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The infinitesimal automorphisms on the tangent bundles of order 2 (**)

K. Yano and S. Ishihara [2] defined lifts of tensor fields, affine connections and pseudo-Riemannian metrics of differentiable manifolds M to the tangent bundles $T_2(M)$ of order 2 over M and proved some properties of $T_2(M)$. On the other hand, K. Yano and S. Kobayashi [3] and S. Tanno [1] study infinitesimal transformations in the tangent bundle T(M) over M. In the present paper, we shall study infinitesimal transformations of $T_2(M)$ and prove some results analogous to those given in [3]. In § 1, we fix notations and terminologies and recall some well known results concerning lifts to $T_2(M)$. Our assertions will be stated in Theorems A and B. These Theorems A and B will be proved in § 2 and § 3 respectively.

§ 1. Introduction

Let M be a C^∞-differentiable manifold of dimension n with an affine connection ∇ and R the real line. The tangent bundle $T_2(M)$ of order 2 over M is the space of equivalence classes of mappings from R into M, the equivalence relation being defined as follows: two mappings F and G from R into M are equivalent to each other if, in a local coordinate neighborhood $(U, (x^i))$ containing a point $p \in M$, they satisfy the conditions

$$F(0) = G(0) = p\,, \quad \frac{d F^i}{d t}(0) = \frac{d G^i}{d t}(0)\,, \quad \frac{d^2 F^i}{d t^2}(0) = \frac{d^2 G^i}{d t^2}(0)\,, \tag{1.1}$$

where $F^i(t)$ and $G^i(t)$ are the coordinates of points F(t) and G(t) in $(U, (x^i))$ respectively, where the indices h, i, j, k, m, p, q, r, s and t run over the range {1, 2, ..., n} and the summation convention will be used with respect to these indices. We call an equivalence class containing F the 2-jet of M determined by F and denote it by $j_p^2(F)$. If we denote by $T_2(M)$ the set of all 2-jets of M, $T_2(M)$ has the natural bundle structure over M, its bundle projection $\pi_2 : T_2(M) \to M$ being defined by $\pi_2(j_p^2(F)) = p$. If we take an 2-jet $j_p^2(F)$ belonging to $\pi_2^{-1}(U)$ and put

$$x^i = F^i(0)\,, \qquad y^i = \frac{d F^i}{d t}(0)\,, \qquad z^i = \frac{d^2 F^i}{d t^2}(0)\,, \tag{1.2}$$

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then we see that 2-jet $j^2_p(F)$ is expressed in a unique way by the set (x^i, y^i, z^i). Thus a system of coordinates (x^i, y^i, z^i) is introduced in the open set $\pi_2^{-1}(U)$ in $T_2(M)$. We call (x^i, y^i, z^i) the induced coordinates in $\pi_2^{-1}(U)$ from $(U, (x^i))$.

Prolongations of tensor fields on M *to* $T_2(M)$ (cf. [2]). For any function f on M, its prolongations f^0, f^I and f^{II} to $T_2(M)$ are functions on $T_2(M)$ and expressed by the form

$$
\begin{aligned}
&f^0 : f(x^i), \qquad f^I : y^s \partial_s f(x^i), \\
&f^{II} : z^s \partial_s f(x^i) + y^s y^t \partial_s \partial_t f(x^i)
\end{aligned} \tag{1.3}
$$

with respect to the induced coordinate system (x^i, y^i, z^i), where $\partial_s = \partial/\partial x^s$.

For any vector field X on M, its prolongations X^0, X^I and X^{II} are vector fields on $T_2(M)$ having the following properties;

$$
\begin{array}{lll}
X^0 f^0 = 0, & X^0 f^I = 0, & X^0 f^{II} = (X f)^0, \\
X^I f^0 = 0, & X^I f^I = \frac{1}{2}(X f)^0, & X^I f^{II} = (X f)^I, \\
X^{II} f^0 = (X f)^0, & X^{II} f^I = (X f)^I, & X^{II} f^{II} = (X f)^{II}
\end{array} \tag{1.4}
$$

for any function f on M. Then X^0, X^I and X^{II} have respectively the following local expressions:

$$
\begin{aligned}
&X^0 : \begin{pmatrix} 0 \\ 0 \\ X^i \end{pmatrix}, \qquad X^I : \begin{pmatrix} 0 \\ \frac{1}{2} X^i \\ (\partial_s X^i) y^s \end{pmatrix}, \\
&X^{II} : \begin{pmatrix} X^i \\ (\partial_s X^i) y^s \\ (\partial_s X^i) z^s + (\partial_s \partial_t X^i) y^s y^t \end{pmatrix},
\end{aligned} \tag{1.5}
$$

in (x^i, y^i, z^i), $X = X^i \partial/\partial x^i$ being the local expressions of X in $(U, (x^i))$. For any 1-form ω on M, its prolongations ω^0, ω^I and ω^{II} are 1-forms on $T_2(M)$ having the following properties:

$$
\begin{array}{lll}
\omega^0(X^0) = 0, & \omega^0(X^I) = 0, & \omega^0(X^{II}) = (\omega(X))^0, \\
\omega^I(X^0) = 0, & \omega^I(X^I) = \frac{1}{2}(\omega(X))^0, & \omega^I(X^{II}) = (\omega(X))^I, \\
\omega^{II}(X^0) = (\omega(X))^0, & \omega^{II}(X^I) = (\omega(X))^I, & \omega^{II}(X^{II}) = (\omega(X))^{II}
\end{array} \tag{1.6}
$$

for any vector field X on M. Then ω^0, ω^I and ω^{II} have respectively local components

$$
\begin{aligned}
&\omega^0 : (\omega_i, 0, 0), \\
&\omega^I : ((\partial_s \omega_i) y^s, \omega_i, 0), \\
&\omega^{II} : ((\partial_s \omega_i) z^s + (\partial_s \partial_t \omega_i) y^s y^t, 2(\partial_s \omega_i) y^s, \omega_i)
\end{aligned} \tag{1.7}
$$

in (x^i, y^i, z^i), $\omega = \omega_i\, d\, x^i$ being the local expressions of ω in $(U, (x^i))$. Taking any two tensor fields P and Q, we have the following laws:

$$(1.8)\qquad \begin{aligned} &(P \bigcirc Q)^0 = P^0 \bigcirc Q^0 \ , \quad (P \bigcirc Q)^{I} = P^{I} \bigcirc Q^0 + P^0 \bigcirc Q^{I} \ , \\ &(P \bigcirc Q)^{II} = P^{II} \bigcirc Q^0 + 2\,(P^{I} \bigcirc Q^{I}) + P^0 \bigcirc Q^{II} \ . \end{aligned}$$

The prolongations P^0, P^{I} and P^{II} of P are called respectively the 0-th, the 1st and the 2nd lifts of P, P being a tensor field on M.

Prolongations of affine connections in M *to* $T_2(M)$ (Cf. [2]). Let there be given an affine connection ∇ in M. Then there exists a unique affine connection ∇^{II} in $T_2(M)$ characterized by the following relation:

$$(1.9)\qquad \nabla^{II}_{X^{II}}(Y^{II}) = (\nabla_X Y)^{II} \ ,$$

X and Y being arbitrary vector fields on M. The connection ∇^{II} is called the lift of ∇ to $T_2(M)$. If we denote respectively by T and R (resp. $\widetilde{T}$ and $\widetilde{R}$) the torsion and the curvature tensor fields of ∇ (resp. ∇^{II}), we have $\widetilde{T} = T^{II}$ and $\widetilde{R} = R^{II}$. Moreover we have the following formulas:

$$(1.10)\qquad \begin{aligned} &\nabla^{II}_{X^0}(Y^0) = 0 \ , && \nabla^{II}_{X^0}(Y^{I}) = 0 \ , && \nabla^{II}_{X^0}(Y^{II}) = (\nabla_X Y)^0 \ , \\ &\nabla^{II}_{X^{I}}(Y^0) = 0 \ , && \nabla^{II}_{X^{I}}(Y^{I}) = \tfrac{1}{2}(\nabla_X Y)^0 \ , && \nabla^{II}_{X^{I}}(Y^{II}) = (\nabla_X Y)^{I} \ , \\ &\nabla^{II}_{X^{II}}(Y^0) = (\nabla_X Y)^0 \ , && \nabla^{II}_{X^{II}}(Y^{I}) = (\nabla_X Y)^{I} \ , && \nabla^{II}_{X^{II}}(Y^{II}) = (\nabla_X Y)^{II} \ . \end{aligned}$$

Let's denote by $\widetilde{\Gamma}^{I}_{JK}$ local components of ∇^{II}, where we have put $x^{\bar i} = y^i$, $x^{\bar{\bar i}} = z^i$. The indices I, J, K and L run over the range $\{1, \ldots, n, \bar 1, \ldots, \bar n, \bar{\bar 1}, \ldots, \bar{\bar n}\}$ and the summation convention will be used with respect to these indices. Then we have

$$(1.11)\qquad \widetilde{\Gamma}^{i}_{JK} = \begin{pmatrix} \Gamma^i_{jk} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\Gamma}^{\bar i}_{JK} = \begin{pmatrix} y^s \partial_s \Gamma^i_{jk} & \Gamma^i_{jk} & 0 \\ \Gamma^i_{jk} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\Gamma}^{\bar{\bar i}}_{Jk} = \begin{pmatrix} z^s \partial_s \Gamma^i_{jk} + y^s y^t \partial_s \partial_t \Gamma^i_{jk} & 2\, y^s \partial_s \Gamma^i_{jk} & \Gamma^i_{jk} \\ 2\, y^s \partial_s \Gamma^i_{jk} & 2\, \Gamma^i_{jk} & 0 \\ \Gamma^i_{jk} & 0 & 0 \end{pmatrix}$$

Γ^i_{jk} being local components of ∇.

Prolongations of pseudo-Riemannian metrics of M *to* $T_2(M)$ (Cf. [2]). Let there be given an pseudo-Riemannian metric g in M. Then there exists a pseudo-Riemannian metric g^{II} with the following components:

$$\begin{pmatrix} z^s \partial_s g_{ij} + y^s y^t \partial_s \partial_t g_{ij} & 2\, y^s \partial_s g_{ij} & g_{ij} \\ 2\, y^s \partial_s g_{ij} & 2\, g_{ij} & 0 \\ g_{ij} & 0 & 0 \end{pmatrix} \tag{1.12}$$

with respect to induced coordinates (x^i, y^i, z^i), g_{ij} being contravariant components of g. The pseudo-Riemannian metric g^{II} has the following properties:

$$L_{X^0}\, g^{II} = (L_X g)^0 , \quad L_{X^I}\, g^{II} = (L_X\, g)^I , \quad L_{X^{II}}\, g^{II} = (L_X\, g)^{II} \tag{1.13}$$

for any vector field X on M, where L_X denotes the Lie derivation with respect to X.

We now state two theorems, which are natural analogies to theorems proved in [3]. To do so, we define two operators ρ_1 and ρ_2 operating on type (1,1) tensor fields U in such a way that $\rho_1 U$ and $\rho_2 U$ are respectively vector fields on $T_2(M)$ defined by

$$\rho_1 U = U^i_s\, y^s\, \partial/\partial y^i + 2\,(U^i_s\, v^s - \Gamma^i_{sj}\, U^j_s\, y^s y^t)\, \partial/\partial z^i , \tag{1.14}$$

$$\rho_2 U = U^i_s\, y^s\, \partial/\partial z^i \tag{1.15}$$

where U^i_j are components of U and $v^s = z^s + \Gamma^s_{jk}\, y^j y^k$.

Theorem A. Let ∇ *be a torsion-free affine connection of* M *and* ∇^{II} *its lift to* $T_2(M)$. *Given arbitrary infinitesimal affine transformations* X, Y *and* Z *and parallel, type* (1,1) *tensor fields* U *and* S *on* M *which are solutions of the following equation with unknown type* (1,1) *tensor field* A *on* M:

$$A \circ R(W, W') = R(AW, W') = R(W, AW') = R(W, W') \circ A \tag{1.16}$$

for any vector fields W *and* W′ *on* M, *where* R *is the curvature tensor field of* ∇ *on* M. *Then the vector field* $\widetilde{X} = X^{II} + Y^I + Z^0 + \rho_1 U + \rho_2 S$ *is an infinitesimal affine transformation on* $T_2(M)$ *with respect to* ∇^{II}.

Conversely, if the following condition [C] *is satisfied*:

[C] M *adimits no nonzero parallel, typ (1,1) tensor field* A *satisfying*

$$A \circ R(W, W') = R(AW, W') = R(W, AW') = R(W, W') \circ A = 0$$

for any vector fields W *and* W′ *on* M,

then any infinitesimal affine transformation $\widetilde{X}$ *of* $T_2(M)$ *with respect to* ∇^{II} *is uniquely written as*

$$\widetilde{X} = X^{II} + Y^I + Z^0 + \rho_1 U + \rho_2 S ,$$

where X , Y *and* Z *are infinitesimal affine transformations on* M *and* U , S *are parallel, type* (1,1) *tensor fields satisfying the equation* (1.16).

Theorem B. *Let* g *be a pseudo-Riemannian metric on* M *and* g^{II} *its lift to* $T_2(M)$. *Then the vector field* $\widetilde{X} = X^{II} + Y^{I} + Z^{0} + \rho_1 U + \rho_2 S$ *is an infinitesimal isometry on* $(T_2(M), g^{II})$, *where* X *and* Y *are infinitesimal affine transformations of* (M, g) *such that* $U = (U^i_j) = \left(-\frac{1}{2} g^{im} L_X g_{mj}\right)$, $S = (S^i_j) = (-g^{im} L_Y g_{mj})$ *are parallel and satisfy the equation* (1.16), *and where* Z *is arbitrary infinitasimal isometry on* (M, g).

Conversely, if M *satisfies the condition* [C] *then any infinitesimal isometry* $\widetilde{X}$ *of* $(T_2(M), g^{II})$ *uniquely written as*

$$\widetilde{X} = X^{II} + Y^{I} + Z^{0} + \rho_1 U + \rho_2 S ,$$

where X *and* Y *are infinitesimal affine transformations on* (M, g) *and* $U = (U^i_j) = \left(-\frac{1}{2} g^{im} L_X g_{jm}\right)$, $S = (S^i_j) = (-g^{im} L_Y g_{jm})$ *are parallel, type* (1,1) *tensor fields satisfying the equation* (1.16), *and where* Z *is arbitrary infinitesimal isometry of* (M, g).

Remark 1. — In [3], there were given some examples of manifolds with affine connection satisfying the condition [C].

Remark 2. — When a pseudo-Riemannian metric g is given in M which is not necessarily requested to satisfy the condition [C], S. Tanno [1] has decomposed, in a unique way, any infinitesimal isometry of $(T(M), g^c)$ into three infinitesimal isometries by using an infinitesimal isometry, an infinitesimal affine transformation and two parallel, type (1,1) tensor fields in (M, g). In $(T_2(M), g^{II})$, it might be true that any infinitesimal isometry in $(T_2(M), g^{II})$ can be uniquely decomposed in a similar way as given in [1].

§ 2. Proof of Theorem A

First, we prove

Proposition 2.1. *If* ∇ *is torsion-free and* M *satisfies the condition* [C] *stated in Theorem* A, *then any infinitesimal affine transformation* $\widetilde{X}$ *of* $(T_2(M), \nabla^{II})$ *is projectable into* $T(M)$ *by* $\pi_{12} : T_2(M) \to T(M)$ *and the image* $\pi_{12}(\widetilde{X})$ *is an infinitesimal affine transformation of* $(T(M), \nabla^c)$, *where* $\pi_{12} : T_2(M) \to T(M)$ *is the projection given by* $\pi_{12}(j^2_p(F)) = j^1_p(F)$ *for any element* $j^2_p(F)$ *of* $T_2(M)$.

Proof Take an infinitesimal affine transformation

$$\widetilde{X} = \xi^i \partial/\partial x^i + \xi^{\bar{i}} \partial/\partial y^i + \xi^{\bar{\bar{i}}} \partial/\partial z^i \text{ on } (T_2(M), \nabla^{II}).$$

From the definition of infinitesimal affine transformation, $\widetilde{X}$ satisfies the equation

$$\partial_J \partial_K \xi^I + \xi^L \partial_L \widetilde{\Gamma}^I_{JK} - \widetilde{\Gamma}^L_{JK} \partial_L \xi^I + \widetilde{\Gamma}^I_{JL} \partial_K \xi^L + \widetilde{\Gamma}^I_{LK} \partial_J \xi^L = 0 , \tag{2.1}$$

where $\widetilde{\Gamma}^I_{JK}$ are local components of ∇^{II}. By setting $(I, J, K) = (i, \bar{\bar{j}}, \bar{\bar{k}}), (i, \bar{j}, \bar{\bar{k}})$, $(i, \bar{j}, \bar{k}), (\bar{i}, \bar{\bar{j}}, \bar{\bar{k}})$ and $(\bar{i}, \bar{j}, \bar{\bar{k}})$ in (2.1), we have respectively

$$\partial_{\bar{\bar{j}}} \partial_{\bar{\bar{k}}} \xi^i = 0 , \quad \partial_{\bar{j}} \partial_{\bar{\bar{k}}} \xi^i = 0 , \quad \partial_{\bar{j}} \partial_{\bar{k}} \xi^i = 2 \Gamma^m_{jk} \partial_{\bar{\bar{m}}} \xi^i ,$$
$$\partial_{\bar{\bar{j}}} \partial_{\bar{\bar{k}}} \xi^{\bar{i}} = 0 , \quad \text{and} \quad \partial_{\bar{j}} \partial_{\bar{\bar{k}}} \xi^{\bar{i}} = - \Gamma^i_{jm} \partial_{\bar{\bar{k}}} \xi^m . \tag{2.2}$$

Therefore ξ^i and $\xi^{\bar{i}}$ can be written as follows:

$$\xi^i = A^i{}_j z^j + \Gamma^m_{jk} A^i{}_m y^j y^k + B^i{}_j y^j + X^i , \tag{2.3}$$

$$\xi^{\bar{i}} = (- \Gamma^i_{jm} A^m{}_k y^j + C^i{}_k) z^k + D^i , \tag{2.4}$$

where $A^i{}_j, B^i{}_j, X^i$ and $C^i{}_j$ depend only on $x^1, \dots, x^n$ and D^i depends only on $x^1, \dots, x^n, y^1, \dots, y^n$. Since $\widetilde{X}$ is a vector field on $T_2(M)$, we can easily show that $A = A^i{}_j \, \partial/\partial x^i \otimes d x^j$, $B = B^i{}_j \, \partial/\partial x^i \otimes d x^j$, $C = C^i{}_j \, \partial/\partial x^i \otimes d x^j$ are all type (1,1) tensor fields on M. By simply calculations (Cf. [3]) we can prove the following properties (a) and (b):

(a) A is parallel, i.e. $\nabla A = 0$.
(b) $A (R (Y , Z) W + \nabla_Z T (Y , W)) = R (Y , A Z) W + \nabla_{AZ} T (Y , W)$

for all vector fields Y, Z and W, where R is the curvature tensor field of ∇.

If ∇ is torsion-free, then from (b) and the equation obtained by putting $(I, J, K) = (\bar{i}, j, \bar{k})$ in (2.1), we know that A satisfies

$$A \circ R (Y , Z) = R (A Y , Z) = R (Y , A Z) = R (Y , Z) \circ A = 0.$$

In particular, if we assume that M satisfies the condition [C] stated in Theorem A, then $A = 0$ holds and so, using (2.3) and (2.4), we have $\xi^i = B^i{}_j y^j + X^i$ and $\xi^{\bar{i}} = C^i{}_j z^j + D^i$.

Next, from equations obtained by putting $(I, J, K) = (\bar{i}, \bar{j}, \bar{k})$ and $(\bar{i}, \bar{j}, \bar{k})$ in (2.1), we can put

$$\xi^{\bar{\bar{i}}} = - (\Gamma^i_{mk} B^m{}_j + 2 \Gamma^i_{jm} C^m{}_k) y^j z^k + Q^i{}_k z^k + F^i , \tag{2.5}$$

where $Q^i{}_k$ depends only on $x^1, \dots, x^n$ and F^i depends only on $x^1, \dots, x^n, y^1, \dots, y^n$. Comparing coefficients of z^s in the equations obtained by putting $(I , J , K) = (\bar{i} , j , \bar{k})$ in (2.1), we have

$$2 C^m{}_s R_{k m j}{}^i = B^m{}_k R_{m j s}{}^i , \tag{2.6}$$

which is equivalent to

$$2 R (Y, C Z) = R (Y, Z) \circ B \tag{2.7}$$

for all vector fields Y and Z on M. On the other hand, as was proved in [3], we have $B \circ R (Y , Z) = R (B Y , Z) = R(Y , B Z) = R (Y , Z) \circ B = 0$ because $(B^i_j y^j +$

$+ X^i)\,\partial/\partial x^i$ is an infinitesimal affine transformation of $(T(M), \nabla^c)$. This shows that $B = 0$ and $R(Y, CZ) = 0$ hold if M satisfies the condition [C]. Moreover, putting $(I, J, K) = (\bar{i}, j, \bar{k})$ and (i, j, k) in (2.1), we see that C is parallel and satisfies

$$C \circ R(Y, Z) = R(CY, Z) = R(Y, CZ) = R(Y, Z) \circ C .$$

Therefore, we have the following statements that if ∇ is torsion-free and M satisfies the condition [C], then $B = C = 0$ holds. And so, any infinitesimal affine transformation $\widetilde{X}$ of $(T_2(M), \nabla^{II})$ is projectable to T(M) by π_{12}. It is easy to show that $\pi_{12}(\widetilde{X}) = X^i\,\partial/\partial x^i + D^i\,\partial/\partial y^i$ is an infinitesimal affine transformation of $(T(M), \nabla^c)$. Q.E.D.

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By means of Theorem 1 proved in [3], we can write uniquely as

$$\pi_{12}\,\widetilde{X} = X^c + \frac{1}{2}\,Y^V + \iota\,U, \tag{2.8}$$

where X and Y are infinitesimal affine transformations of (M, ∇) and U is a parallel, type (1,1) tensor field on M satisfying $U \circ R(Y, Z) = R(UY, Z) = R(Y, UZ) = R(Y, Z) \circ U$ for all vector fields Y and Z on M.

On the other hand, we have the following properties (Cf. [2]):

(c) For the vector fields X and Y on M, we have

$$\pi_{12}(X^{II}) = X^c \quad \text{and} \quad \pi_{12}(2\,Y^I) = Y^V .$$

(d) If X is an infinitesimal affine transformation of (M, ∇), then X^{II}, X^I and X^0 are also infinitesimal affine transformations of $(T_2(M), \nabla^{II})$.

Because of (c), (d) and (2.8), it is sufficient for our present purpose to determine a projectable infinitesimal affine transformation $\widetilde{Y}$ of $(T_2(M), \nabla^{II})$ satisfying $\pi_{12}(\widetilde{Y}) = r\,U$. To do so, we put

$$\widetilde{Y} = \iota\,U + \eta^{\bar{i}}\,\partial/\partial z^i = U^i{}_j\,y^j\,\partial/\partial y^i + \eta^{\bar{i}}\,\partial/\partial z^i, \tag{2.9}$$

where $\eta^{\bar{i}}$ will be determined. Since $\widetilde{Y}$ is an infinitesimal affine transformation of $(T_2(M), \nabla^{II})$, $\widetilde{Y}$ satisfies the equation (2.1).

Putting $(I, J, K) = (\bar{i}, \bar{j}, \bar{k})$ and $(\bar{i}, \bar{j}, \bar{k})$ in (2.1), we have respectively

$$\partial_{\bar{j}}\,\partial_{\bar{k}}\,\eta^{\bar{i}} = 0, \qquad \partial_{\bar{j}}\,\partial_{\bar{k}}\,\eta^{\bar{i}} = 0 . \tag{2.10}$$

Therefore we can put

$$\eta^{\bar{i}} = N^i{}_j\,z^j + L^i, \tag{2.11}$$

where $N^i{}_j$ depends only on $x^1, \ldots, x^n$, and where L^i depends only on $x^1, \ldots, x^n$, $y^1, \ldots, y^n$. Also putting $(I, J, K) = (\bar{i}, \bar{j}, \bar{k})$ in (2.1) we have

$$\partial_{\bar{j}}\,\partial_{\bar{k}}\,L^i - 2\,\Gamma^m_{jk}\,N^i{}_m + 2\,\Gamma^i_{jm}\,U^m{}_k + 2\,\Gamma^i_{km}\,U^m{}_j = 0$$

and so we can put

$$L^i = (\Gamma^m_{jk} N^i{}_m - \Gamma^i_{jm} U^m{}_k - \Gamma^i_{km} U^m{}_j) y^j y^k + S^i_k y^k + Z^i , \tag{2.12}$$

where $S^i{}_k$, Z^i depend only on $x^1, \ldots, x^n$. Now $N = N^i{}_j \partial/\partial x^i \bigcirc d x^j$, $S = S^i{}_j \partial/\partial x^i \bigcirc d x^j$ and $Z = Z^i \partial/\partial x^i$ are well-defined tensor fields on M. If we put $(I, J, K) = (\bar{i}, j, \bar{k})$, $(\bar{i}, j, \bar{k})$ and $(\bar{i}, j, k)$ in (2.1) for $\widetilde{Y}$, we can show that the following properties (e), (f) and (g) hold:

(e) N and S satisfy (a) and (b)

(f) $N = 2U$

(g) Z is the infinitesimal affine transformation of (M, ∇).

Now, using (e), we have

$$\widetilde{Y} = U^i{}_j y^j \partial/\partial y^i + 2 [U^i{}_j z^j + (\Gamma^m_{jk} U^i{}_m - \Gamma^i_{jm} U^m{}_k) y^j y^k] \partial/\partial z^i + S^i_k y^k \partial/\partial z^i + Z^i \partial/\partial z^i . \tag{2.13}$$

If we put $\rho_1 U = U^i{}_j y^j \partial/\partial y^i + 2 (U^i{}_j v^j - \Gamma^i_{jm} U^m{}_k y^j y^k) \partial/\partial z^i$, $\rho_2 S = S^i{}_j y^j \partial/\partial z^i$ where $v^i = z^i + \Gamma^i_{jk} y^j y^k$, we see that $\rho_1 U$ and $\rho_2 S$ are the vector fields on $T_2(M)$ and that $\widetilde{Y}$ can be written as

$$\widetilde{Y} = Z^0 + \rho_1 U + \rho_2 S. \tag{2.14}$$

Therefore, when the condition [C] is satisfied, we have determined the form of all infinitesimal affine transformations $\widetilde{X}$ on $T_2(M)$. That is, for any infinitesimal affine transformation $\widetilde{X}$, we have

$$\widetilde{X} = X^{II} + Y^I + Z^0 + \rho_1 U + \rho_2 S , \tag{2.15}$$

where X, Y and Z are infinitesimal affine transformations of (M, ∇) and U, S are parallel, type (1,1) tensor fields satisfying the equation $A R(W, W') = R(AW, W') = R(W, AW') = R(W, W') \circ A$ with unknown type (1,1) tensor field A on M. Summing up, we have here proved the last half part of Theorem A. It will be easy to prove that the decomposition of an infinitesimal affine transformation in the form (2.15) is unique and that any vector field having expression (2.15) on $T_2(M)$ is an infinitesimal affine transformation in $(T_2(M), \nabla^{II})$. Thus the 1st part of Theorem A will be established. Therefore the proof of Theorem A is complete.

§ 3. — Proof of Theorem B

Take any infinitesimal isometry $\widetilde{X}$ of $(T_2(M), g^{II})$. Since $\widetilde{X}$ is the infinitesimal affine transformation of $(T_2(M), \nabla^{II}_g)$, ∇^{II}_g being the lift to $T_2(M)$ of Riemannian connection ∇_g of g, as a consequence of Theorem A, we can uniquely write as

$$\widetilde{X} = X^{II} + Y^I + Z^0 + \rho_1 U + \rho_2 S , \tag{3.1}$$

where X , Y and Z are infinitesimal affine transformations of (M, ∇_g) and U , S are parallel, type (1,1) tensor fields satisfying $A \circ R(W, W') = R(AW, W') = R(W, AW') = R(W, W') \circ A$ with unknown type (1,1) tensor field A on M. We have by (1.13)

$$\text{(3.2)} \qquad L_{\tilde{X}}\, g^{II} = L_{X^{II}}\, g^{II} + L_{Y^{I}}\, g^{II} + L_{Z^{0}}\, g^{II} + L_{\varphi_1 U}\, g^{II} + L_{\varphi_2 S}\, g^{II}$$

$$= (L_X\, g)^{II} + (L_Y\, g)^{I} + (L_Z\, g)^{0} + L_{\varphi_1 U}\, g^{II} + L_{\varphi_2 S}\, g^{II} = 0\ .$$

Also, if we put locally $L_X\, g = H_{ij}\, d x^i\, d x^j$, $L_Y\, g = K_{ij}\, d x^i\, d x^j$ and $L_Z\, g = L_{ij}\, d x^i\, d x^j$, then we have their local expressions respectively (Cf. [2])

$$\text{(3.3)} \qquad (L_X\, g)^{II} = \begin{pmatrix} z^s \partial_s H_{ij} + y^s y^t \partial_s \partial_t H_{ij} & 2\, y^s \partial_s H_{ij} & H_{ij} \\ 2\, y^s \partial_s H_{ij} & 2\, H_{ij} & 0 \\ H_{ij} & 0 & 0 \end{pmatrix},$$

$$\text{(3.4)} \qquad (L_Y\, g)^{I} = \begin{pmatrix} y^s \partial_s K_{ij} & K_{ij} & 0 \\ K_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\text{(3.5)} \qquad (L_Z\, g)^{0} = \begin{pmatrix} L_{ij} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Next we can easily calculate components of $L_{\varphi_1 U}\, g^{II}$ and $L_{\varphi_2 S}\, g^{II}$, which are given respectively by

$$\text{(3.6)} \qquad L_{\varphi_1 U}\, g^{II} = 2 \begin{pmatrix} z^s \partial_s U_{ij} + y^s y^t \partial_s \partial_t U_{ij} & 2\, y^s \partial_s U_{ij} & U_{ij} \\ 2\, y^s \partial_s U_{ij} & g_{mi}\, U^m{}_j + g_{mj}\, U^m{}_i & 0 \\ U_{ij} & 0 & 0 \end{pmatrix}$$

and

$$\text{(3.7)} \qquad L_{\varphi_2 U}\, g^{II} = \begin{pmatrix} y^t \partial_t S_{ij} & g_{mi}\, S^m{}_j & 0 \\ g_{mj}\, S^m{}_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $U_{ij} = g_{jm}\, U^m{}_i$ and $S_{ij} = g_{jm}\, S^m{}_i$. From (3.2) ~ (3.7) we have

$$\text{(3.8)} \qquad H_{ij} = -2\, U_{ij} = -2\, g_{jm}\, U^m{}_i\,,$$

$$K_{ij} = -S_{ij} = -g_{jm}\, S^m{}_j\,, \quad L_{ij} = 0$$

which prove the 2nd half of Theorem B.

The 1st half of Theorem B is easily proved by reversing the above reason. Therefore the proof of Theorem B is complete.

Remark 3. — In Theorem B, $X^{II} + \varphi_1 U$ and $Y^{I} + \varphi_2 S$ are infinitesimal isometries of $(T_2(M), g^{II})$.

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