On pseudo Möbius planes over fields of characteristic $> 2$ (**)

**INTRODUCTION**

The purpose of this paper is to construct and study a class of the so called Minkowski geometries (see bibliography [1]).

To do this, given a projective plane $\pi$ over a field $K$ of characteristic $> 2$ and order $q$, let $I$ and $J$ be two distinct points of $\pi$ and denote by $r$ the line through $I$ and $J$. As points of an incidence structure $\mathcal{V}$ we assume the points of $\pi$, not belonging to $r$, and the lines of $\pi$ through $I$ or $J$; certain conics of $\pi$ through $I$ and $J$ are assumed as blocks of $\mathcal{V}$ and called circles of $\mathcal{V}$.

By construction, an involutorial automorphism of $\mathcal{V}$ is associated to every circle. Such an automorphism, called inversion, gives rise to the definition of orthogonality between two circles.

On the planes $\mathcal{V}$, constructed in this way, the quadruplets of mutually orthogonal circles may be of four types and two of them occur when $\frac{1}{2} (q + 1)$ is odd, while the other ones when $\frac{1}{2} (q + 1)$ is even. It follows that, with respect to the types of such quadruplets of circles, the planes $\mathcal{V}$ may be divided into two classes, according as $\frac{1}{2} (q + 1)$ is odd or even. It is to recall that on the Möbius planes over fields of characteristic $> 2$ and order $q$ there are only two types of such quadruplets of circles and one of them occurs when $\frac{1}{2} (q + 1)$ is odd, while the other one when $\frac{1}{2} (q + 1)$ is even (see [2]).

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Successively we study the group $G$ generated by the inversions of $\Psi$ and demonstrate that every element of $G$ may be represented as product of at most five generators. This property is true also for the pseudo M"obius plane over the real numbers (see [4] n. 4, where such a plane is called hyperbolic inversive plane).

At last by the circles, the pencils and bundles of $\Psi$ we construct a projective 3-space $\Sigma$, such that the points and the circles of $\Psi$ represent a ruled quadric of $\Sigma$ and hence $\Psi$ is a Minkowski geometry.

1. Points and circles of a pseudo M"obius plane

Let $K'$ be a field of characteristic $p > 2$ and order $q' = p^m$ and denote by $T$ an involutorial automorphism of $K'$; such a field $K'$ contains a subfield $K$ of order $q = p^k$ consisting of the self-conjugate elements of $K'$.

Moreover let $\pi'$ be a projective plane over $K'$ and $\pi$ the subplane of $\pi'$ defined over $K$.

In this work we shall refer to $\pi$, but sometimes it will be convenient to consider the plane $\pi'$.

Now we denote by $I$ and $J$ two distinct points of $\pi$ and by $r$ the line through $I$ and $J$.

As points of an incidence structure $\Psi$ we assume the points of $\pi$ not belonging to $r$, the line $r$ and the $2q$ lines of $\pi$ through $I$ or $J$; in this way $\Psi$ contains $(q + 1)^2$ points.

Furthermore we consider the system $\Gamma$ of the $q^2 + q^2 + q + 1$ conics of $\pi$ through $I$ and $J$. $\Gamma$ contains $2q^2 + q + 1$ degenerate conics: $q(q - 1)$ consisting of $r$ and a line not through $I$ nor $J$, and $(q + 1)^2$ consisting of two lines, distinct or not, through $I$ and $J$.

As blocks (called circles) of $\Psi$ we assume the $q(q-1)$ conics of $\Gamma$ consisting of $r$ and a line not through $I$ nor $J$ and the $q^2 - q^2$ non degenerate conics of $\Gamma$, each one together with the tangent lines at $I$ and $J$. In this way in $\Psi$ there are $q(q-1) + q^2 - q^2 = q^2 - q - q$ circles.

A point $P$ of $\Psi$ is said to belong to a circle $A$, if the image of $P$ on $\pi$ belongs to the image of $A$.

For the incidence on $\Psi$ there are the following properties.

Every circle contains $q + 1$ points.

In fact for a circle represented by a degenerate conic there are $q$ points of the line different from $r$ and the point represented by $r$; for a circle represented by a non-degenerate conic $C$, there are $q - 1$ points of $C$, different from $I$ and $J$ and two points represented by the lines $a$ and $b$, tangent to $C$ at $I$ and $J$ respectively.

There are $q(q-1)$ circles passing through every point.

In fact, if the point of $\Psi$ is represented by the line $r$, there are $q$ lines different from $r$ and passing through a same point of $r$ different from $I$ and $J$.

When the point of $\Psi$ is represented by a point $P$ of $\pi$ not on $r$, there are $q^2 + q + 1$ conics of $\Gamma$ through $P$: one consisting of the lines $PI$ and $PJ$, and $2q$...
consisting of the line \( PI \) and a line through \( J \) different from \( PJ \) or of the line \( PJ \) and a line through \( I \) different from \( PI \). In this case we obtain again \( q (q - 1) \).

When the point of \( \Psi \) is represented by a line \( a \) through \( J \) different from \( r \), there are \( q^2 + q + 1 \) conics tangent to \( a \) at \( J \), but \( q + 1 \) of them consist of the line \( a \) and a line through \( I \) and other \( q \) of them consist of the line \( r \) and a line through \( J \) different from the line \( a \). We obtain again \( q (q - 1) \). When the point of \( \Psi \) is represented by a line \( b \) through \( I \), it happens the same.

Two circles of \( \Psi \) are said to be disjoint, tangent or intersecting, according as the corresponding conics have 0, 1 or 2 common points, besides \( I \) and \( J \).

2. Construction of the Inversion in a Circle

Now to each circle \( C \) of \( \Psi \) we are going to associate an inversion \( \gamma \), that is, an involutorial automorphism of \( \Psi \) leaving the points of \( C \) fixed.

If \( C \) is represented by the line \( r \) and a line \( c \) of \( \pi \) not through \( I \) nor \( J \) as inversion \( \gamma \) in \( C \) we assume the harmonic homology of \( \pi \), with axis \( c \) and the points \( I \) and \( J \) mutually corresponding. Such a harmonic homology is an involutorial automorphism of \( \Psi \), since it leaves the line \( c \) pointwise fixed and the line \( r \) invariant, and transforms lines and conics onto lines and conics, mapping \( I \) and \( J \) one into another and preserving incidence.

If \( C \) is represented by a non-degenerate conic \( \mathcal{C} \) of \( \pi \), we consider the following construction. Given two lines \( s \) and \( s' \) of \( \pi \), not conjugate with respect to \( \mathcal{C} \), by the pencil \( F \) of the polar lines of the points on \( s \) we obtain a projectivity \( \omega \) between \( s \) and \( s' \), since there is a projectivity between \( s \) and \( F \); notice that \( \omega \) is a perspectivity, if the point \( s \cap s' \) is self-conjugate with respect to \( \mathcal{C} \), that is, \( s \cap s' \in C \). Given a point \( A \) of \( \mathcal{C} \), different from \( I \) and \( J \), if \( s \) and \( s' \) are the lines \( AI \) and \( AJ \) respectively, \( s \) and \( s' \) are not conjugate with respect to \( \mathcal{C} \) and hence there is a perspectivity \( \omega \) between \( s \) and \( s' \). Denoting by \( t \) and \( t' \) the lines tangent to \( \mathcal{C} \) at \( I \) and \( J \) respectively, the perspectivity \( \omega \) sends the point \( t \cap s' \) into \( I \) and the point \( J \) into the point \( t' \cap s \). It follows that the center of \( \omega \) is the point \( C = t \cap t' \), that is, the pole of the line \( r \); moreover the points of \( s \) and \( s' \) collinear with \( C \) are conjugate with respect to \( \mathcal{C} \).

Now let us denote by \( \gamma \) the correspondence sending a point \( P \) of \( \pi \) into a point \( P' \), such that \( P \) and \( P' \) are conjugate with respect to \( C \) and collinear with \( C \). From the construction if follows that \( \gamma \) transforms a line through \( I \), different from \( t \), onto a line through \( J \), different from \( t' \), and conversely; moreover by \( \gamma \) the points of \( t \) correspond to \( I \) and the ones of \( t' \) correspond to \( J \), and the line \( r \) corresponds to \( C \).

It is easy to verify that by \( \gamma \) a projectivity or perspectivity \( \varphi \) between the pencil of lines through \( I \) and the one of lines through \( J \) is sent into a projectivity \( \varphi' \) between the corresponding pencils and \( \varphi' \) is a perspectivity, when \( \varphi \) transforms \( t \) into \( t' \). This means that \( \gamma \) transforms lines of \( \pi \) not through \( I \) nor \( J \) and the conics of \( \pi \)
through I and J into conics through I and J, and the conics through I, J and C into lines not through I nor J. It follows that $\gamma$ is an involutorial automorphism of $\mathfrak{P}$ mapping circles onto circles and leaving the points of C fixed.

The involutorial automorphism $\gamma$ of $\mathfrak{P}$ is assumed as inversion in the circle C.

3. **Orthogonal circles and orthogonal pencils**

Denoting by $\alpha$ the inversion in a circle $A$ of $\mathfrak{P}$, represented by a conic $A$, let $P$ and $Q$ be two distinct points of $\pi$ corresponding by $\alpha$, such that $P$, $Q$, $I$, $J$ are not triply collinear.

The conics through $P$, $Q$, $I$, $J$ form a pencil $F$ and are invariant by $\alpha$. On every conic $B$ of $F$ $\alpha$ determines an involution, which is hyperbolic or elliptic, according as $A$ intersects $B$, besides I and J, in other two points or not. Denoting by $B$ the circle represented by $B$, this means that the circles $A$ and $B$ are intersecting or disjoint.

The lines $I P$, $J Q$, $I Q$, $J P$ form a complete quadrilateral $\Delta$; denoting by $R$ and $S$ the other vertices of $\Delta$, we observe that $R$ and $S$ belong to $A$ and by the inversion $\gamma$ in a circle $C$ of $F$ the lines $I P$ and $I Q$ commute with $J P$ and $J Q$ respectively and hence $R$ and $S$ commute with each other. This means that the conics of the pencil $F^*$ determined by the points $R$, $S$, $I$, $J$ are invariant by $\gamma$. It follows that as the circle $B$ is invariant by $\alpha$, the circle $A$ is invariant by the inversion $\beta$ in $B$.

The circles $A$ and $B$ are called **orthogonal** and the pencils $F$ and $F^*$ are said to be **orthogonal** too.

When the circle $A$ is represented by a degenerate conic, it is easy to verify that the result is the same.

**Notice** that the orthogonal circles of $\mathfrak{P}$ are intersecting or disjoint.

Let $A$ and $B$ be two conics of $\Gamma$, intersecting in two points $M$ and $N$, besides $I$ and $J$, and representing two orthogonal circles $A$ and $B$.

Denoting by $A$ the center of the inversion $\alpha$ in $A$ and by $B$ the one of the inversion $\beta$ in $B$, it is easy to verify that the lines $A M$ and $A N$ are tangent to $B$, and the lines $B M$ and $B N$ are tangent to $A$.

Now let $L$ be a point of $A$, different from $I$ and $J$, and denote by $l$ the line $A L$ and by $C$ a conic of $\Gamma$ tangent to $l$ at $L$. Obviously $\alpha$ leaves $C$ invariant. If $C$ does not consist of the lines $I L$ and $J L$, there is an inversion $\gamma$ in the circle $C$ represented by $C$: $\alpha$ leaves $C$ invariant as $\gamma$ leaves $A$ invariant.

Denoting by $I'$ the line tangent to $A$ at $L$, $I'$ passes through the center $C$ of $\gamma$. It follows that any conic of $\Gamma$, tangent to $I'$ at $L$, is invariant by $\gamma$ and hence the conics of $\Gamma$ tangent to $l$ at $L$ and the ones of $\Gamma$ tangent to $I'$ at $L$ form two pencils $F$ and $F^*$, such that the circles represented by elements of $F$ are orthogonal to the ones represented by elements of $F^*$ and conversely. Also these pencils $F$ and $F^*$ are called **orthogonal**.
Notice that the lines I, I' and I L, J L form a harmonic set.

At last we consider two disjoint circles of $\mathfrak{H}$; as we know, they are represented by two conics $\mathcal{A}$ and $\mathcal{B}$ intersecting only in $I$ and $J$. We can say that on $\pi' \mathcal{A}$ and $\mathcal{B}$ intersect, besides $I$ and $J$, in two points $P$ and $Q$ conjugate with respect to $T$. Also the other two vertices $R$ and $S$ of the complete quadrilateral $(I P, J Q, I Q, J P)$ are conjugate with respect to $T$. This means that two disjoint circles of $\mathfrak{H}$ determine a pencil $F$ orthogonal to another pencil $F^*$, whose circles are mutually disjoint.

A pencil of circles of $\mathfrak{H}$ is called hyperbolic, parabolic or elliptic, according as its circles are intersecting, tangent or disjoint.

Observe that two circles represented by degenerate conics determine a hyperbolic pencil $F$ since the circles of the orthogonal pencil $F^*$ are represented by conics tangent at $I$ and $J$.

4. BUNDLES OF CIRCLES

Given two orthogonal circles $\mathcal{A}$ and $\mathcal{B}$ represented by the conics $\mathcal{A}$ and $\mathcal{B}$ respectively, as we know, the inversion $\beta$ in $\mathcal{B}$ determines an involution $\iota$ on $\mathcal{A}$. It follows that the product $x \beta \alpha$ leaves the conic $\mathcal{B}$ pointwise fixed and the involution $\iota$ on $\mathcal{A}$ invariant. This means that $x \beta \alpha = \beta$, that is, $x$ commutes with $\beta$.

Moreover, given an involution $\iota$ on $\mathcal{A}$, it is easy to verify that there is a unique conic $\mathcal{B}$ of $\mathfrak{H}$ such that the inversion $\beta$ in $\mathcal{B}$ determines on $\mathcal{A}$ the involution $\iota$.

It follows that the circles orthogonal to a circle of $\mathfrak{H}$ are as many as the involutions on a conic, that is, $q^3$. Among these involutions the hyperbolic ones are $\frac{1}{2} q (q + 1)$ and the elliptic ones are $\frac{1}{2} q (q - 1)$ (see [3]). This means that among the $q^3$ circles orthogonal to a given circle $\mathcal{A}$ the ones intersecting $\mathcal{A}$ are $\frac{1}{2} q (q + 1)$, and the ones disjoint with respect to $\mathcal{A}$ are $\frac{1}{2} q (q - 1)$.

Any set of circles orthogonal to a given circle is called nonsingular bundle, and any set of circles through a given point of $\mathfrak{H}$ is called singular bundle.

Obviously there are $(q + 1)^3$ singular bundles and $q^3 - q$ nonsingular bundles and moreover every singular bundle contains $q (q - 1)$ circles, and every nonsingular bundle contains $q^3$ circles.

Three circles, not in a pencil, determine a either singular or nonsingular bundle.

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be three circles of $\mathfrak{H}$, not in a pencil. If they are represented by three degenerate conics, they determine a singular bundle, since the line $r$ is a point of $\mathfrak{H}$. Therefore we can suppose that at least one circle, for instance $\mathcal{A}$, is represented by a non-degenerate conic $\mathcal{A}$. In this case, if the circles do not pass through a common point, we can say that on $\pi'$ the conics corresponding to $\mathcal{B}$ and $\mathcal{C}$ intersect $\mathcal{A}$ in two point pairs $(\mathcal{B}, \mathcal{B}')$ and $(\mathcal{C}, \mathcal{C}')$ respectively, besides $I$ and $J$. Denoting by $b$ and $c$ the lines $B B'$ and $C C'$ respectively, the point $P = b \cap c$ belongs to $\pi$ as $b$ and $c$. Let $R$ and $S$ be the points $b \cap r$ and $c \cap r$. The point pairs $(b,
B') and (P, R) determine an involution whose fixed points (certainly existing in π') are denoted by D and D': likewise the point pairs (C, C') and (P, S) determine another involution, whose fixed points (in any case in π') are denoted by E and E'. The conic of Γ through D, D' and E passes through E' and represents a circle orthogonal to A, B, C. This means that A, B, C determine a nonsingular bundle.

Now we determine some numerical properties of the pencils.

Let R(A) be the nonsingular bundle relative to the circle A and F a pencil of R(A).

If F is parabolic, the inversion in circles of F determine on A a pencil of q involutions with a common fixed point: it follows that a parabolic pencil of circles contains q circles.

If F is hyperbolic, the inversions in circles of F determine on A a pencil F of involutions with a common pair of corresponding points R and S. The involutions of F commute with the one, whose fixed points are R and S, that is, they form a hyperbolic pencil and hence are q 1; moreover among these, there are \( \frac{1}{2} (q - 1) \) hyperbolic involutions and as many elliptic ones (see [3]). It follows that a hyperbolic pencil of circles contains \( q - 1 \) circles.

At last, if F is elliptic, the inversions in circles of F determine on A a pencil F of involutions, without a common pair of corresponding points. As it is known (see [3]), in this case F contains \( q + 1 \) involutions, and \( \frac{1}{2} (q + 1) \) of them are hyperbolic, and the other ones are elliptic. It follows that there are \( q + 1 \) circles in an elliptic pencil.

Since there are \( \frac{1}{2} q (q + 1) \) hyperbolic pencils containing a given circle, \( \Psi^' \) contains \( \frac{1}{2} q (q + 1) (q^2 - q) / (q - 1) = \frac{1}{2} q^3 (q + 1)^2 \) hyperbolic pencils.

Moreover, since there are \( q + 1 \) parabolic pencils containing a given circle, \( \Psi^' \) contains \( (q + 1) (q^2 - q) / (q - 1) = (q + 1)^2 (q - 1) \) parabolic pencils.

At last since there are \( \frac{1}{2} q (q - 1) \) elliptic pencils containing a given circle, \( \Psi^' \) contains \( \frac{1}{2} q (q - 1) (q^2 - q) / (q + 1) = \frac{1}{2} q^2 (q - 1)^2 \) elliptic pencils.

5. Triplets and quadruplets of mutually orthogonal circles

Now let F be a nonparabolic pencil of circles and \( F^* \) the pencil orthogonal to F. As we know, the inversions in circles of F determine on a circle A of \( F^* \) involutions commuting with an involution j, which is hyperbolic or elliptic according as F is hyperbolic or elliptic too.

Since on a conic (or projective line) there are triplets of mutually commuting involutions, for every circle B of F there is another circle C of F orthogonal to B.
Now we recall that on a conic (or projective line) in a projective plane over a field of characteristic $> 2$ and order $q = p^h$ a triplet of mutually commuting involutions contains an odd or even number of hyperbolic involutions, according as $\frac{1}{2}(q + 1)$ is odd or even (see [3]).

Because of this property there are the following cases.

Given a circle $A$, let $B$ and $C$ be two orthogonal circles intersecting and belonging to the bundle $R(A)$. If $\frac{1}{2}(q + 1)$ is odd, $B$ and $C$ are both intersecting or disjoint with respect to $A$; if $\frac{1}{2}(q + 1)$ is even one of them is intersecting, and the other one is disjoint, with respect to $A$.

If $B$ and $C$ are disjoint and orthogonal to $A$, they are both intersecting or disjoint with respect to $A$, when $\frac{1}{2}(q + 1)$ is even, but one of them is intersecting and the other one is disjoint with respect to $A$, when $\frac{1}{2}(q + 1)$ is odd.

Now let $A, B, C, D$ be four circles mutually orthogonal: two by two, they determine six pencils, which may be divided into three unordered pairs of orthogonal pencils. From the previous results it follows that all these pencils or only two of them are hyperbolic or elliptic, according as $\frac{1}{2}(q + 1)$ is odd or even. In other words there can be four types of quadruplets of mutually orthogonal circles and two of them occur when $\frac{1}{2}(q + 1)$ is odd, and the other two when $\frac{1}{2}(q + 1)$ is even.

It follows that, with respect to the types of quadruplets of mutually orthogonal circles, the pseudo inversive planes over fields of characteristic $> 2$ and order $q$ may be divided into two classes, according as $\frac{1}{2}(q + 1)$ is odd or even.

6. The group generated by the inversions

Now we study some properties of the group $G$ generated by the inversions in circles of $\mathcal{P}$.

Since the product of three involutions in a pencil $F$ is an involution of $F$ we obtain that the product of the inversions in three circles in a pencil $F$ is the inversion in a circle of $F$ (three inversions theorem).

It follows that the products of two inversions belonging to the same pencil form an abelian group. In fact, if $A, B, C, D$ are circles in a pencil and $\alpha, \beta, \gamma, \delta$ the corresponding involutions: $(\alpha \beta) (\gamma \delta) = (\alpha \beta \gamma \delta) = (\gamma \beta)(\alpha \delta) = (\beta \alpha)(\gamma \delta) = (\delta \beta)(\alpha \gamma) = (\gamma \delta)(\alpha \beta)$.

Now let us consider the inversions $\alpha, \beta, \gamma, \delta$ in four circles $A, B, C, D$ mutually orthogonal, that is, such inversions commute mutually two by two.
Three of them, for instance $\alpha$, $\beta$, $\gamma$, determine on the circle $D$ three involutions $I_1$, $I_2$, $I_3$ mutually commuting and hence such that their product is the identity. This means that the product $\alpha \beta \gamma$ is the identity or the inversion $\delta$. Since the product of two distinct involutions cannot be an inversion, $\alpha \beta \gamma = \delta$, that is, the product of three involutions mutually commuting is an inversion.

Let $R(C)$ be the nonsingular bundle relative to the circle $C$. As we know, every pencil of $R(C)$ determines on $C$ a pencil of involutions. Since on a conic (or projective line) an elliptic pencil of involutions and another one have a common involution, an elliptic pencil and another one, both of $R(C)$, contain a common circle.

Now let $F_1$ and $F_2$ be two non-elliptic pencils of $R(C)$. The corresponding pencils of conics on $\pi$ contain a common conic, which cannot represent a circle of $\Psi$: this means that two non-elliptic pencils of $R(C)$ may have no common circle.

Observe that a pencil of conics of $\Gamma$ representing a hyperbolic pencil of $\Psi$ contains two degenerate conics representing no circle of $\Psi$, and a pencil of conics of $\Gamma$ representing a parabolic pencil of $\Psi$ contains one degenerate conic, representing no circle of $\Psi$.

It follows that, if a pencil $F_1$ and three circles $A$, $B$, $C$, not in a pencil, belong to a same bundle of $\Psi$, the pencil $F_1$ and at least one of the three pencils $F(A, B)$, $F(A, C)$, $F(B, C)$ contain a common circle.

Now we can demonstrate that every element of the group $G$ may be represented as product of at most five generators. This result depends on the following properties.

a) The product of the involutions in four circles of a nonsingular bundle $R$ is equal to the product of the involutions in two circles of $R$.

Let $A$, $B$, $C$, $D$ be four circles of $R$ and $\alpha$, $\beta$, $\gamma$, $\delta$ the corresponding involutions and denote by $F_1$ an elliptic pencil of $R$. There is a circle $E \in F_1 \cap F(A, B)$. Denoting by $\varepsilon$ the inversion in $E$, $\varepsilon \alpha \beta = \xi$, that is, $\alpha \beta = \varepsilon \xi$. If $Z$ is the circle corresponding to $\xi$, there is a circle $H \in F_1 \cap F(Z, C)$, such that $\xi \gamma = \gamma h$, where $\gamma$ is the inversion in $H$; and at last, denoting by $L$ the circle corresponding to $\xi$, there is a circle $M \in F_1 \cap F(L, D)$, such that $\lambda \delta = \mu \gamma$, where $\mu$ is the inversion in $M$. In this way $\alpha \beta \gamma \delta = \varepsilon \gamma \delta = \xi \gamma \lambda \delta = \xi \gamma \mu \gamma = \delta \gamma \gamma$, since $E, H, M \in F_1$.

It follows that any product of the involutions in circles belonging to a nonsingular bundle may be represented as product of at most three involutions.

b) Given three involutions $\alpha$, $\beta$, $\gamma$ their product is equal to the product of three involutions, such that the first two of them are any two of $\alpha$, $\beta$, $\gamma$.

In fact $\alpha \beta \gamma = \xi \gamma \beta \gamma = \xi \gamma \delta$, where $\delta = \gamma \beta \gamma$, and moreover $\alpha \beta \gamma = \xi \beta \gamma = \xi \gamma \varepsilon \gamma = \delta \gamma \gamma$, where $\varepsilon = \beta \times \beta \gamma$ and $\varepsilon = \gamma \xi \gamma$.

c) A nonsingular bundle and another one contain a common pencil.

At first let $R(A)$ and $R(B)$ be two nonsingular bundles. The circles $A$ and $B$ determine a pencil $F$. The pencil $F^*$ orthogonal to $F$, belongs to $R(A) \cap R(B)$.

Now, besides the nonsingular bundle $R(A)$, let $R(P)$ be the singular bundle, whose circles pass through the point $P$. If $P$ does not belong to the circle $A$, we de-
note by $P'$ the point corresponding to $P$ by the inversion $z$ in $A$. The circles through $P$ and $P'$ form a pencil belonging to $R(A) \cap R(P)$. If $P$ belongs to $A$, the parabolic pencil of $R(A)$, whose circles are tangent at $P$, is a pencil of $R(P)$ too.

Now let $z$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, $\zeta$ be six inversions triply not in a pencil. Denoting by $R_i$ a nonsingular bundle of inversions, there is a pencil $F_i$ common to the bundles $R_i$ and $R(x, \beta, \gamma)$. As we know, there is an inversion belonging to $F_i$ and one of the pencils $F(x, \beta), F(x, \gamma), F(\beta, \gamma)$. Because of property b) we can suppose that there is an inversion $\eta \in F_i \cap F(x, \beta)$, such that $\eta \cdot \beta = \gamma$, that is, $x \beta = \gamma \delta$. In the same way there is an inversion $\lambda \in R_i \cap R(\delta, \gamma, \delta)$ such that $\lambda \gamma = \mu \delta$. By this proceeding we obtain $x \beta \gamma \delta \varepsilon \zeta = \eta \delta \gamma \delta \varepsilon \zeta = \eta \lambda \nu \pi \rho \sigma = \varphi \omega \rho \sigma$, where $\varphi \omega = \eta \lambda \nu \pi$ since $\nu \lambda, \nu, \pi \in R_i$.

It follows that the product of seven inversions of $\Psi$ may be represented as product of five inversions and hence that any element of $G$ may be represented as product of at most five inversions (Reduction theorem).

To complete the study of $G$ we consider a singular bundle $R$.

Let $z$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, $\zeta$ be five inversions of $R$. As we know, there is an inversion $\zeta$ common to the pencil $F(x, \beta)$ and one of the pencils $F(\gamma, \delta), F(\gamma, \varepsilon), F(\delta, \varepsilon)$. Because of property b) we can suppose $\zeta \in F(x, \beta) \cap F(\gamma, \delta)$. Since $\zeta \gamma \delta = \gamma_\nu$, that is, $\gamma \delta = \zeta \gamma_\nu$, we can write $x \beta \gamma \delta \varepsilon \zeta = x \beta \gamma \delta \varepsilon \zeta = \eta \gamma \delta$. It follows that the product of six inversions of a singular bundle $R$ is equal to the product of four inversions of $R$ and hence that any element of the group generated by the inversions of a singular bundle may be represented as product of at most four inversions.

7. THE PSEUDO MÖBIUS PLANES AS A CLASS OF MINKOWSKI GEOMETRIES

As we saw in preceding section, two singular bundles $R_1$ and $R_2$ may have no common pencil. This happens when $R_1$ and $R_2$ correspond to two points of $\Psi$, represented by two points of $\pi$ collinear with $I$ or $J$, or by two lines through $I$ or $J$, or by a point $P$ of $\pi$ and the line $P I$ or $P J$.

In these cases the bundles $R_1$ and $R_2$ are called disjoint and precisely right-disjoint (or left-disjoint), if they correspond to points of $\Psi$ represented by two points of $\pi$ collinear with $I$ (or $J$), or by two lines through $I$ (or $J$), or by a point $P$ and the line $P I$ (or $P J$).

If two singular bundles $R_1$ and $R_2$ are right-disjoint (or left-disjoint), we call right-isotropic line (or left-isotropic line) the set consisting of $R_1$ and $R_2$ and the other singular bundles disjoint from $R_1$ and $R_2$. To every right-isotropic line (or left-isotropic line) it is associated a line of $\pi$ through $I$ (or $J$) different from $r$, or the pencil of lines of $\pi$ through $I$ (or $J$). Therefore there are $q + 1$ right-isotropic lines and as many left-isotropic lines.

Moreover we call isotropic circle any unordered pair of isotropic lines of different type: obviously there are $(q + 1)^2$ isotropic circles. From now on the circles of $\Psi$ will be called nonisotropic circles.
Notice that two isotropic lines of different type contain a common singular bundle, which is called the center of the corresponding isotropic circle. Conversely, for every singular bundle \( R \), there is a unique isotropic circle, whose center is \( R \).

Given a singular bundle \( R \), it is called isotropic plane relative to \( R \), the set consisting of the isotropic circle with center \( R \) and the nonsingular bundles associated to the nonisotropic circles of \( R \).

Now, in order to construct a space \( \Sigma \) we assume as planes of \( \Sigma \) the nonisotropic circles of \( \Psi \) and the isotropic planes, moreover as lines of \( \Sigma \) the pencils of \( \Psi \) and the isotropic lines, and at last as points of \( \Sigma \) the bundles of \( \Psi \).

Because of these definitions \( \Sigma \) contains \( q^3 + q^2 + q + 1 \) planes and as many points and moreover \( (q^3 + 1)(q^2 + q + 1) \) lines.

For the incidence in \( \Sigma \) we give the following definitions.

A point \( P \) of \( \Sigma \) is said to belong to a plane \( z \) or a line \( b \), if the bundle of \( \Psi \) representing \( P \) either contains the circle or the pencil corresponding to \( z \) or \( b \) respectively, or belongs to the isotropic plane or the isotropic line corresponding to \( z \) or \( b \) respectively.

To define the inclusion of a line \( b \) in a plane \( z \), it is convenient to distinguish the several cases.

When \( z \) is represented by a nonisotropic circle of \( \Psi \) and \( b \) by a pencil of \( \Psi \), the line \( b \) is said to belong to \( z \), if the circle corresponding to \( z \) belongs to the pencil representing \( b \); when \( z \) is represented by an isotropic plane relative to a bundle \( R \) and \( b \) by hyperbolic pencil \( F_1 \) (or parabolic pencil \( F_2 \)), \( b \) is said to belong to \( z \), if \( F_i \) belongs to two singular bundles of \( z \) (or \( F_2 \) belongs to \( R \); when \( z \) is represented by an isotropic plane and \( b \) by an isotropic line, \( b \) is said to belong to \( z \), if \( z \) includes \( b \).

Now it is easy to verify that:

a) every plane of \( \Sigma \) contains \( q^3 + q + 1 \) points and as many lines.

b) every line of \( \Sigma \) contains \( q + 1 \) points and belongs to as many planes.

This means that \( \Sigma \) is a projective 3-space of order \( q \) over \( K \).

The orthogonality on \( \Psi \) allows to introduce a polarity in \( \Sigma \) in the following way.

To every plane \( z \) of \( \Sigma \) represented by a nonisotropic circle \( A \) we associate the point \( A \) of \( \Sigma \) represented by the nonsingular bundle \( R \) (\( A \)) and moreover to every plane \( \beta \) of \( \Sigma \) represented by an isotropic plane \( \beta \) we associate the point \( B \) of \( \Sigma \) represented by the singular bundle relative to \( \beta \).

It follows that to every plane through \( A \) (or \( B \)) it is associated a point of \( z \) (or \( \beta \)) ; moreover the points of \( \Sigma \) associated to the planes passing through a line \( l \) of \( \Sigma \) represented by a pencil \( F \) of \( \Psi \) belong to the line \( l^* \) represented by the pencil \( F^* \) orthogonal to \( F \); and at last the points of \( \Sigma \) associated to the planes passing through a line \( l \) of \( \Sigma \) represented by an isotropic line belong to the line \( l \).

This means that we obtain an inclusion reversing permutation \( \Phi \) of the subspaces of \( \Sigma \), that is, a projective correlation of \( \Sigma \). Since there are points of \( \Sigma \) not belonging to the corresponding planes, \( \Phi \) is not symplectic; moreover \( \Phi \) is of order 2 and hence \( \Phi \) is a polarity of \( \Sigma \). As it is known, the self-conjugate points of \( \Sigma \) by
Φ form a quadric Q, which is ruled, since Φ is symplectic on the lines of Σ represented by isotropic lines.

As we know, on Ψ there is a one-to-one correspondence between the set of the points and the set of the singular bundles of Ψ: it follows that there is a one-to-one correspondence between the set of points of Ψ and the set of points of Q. Since every nonisotropic circle A of Ψ does not belong to the associated nonsingular bundle R(A), the plane of Σ corresponding to A is intersecting Q and hence there is a one-to-one correspondence between the set of the nonisotropic circles of Ψ and the set of the nondegenerate conics of Q. Moreover these two one-to-one correspondences preserve the incidence: this means that Ψ with its points and nonisotropic circles represents a ruled quadric of Σ and hence a Minkowski geometry.

BIBLIOGRAPHY


