Semi-Symmetric connections in a Finsler space

Recently Imai [5] has studied the semi-symmetric metric connections in a Riemannian space. Similar to his definition, we define the connections with respect to Berwald’s connections in a Finsler space. The covariant derivative of a metric tensor with respect to these connections do not vanish (cf. corollary (2.1)). So we call these connections as semi-symmetric connections in the Finsler space. In this paper we have defined the covariant derivative of a vector field for these connections and studies its further effect. Different types of commutation formulae are reduced and at the end, the relation of $R^i_{jkh}$ (1) and $H_{jkh}$ is taken into consideration.

1. Preliminaries

Let $F_n$ be an n-dimensional Finsler space in which the metric function $F(x, \dot{x})$ (1) satisfies the requisite conditions. The entities are given by

\[ g_0 (x, \dot{x}) = \frac{1}{2} \delta_j \delta_i F, \quad g_0^0 g_0 = \delta_i, \]

constitute a tensor-field which is symmetric and positively homogeneous function of degree zero in the directional arguments. On transvection, it gives

\[ g_0 \dot{x}^i = F \delta_i F. \]

The partial differentiation of $g_0$ with respect to $\dot{x}^h$ becomes a tensor-field $C_{hij}$ which on transvection with $g_\alpha^\beta$ defines another tensor-field $C_{\alpha j}^i$, symmetric in its indices. For these entities the covariant differentiation of a vector-field $X^i (x, \dot{x})$ with respect to $\dot{x}^1$ has been defined by ([2], p. 187)

\[ \dot{\nabla}_j X^i = \delta_j X^i + C_{ij}^k X^k. \]

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(1) $R^i_{jkh}$ is the curvature tensor-type quantities defined in (3.8) and $H^i_{jkh}$ is the curvature tensor-field due to the Berwald’s connections.

(*) The line-element $(x^i, \dot{x}^1)$ is briefly denoted as $(x, \dot{x})$ and $\dot{x}^1$ stands for $\frac{dx^i}{dt}$.
Berwald introduces the connection parameters \( G^i_{jk}(x, \dot{x}) \) which are homogeneous functions of degree zero in the directional arguments and symmetric in their indices. The covariant derivative of a vector-field \( X^i(x, \dot{x}) \) with respect to \( G^i_{jk} \) is given by

\[
B_j X^i = \dot{\lambda}_j X^i - (\dot{\lambda}_j X^i) G^i_{jk} \dot{x}^k + X^k G^i_{jk}.
\]

It can be easily verified that the covariant derivatives of the element of support \( \dot{x}^i \), unit vector-field \( f(x, \dot{x}) \) and the metric function \( F(x, \dot{x}) \) vanish whereas the metric tensor-field \( g_{ij} \) does not.

The commutation formula for the operators \( B_j \) and \( \dot{\lambda}_j \) is related according to ([2], eqn. (4.11 a))

\[
(B_j \dot{\lambda}_k - \dot{\lambda}_j B_k) X^i = -X^b G^i_{bjk},
\]

where \( G^i_{jk} = \dot{\lambda}_k G^i_{jk} \), and for \( B_j \)

\[
2 B_j B_k X^i = X^b H^i_{jkb} - (\dot{\lambda}_k X^i) H^i_{jbk},
\]

where the tensor-field \( H^i_{jkb} \) is defined by

\[
H^i_{jkb} = 2 \{ \delta^i_k G^i_{jbk} - \dot{x}^i G^i_{jbk} G^i_{klb} + G^i_{ijl} G^i_{klb} \} \tag{\*},
\]

and related with \( H^i_{jkb} \) by \( H^i_{jkb} = H^i_{kb} \dot{x}^b \).

2. Covariant Differentiation

Similar to the definition of Imai [5] we define

\[
\pi^i_{jk} = G^i_{jk} + U^i_{jk},
\]

as a semi-symmetric connections in \( F_\pi \). We have noted here

\[
U^i_{jk} = \delta^i_k p_j - g_{jk} p^i,
\]

where \( p_j \) is a covariant vector related with the contravariant vector \( p^i \) by \( p_j = g_{jk} p^i \). The vector-field \( p_j \), in general, is not a homogeneous function in the directional arguments. Thus, the connection parameters \( \pi^i_{jk} \) are not homogeneous functions in \( \dot{x}^i \)’s and also not symmetric in its indices. The skew-symmetric part of (2.2) is given by

\[
U^i_{[jk]} = \delta^i_{[j} p_{k]},
\]

because the metric tensor is symmetric in its indices.

\( ^* \) The symmetric and skew-symmetric parts of a geometric object \( \Omega_{ij} \) with respect to the indices i, j have been defined by \( \Omega_{[i|j]} = \frac{1}{2} (\Omega_{ij} + \Omega_{ji}) \) and \( \Omega_{i|j]} = \frac{1}{2} (\Omega_{ij} - \Omega_{ji}) \) respectively where the index \( [\cdot] \) enclosed within two vertical bars contributes nothing towards the symmetric and skew-symmetric parts of the object.
The covariant derivative with respect to \( \pi_{ik} \) is defined in the manner analogous to (1.4). Denoting by \( s_j X^i \) the corresponding covariant derivative of a vectorfield \( X^i (x, \hat{x}) \) we thus have

\[
s_j X^i = \partial_j X^i - (\hat{\partial}_j X^i) \pi_{jr}^i \hat{x}^r + X^i \pi_{jr}^i \hat{x}^r,
\]

which can be reduced, by the help of (1.4) and (2.1), to

\[
s_j X^i = B_i X^i + X^i U^j_{\langle j} \hat{x}^r \rangle - (\hat{\partial}_j X^i) U^j_{\langle j} \hat{x}^r \rangle.
\]

**Theorem 2.1.** — *The covariant derivatives \( s_j \) of \( \hat{x}^i \), the unit vector \( \hat{t} (x, \hat{x}) \) and the metric function \( F (x, \hat{x}) \) vanish.*

**Proof.** — We have already mentioned earlier that the covariant derivatives \( B_j \) of \( \hat{x}^i \), \( \hat{t} \) and \( F \) vanish. Applying (2.4) we can easily see that \( s_j \hat{x}^i = 0 \). Considering (2.5) for \( \hat{t} \) we obtain

\[
s_j \hat{t} = \hat{t} U^j_{\langle j} \hat{x}^r \rangle - (\hat{\partial}_j \hat{t}) U^j_{\langle j} \hat{x}^r \rangle,
\]

where \( B_j \hat{t} = 0 \). Putting \( \hat{t} = \frac{\hat{x}^i}{F} \) and using (2.2), the above relation is reduced to

\[
s_j \hat{t} = F^{-3} \left( \hat{\partial}_j F \right) \hat{x}^i \partial^i \delta^m_p g_{mr} \hat{x}^r - p^m g_{pr} \hat{x}^r \hat{x}^i.
\]

Applying (1.2) we can see \( s_j \hat{t} = 0 \). In the same manner, with the help of (2.1), (2.2) and (1.2), we get the truth of the above theorem.

The derivative of (2.5) may be generalised to an arbitrary tensor-field. Let \( T \) be a tensor of rank \( (r, s) \). Its covariant derivative \( s_k \) is given by

\[
s_k T = B_k T + \sum_m T U^i_{km} U^m_{\langle k} U^r_{\langle r} \hat{x}^m.
\]

**Corollary 2.1.** — *The semi-symmetric connections are not metric in \( F_a \).*

**Proof.** — Using (2.7) for \( g_{jk} \), we have

\[
s_k g_{jk} = B_k g_{jk} - C_{jk} U^i_{lr} \hat{x}^r - g_{ik} U^i_{jl} - g_{ji} U^i_{hk},
\]

which, with the help of (2.2) and \( p_i = g_{ij} p^j \), we obtain

\[
s_k g_{jk} = B_k g_{jk} - C_{jk} p_l \hat{x}^r + C_{jk} p^l g_{lr} \hat{x}^r.
\]

Thus, \( s_k g_{jk} \) do not vanish in \( F_a \). So the metric tensor-field \( g_{jk} \) is not a covariant constant with respect to \( \pi^i_{jk} \).
3. - Commutation Formulae

It is noted earlier that the semi-symmetric connections $\pi^k_{jk}$ are not homogeneous functions in $\dot{x}^i$. The commutation formulae evolving $s_i$ and the operators $\dot{s}_k$, $\ddot{\gamma}_k$ and $B_k$ possess some more terms due to the presence of the vector-field $p^1$ in $\pi^k_{jk}$.

**Theorem 3.1.** — For the vector-field $X^i(x, \dot{x})$ the operators $s_i$ and $\dot{s}_k$ commute according to

$$\left(s_i \dot{s}_k - \dot{s}_k s_i\right) X^j = -X^j \pi^i_{jk} + \left(\dot{\gamma}_i \pi^i_{jk}\right) \dot{x}^j.$$

**Proof.** — Applying (2.4) for the vector-field $\dot{x} X^i$ we may obtain

$$s_j \dot{s}_k X^i = \dot{s}_j \dot{s}_k X^i - \left(\dot{s}_k \pi^j_{ik}\right) \dot{x}^j + \left(\dot{s}_k X^j\right) \pi^i_{jk} - \left(\dot{s}_i \pi^j_{ik}\right) \dot{x}^j.$$

Taking the partial differentiation of (2.4) with respect to $\dot{x}^i$, we get

$$\dot{s}_j X^i = \dot{s}_j \dot{s}_k X^i - \left(\dot{s}_k \pi^j_{ik}\right) \dot{x}^j - \left(\dot{s}_i \pi^j_{ik}\right) \dot{x}^i,$$

where we have noted $\pi^i_{jk} = \pi_{jk}$. Subtracting (3.3) from (3.2) we obtain (3.1).

Theorem 3.1 can be generalised for a tensor $T(x, \dot{x})$ of rank $(m, r)$ and can be written directly from (3.1):

$$(s_k \dot{s}_k - \dot{s}_k s_k) T = -\sum_{t}^{m} T_{ik} \pi^i_{kt} X^t + \sum_{t}^{r} T_{ik} \pi^i_{kt} \pi^k_{jt} \dot{x}^t.$$

**Theorem 3.2.** — For a vector-field $X^i(x, \dot{x})$ the operators $s_i$ and $\dot{\gamma}_k$ commute according to

$$\left(s_i \dot{\gamma}_k - \dot{\gamma}_k s_i\right) X^i = -X^i \left(\pi^j_{ik} - s_j C^j_{ik}\right)$$

$$+ \left(s_i \pi^j_{ik}\right) \dot{x}^j + \pi^j_{ik} \dot{x}^j.$$

**Proof.** — $\dot{\gamma}_k X^i$ is a tensor-field and defined by (1.3). Applying (1.3) by $s_i$, we get

$$s_i \dot{\gamma}_k X^i = s_i \dot{\gamma}_k X^i + \left(s_i C^j_{ik}\right) X^i + C^j_{ik} s_j X^i.$$

Using (1.3) for $s_i X^i$ yields

$$\dot{\gamma}_k s_i X^i = \dot{s}_k s_i X^i + C^j_{ik} s_j X^i - C^j_{ik} s_i X^i.$$

Taking the consideration of (3.5), (3.6) and (3.1) we get (3.4).
Due to the presence of the vector-field $U_{jh}$, the connection parameters $\gamma^j_{kh}$ are not symmetric in its indices. So the tensor-type quantities $R^i_{jkh}$ evolved due to the commutation formula $s_j$ are quite different from the curvature tensor-field $H^i_{kh}$.

**Theorem 3.3.** — For the operator $s_j$ the commutation formula of a vector-field $X^j(x, \dot{x})$ is given by

\[
2 s_j s_k X^l = X^h R^l_{jkh} - (\dot{\gamma}^l_{kh}) R^h_{jkh} \dot{x}^l - 2 ( \dot{\gamma}^l_{kh} ) \gamma^m_{jkl} ,
\]

where

\[
R^l_{jkh} = 2 \left[ \gamma^l_{kh} \gamma^m_{jkl} \dot{x}^m + \gamma^m_{jkl} \gamma^m_{kln} + \gamma^m_{jkl} \gamma^m_{kn} \right] .
\]

**Proof.** — The covariant derivative of a vector-field $X^j(x, \dot{x})$ with respect to $\gamma^j_{kh}$ is given in (2.4). Taking the covariant derivative of $s_k X^l$ with respect to $x^i$, we get

\[
s_j (s_k X^l) = \partial_j s_k X^l - (\dot{\gamma}^l_{kh}) s_k \gamma^m_{jkl} \dot{x}^m
\]

\[
+ (s_k X^l) \gamma^m_{jkl} - (s_k X^l) \gamma^m_{jkl} .
\]

Substituting (2.4) in (3.9) and carrying out the operation as indicated in (3.9) and collecting the terms we obtain

\[
s_j s_k X^l = X^h \left[ \partial_j \gamma^l_{kh} - \gamma^m_{kh} \gamma^m_{jkl} \dot{x}^m + \gamma^m_{kh} \gamma^m_{jkl} - \gamma^m_{kh} \gamma^m_{jkl} \right]
\]

\[
- (\dot{\gamma}^l_{kh}) \left[ \partial_j \gamma^m_{jkl} \dot{x}^m - \gamma^m_{kh} \gamma^m_{jkl} + \gamma^m_{kh} \gamma^m_{jkl} \right]
\]

\[
+ \partial_j \gamma^l_{kh} X^l - 2 \left( (\dot{\gamma}^l_{kh}) \gamma^m_{jkl} \dot{x}^m + (\dot{\gamma}^l_{kh}) \gamma^m_{jkl} \gamma^m_{jkl} \dot{x}^m \right)
\]

\[
- (\dot{\gamma}^l_{kh}) \gamma^m_{jkl} \gamma^m_{jkl} \dot{x}^m - (\dot{\gamma}^l_{kh}) \gamma^m_{jkl} .
\]

Inspection of the above relation shows that the expression contains in the curly bracket is symmetric in its indices $j, k$.

Thus, taking the skew-symmetric part of (3.10), in view of (3.8), we get (3.7).

**Theorem 3.4.** — The tensor-type quantities $R^i_{jkh}$ and the tensor-field $H^i_{kh}$ are related by

\[
R^i_{jkh} = H^i_{jkh} + 2 \left[ \delta^i_{kh} (L^j_{lh} p_l + g^j_{lh} p^l) + L^i_{kh} (g^j_{lh} p^l) \right]
\]

\[
+ 2 \left[ g^j_{lh} G^i_{klh} (p^l \dot{x}^m + p_l G^i_{klh}) \right] ,
\]

where the operator $L^i_j$ of a tensor-field $T^i_j(x, \dot{x})$ is given by

\[
L^i_j T^j_k = B^i_j T^j_k - (\dot{\gamma}^i_{kh}) p_l \dot{x}^l + (\dot{\gamma}^i_{kh}) g^j_{lh} p^l \dot{x}^l .
\]
Proof. - The tensor-type quantities $R_{ab}^i$ are defined in (3.8). Substituting (2.1) and its partial differentiation of (2.1) in (3.8), carrying out the operation as indicated in (3.8) and arranging the terms we get

$$R_{ab} = H_{ab}^i + 2 \left( \delta_{ij} (\delta_b p_i) \right) \left( \delta_{ij} (\delta_b p_i) \right) G_{ij}^i \delta^{\alpha} \delta^{\beta} + \left( \delta_{ij} (\delta_b p_i) \right) \left( \delta_{ij} (\delta_b p_i) \right) \delta^{\alpha} \delta^{\beta}$$

$$+ \left( \delta_{ij} (\delta_b p_i) \right) \left( \delta_{ij} (\delta_b p_i) \right) G_{ij}^i \delta^{\alpha} \delta^{\beta} + \left( \delta_{ij} (\delta_b p_i) \right) \left( \delta_{ij} (\delta_b p_i) \right) \delta^{\alpha} \delta^{\beta}$$

$$- 2 \left( \delta_{ij} (g_{ijb} p^i) \right) \left( \delta_{ij} (g_{ijb} p^i) \right) G_{ij}^i \delta^{\alpha} \delta^{\beta} + \left( \delta_{ij} (g_{ijb} p^i) \right) \left( \delta_{ij} (g_{ijb} p^i) \right) \delta^{\alpha} \delta^{\beta}$$

$$+ \left( \delta_{ij} (g_{ijb} p^i) \right) \left( \delta_{ij} (g_{ijb} p^i) \right) G_{ij}^i \delta^{\alpha} \delta^{\beta} + \left( \delta_{ij} (g_{ijb} p^i) \right) \left( \delta_{ij} (g_{ijb} p^i) \right) \delta^{\alpha} \delta^{\beta}$$

where we have taken the consideration of (1.7). The above relation, in view of (3.12) and (1.4) reduces to (3.11).

REFERENCES