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# On pseudo Möbius planes over fields of characteristic > 2 (**)

Riassunto: Oggetto di questa nota è la costruzione e lo studio di una classe di piani finiti di inversione che non sono piani di Möbius.

## Introduction

The purpose of this paper is to construct and study a class of the so called Minkowski geometries (see bibliography [1]).

To do this, given a projective plane $\pi$ over a field K of characteristic $> 2$ and order q, let I and J be two distinct points of $\pi$ and denote by r the line through I and J. As points of an incidence structure $\Psi$ we assume the points of $\pi$, not belonging to r, and the lines of $\pi$ through I or J: certain conics of $\pi$ through I and J are assumed as blocks of $\Psi$ and called circles of $\Psi$.

By construction, an involutorial automorphism of $\Psi$ is associated to every circle. Such an automorphism, called inversion, gives rise to the definition of orthogonality between two circles.

On the planes $\Psi$, constructed in this way, the quadruplets of mutually orthogonal circles may be of four types and two of them occur when $\frac{1}{2}(q+1)$ is odd, while the other ones when $\frac{1}{2}(q+1)$ is even. It follows that, with respect to the types of such quadruplets of circles, the planes $\Psi$ may be divided into two classes, according as $\frac{1}{2}(q+1)$ is odd or even. It is to recall that on the Möbius planes over fields of characteristic $> 2$ and order q there are only two types of such quadruplets of circles and one of them occurs when $\frac{1}{2}(q+1)$ is odd, while the other one when $\frac{1}{2}(q+1)$ is even (see [2]).

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Successively we study the group $G$ generated by the inversions of $\Psi$ and demonstrate that every element of $G$ may be represented as product of at most five generators. This property is true also for the pseudo Möbius plane over the real numbers (see [4] n. 4, where such a plane is called hyperbolic inversive plane).

At last by the circles, the pencils and bundles of $\Psi$ we construct a projective 3- space $\Sigma$, such that the points and the circles of $\Psi$ represent a ruled quadric of $\Sigma$ and hence $\Psi$ is a Minkowski geometry.

## 1. Points and circles of a pseudo Möbius plane

Let $K'$ be a field of characteristic $p > 2$ and order $q' = p^{2h}$ and denote by $T$ an involutorial automorphism of $K'$; such a field $K'$ contains a subfield $K$ of order $q = p^h$ consisting of the self-conjugate elements of $K'$.

Moreover let $\pi'$ be a projective plane over $K'$ and $\pi$ the subplane of $\pi'$ defined over $K$.

In this work we shall refer to $\pi$, but sometimes it will be convenient to consider the plane $\pi'$.

Now we denote by I and J two distinct points of $\pi$ and by r the line through I and J.

As *points of an incidence structure* $\Psi$ we assume *the points of* $\pi$ *not belonging to* r, *the line* r *and the* 2q *lines of* $\pi$ *through* I *or* J : in this way $\Psi$ contains $(q+1)^2$ points.

Furthermore we consider the system $\Gamma$ of the $q^3 + q^2 + q + 1$ conics of $\pi$ through I and J. $\Gamma$ contains $2q^2 + q + 1$ degenerate conics: $q(q-1)$ consisting of r and a line not through I nor J, and $(q+1)^2$ consisting of two lines, distinct or not, through I and J.

As *blocks* (called *circles*) *of* $\Psi$ we assume *the* $q(q-1)$ *conics of* $\Gamma$ *consisting of* r *and a line not through* I *nor* J *and the* $q^3 - q^2$ *non degenerate conics of* $\Gamma$, *each one toghether with the tangent lines at* I *and* J. In this way in $\Psi$ there are $q(q-1) + q^3 - q^2 = q^3 - q$ circles.

*A point $P$ of $\Psi$ is said to belong to a circle $A$, if the image of* P *on* $\pi$ *belongs to the image of $A$.*

For the incidence on $\Psi$ there are the following properties.

*Every circle contains* $q+1$ *points.*

In fact for a circle represented by a degenerate conic there are q points of the line different from r and the point represented by r; for a circle represented by a non-degenerate conic $C$, there are $q-1$ points of $C$, different from I and J and two points represented by the lines $a$ and $b$, tangent to $C$ at I and J respectively.

*There are* $q(q-1)$ *circles passing through every point.*

In fact, if the point of $\Psi$ is represented by the line r, there are q lines different from r and passing through a same point of r different from I and J.

When the point of $\Psi$ is represented by a point P of $\pi$ not on r, there are $q^2 + q + 1$ conics of $\Gamma$ through P: one consisting of the lines P I and P J, and 2 q consisting of the line P I and a line through J different from P J or of the line P J and a line through I different from P I. In this case we obtain again $q(q-1)$.

When the point of $\Psi$ is represented by a line $a$ through J different from r, there are $q^2+q+1$ conics tangent to $a$ at J, but $q+1$ of them consist of the line $a$ and a line through I and other q of them consist of the line r and a line through J different from the line $a$. We obtain again $q(q-1)$. When the point of $\Psi$ is represented by a line $b$ through I, it happens the same.

*Two circles of* $\Psi$ are said to be *disjoint*, *tangent* or *intersecting*, according as the corresponding conics have 0, 1 or 2 common points, besides I and J.

## 2. Construction of the inversion in a circle

Now to each circle $C$ of $\Psi$ we are going to associate an inversion $\gamma$, that is, an involutorial automorphism of $\Psi$ leaving the points of $C$ fixed.

If $C$ is represented by the line r and a line $c$ of $\pi$ not through I nor J as inversion $\gamma$ in $C$ we assume the harmonic homology of $\pi$, with axis $c$ and the points I and J mutually corresponding. Such a harmonic homology is an involutorial automorphism of $\Psi$, since it leaves the line $c$ pointwise fixed and the line r invariant, and transforms lines and conics onto lines and conics, mapping I and J one into another and preserving incidence.

If $C$ is represented by a non-degenerate conic $\bar{C}$ of $\pi$, we consider the following construction. Given two lines s and s′ of $\pi$, not conjugate with respect to $\bar{C}$, by the pencil $F$ of the polar lines of the points on s we obtain a projectivity $\omega$ between s and s′, since there is a projectivity between s and $F$: notice that $\omega$ is a perspectivity, if the point $s \cap s'$ is self-conjugate with respect to $\bar{C}$, that is, $s \cap s' \in \bar{C}$. Given a point A of $\bar{C}$, different from I and J, if s and s′ are the lines A I and A J respectively, s and s′ are not conjugate with respect to $\bar{C}$ and hence there is a perspectivity $\omega$ between s and s′. Denoting by t and t′ the lines tangent to $\bar{C}$ at I and J respectively, the perspectivity $\omega$ sends the point $t \cap s'$ into I and the point J into the point $t' \cap s$. It follows that the center of $\omega$ is the point $C = t \cap t'$, that is, the pole of the line r; moreover the points of s and s′ collinear with C are conjugate with respect $\bar{C}$.

Now let us denote by $\gamma$ the correspondence sending a point P of $\pi$ into a point P′, such that P and P′ are conjugate with respect to $C$ and collinear with C. From the construction if follows that $\gamma$ transforms a line through I, different from t, onto a line through J, different from t′, and conversely; moreover by $\gamma$ the points of t correspond to I and the ones of t′ correspond to J, and the line r corresponds to C.

It is easy to verify that by $\gamma$ a projectivity or perspectivity $\varphi$ between the pencil of lines through I and the one of lines through J is sent into a projectivity $\varphi'$ between the corresponding pencils and $\varphi'$ is a perspectivity, when $\varphi$ transforms t into t′. This means that $\gamma$ transforms lines of $\pi$ not through I nor J and the conics of $\pi$

through I and J into conics through I and J, and the conics through I, J and C into lines not through I nor J. It follows that $\gamma$ is an involutorial automorphism of $\Psi$ mapping circles onto circles and leaving the points of C fixed.

*The involutorial automorphism* $\gamma$ *of* $\Psi$ is assumed as *inversion in the circle* $C$.

### 3. Orthogonal circles and orthogonal pencils

Denoting by $\alpha$ the inversion in a circle $A$ of $\Psi$, represented by a conic $\bar{A}$, let P and Q be two distinct points of $\pi$ corresponding by $\alpha$, such that P, Q, I, J are not triply collinear.

The conics through P, Q, I, J form a pencil $F$ and are invariant by $\alpha$. On every conic $\bar{B}$ of $F$ $\alpha$ determines an involution, which is hyperbolic or elliptic, according as $\bar{A}$ intersects $\bar{B}$, besides I and J, in other two points or not. Denoting by $B$ the circle represented by $\bar{B}$, this means that the circles $A$ and $B$ are intersecting or disjoint.

The lines I P, J Q, I Q, J P form a complete quadrilateral $\Delta$; denoting by R and S the other vertices of $\Delta$, we observe that R and S belong to $A$ and by the inversion $\gamma$ in a circle $C$ of $F$ the lines I P and I Q commute with J P and J Q respectively and hence R and S commute with each other. This means that the conics of the pencil $F^*$ determined by the points R, S, I, J are invariant by $\gamma$. It follows that as the circle $B$ is invariant by $\alpha$, the circle $A$ is invariant by the inversion $\beta$ in $B$.

The *circles* $A$ and $B$ are called *orthogonal* and the *pencils* $F$ and $F^*$ are said to be *orthogonal* too.

When the circle $A$ is represented by a degenerate conic, it is easy to verify that the result is the same.

Notice that *two orthogonal circles of* $\Psi$ *are intersecting or disjoint.*

Let $\bar{A}$ and $\bar{B}$ be two conics of $\Gamma$, intersecting in two points M and N, besides I and J, and representing two orthogonal circles $A$ and $B$.

Denoting by A the center of the inversion $\alpha$ in $A$ and by $B$ the one of the inversion $\beta$ in $B$, it is easy to verify that the lines A M and A N are tangent to $\bar{B}$, and the lines B M and B N are tangent to $\bar{A}$.

Now let L be a point of $A$, different from I and J, and denote by l the line A L and by $\bar{C}$ a conic of $\Gamma$ tangent to l at L. Obviously $\alpha$ leaves $\bar{C}$ invariant. If $\bar{C}$ does not consist of the lines L I and L J, there is an inversion $\gamma$ in the circle $C$ represented by $\bar{C}$ : $\alpha$ leaves $C$ invariant as $\gamma$ leaves $A$ invariant.

Denoting by l′ the line tangent to $A$ at L, l′ passes through the center C of $\gamma$. It follows that any conic of $\Gamma$, tangent to l′ at L, is invariant by $\gamma$ and hence the conics of $\Gamma$ tangent to l at L and the ones of $\Gamma$ tangent to l′ at L form two pencils $F$ and $F^*$, such that the circles represented by elements of $F$ are orthogonal to the ones represented by elements of $F^*$ and conversely. Also these *pencils* $F$ and $F^*$ are called *orthogonal.*

Notice that the lines l, l′ and I L, J L form a harmonic set.

At last we consider two disjoint circles of $\Psi$; as we know, they are represented by two conics $\bar{A}$ and $\bar{B}$ intersecting only in I and J. We can say that on $\pi'$ $\bar{A}$ and $\bar{B}$ intersect, besides I and J, in two points P and Q conjugate with respect to $T$. Also the other two vertices R and S of the complete quadrilateral (I P, J Q, I Q, J P) are conjugate with respect to $T$. This means that *two disjoint circles of $\Psi$ determine a pencil $F$ orthogonal to another pencil $F^*$, whose circles are mutually disjoint.*

A *pencil* of circles of $\Psi$ is called *hyperbolic*, *parabolic* or *elliptic*, according as its circles are intersecting, tangent or disjoint.

Observe that two circles represented by degenerate conics determine a hyperbolic pencil $F$ since the circles of the orthogonal pencil $F^*$ are represented by conics tangent at I and J.

## 4. Bundles of circles

Given two orthogonal circles $A$ and $B$ represented by the conics $\bar{A}$ and $\bar{B}$ respectively, as we know, the inversion $\beta$ in $B$ determines an involution j on $\bar{A}$. It follows that the product $\alpha\,\beta\,\alpha$ leaves the conic $\bar{B}$ pointwise fixed and the involution j on $\bar{A}$ invariant. This means that $\alpha\,\beta\,\alpha = \beta$, that is, $\alpha$ commutes with $\beta$.

Moreover, given an involution j on $\bar{A}$, it is easy to verify that there is a unique conic $\bar{B}$ of $\Gamma$ such that the inversion $\beta$ in $B$ determines on $\bar{A}$ the involution j.

It follows that *the circles orthogonal to a circle of $\Psi$ are as many as the involutions on a conic*, that is, $q^2$. Among these involutions the hyperbolic ones are $\frac{1}{2}\,q\,(q+1)$ and the elliptic ones are $\frac{1}{2}\,q\,(q-1)$ (see [3]). This means that *among the $q^2$ circles orthogonal to a given circle $A$ the ones intersecting $A$ are $\frac{1}{2}\,q\,(q+1)$, and the ones disjoint with respect to $A$ are $\frac{1}{2}\,q\,(q-1)$.*

*Any set of circles orthogonal to a given circle* is called *nonsingular bundle*, and *any set of circles through a given point of $\Psi$* is called *singular bundle*.

Obviously there are $(q+1)^2$ singular bundles and $q^3 - q$ nonsingular bundles and moreover every singular bundle contains $q\,(q-1)$ circles, and every nonsingular bundle contains $q^2$ circles.

*Three circles, not in a pencil, determine a either singular or nonsingular bundle.*

Let $A$, $B$, $C$ be three circles of $\Psi$, not in a pencil. If they are represented by three degenerate conics, they determine a singular bundle, since the line r is a point of $\Psi$. Therefore we can suppose that at least one circle, for instance $A$, is represented by a non-degenerate conic $\bar{A}$. In this case, if the circles do not pass through a common point, we can say that on $\pi'$ the conics corresponding to $B$ and $C$ intersect $\bar{A}$ in two point pairs (B, B′) and (C, C′) respectively, besides I and J. Denoting by b and c the lines B B′ and C C′ respectively, the point $P = b \cap c$ belongs to $\pi$ as b and c. Let R and S be the points $b \cap r$ and $c \cap r$. The point pairs (B, B′) and (P, R) determine an involution whose fixed points (certainly existing in $\pi'$) are denoted by D and D′ : likewise the point pairs (C, C′) and (P, S) determine another involution, whose fixed points (in any case in $\pi'$) are denoted by E and E′. The conic of $\Gamma$ through D, D′ and E passes through E′ and represents a circle orthogonal to $A$, $B$, $C$. This means that $A$, $B$, $C$ determine a nonsingular bundle.

Now we determine some numerical properties of the pencils.

Let $R(A)$ be the nonsingular bundle relative to the circle $A$ and $F$ a pencil of $R(A)$.

If $F$ is parabolic, the inversion in circles of $F$ determine on $A$ a pencil of q involutions with a common fixed point: it follows that *a parabolic pencil of circles contains* q *circles.*

If $F$ is hyperbolic, the inversions in circles of $F$ determine on $A$ a pencil $\bar{F}$ of involutions with a common pair of corresponding points R and S. The involutions of $\bar{F}$ commute with the one, whose fixed points are R and S, that is, they form a hyperbolic pencil and hence are $q-1$; moreover among these, there are $\frac{1}{2}(q-1)$ hyperbolic involutions and as many elliptic ones (see [3]). It follows that *a hyperbolic pencil of circles contains* $q-1$ *circles.*

At last, if $F$ is elliptic, the inversions in circles of $F$ determine on $A$ a pencil $\bar{F}$ of involutions, without a common pair of corresponding points. As it is known (see [3]), in this case $\bar{F}$ contains $q+1$ involutions, and $\frac{1}{2}(q+1)$ of them are hyperbolic, and the other ones are elliptic. It follows that *there are* $q+1$ *circles in an elliptic pencil.*

Since there are $\frac{1}{2}q(q+1)$ hyperbolic pencils containing a given circle, $\Psi$ contains $\frac{1}{2}q(q+1)(q^3-q)/(q-1)=\frac{1}{2}q^2(q+1)^2$ hyperbolic pencils.

Moreover, since there are $q+1$ parabolic pencils containing a given circle, $\Psi$ contains $(q+1)(q^3-q)/(q-1)=(q+1)^2(q-1)$ parabolic pencils.

At last since there are $\frac{1}{2}q(q-1)$ elliptic pencils containing a given circle, $\Psi$ contains $\frac{1}{2}q(q-1)(q^3-q)/(q+1)=\frac{1}{2}q^2(q-1)^2$ elliptic pencils.

## 5. Triplets and quadruplets of mutually orthogonal circles

Now let $F$ be a *nonparabolic pencil of circles* and $F^*$ the pencil orthogonal to $F$. As we know, the inversions in circles of $F$ determine on a circle $A$ of $F^*$ involutions commuting with an involution j, which is hyperbolic or elliptic according as $F$ is hyperbolic or elliptic too.

Since on a conic (or projective line) there are triplets of mutually commuting involutions, *for every circle $B$ of $F$ there is another circle $C$ of $F$ orthogonal to $B$.*

Now we recall that on a conic (or projective line) in a projective plane over a field of characteristic $> 2$ and order $q = p^h$ a triplet of mutually commuting involutions contains an odd or even number of hyperbolic involutions, according as $\frac{1}{2}(q+1)$ is odd or even (see [3]).

Because of this property there are the following cases.

Given a circle $A$, let $B$ and $C$ be *two orthogonal circles intersecting and belonging to the bundle* $R(A)$. *If* $\frac{1}{2}(q+1)$ *is odd,* $B$ and $C$ *are both intersecting or disjoint with respect to* $A$; *if* $\frac{1}{2}(q+1)$ *is even one of them is intersecting, and the other one is disjoint, with respect to* $A$.

If $B$ *and* $C$ are *disjoint and orthogonal to* $A$, *they are both intersecting or disjoint with respect to* $A$, *when* $\frac{1}{2}(q+1)$ *is even, but one of them is intersecting and the other one is disjoint with respect to* $A$, *when* $\frac{1}{2}(q+1)$ *is odd.*

Now let $A$, $B$, $C$, $D$ be four circles mutually orthogonal: two by two, they determine six pencils, which may be divided into three unordered pairs of orthogonal pencils. From the previous results it follows that *all these pencils or only two of them are hyperbolic or elliptic, according as* $\frac{1}{2}(q+1)$ *is odd or even.* In other words *there can be four types of quadruplets of mutually orthogonal circles and two of them occur when* $\frac{1}{2}(q+1)$ *is odd, and the other two when* $\frac{1}{2}(q+1)$ *is even.*

It follows that, *with respect to the types of quadruplets of mutually orthogonal circles, the pseudo inversive planes over fields of characteristic* $> 2$ *and order* $q$ *may be divided into two classes, according as* $\frac{1}{2}(q+1)$ *is odd or even.*

## 6. The group generated by the inversions

Now we study some properties of the group $G$ generated by the inversions in circles of $\Psi$.

Since the product of three involutions in a pencil $\bar{F}$ is an involution of $\bar{F}$ we obtain that *the product of the inversions in three circles in a pencil* $F$ *is the inversion in a circle of* $F$ (*three inversions theorem*).

It follows that *the products of two inversions belonging to the same pencil form an abelian group.* In fact, if $A$, $B$, $C$, $D$ are circles in a pencil and $\alpha$, $\beta$, $\gamma$, $\delta$ the corresponding inversions : $(\alpha\,\beta)(\gamma\,\delta) = (\alpha\,\beta\,\gamma)\,\delta = (\gamma\,\beta\,\alpha)\,\delta = \gamma\,(\beta\,\alpha\,\delta) = \gamma\,(\delta\,\alpha\,\beta) = (\gamma\,\delta)(\alpha\,\beta)$.

Nov let us consider the inversions $\alpha$, $\beta$, $\gamma$, $\delta$ in four circles $A$, $B$, $C$, $D$ mutually orthogonal, that is, such inversions commute mutually two by two.

Three of them, for instance $\alpha$, $\beta$, $\gamma$, determine on the circle $D$ three involutions $j_1$, $j_2$, $j_3$ mutually commuting and hence such that their product is the identity. This means that the product $\alpha\beta\gamma$ is the identity or the inversion $\delta$. Since the product of two distinct inversions cannot be an inversion, $\alpha\beta\gamma=\delta$, that is, *the product of three inversions mutually commuting is an inversion.*

Let $R(C)$ be the *nonsingular bundle* relative to the circle $C$. As we know, every pencil of $R(C)$ determines on $C$ a pencil of involutions. Since on a conic (or projective line) an elliptic pencil of involutions and another one have a common involution, *an elliptic pencil and another one, both of $R(C)$, contain a common circle.*

Now let $F_1$ and $F_2$ be two non-elliptic pencils of $R(C)$. The corresponding pencils of conics on $\pi$ contain a common conic, which cannot represent a circle of $\Psi$: this means that *two non-elliptic pencils of $R(C)$ may have no common circle.*

Observe that a pencil of conics of $\Gamma$ representing a hyperbolic pencil of $\Psi$ contains two degenerate conics representing no circle of $\Psi$, and a pencil of conics of $\Gamma$ representing a parabolic pencil of $\Psi$ contains one degenerate conic, representing no circle of $\Psi$.

It follows that, *if a pencil $F_1$ and three circles $A$, $B$, $C$, not in a pencil, belong to a same bundle of $\Psi$, the pencil $F_1$ and at least one of the three pencils $F(A, B)$, $F(A, C)$, $F(B, C)$ contain a common circle.*

Now we can demonstrate that every element of the group $G$ may be represented as product of at most five generators. This result depends from the following properties.

a) *The product of the inversions in four circles of a nonsingular bundle $R$ is equal to the product of the inversions in two circles of $R$.*

Let $A$, $B$, $C$, $D$ be four circles of $R$ and $\alpha$, $\beta$, $\gamma$, $\delta$ the corresponding inversions and denote by $F_1$ an elliptic pencil of $R$. There is a circle $E\in F_1\cap F(A,B)$. Denoting by $\varepsilon$ the inversion in $E$, $\varepsilon\alpha\beta=\zeta$, that is, $\alpha\beta=\varepsilon\zeta$. If $Z$ is the circle corresponding to $\zeta$, there is a circle $H\in F_1\cap F(Z,C)$, such that $\zeta\gamma=\eta\lambda$, where $\eta$ is the inversion in $H$; and at last, denoting by $L$ the circle corresponding to $\lambda$, there is a circle $M\in F_1\cap F(L,D)$, such that $\lambda\delta=\mu\nu$, where $\mu$ is the inversion in $M$. In this way $\alpha\beta\gamma\delta=\varepsilon\zeta\gamma\delta=\varepsilon\eta\lambda\delta=\varepsilon\eta\mu\nu=\omega\nu$, since $E, H, M\in F_1$.

It follows that *any product of the inversions in circles belonging to a nonsingular bundle may be represented as product of at most three inversions.*

b) *Given three inversions $\alpha$, $\beta$, $\gamma$ their product is equal to the product of three inversions, such that the first two of them are any two of $\alpha$, $\beta$, $\gamma$.*

In fact $\alpha\beta\gamma=\alpha\gamma\gamma\beta\gamma=\alpha\gamma\delta$, where $\delta=\gamma\beta\gamma$, and moreover $\alpha\beta\gamma=\beta\beta\alpha\beta\gamma=\beta\varepsilon\gamma=\beta\gamma\gamma\varepsilon\gamma=\beta\gamma\zeta$, where $\varepsilon=\beta\alpha\beta$ and $\zeta=\gamma\varepsilon\gamma$.

c) *A nonsingular bundle and another one contain a common pencil.*

At first let $R(A)$ and $R(B)$ be two nonsingular bundles. The circles $A$ and $B$ determine a pencil $F$. The pencil $F^*$ orthogonal to $F$, belongs to $R(A)\cap R(B)$.

Now, besides the nonsingular bundle $R(A)$, let $R(\mathrm{P})$ be the singular bundle, whose circles pass through the point P. If P does not belong to the circle $A$, we denote by P′ the point corresponding to P by the inversion $\alpha$ in $A$. The circles through P and P′ form a pencil belonging to $R(A) \cap R(\mathrm{P})$. If P belongs to $A$, the parabolic pencil of $R(A)$, whose circles are tangent at P, is a pencil of $R(\mathrm{P})$ too.

Now let $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ be six inversions triply not in a pencil. Denoting by $R_1$ a nonsingular bundle of inversions, there is a pencil $F_1$ common to the bundles $R_1$ and $R(\alpha, \beta, \gamma)$. As we know, there is an inversion belonging to $F_1$ and one of the pencils $F(\alpha, \beta), F(\alpha, \gamma), F(\beta, \gamma)$. Because of property b) we can suppose that there is an inversion $\eta \in F_1 \cap F(\alpha, \beta)$, such that $\eta\,\alpha\,\beta = \vartheta$, that is, $\alpha\,\beta = \eta\,\vartheta$. In the same way there is an inversion $\lambda \in R_1 \cap R(\vartheta, \gamma, \delta)$ such that $\vartheta\,\gamma = \lambda\,\mu$. By this proceeding we obtain $\alpha\,\beta\,\gamma\,\delta\,\varepsilon\,\zeta = \eta\,\vartheta\,\gamma\,\delta\,\varepsilon\,\zeta = \eta\,\lambda\,\nu\,\pi\,\rho\,\sigma = \varphi\,\omega\,\rho\,\sigma$, where $\varphi\,\omega = \eta\,\lambda\,\nu\,\pi$ since $\eta, \lambda, \nu, \pi \in R_1$.

It follows that the product of seven inversions of $\Psi$ may be represented as product of five inversions and hence that *any element of $G$ may be represented as product of at most five inversions (Reduction theorem).*

To complete the study of $G$ we consider a singular bundle $R$.

Let $\alpha, \beta, \gamma, \delta, \varepsilon$, be five inversions of $R$. As we know, there is an inversion $\zeta$ common to the pencil $F(\alpha, \beta)$ and one of the pencils $F(\gamma, \delta), F(\gamma, \varepsilon), F(\delta, \varepsilon)$. Because of property b) we can suppose $\zeta \in F(\alpha, \beta) \cap F(\gamma, \delta)$. Since $\zeta\,\gamma\,\delta = \eta$, that is, $\gamma\,\delta = \zeta\,\eta$, we can write $\alpha\,\beta\,\gamma\,\delta\,\varepsilon = \alpha\,\beta\,\zeta\,\eta\,\varepsilon = \vartheta\,\eta\,\varepsilon$. It follows that the product of six inversions of a singular bundle $R$ is equal to the product of four inversions of $R$ and hence that *any element of the group generated by the inversions of a singular bundle may be represented as product of at most four inversions.*

## 7. The pseudo Möbius planes as a class of Minkowski geometries

As we saw in preceding section, two singular bundles $R_1$ and $R_2$ may have no common pencil. This happens when $R_1$ and $R_2$ correspond to two points of $\Psi$, represented by two points of $\pi$ collinear with I or J, or by two lines through I or J, or by a point P of $\pi$ and the line P I or P J.

In these cases the *bundles* $R_1$ and $R_2$ are called *disjoint* and precisely *right-disjoint* (or *left-disjoint*), if they correspond to points of $\Psi$ represented by two points of $\pi$ collinear with I (or J), or by two lines through J (or I), or by a point P and the line P I (or P J).

If two singular bundles $R_1$ and $R_2$ are right-disjoint (or left-disjoint), we call *right-isotropic line* (or *left-isotropic line*) *the set consisting of $R_1$ and $R_2$ and the other singular bundles disjoint from $R_1$ and $R_2$.* To every right-isotropic line (or left-isotropic line) it is associated a line of $\pi$ through I (or J) different from r, or the pencil of lines of $\pi$ through I (or J). Therefore there are $q + 1$ right-isotropic lines and as many left-isotropic lines.

Moreover we call *isotropic circle any unordered pair of isotropic lines of different type*: obviously there are $(q + 1)^2$ isotropic circles. From now on the circles of $\Psi$ will be called nonisotropic circles.

Notice that two isotropic lines of different type contain a common singular bundle, which is called the center of the corresponding isotropic circle. Conversely, for every singular bundle $R$, there is a unique isotropic circle, whose center is $R$.

*Given a singular bundle* $R$, it is called *isotropic plane relative to* R, *the set consisting of the isotropic circle with center* R *and the nonsingular bundles associated to the nonisotropic circles of* $R$.

Now, in order to construct a space $\Sigma$ we assume as *planes of* $\Sigma$ *the nonisotropic circles of* $\Psi$ *and the isotropic planes*, moreover as *lines of* $\Sigma$ *the pencils of* $\Psi$ *and the isotropic lines*, and at last as *points of* $\Sigma$ *the bundles of* $\Psi$.

Because of these definitions $\Sigma$ contains $q^3 + q^2 + q + 1$ planes and as many points and moreover $(q^2 + 1)(q^2 + q + 1)$ lines.

For the incidence in $\Sigma$ we give the following definitions.

A point P of $\Sigma$ is said to belong to a plane $\alpha$ or a line $b$, if the bundle of $\Psi$ representing P either contains the circle or the pencil corresponding to $\alpha$ or $b$ respectively, or belongs to the isotropic plane or the isotropic line corresponding to $\alpha$ or $b$ respectively.

To define the inclusion of a line $b$ in a plane $\alpha$, it is convenient to distinguish the several cases.

When $\alpha$ is represented by a nonisotropic circle of $\Psi$ and $b$ by a pencil of $\Psi$, the line $b$ is said to belong to $\alpha$, if the circle corresponding to $\alpha$ belongs to the pencil representing $b$; when $\alpha$ is represented by an isotropic plane relative to a bundle $R$ and $b$ by hyperbolic pencil $F_1$ (or parabolic pencil $F_2$), $b$ is said to belong to $\alpha$, if $F_1$ belongs to two singular bundles of $\alpha$ (or $F_2$ belongs to $R$); when $\alpha$ is represented by an isotropic plane and $b$ by an isotropic line, $b$ is said to belong to $\alpha$, if $\alpha$ includes $b$.

Now it is easy to verify that:

*a*) every plane of $\Sigma$ contains $q^2 + q + 1$ points and as many lines.

*b*) every line of $\Sigma$ contains $q + 1$ points and belongs to as many planes.

This means that $\Sigma$ is a projective 3-space of order q over K.

The orthogonality on $\Psi$ allows to introduce a polarity in $\Sigma$ in the following way.

To every plane $\alpha$ of $\Sigma$ represented by a nonisotropic circle $A$ we associate the point A of $\Sigma$ represented by the nonsingular bundle $R(A)$ and moreover to every plane $\beta$ of $\Sigma$ represented by an isotropic plane $\beta$ we associate the point B of $\Sigma$ represented by the singular bundle relative to $\beta$.

It follows that to every plane through A (or B) it is associated a point of $\alpha$ (or $\beta$); moreover the points of $\Sigma$ associated to the planes passing through a line $l$ of $\Sigma$ represented by a pencil $F$ of $\Psi$ belong to the line $l^*$ represented by the pencil $F^*$ orthogonal to $F$; and at last the points of $\Sigma$ associated to the planes passing through a line $l$ of $\Sigma$ represented by an isotropic line belong to the line $l$.

This means that we obtain an inclusion reversing permutation $\Phi$ of the subspaces of $\Sigma$, that is, a projective correlation of $\Sigma$. Since there are points of $\Sigma$ not belonging to the corresponding planes, $\Phi$ is not sympletic: moreover $\Phi$ is of order 2 and hence $\Phi$ is a polarity of $\Sigma$. As it is known, the self-conjugate points of $\Sigma$ by

$\Phi$ form a quadric $Q$, which is ruled, since $\Phi$ is sympletic on the lines of $\Sigma$ represented by isotropic lines.

As we know, on $\Psi$ there is a one-to-one correspondence between the set of the points and the set of the singular bundles of $\Psi$: it follows that there is a one-to-one correspondence between the set of points of $\Psi$ and the set of points of $Q$. Since every nonisotropic circle $A$ of $\Psi$ does not belong to the associated nonsingular bundle $R(A)$, the plane of $\Sigma$ corresponding to $A$ is intersecting $Q$ and hence there is a one-to-one correspondence between the set of the nonisotropic circles of $\Psi$ and the set of the nondegenerate conics of $Q$. Moreover these two one-to-one correspondences preserve the incidence: this means that $\Psi$ with its points and nonisotropic circles represents a ruled quadric of $\Sigma$ and hence a Minkowski geometry.

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