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Numerical solution of a Goursat problem for a polyvibrating equation of Mangeron (*)

Dedichiamo questo lavoro a MAURO PICONE *nel quinquennio della* « Presentazione di pubblicazioni riguardanti l'attività dell'Istituto per le applicazioni del Calcolo, dal 1927, anno della sua fondazione, al 1960, in cui fu sottratto alla direzione del suo ideatore » [6].

Sunto: Gli AA., continuando la loro serie di ricerche concernenti l'estensione dei problemi di Goursat studiati dall'Illustre Accademico Linceo Mauro Picone nelle Sue geniali Memorie pubblicati più di sessant'anni fa, presentano in ciò che segue soluzione numerica del problema di Goursat per un'equazione polivibrante di Mangeron, allorché una simulazione computeriale dell'analisi eseguita e varie figure illustrativi tridimensionali delle soluzioni di alcuni problemi di Goursat di forma (I.3)-(I.4) si trovano esposti in un'altra Nota.

I. Let $f(x)$ and $g(y)$ be two functions monotonic and sufficiently smooth in the intervals $0 \leq x \leq \alpha$ and $0 \leq y \leq \alpha$, respectively, and such that

$$\left\{ \begin{array}{l} f(0) = g(0) = 0 \;\; , \;\; 0 \leq f'(0)\, g'(0) < 1 \, , \\ 0 \leq f(x) \leq \dfrac{1}{2} x \;\; , \;\; 0 \leq g(y) \leq \dfrac{1}{2} y \, . \end{array} \right. \tag{1.1}$$

Further, let $A = A(x, y)$, $B = B(x, y)$, $C = C(x, y)$, $D = D(x, y)$, $F_i = F_i(x)$ and $G_i(y)$ $(i = 0,1)$ be certain sufficiently smooth functions in $0 \leq x \leq \alpha$, $0 \leq y \leq \alpha$. We assume that

$$F_i(0) = G_i(0) \quad (i = 0,1) \, . \tag{1.2}$$

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Consider the following polyvibrating equation of D. Mangeron (³)

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = A \frac{\partial^2 u}{\partial x \partial y} + B u + C \tag{1.3}$$

in $S : 0 < x < \alpha, 0 < y < \alpha$, subject to the conditions

$$\begin{cases} u(x,y)\Big|_{y=f(x)} = F_0(x) \,, \ \dfrac{\partial^2 u(x,y)}{\partial x \partial y}\Big|_{y=f(x)} = F_1(x) \ (0 \le x \le \alpha) \,, \\ u(x,y)\Big|_{x=g(y)} = G_0(y) \,, \ \dfrac{\partial^2 u(x,y)}{\partial x \partial y}\Big|_{x=g(y)} = G_1(y) \ (0 \le y \le \alpha \,, \end{cases} \tag{1.4}$$

where $u = u(x, y)$ is the unknown function of class C^4 in S. The existence and uniqueness of the solution of the Goursat problem (1.3)-(1.4), with $A \equiv 0$, have been demonstrated in [1] by use of multidimensional integral equations of more complex structures then of the usual Volterra integral equations and by employing certain results on the solutions of Schröder's functional equation (cf. [2]). The case $A(x, y) \neq 0$ can be treated similarly by certain minor modifications in [1].

The problem (1.3)-(1.4) is closely connected with the following Goursat problem which has been considered by M. Picone in his famous Memoires [3] (⁴) sixty years ago:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = A \dfrac{\partial u}{\partial x} + B \dfrac{\partial u}{\partial y} + C u + D \quad ((x,y) \in S) \,, \\ u(x,y)\Big|_{y=f(x)} = F_0(x) \ (0 \le x \le \alpha), u(x,y)\Big|_{x=g(y)} = G_0(y) \ (0 \le y \le \alpha) \,. \end{cases} \tag{1.5}$$

In [4] we have given a numerical scheme for the solution of the problem (1.5). Let us note that the Goursat problem (1.3)-(1.4) is equivalent to the following problem which can be considered as an extension of the Picone's problem (1.5):

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = v \quad , \quad \dfrac{\partial^2 u}{\partial x \partial y} = A v + B u + C \quad ((x,y) \in S) \,, \\ u(x,y)\Big|_{y=f(x)} = F_0(x) \,, \ v(x,y)\Big|_{y=f(x)} = F_1(x) \ (0 \le x \le \alpha) \,, \\ u(x,y)\Big|_{x=g(y)} = G_0(y) \,, \ v(x,y)\Big|_{x=g(y)} = G_1(y) \ (0 \le y \le \alpha) \,. \end{cases} \tag{1.6}$$

(³) Further literature concerning Mangeron equations is to be found in [7]-[13].
(⁴) Cf. also [6].

In the following we shall present a numerical scheme ([5]) for the solution of the Goursat problem (1.3)-(1.4).

II. Let N be a sufficiently large positive integer. Put $h = \dfrac{\alpha}{N}$. Let us divide S into N^2 square elements of size h^2 by the lines $x = r\,h$, $y = s\,h$ $(r, s = 1, 2, \ldots, N-1)$. We shall denote the grid points by $P_{r,s}$ or $P^{(i)}$ with $i = i\,(r, s) = (s-1)\,(N-1) + r$. For convenience we also write $P^{(0)} = P_{0,0} = (0, 0)$.

We shall denote the value of a function $z = z\,(x, y)$ at $P_{r,s}$ by one of the following notation: $z_{r,s} = z\,(r\,h, s\,h) = z\,(P_{r,s}) = z_i = z\,(P^{(i)})$. Further we shall use the symbols $z_{(r,s)}^{(m,n)}$ or $z_i^{(m,n)}$ to denote the values $\dfrac{\partial^{m+n} u}{\partial x^m\,\partial y^n}$ at the node $P^{(i)}$:

$$z_{r,s}^{(m,n)} = z_{,s}^{(m,n)} = \left(\frac{\partial^{m+n} z}{\partial x^m\,\partial y^n}\right)_{r,s}, \quad z_i^{(0,0)} = z_i\,. \tag{II.1}$$

Let us note that

$$\begin{aligned}(4 - h^2 A_{r,s})\,(u_{r+1,s+1} + u_{r-1,s-1}) + (4 + h^2 A_{r,s})\,(u_{r+1,s-1} + u_{r-1,s+1}) \,-\\ - 8\,(u_{r+1,s} + u_{r-1,s} + u_{r,s+1} + u_{r,s-1}) + 4\,(4 - h^2 B_{r,s})\,u_{r,s} = 4\,h^2\,C_{r,s}\end{aligned} \tag{II.2}$$

for $r, s = 1, 2, \ldots, N-1$, which can be easly proved. This 9-points formula provides us the value of u at one of the points $P_{r,s}$, $P_{r-1,s}$, $P_{r,s-1}$, $P_{r,s+1}$, $P_{r+1,s}$, $P_{r-1,s-1}$, $P_{r+1,s-1}$, $P_{r-1,s+1}$ and $P_{r+1,s+1}$ if its values are known at the remaining eight points.

III. We begin with the calculations of the partial derivatives $u_0^{(m,n)}$ of u at $P^{(0)}$, the origin, with $0 < m + n \leq 4$. Let us divide the square element of corners $P_{0,0}$, $P_{1,0}$, $P_{0,1}$, $P_{1,1}$ into nine square elements by the lines $x = \dfrac{h}{3}$, $x = \dfrac{2\,h}{3}$, $y = \dfrac{h}{3}$ and $y = \dfrac{2\,h}{3}$. Let $Q_r\,(h_r, k_r)$ be the curve points on $y = f\,(x)$ for $r = 1, 2, 3$, and $P_r\,(h_r, k_r)$ be the curve points on $x = g\,(y)$ for $r = 4, 5, 6$, where

$$\left\{\begin{aligned} & h_1 = \frac{h}{3}\,, \ h_2 = \frac{2\,h}{3}\,, \ h_3 = h\,; \ h_4 = g\left(\frac{h}{3}\right), \ h_5 = g\left(\frac{2\,h}{3}\right), \ h_6 = g\,(h)\,, \\ & k_1 = f\left(\frac{h}{3}\right), \ k_2 = f\left(\frac{2\,h}{3}\right), \ k_3 = f\,(h)\,; \ k_4 = \frac{h}{3}\,, \ k_5 = \frac{2\,h}{3}\,, \ k_6 = h\,. \end{aligned}\right. \tag{III.1}$$

Let U_r be the value of u at the point Q_r $(r = 1, \ldots, 6)$. We have

$$\begin{aligned} & U_1 = F_0\left(\frac{h}{3}\right), \ U_2 = F_0\left(\frac{2\,h}{3}\right), \ U_3 = F_0\,(h)\,; \\ & U_4 = G_0\left(\frac{h}{3}\right), \ U_5 = G_0\left(\frac{2\,h}{3}\right), \ U_6 = G_0\,(h)\,, \end{aligned} \tag{III.2}$$

([5]) A large set of valuable numerical methods contains a very comprehensive M. A. Sneider Ludovici book [5] dedicated to MAURO PICONE, *Padre del Calcolo numerico*.

by virtue of Eqs (I.4). Then we have the approximate equations

$$U_r = (E_4\, u_0)_r\,,\quad U_r^{(1,1)} = (E_2\, u_0)_r \quad (r = 1, \ldots, 6) \tag{III.3}$$

where

$$(E_j\, z)_i = z + \frac{1}{1!}\,[h_i\, z^{(1,0)} + k_i\, z^{(0,1)}] + \frac{1}{2!}\,[h_i^2\, z^{(2,0)} + 2\, h_i\, k_i\, z^{(1,1)} + k_i^2\, z^{(0,2)}] + \tag{III.4}$$

$$+ \ldots \frac{1}{j!}\,[h_i^j\, z^{(j,0)} + j\, h_i^{j-1}\, k_i\, z^{(j-1,1)} + \frac{j\,(j-1)}{2!}\, h_i^{j-2}\, k_i^2\, z^{(j-2,2)} + \ldots + k_i^j\, z^{(0,j)}]\,.$$

From the 12 equations (III.3) we can compute $u_0^{(1,0)}$, $u_0^{(0,1)}$, $u_0^{(2,0)}$, $u_0^{(0,2)}$, $u_0^{(3,0)}$, $u_0^{(2,1)}$, $u_0^{(1,2)}$, $u_0^{(0,3)}$, $u_0^{(4,0)}$, $u_0^{(3,1)}$, $u_0^{(1,3)}$, $u_0^{(0,4)}$ in terms of the known quantities u_0, $u_0^{(1,1)}$, $u_0^{(2,2)}$, $U_1, \ldots, U_6$, $U_1^{(1,1)}, \ldots, U_6^{(1,1)}$, where

$$u_0^{(2,2)} = A_0\, u_0^{(1,1)} + B_0\, u_0 + C_0 \tag{III.5}$$

by Eqs. (I.3).

We now calculate $u_{1,0}$, $u_{0,1}$ and $u_{1,1}$ by the Taylor expansions:

$$\text{(III.6)}\quad \begin{cases} u_{1,0} = u_0 + \dfrac{h}{1!}\, u_0^{(1,0)} + \dfrac{h^2}{2!}\, u_0^{(2,0)} + \dfrac{h^3}{3!}\, u_0^{(3,0)} + \dfrac{h^4}{4!}\, u_0^{(4,0)}, \\[2ex] u_{0,1} = u_0 + \dfrac{h}{1!}\, u_0^{(0,1)} + \dfrac{h^2}{2!}\, u_0^{(0,2)} + \dfrac{h^3}{3!}\, u_0^{(0,3)} + \dfrac{h^4}{4!}\, u_0^{(0,4)}, \\[2ex] u_{1,1} = u_0 + \dfrac{h}{1!}[u_0^{(1,0)} + u_0^{(0,1)}] + \dfrac{h^2}{2!}[u_0^{(2,0)} + 2\, u_0^{(1,1)} + u_0^{(0,2)}] \end{cases}$$

$$+ \frac{h^3}{3!}[u_0^{(3,0)} + 3\, u_0^{(2,1)} + 3\, u_0^{(1,2)} + u_0^{(0,3)}]$$

$$+ \frac{h^4}{4!}[u_0^{(4,0)} + 4\, u_0^{(3,1)} + 6\, u_0^{(2,2)} + 4\, u_0^{(1,3)} + u_0^{(0,4)}]\,.$$

Next we compute the partial derivatives $U_r^{(m,n)}$ of u at the points $Q^{(r)}$ $(r = 3,6)$ by the following formulas:

$$\text{(III.7)}\quad \begin{cases} U_r^{(1,0)} = (E_3\, u_0^{(1,0)})_r,\; U_r^{(0,1)} = (E_3\, u_0^{(0,1)})_r,\; U_r^{(2,0)} = (E_2\, u_0^{(2,0)})_r,\; U_r^{(0,2)} = (E_2\, u_0^{(0,2)})_r \\ U_r^{(3,0)} = (E_1\, u_0^{(3,0)})_r,\; U_r^{(2,1)} = (E_1\, u_0^{(2,1)})_r,\; U_r^{(1,2)} = (E_1\, u_0^{(1,2)})_r,\; U_r^{(0,3)} = (E_1\, u_0^{(0,3)})_r. \end{cases}$$

Note that

$$\text{(III.8)}\quad \begin{cases} U_3 = F_0\,(h)\,,\; U_3^{(1,1)} = F_1\,(h)\,,\; U_3^{(2,2)} = A_3\, F_1\,(h) + B_3\, F_0\,(h) + C_3\,, \\ U_6 = G_0\,(h)\,,\; U_6^{(1,1)} = G_1\,(h)\,,\; U_6^{(2,2)} = A_6\, G_1\,(h) + B_6\, G_0\,(h) + C_6\,, \end{cases}$$

by virtue of Eqs (I.3), (I.4).

Let $Q_7\,(h_7\,,k_7)$ be the curve point at which the curve $y=f(x)$ first leaves the subsquare $\{h \leq x \leq 2\,h, 0 \leq y \leq h\}$, and $Q_8\,(h_8\,,k_8)$ be the curve point at which the curve $x=g(y)$ first leaves the subsquare $\{0 \leq x \leq h, h \leq y \leq 2\,h\}$. By Eqs (I.3), (1.4) and by the above calculations the numbers $U_6^{(m,n)}$ $(0 \leq m+n \leq 3)$, U_r, $U_r^{(1,1)}$ and $U_r^{(2,2)}$ $(r=7{,}8)$ are all known. Further we have

$$\text{(III.10)} \quad \begin{cases} U_r^{(1,0)} = (E_3\,U_6^{(1,0)})_r\,,\; U_r^{(0,1)} = (E_3\,U_6^{(0,1)})_r\,, \\ U_r^{(2,0)} = (E_2\,U_6^{(2,0)})_r\,,\; U_r^{(0,2)} = (E_2\,U_r^{(0,2)})_r\,, \\ U_r^{(3,0)} = (E_1\,U_6^{(3,0)})_r\,,\; U_r^{(2,1)} = (E_1\,U_6^{(2,1)})_r\,,\; U_r^{(1,2)} = (E_1\,U_6^{(1,2)})_r\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad U_r^{(0,3)} = (E_1\,U_6^{(0,3)}\,)_r \end{cases}$$

for $r=7{,}8$.

Clearly, the point Q_7 will be between the nodes $P_{2,0}$ and $P_{2,1}$ and the point Q_8 will be between $P_{0,2}$ and $P_{1,2}$. The values of $u\,(x\,,y)$ at these nodes can be calculated by the Taylor expansions:

$$\text{(III.11)} \quad \begin{cases} u_{2,0} = U_7 - k_7\,U_7^{(0,1)} + \dfrac{1}{2}\,k_7^2\,U_7^{(2,0)} - \dfrac{1}{6}\,k_7^3\,U_7^{(0,3)}\,, \\[2ex] u_{2,1} = U_7 + (h-k_7)\,U_7^{(0,1)} + \dfrac{1}{2}\,(h-k_7)^2\,U_7^{(0,2)} + \dfrac{1}{6}\,(h-k_7)^3\,U_7^{(0\,3)}\,, \\[2ex] u_{1,2} = U_8 - h_8\,U_8^{(1,0)} + \dfrac{1}{2}\,h_8^2\,U_8^{(2,0)} - \dfrac{1}{6}\,h_8^3\,U_8^{(3,0)}\,, \\[2ex] u_{2,2} = U_8 + (h-h_8)\,U_8^{(1,0)} + \dfrac{1}{2}\,(h-h_8)^2\,U_8^{(2,0)} + \dfrac{1}{6}\,(h-h_8)^3\,U_8^{(3,0)})\,. \end{cases}$$

Continuing this process until we reach the boundary of S we can evaluate step by step $u\,(x, y)$ at the nearest corners $P_{r,s}$ intercepting curve points Q_r, that is the values of $u\,(x, y)$ at $4\,N-1$ nodes surrounding the curves $y=f\,(x)$ and $x=g\,(y)$. The values of $u\,(x, y)$ at the remaining $(N-1)^2$ can be easily calculated by the 9-points formula (II.2). For example, we have

$$\text{(III.12)} \qquad u_{2,2} = \frac{1}{4-h^2\,A_{1,1}}\,[4\,h^2\,C_{1,1} + 8\,(u_{2,1}+u_{1,2}+u_{0,1}+u_{1,0}) -$$

$$-4\,(4-h^2\,B_{1,1})\,u_{1,1} - (4+h^2\,A_{1,1})\,(u_{2,0}+u_{0,2})] - u_{0,0}\,,$$

for $r=s=1$.

Let us note that the formulas (III.11) can be made more accurate considering intermediate points between Q_2 and Q_7, and Q_6 and Q_8, respectively, in continuity with the method given an the beginning of this section.

IV. Clearly, the accuracy of the above described scheme depends on the magnitude of h. For $\alpha=1$ and $N=250$, the error bound is 0.01%. A computer simulation of the above analysis will be presented in a separate paper where we shall also give several three dimensional illustrative figures of the solutions of some Goursat problems of the form (I.3)-(I.4).

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