cdots m \cdots}_{\cdots\cdots}\, U^l_{km} \\ &- \sum_m^s T^{\cdots\cdots}_{\cdots m \cdots}\, U^m_{kj} - (\dot{\partial}_t T^{\cdots\cdots}_{\cdots\cdots})\, U^t_{km}\, \dot{x}^m\,. \end{aligned} \tag{2.7}$$

COROLLALLY 2.1. — *The semi-symmetric connections are not metric in* F_n.

Proof. - Using (2.7) for g_{jk}, we have

$$\boldsymbol{s}_h g_{jk} = \boldsymbol{B}_h g_{jk} - C_{sjk}\, U^s_{hr}\, \dot{x}^r - g_{sk}\, U^s_{hj} - g_{js}\, U^s_{hk}\,,$$

which, with the help of (2.2) and $p_j = g_{js} p^s$, we obtain

$$\boldsymbol{s}_h g_{jk} = \boldsymbol{B}_h g_{jk} - C_{jkh}\, p_r\, \dot{x}^r + C_{jks}\, p^s g_{hr}\, \dot{x}^r\,.$$

Thus, $\boldsymbol{s}_h g_{jk}$ donot vanish in F_n. So the metric tensor-field g_{jk} is not a covariant consant with respect to π^i_{jk}.

3. - Commutation formulae

It is noted earlier that the semi-symmetric connections π^i_{jk} are not homogeneous functions in $\dot{x}^{i'}$ s. The commutation formulae evolving $\boldsymbol{s}_j$ and the operators $\dot{\partial}_k$, $\dot{\nabla}_k$ and $\boldsymbol{B}_k$ possess some more terms due to the presence of the vector-field p^i in π^i_{jk}.

Theorem 3.1. — *For the vector-field* $X^i(x, \dot{x})$ *the operators* $\boldsymbol{s}_j$ *and* $\dot{\partial}_k$ *commute according to*

$$(\boldsymbol{s}_j \dot{\partial}_k - \dot{\partial}_k \boldsymbol{s}_j) X^i = - X^s \pi^i_{kjs} + (\dot{\partial}_s X^i) \pi^s_{kjr} \dot{x}^r . \tag{3.1}$$

Proof. - Applying (2.4) for the vector-field $\dot{\partial}_k X^i$ we may obtain

$$\boldsymbol{s}_j \dot{\partial}_k X^i = \partial_j \dot{\partial}_k X^i - (\dot{\partial}_s \dot{\partial}_k X^i) \pi^s_{jr} \dot{x}^r + (\dot{\partial}_k X^s) \pi^i_{js} - (\dot{\partial}_s X^i) \pi^s_{jk} . \tag{3.2}$$

Taking the partial differentiation of (2.4) with respect to $\dot{x}^k$, we get

$$\begin{aligned} \dot{\partial}_k \boldsymbol{s}_j X^i = {} & \dot{\partial}_k \partial_j X^i - (\dot{\partial}_k \dot{\partial}_s X^i) \pi^s_{jr} \dot{x}^r - (\dot{\partial}_s X^i) \pi^s_{kjr} \dot{x}^r \\ & - (\dot{\partial}_s X^i) \pi^s_{jk} + (\dot{\partial}_k X^s) \pi^i_{js} + X^s \pi^i_{kjs} , \end{aligned} \tag{3.3}$$

where we have noted $\pi^i_{kjs} = \dot{\partial}_k \pi^i_{js}$. Substracting (3.3) from (3.2) we obtain (3.1).

Theorem 3.1. can be generalised for a tensor $T^{\cdots\cdots}_{\cdots\cdots}(x, \dot{x})$ of rank (m, r) and can be written directly from (3.1):

$$(\boldsymbol{s}_h \dot{\partial}_k - \dot{\partial}_k \boldsymbol{s}_h) T^{\cdots\cdots}_{\cdots\cdots} = - \sum_t^m T^{\cdots t \cdots}_{\cdots\cdots} \pi^i_{khs} + \sum_t^r T^{\cdots\cdots}_{\cdots t \cdots} \pi^i_{khj} + (\dot{\partial}_s T^{\cdots\cdots}_{\cdots\cdots}) \pi^s_{kht} \dot{x}^t .$$

Theorem 3.2. — *For a vector-field* $X^i(x, \dot{x})$ *the operators* $\boldsymbol{s}_j$ *and* $\dot{\nabla}_k$ *commute according to*

$$\begin{aligned} (\boldsymbol{s}_j \dot{\nabla}_k - \dot{\nabla}_k \boldsymbol{s}_j) X^i = {} & - X^s (\pi^i_{kjs} - \boldsymbol{s}_j C^i_{ks}) \\ & + (\boldsymbol{s}_s X^i) C^s_{kj} + (\dot{\partial}_s X^i) \{ \pi^s_{kjr} \dot{x}^r + \pi^s_{jk} \} . \end{aligned} \tag{3.4}$$

Proof. - $\dot{\nabla}_k X^i$ is a tensor-field and defined by (1.3). Applying (1.3) by $\boldsymbol{s}_j$, we get

$$\boldsymbol{s}_j \dot{\nabla}_k X^i = \boldsymbol{s}_j \dot{\partial}_k X^i + (\boldsymbol{s}_j C^i_{kr}) X^r + C^i_{kr} \boldsymbol{s}_j X^r . \tag{3.5}$$

Using (1.3) for $\boldsymbol{s}_j X^i$ yields

$$\dot{\nabla}_k \boldsymbol{s}_j X^i = \dot{\partial}_k \boldsymbol{s}_j X^i + C^i_{kr} \boldsymbol{s}_j X^r - C^s_{kj} \boldsymbol{s}_s X^i . \tag{3.6}$$

Taking the consideration of (3.5), (3.6) and (3.1) we get (3.4).

Due to the presence of the vector-field U^i_{jk}, the connection parameters π^i_{jk} are not symmetric in its indices. So the tensor-type quantities R^i_{jkh} evolved due to the commutation formula $\boldsymbol{s}_j$ are quite different from the curvature tensor-field H^i_{jkh}.

THEOREM 3.3. — *For the operator* $\boldsymbol{s}_j$ *the commutation formula of a vector-field* $X^i(x, \dot{x})$ *is given by*

$$2\, \boldsymbol{s}_{[j}\, \boldsymbol{s}_{k]}\, X^i = X^h R^i_{jkh} - (\dot{\partial}_s X^i)\, R^s_{jkh}\, \dot{x}^h - 2\, (\dot{\partial}_s X^i)\, \pi^s_{[jk]}\,, \tag{3.7}$$

where

$$R^i_{jkh} = 2 \left\{ \partial_{[j}\, \pi^i_{k]h} - \pi^i_{m[k|h|}\, \pi^m_{j]r}\, \dot{x}^r + \pi^m_{[j|h|}\, \pi^i_{k]m} + \pi^i_{mh}\, \pi^m_{[kj]} \right\} . \tag{3.8}$$

Proof. - The covariant derivative of a vector-field $X^i(x, \dot{x})$ with respect to π^i_{jk} is given in (2.4). Taking the covariant derivative of $\boldsymbol{s}_k X^i$ with respect to x^j, we get

$$\begin{aligned} \boldsymbol{s}_j(\boldsymbol{s}_k X^i) &= \partial_j\, \boldsymbol{s}_k X^i - (\dot{\partial}_s\, \boldsymbol{s}_k X^i)\, \pi^s_{jr}\, \dot{x}^r \\ &\quad + (\boldsymbol{s}_k X^s)\, \pi^i_{js} - (\boldsymbol{s}_s X^i)\, \pi^s_{jk} . \end{aligned} \tag{3.9}$$

Substituting (2.4) in (3.9) and carrying out the operation as indicated in (3.9) and collecting the terms we obtain

$$\begin{aligned} \boldsymbol{s}_j\, \boldsymbol{s}_k\, X^i &= X^h \left[\partial_j\, \pi^i_{kh} - \pi^i_{mkh}\, \pi^m_{jr}\, \dot{x}^r + \pi^m_{kh}\, \pi^i_{jm} - \pi^i_{mh}\, \pi^m_{jk} \right] \\ &\quad - (\dot{\partial}_s X^i) \left[\partial_j\, \pi^s_{kh} - \pi^s_{mkh}\, \pi^m_{jr}\, \dot{x}^r - \pi^m_{jh}\, \pi^s_{km} - \pi^s_{mh}\, \pi^m_{jk} \right] \\ &\quad + \left\{ \partial_j\, \partial_k X^i - 2\, \left((\dot{\partial}_s\, \partial_{(j} X^i)\, \pi^s_{k)r}\, \dot{x}^r + (\partial_{(j} X^s)\, \pi^i_{k)s} - (\dot{\partial}_s X^m)\, \pi^i_{(k|m|}\, \pi^s_{j)t}\, \dot{x}^t \right) \right. \\ &\quad \left. - (\dot{\partial}_s\, \dot{\partial}_m X^i)\, \pi^m_{kr}\, \pi^s_{jt}\, \dot{x}^r\, \dot{x}^t \right\} - (\dot{\partial}_s X^i)\, \pi^s_{jk} . \end{aligned} \tag{3.10}$$

Inspection of the above relation shows that the expression contains in the curly bracket is symmetric in its indices j, k.

Thus, taking the skew-symmetric part of (3.10), in view of (3.8), we get (3.7).

THEOREM 3.4. — *The tensor-type quantities* R^i_{jkh} *and the tensor-field* H^i_{jkh} *are related by*

$$\begin{aligned} R^i_{jkh} &= H^i_{jkh} + 2 \left[\delta^i_{[k} \left(L_{j]}\, p_h + g_{j]h}\, p^s\, p_s \right) + L_{[k} \left(g_{j]h}\, p^i \right) \right] \\ &\quad + 2 \left[g_{r[j}\, G^i_{k]hs}\, p^s\, \dot{x}^r + p_{[j}\, G^i_{k]h} \right], \end{aligned} \tag{3.11}$$

where the operator L_j *of a tensor-field* $T^i_k(x, \dot{x})$ *is given by*

$$L_j\, T^i_k = \boldsymbol{B}_j\, T^i_k - (\dot{\partial}_j\, T^i_k)\, p_r\, \dot{x}^r + (\dot{\partial}_s\, T^i_k)\, g_{jr}\, p^s\, \dot{x}^r . \tag{3.12}$$

Proof. - The tensor-type quantities R^{i}_{jkh} are defined in (3.8). Substituting (2.1) and its partial differentiation of (2.1) in (3.8), carrying out the operation as indicated in (3.8) and arranging the terms we get

$$\begin{aligned} R^{i}_{jkh} = H^{i}_{jkh} &+ 2 \{ \delta^{i}_{[k} (\partial_{j]} p_h - (\dot{\partial}_{|s} p_{h|}) G^{s}_{j]r} \dot{x}^r - (\dot{\partial}_{j]} p_h) p_r \dot{x}^r \\ &+ g_{j]r} (\dot{\partial}_s p_h) p^s \dot{x}^r - G^{s}_{j]h} p_s + g_{j]h} p^s p_s) \} \\ &- 2 \{ \partial_{[j} (g_{k]h} p^i) - (\dot{\partial}_s p^i g_{h[k}) G^{s}_{j]r} \dot{x}^r - (\dot{\partial}_{[j} g_{k]h} p^i) p_r \dot{x}^r \\ &+ (\dot{\partial}_s p^i g_{h[k}) g_{j]r} p^s \dot{x}^r + p^i g_{s[j} G^{s}_{k]h} + p^s g_{h[k} G^{i}_{j]s} \}, \end{aligned}$$

where we have taken the consideration of (1.7). The above relation, in view of (3.12) and (1.4) reduces to (3.11).

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