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Absolute summability factors for Fourier series (**)

Summary: In this paper we have proved three theorems. Theorem 1 improves a result of M. K. Nayak [7] while Theorem 2 improves the result of M. T. Cheng [2]. Theorem 3 is of general nature which generalises a number of results due to M. T. Cheng [2], G. D. Dikshit [3], P. Chandra [1] and S. N. Lal [5] and also gives, as particular case, the following interesting result:

Let $\varepsilon > 0$ and $k > \pi e^2$. Then $\dfrac{\phi_1(t)}{\left(\log_2 \frac{k}{t}\right)^{\varepsilon}} \in BV(0, \pi)$ implies that

$$\sum_{n=1}^{\infty} \frac{A_n(x)}{(\log_2 (n+2))^{2+\varepsilon}} \in |R, \log n, 1| .$$

1. Definitions and Notations. — Let $L(w)$ be a continuous, differentiable and monotonic increasing (↑) function of w, and let it tend to infinity with w. Suppose that $\sum_{n=1}^{\infty} a_n$ be a given infinite series then $\sum_{n=1}^{\infty} a_n$ is summable $|R, L(w), r|$ $(r > 0)$ or symbolically $\sum_{n=1}^{\infty} a_n \in |R, L(w), r|$ $(r > 0)$, if

$$\int_h^{\infty} \frac{L'(w)}{(L(w))^{r+1}} \left| \sum_{n<w} \{L(w) - L(n)\}^{r-1} L(n)\, a_n \right| dw$$

is convergent, where h is a finite number (Obrechkoff [8], [9]).

The above definition has been used in the proof of Theorem 1.

Summability $|C, \alpha| \sim |R, n, \alpha|$ $(\alpha > 0)$. See Hyslop [4]. This has been used in the proof of Theorem 2.

It has been observed by Bosanquet (See Mohanty [6], footnote to the page 298) that

$$|R, L_n, 1| \sim |R', L_n, 1| ,$$

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where $(R', L_n, 1)$ mean of $\sum_{n=1}^{\infty} a_n$ is

$$t_n = \frac{1}{L_{n+1}} \sum_{m=1}^{\infty} (L_{m+1} - L_m) s_m ,$$

where
$$s_n = a_1 + a_2 + a_3 + \ldots + a_n .$$

By Abel's transformation, we have

$$t_n = -\frac{1}{L_{n+1}} \sum_{m=1}^{n-1} a_{m+1} \sum_{p=1}^{m} (L_{p+1} - L_p)$$

$$+ \frac{s_n}{L_{n+1}} \sum_{m=1}^{n} (L_{m+1} - L_m) = s_n - \frac{1}{L_{n+1}} \sum_{m=1}^{n} L_m a_m .$$

Thus
$$\sum_{n=1}^{\infty} a_n \in | R', L_n, 1 | , \text{ if } \sum_{n=1}^{\infty} \Delta \left(\frac{1}{L_{n+1}} \right) \left| \sum_{m=1}^{n+1} L_m a_m \right| < \infty ,$$

where, for any function of n, we write

$$\Delta f_n = f_n - f_{n+1}.$$

This definition has been used in the proof of Theorem 3.

Let $f(t)$ be 2π-periodic and Lebesgue-integrable over $(-\pi, \pi)$. Without any loss of generality, the constant term of the Fourier series of $f(t)$ can be taken to be zero so that its Fourier series, at $t = x$, be

$$\sum_{n=1}^{\infty} (a_n \cos n x + b_n \sin n x) = \sum_{n=1}^{\infty} A_n (x) .$$

We use the following notations throughout this paper:

$\bar{\Delta} > 1, \varepsilon > 0, 0 < \alpha < 1$ and in (1.8) and (1.9)

$$k \geqslant \max (\pi e^{2\varepsilon+2}, e^{2\varepsilon/\alpha}) .$$

(1.1)
$$\varnothing (t) = \frac{1}{2} \{ f(x+t) + f(x-t) \} .$$

(1.2)
$$\varnothing_\beta (t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-u)^{\beta-1} \varnothing(u) \, du \quad (\beta > 0) .$$

(1.3)
$$\varnothing_\beta (t) = \Gamma(1+\beta) t^{-\beta} \varnothing_\beta (t) \quad (\beta \geqslant 0) .$$

(1.4)
$$e(w) = \exp \{ (\log w)^{2\bar{\Delta}-1} \} .$$

(1.5)
$$M(w, t) = \sum_{n < w} (w-n)^{\alpha-1} \frac{n \cos n t}{(\log (n+1))^{1+\varepsilon}}$$

(1.6) $$K(w,t) = \sum_{n<w} (w-n)^{\alpha-1} \frac{n^\alpha}{\log(n+1))^{1+\varepsilon}} \frac{\sin}{\cos}(n\,t)\,.$$

(1.7) $$g(w,u) = \frac{1}{\Gamma(1-\alpha)} \int_u^\pi (t-u)^{-\alpha} M(w,t)\,dt\,.$$

(1.8) $$G(w,u) = \frac{1}{\Gamma(1+\alpha)} \int_0^u x^\alpha \left(\log\frac{k}{x}\right)^\varepsilon \frac{\partial}{\partial x}(g(w,x))\,dx\,.$$

(1.9) $$H(w,u) = \frac{1}{\Gamma(1+\alpha)} \int_u^\pi x^\alpha \left(\log\frac{k}{x}\right)^\varepsilon \frac{\partial}{\partial x}(g(w,x))\,dx\,.$$

(1.10) $$\log_1 = \log\,,\ \log_b = \log\log_{b-1}\ (b>1)\,.$$

We write $T = \left[\frac{k}{t}\right]$, where k is suitable positive constant not necessarily the same in each case and [x] denotes the integral part of x. We take $k = 1$ in Theorem 3.

By « $\downarrow$ » we mean « monotonic decreasing ».

Any sequence $\{x_n\} \in (A)$ if x_n stands for any one of the following sequences:

$$\frac{1}{(\log n)^{1+\varepsilon}}\,,\ \frac{1}{\log n\,(\log_2 n)^{1+\varepsilon}}\,,\ \dots,\ \frac{1}{\log n\dots\log_{b-1} n\,(\log_b n)^{1+\varepsilon}}\,,$$

where $\varepsilon > 0$.

2. Introduction. — Recently Nayak [7] proved the following:

Theorem A. — Let $k > \pi$ and $\bar{\Delta} > 1$. Then $\dfrac{\emptyset(t)}{\log\frac{k}{t}} \in BV(0,\pi)$ implies that $\sum_{n=1}^{\infty} \dfrac{A_n(x)}{(\log(n+1))^{\bar{\Delta}}} \in |R, \exp\{(\log w)^{2\bar{\Delta}-1}\}, 1|$.

In this context we prove the following theorem by sharping the order of the absolute Riesz summability of the above theorem.

THEOREM 1. — *Let* $\bar{\Delta} > 1$ *and* $k > \pi$. *Then* $\dfrac{\emptyset(t)}{\log\frac{k}{t}} \in BV(0,\pi)$ *implies that* $\sum_{n=1}^{\infty} \dfrac{A_n(x)}{(\log(n+1))^{\bar{\Delta}}} \in |R, \exp\{(\log w)^{2\bar{\Delta}-1}\}, \delta|\ (\delta > 0)$.

In 1948, Cheng [2] proved the following:

Theorem B. — Let $0 \leqslant \alpha \leqslant 1$ and $\varepsilon > 0$. Then $\emptyset_\alpha^{(t)} \in BV(0,\pi)$ implies that $\sum_{n=1}^{\infty} \dfrac{A_n(x)}{(\log(n+1))^{1+\varepsilon}} \in |R, n, \alpha|$.

The following theorem gives an improvement of the above theorem :

THEOREM 2. — *Let* $\varepsilon > 0$ *and* $0 \leqslant \alpha \leqslant 1$. *Then* $\dfrac{\varnothing_\alpha(t)}{\left(\log \dfrac{k}{t}\right)^{\varepsilon}} \in BV(0, \pi)$ *implies that* $\sum_{n=1}^{\infty} \dfrac{A_n(x)}{(\log(n+1))^{1+\varepsilon}} \in |C, \alpha|$, *whenever* k *is suitable positive constant, for example,* $k = \max(\pi e^{2\varepsilon+2}, e^{2\varepsilon/\alpha}(0 < \alpha < 1))$.

REMARK 1. — In the above theorem $\dfrac{1}{\log n (\log_2 n)^{1+\varepsilon}}$, $\dfrac{1}{\log n \log_2 n (\log_3 n)^{1+\varepsilon}}, \ldots, \dfrac{1}{\log n \ldots \log_{b-1} n (\log_b n)^{1+\varepsilon}}$ are also the summability factors whenever we replace the factor of $\varnothing_\alpha(t)$, respectively, by $\left(\log_2 \dfrac{k}{t}\right)^{-\varepsilon}$, $\left(\log_3 \dfrac{k}{t}\right)^{-\varepsilon}, \ldots, \left(\log_b \dfrac{k}{t}\right)^{-\varepsilon}$, where $\varepsilon > 0$ and k is a suitable positive constant taken for the convenience in the analysis and not necessarily the same et each occurrence.

Lastly we prove the following general theorem which gives a number of interesting results :

THEOREM 3. — *Let* $0 < p(x)$, $0 < y(x) \downarrow$ *with* $x > 0$ *and satisfy the following conditions :*

(2.1) $$\{x^{\delta} y(x)\} \uparrow \text{ with } x > 0 \text{ for } 1 > \delta > 0.$$

(2.2) $$\left\{\frac{1}{p(1/t)} + \left|\frac{d}{dt}\left(\frac{t}{p(1/t)}\right)\right|\right\} = O\left\{\frac{1}{y(1/t)}\right\}, \text{ as } t \longrightarrow 0 .$$

(2.3) $$\left| x^{1+\delta}\left(\frac{d}{dx}\right)^2 \left(\frac{x}{p(1/x)}\right)\right| \uparrow \text{ with } x > 0 \text{ for } 1 > \delta > 0$$

(2.4) $$\left|\left(\frac{d}{dx}\right)^2 \left(\frac{x}{p(1/x)}\right)\right| \downarrow \text{ with } x > 0.$$

(2.5) $$\sum_{n=1}^{\infty} \frac{y(n)}{n} \left| \left\{ x \left(\frac{d}{dx}\right)^2 \left(\frac{x}{(p(1/x)}\right)\right\}_{x=\frac{1}{n}} \right| < \infty .$$

And let L_n *statisfy the following :*

(2.6) $$\{L_n y(n)\} \uparrow \text{ with } n \geqslant n_0$$

(2.7) $$\left\{\frac{1}{p(1/t)} + \left|\frac{d}{dt}\left(\frac{t}{p(1/t)}\right)\right|\right\} \left\{\sum_{n=\tau}^{\infty} \frac{y(n)}{L_n} |\Delta L_n|\right\} = O(1),$$

uniformly in $0 < t \leqslant \pi$. *Then* $\varnothing_1(t)\, p\left(\dfrac{1}{t}\right) \in BV(0, \pi)$ *implies that* $\sum_{n=1}^{\infty} A_n(x)\, y(n) \in |R', L_n, 1|$.

REMARK 2. — It will be interesting to note that Theorem 3 holds good even if we replace the conditions (2.1) and (2.2) by more general conditions: as $t \rightarrow 0$

$$\text{(2.8)} \qquad \text{(i)} \sum_{m=1}^{T} y(m) = O\left\{t^{-1} p\left(\frac{1}{t}\right)\right\}; \text{ (ii)} \frac{d}{dt}\left(\frac{t}{p\left(\frac{1}{t}\right)}\right) \sum_{m=1}^{T} y(m) = O\left(\frac{1}{t}\right),$$

and

$$\text{(2.9)} \qquad \text{(i)}\, t = o\left\{p\left(\frac{1}{t}\right)\right\}; \text{ (ii)}\; t \frac{d}{dt}\left(\frac{t}{p\left(\frac{1}{t}\right)}\right) = o(1) .$$

The condition (2.7) is the combined form of the conditions:

$$\text{(2.10)} \qquad \begin{cases} \text{(i)} \displaystyle\sum_{n=T}^{\infty} \frac{|\Delta L_n|}{L_n} y(n) = O\left\{p\left(\frac{1}{t}\right)\right\} \\ \text{and} \\ \text{(ii)} \displaystyle\frac{d}{dt}\left(\frac{t}{p\left(\frac{1}{t}\right)}\right) \sum_{n=T}^{\infty} \frac{|\Delta L_n|\, y(n)}{L_n} = O(1) , \end{cases}$$

which have been used explicitly in this paper. Therefore (2.7) can be replaced by (2.10).

Now, if we restrict $p(x)$ a little more by the condition

$$\text{(2.11)} \qquad p\left(\frac{1}{t}\right) \left| \frac{d}{dt}\left(\frac{t}{p\left(\frac{1}{t}\right)}\right)\right| = O(1) ; \text{ as } t \rightarrow 0 ,$$

then (2.1), (2.2) and (2.7) can be replaced by (2.8) (i), (2.9) (i) and (2.10) (i) for the proof of Theorem 3.

3. — In this paper we shall use the following order estimates for $0 < \alpha < 1$, $0 < \delta \leqslant 1$, $\bar{\Delta} > 1$, $\varepsilon > 0$ and large w:

$$\text{(3.1)} \qquad \sum_{n<w} \{e(w) - e(n)\}^{\delta-1} \frac{e(n) \sin nt}{n(\log(n+1))^{\bar{\Delta}}}$$
$$= O\{t^{-\delta} w^{-\delta} e^{\delta}(w) (\log w)^{\alpha(\bar{\Delta}-1)(\delta-1)-\bar{\Delta}}\} .$$

$$\text{(3.2)} \qquad \sum_{n<w} (w-n)^{\alpha-1} \frac{\sin nt}{(\log(n+1))^{1+\varepsilon}} = O\{w^{\alpha} (\log w)^{-1-\varepsilon}\} .$$

$$\text{(3.3)} \qquad K(w, t) = O\{t^{-\alpha} w^{\alpha} (\log w)^{-1-\varepsilon}\} .$$

$$\text{(3.4)} \qquad g(w, t) = O\{t^{-\alpha} w^{\alpha} (\log w)^{-1-\varepsilon}\} .$$

Proof of (3.1). — For $w > t^{-1}$, we write

$$\sum_{n<w} \{e(w) - e(n)\}^{\delta-1} \cdot \frac{e(n)}{(\log(n+1))^{\bar{\Delta}}} \cdot \frac{\sin nt}{n}$$

$$\sum_{n=1}^{\left[w-\frac{1}{t}\right]} + \sum_{\left[w-\frac{1}{t}\right]+1}^{[w]} = \Sigma_1 + \Sigma_2 \text{, say.}$$

Now, for $p = \left[\exp\left\{\frac{\bar{\Delta}}{\bar{\Delta}-1}\right\}\right]$, we write

$$\Sigma_1 = \sum_{n=1}^{p} + \sum_{p+1}^{\left[w-\frac{1}{t}\right]} = \Sigma_{1,1} + \Sigma_{1,2} \text{, say .}$$

Since $(e(w) - e(n))^{\delta-1} \uparrow$ with n, where $0 < \delta \leqslant 1$, and $\left\{\frac{e(n)}{n(\log(n+1))^{\bar{\Delta}}}\right\} \uparrow$ with $n \geqslant p$, we have, by Abel's lemma, that

$$\Sigma_{1,2} = O\left\{\left\{e(w) - e\left(w-\frac{1}{t}\right)\right\}^{\delta-1} \frac{e\left(w-\frac{1}{t}\right)}{\left(w-\frac{1}{t}\right)\left(\log\left(w-\frac{1}{t}\right)\right)^{\bar{\Delta}}} \cdot \max_{1+p \leqslant p' \leqslant \left[w-\frac{1}{t}\right]} \left|\sum_{n=p'}^{\left[w-\frac{1}{t}\right]} \sin nt\right|\right\}$$

$$= O\left\{t^{-1}\left\{t^{-1}e'\left(w-\frac{1}{t}\right)\right\}^{\delta-1} e'\left(w-\frac{1}{t}\right)\left(\log\left(w-\frac{1}{t}\right)\right)^{2-3\bar{\Delta}}\right\}$$

$$= O\left\{t^{-\delta}\left\{e'\left(w-\frac{1}{t}\right)\right\}^{\delta}\left(\log\left(w-\frac{1}{t}\right)\right)^{2-3\bar{\Delta}}\right\}$$

$$= O\left\{t^{-\delta} w^{-\delta} e^{\delta}(w) (\log w)^{2(\bar{\Delta}-1)(\delta-1)-\bar{\Delta}}\right\},$$

uniformly in $0 < t \leqslant \pi$. And

$\Sigma_{1,1} = O(1)$.

Now

$$\Sigma_2 = O\left\{\int_{w-\frac{1}{t}}^{w} \{e(w) - e(x)\}^{\delta-1} e'(x) (\log x)^{2-3\bar{\Delta}}\, dx\right\}$$

$$= O\left\{(\log w)^{2-3\bar{\Delta}}\left\{e(w) - e\left(w-\frac{1}{t}\right)\right\}^{\delta}\right\}$$

$$= O\{t^{-\delta} w^{-\delta} e^{\delta}(w) (\log w)^{2(\bar{\Delta}-1)(\delta-1)-\bar{\Delta}}\} \text{, uniformly in } 0 < t \leqslant \pi .$$

Combining $\Sigma_{1,1}$, $\Sigma_{1,2}$, and Σ_2 we follow the result.

Proof of (3.2). — We write

$$\sum_{n<w} (w-n)^{\alpha-1} \frac{\sin n t}{(\log(n+1))^{1+\varepsilon}} = \sum_{n=1}^{\left[\frac{w}{2}\right]} + \sum_{\left[\frac{w}{2}\right]+1}^{[w]} = \Sigma_1 + \Sigma_2 \text{ , say.}$$

By Abel's lemma, we have

$$\Sigma_1 = O \left\{ \frac{w^\alpha}{(\log w)^{1+\varepsilon}} \max_{1 \leq n' \leq \left[\frac{w}{2}\right]} \left| \sum_{n=n'}^{\left[\frac{w}{2}\right]} \frac{\sin n t}{n} \right| \right\} .$$

$$= O \{ w^\alpha (\log w)^{-1-\varepsilon} \} .$$

And

$$\Sigma_2 = O \left\{ (\log w)^{-1-\varepsilon} \int_{\frac{w}{2}}^{w} (w-x)^{\alpha-1} \, d x \right\}$$

$$= O \{ w^\alpha (\log w)^{-1-\varepsilon} \} .$$

Hence, on combining Σ_1 and Σ_2 , we follow the proof of (3.2).

Proof of (3.3). — Its proof, being simple, has been omitted.

Proof of (3.4). — We have, by using the first and second mean value theorem,

$$\Gamma(1-\alpha)\, g(w, t) = \sum_{n<w} (w-n)^{\alpha-1} \frac{n}{(\log(n+1))^{1+\varepsilon}} \cdot$$

$$\cdot \left\{ \int_t^{t+\frac{1}{n}} (u-t)^{-\alpha} \cos n u \, d u + \int_{t+\frac{1}{n}}^{\pi} (u-t)^{-\alpha} \cos n u \, d u \right\}$$

$$= \sum_{n<w} (w-n)^{\alpha-1} \frac{n}{(\log(n+1))^{1+\varepsilon}} \cdot$$

$$\cdot \left\{ \cos n \vartheta \int_t^{t+\frac{1}{n}} (u-t)^{-\alpha} \, d u + n^\alpha \int_{t+\frac{1}{n}}^{t'} \cos n u \, d u \right\}$$

$$\left(t \leq \vartheta \leq t + \frac{1}{n} \text{ , } t + \frac{1}{n} < t' < \pi \right)$$

(3.4.1) $$= \sum_{n<w} (w-n)^{\alpha-1} \frac{n^\alpha}{(\log(n+1))^{1+\varepsilon}} \{ \cos n \vartheta + \sin n t' - \sin(n t + 1) \}$$

$$= O \{ t^{-\alpha} w^\alpha (\log w)^{-1-\varepsilon} \} \text{ , by (3.3).}$$

4. — For the proof of the theorems we require the following lemmas:

LEMMA 1. — *If* $\sum_{n=1}^{\infty} a_n \in |R, \lambda_n, r| \ (r \geqslant 0)$ *then* $\sum_{n=1}^{\infty} a_n \in |R, \lambda_n, r'|$ *for* $r' > r$.

This is due to Obrechkoff [8], [9].

LEMMA 2. — $0 \leqslant u < \pi$, $0 < \alpha < 1$, $q(x) = \left(\log \frac{k}{x}\right)^{\varepsilon} \left(\alpha - \frac{\varepsilon}{\log \frac{k}{x}}\right)$ $(\varepsilon > 0)$

and $k = \max(\pi e^{2\varepsilon+2}, e^{2\varepsilon/\alpha})$. *Then, for* $0 \leqslant a < b \leqslant \infty$ *and* $0 < y \leqslant 1$,

(4.1)
$$\int_a^b \frac{\sin nt}{t} \left(\log \frac{k}{ty}\right)^{\varepsilon-2} dt = O\left\{\left(\log \frac{nk}{y}\right)^{\varepsilon-2}\right\}.$$

and

(4.2)
$$\int_u^{\pi} x^{\alpha-1} q(x)\, dx \int_x^{\pi} (t-x)^{-\alpha} \cos nt\, dt$$
$$= \frac{q(u)}{n} \int_{\frac{u}{\pi}}^{1} y^{\alpha-1} (1-y)^{-\alpha} \sin\left(\frac{nu}{y}\right) dy + O\{n^{-1} (\log n)^{\varepsilon-1}\}.$$

Proof of Lemma 2.

Proof of (4.1). — Let $b > \frac{1}{n}$. Then we write

$$\int_a^b \frac{\sin nt}{t} \left(\log \frac{k}{ty}\right)^{\varepsilon-2} dt = \int_a^{\frac{1}{n}} + \int_{\frac{1}{n}}^{b} = I_1 + I_2, \text{ say.}$$

$$I_1 \leqslant n \int_a^{\frac{1}{n}} \left(\log \frac{k}{ty}\right)^{\varepsilon-2} dt$$
$$= O\left\{n^{\frac{1}{2}} \left(\log \frac{kn}{y}\right)^{\varepsilon-2} \int_a^{\frac{1}{n}} t^{-\frac{1}{2}}\, dt\right\}$$
$$= O\left\{\left(\log \frac{kn}{y}\right)^{\varepsilon-2}\right\}.$$

And, since $\left\{t^{-1}\left(\log \frac{k}{ty}\right)^{\varepsilon-2}\right\} \downarrow$ with t, therefore by the second mean value theorem, we have

$$I_2 = n \left(\log \frac{kn}{y}\right)^{\varepsilon-2} \int_{\frac{1}{n}}^{b'} \sin nt\, dt \left(\frac{1}{n} < b' < b\right)$$
$$= O\left\{\left(\log \frac{kn}{y}\right)^{\varepsilon-2}\right\}.$$

Combining I_1 and I_2, we get, for $b > \frac{1}{n}$,

$$\int_a^b \frac{\sin nt}{t}\left(\log\frac{k}{ty}\right)^{\varepsilon-z} dt = O\left\{\left(\log\frac{kn}{y}\right)^{\varepsilon-z}\right\}.$$

And, in the case $b \leqslant \frac{1}{n}$, we have

$$\int_a^b \frac{\sin nt}{t}\left(\log\frac{k}{ty}\right)^{\varepsilon-z} dt \leqslant \int_a^{\frac{1}{n}} \frac{\sin nt}{t}\left(\log\frac{k}{ty}\right)^{\varepsilon-z} dt = O\left\{\left(\log\frac{kn}{y}\right)^{\varepsilon-z}\right\},$$

from I_1. This completes the proof of (4.1).

Proof of (4.2). — Set, for $z = tp > 0$,

$$I_z = \int_u^\pi x^{z-1} q(x)\, dx \int_x^\pi (t + z - x)^{-z} \cos nt\, dt$$

We then have, by changing the order of integration,

$$I_z = \int_u^\pi \cos nt\, dt \int_u^t x^{z-1}(t + z - x)^{-z} q(x)\, dx$$

$$= \int_u^\pi \cos nt\, dt \int_{\frac{u}{t}}^1 y^{z-1}\left(1 + \frac{z}{t} - y\right)^{-z} q(ty)\, dy$$

$$= \int_{\frac{u}{\pi}}^1 y^{z-1}\left(1 + \frac{z}{t} - y\right)^{-z} dy \int_{\frac{u}{y}}^\pi q(ty) \cos nt\, dt\,.$$

(by changing the order of integration)

Now

$$\int_u^\pi x^{z-1} q(x)\, dx \int_x^\pi (t - x)^{-z} \cos nt\, dt$$

$$= \underset{p \to 0}{\mathrm{Lt}}\ I_z$$

$$= \int_{\frac{u}{\pi}}^1 y^{z-1}(1 - y)^{-z}\, dy \int_{\frac{u}{y}}^\pi q(ty) \cos nt\, dt$$

$$= \frac{q(u)}{n} \int_{\frac{u}{\pi}}^1 y^{z-1}(1 - y)^{-z} \sin(nu/y)\, dy$$

$$- \frac{1}{n} \int_{\frac{u}{\pi}}^1 y^{z-1}(1 - y)^{-z}\, dy \int_{\frac{u}{y}}^\pi \sin nt \frac{\partial}{\partial t} q(ty)\, dt$$

$$= \frac{q(u)}{n} \int_{\frac{u}{\pi}}^{1} y^{\alpha-1}(1-y)^{-\alpha} \sin(nu/y)\, dy$$

$$+ O\left\{ \frac{1}{n} \int_{\frac{u}{\pi}}^{1} y^{\alpha-1}(1-y)^{-\alpha} \left(\log \frac{k\, n}{y}\right)^{\varepsilon-1} dy \right\} \text{ (by (4.1))}$$

$$= \frac{q(u)}{n} \int_{\frac{u}{\pi}}^{1} y^{\alpha-1}(1-y)^{-\alpha} \sin\left(\frac{n\, u}{y}\right) dy + O\{n^{-1}(\log n)^{\varepsilon-1}\}$$

This completes the proof of (4.2).

5. Proof of the theorems. — In view of Lemma 1, we take $0 < \delta < 1$ for the proof of Theorem 1. Also, since we deduce the case $\alpha = 1$ of Theorem 2 from Theorem 3, the proof of Theorem 2 shall be given after Theorem 3.

5.1. *Proof of Theorem* 1. — We have

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \frac{\emptyset(t)}{\log \frac{k}{t}} \log \frac{k}{t} \cos nt\, dt\, .$$

Therefore integrating by parts

$$A_n(x) = \frac{2}{\pi} \frac{\emptyset(\pi)}{\log \frac{k}{\pi}} \int_0^{\pi} \log \frac{k}{t} \cos nt\, dt$$

$$- \frac{2}{\pi} \int_0^{\pi} d\left\{ \frac{\emptyset(t)}{\log \frac{k}{t}} \right\} \int_0^{t} \log \frac{k}{u} \cos nu\, du\, .$$

Now, for $0 < t \leqslant \pi$, we have integrating by parts

$$\int_0^{t} \log \frac{k}{u} \cos nu\, du = \frac{\sin nt}{n} \log \frac{k}{t} + \frac{1}{n} \int_0^{t} \frac{\sin nu}{u} du$$

$$= \frac{\sin nt}{n} \log \frac{k}{t} + O\left(\frac{1}{n}\right).$$

The series $\sum_{n=1}^{\infty} \frac{A_n(x)}{(\log(n+1))^{\Delta}} \in |R, e(w), \delta|$, where $\bar{\Delta} > 1$ and $0 < \delta < 1$, if

$$I_1 = \int_1^{\infty} \frac{e'(w)}{(e(w))^{1+\delta}} \left| \sum_{n<w} \{e(w) - e(n)\}^{\delta-1} \frac{e(n)}{n(\log(n+1))^{\bar{\Delta}}} \right| dw$$

is convergent and

$$I_2 = \log \frac{k}{t} \int_1^{\infty} \frac{e'(w)}{(e(w))^{1+\delta}} \left| \sum_{n<w} \{e(w) - e(n)\}^{\delta-1} \frac{e(n) \sin nt}{n(\log(n+1))^{\bar{\Delta}}} \right| dw = O(1),$$

uniformly in $0 < t < \pi$.

The convergence of I_1 follows from Lemma 1, since $\sum_{n=2}^{\infty} \frac{1}{n (\log n)^{\bar{\Delta}}} \in | C, 0 |$ for $\bar{\Delta} > 1$. Therefore, for the proof of the theorem, we only require to prove the boundedness of I_2 for $0 < t < \pi$.

For $\tau = \frac{k}{t}\left(\log \frac{k}{t}\right)^{\lambda}$, where $\lambda = (\bar{\Delta} - 1)\left(2 - \frac{1}{\delta}\right)$, we write

$$I_2 = \log \frac{k}{t}\left(\int_1^{\tau} + \int_{\tau}^{\infty}\right) = I_{2,1} + I_{2,2}, \text{ say }.$$

Since $|\sin nt| \leqslant nt$, we have

$$I_{2,1} = O\left\{t \log \frac{k}{t} \int_1^{\tau} \frac{e'(w)}{(e(w))^{1+\delta}} dw \int_1^{w} \{e(w) - e(x)\}^{\delta-1} \cdot\right.$$

$$\left. \cdot\, e(x) (\log (x+1))^{-\bar{\Delta}}\, dx\right\} + O(1)$$

$$= O\left\{t \log \frac{k}{t} \int_1^{\tau} e(x) (\log (x+1))^{-\bar{\Delta}}\, dx \,\cdot\right.$$

$$\left. \cdot \int_x^{\tau} \frac{(e(w) - (x))^{\delta-1} e'(w)}{(e(w))^{1+\delta}}\, dw\right\} + O(1).$$

Now the inner integral, integrating by parts, is equal to

$$\frac{1}{\delta}\left[(e(w) - e(x))^{\delta} (e(w))^{-1-\delta}\right]_x^{\tau} + \frac{1+\delta}{\delta}\int_x^{\tau} \frac{\{e(w) - e(x)\}^{\delta} e'(w)}{(e(w))^{2+\delta}}\, dw$$

$$= O\,\{(e(\tau))^{-1}\} + O\left\{\int_x^{\tau} \frac{e'(w)}{(e(w))^2}\, dw\right\} = O\,\{(e(x)^{-1}\}.$$

Therefore

$$I_{2,1} = O\left\{t \log \frac{k}{t} \int_1^{\tau} (\log (x+1))^{-\bar{\Delta}}\, dx\right\} + O(1)$$

$$= O\left\{t\,\tau\left(\log \frac{k}{t}\right)^{1-\bar{\Delta}}\right\} + O(1)$$

$$= O\left\{\left(\log \frac{k}{t}\right)^{1+\lambda-\bar{\Delta}}\right\} + O(1)$$

$$= O(1).$$

And, by (3.1), we have

$$I_{2.2} = O\left\{ t^{-\delta} \log \frac{k}{t} \int_{\tau}^{\infty} \frac{e'(w)}{e(w)} w^{-\delta} (\log w)^{2(\bar{\Delta}-1)(\delta-1)-\bar{\Delta}} \, dw \right\}$$

$$= O\left\{ t^{-\delta} \log \frac{k}{t} \int_{\tau}^{\infty} \frac{(\log w)^{2(\bar{\Delta}-1)\delta-\bar{\Delta}}}{w^{1+\delta}} \, dw \right\}$$

$$= O\left\{ t^{-\delta} \tau^{-\delta} \left(\log \frac{k}{t} \right)^{(\bar{\Delta}-1)(2\delta-1)} \right\}$$

$$= O(1) .$$

This completes the proof of Theorem 1.

5.2 *Proof of Theorem* 3. — We have

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \emptyset(t) \cos nt \, dt .$$

Integrating by parts and using $\emptyset_1(\pi) = 0$, we have

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \emptyset_1(t) \, nt \sin nt \, dt$$

$$= \frac{2}{\pi} \int_0^{\pi} \emptyset_1(t) \, p\left(\frac{1}{t}\right) \frac{nt}{p\left(\frac{1}{t}\right)} \sin nt \, dt$$

$$= \frac{2}{\pi} \emptyset_1(\pi) \, p\left(\frac{1}{\pi}\right) \int_0^{\pi} \frac{nt}{p\left(\frac{1}{t}\right)} \sin nt \, dt$$

$$- \frac{2}{\pi} \int_0^{\pi} d\left\{ \emptyset_1(t) \, p\left(\frac{1}{t}\right) \right\} \int_0^{t} \frac{nu}{p\left(\frac{1}{u}\right)} \sin nu \, du ,$$

integration by parts again.

The series $\sum_{n=1}^{\infty} A_n(x) \, y(n) \in | R', L_n, 1 |$, if

$$\sum_{n=1}^{\infty} \Delta\left(\frac{1}{L_{n+1}}\right) \left| \sum_{m=1}^{n+1} L_m \, y(m) \int_0^{t} \frac{mu}{p\left(\frac{1}{u}\right)} \sin mu \, du \right| = O(1) ,$$

uniformly in $0 < t \leqslant \pi$, since $\emptyset_1(\pi) \, p\left(\frac{1}{\pi}\right)$ and $\frac{2}{\pi} \int_0^{\pi} \left| d\left\{ \emptyset_1(t) \, p\left(\frac{1}{t}\right) \right\} \right|$ are finite.

Now, integrating by parts two times and using (2.1) and (2.2), we have

$$\int_0^t \frac{m u}{p\left(\frac{1}{u}\right)} \sin m u \, du = -\frac{t}{p\left(\frac{1}{t}\right)} \cos m t + \frac{\sin m t}{m} \frac{d}{dt}\left(\frac{t}{p\left(\frac{1}{t}\right)}\right)$$

$$-\int_0^t \frac{\sin m u}{m} \left(\frac{d}{du}\right)^2 \left(\frac{u}{p\left(\frac{1}{u}\right)}\right) du .$$

And, by proceeding parallel to that of (4.1) of Lemma 2 and using (2.3) and (2.4), it is easy to follow that

$$\int_0^t \frac{\sin m u}{m} \left(\frac{d}{du}\right)^2 \left(\frac{u}{p\left(\frac{1}{u}\right)}\right) du = O\left\{\frac{1}{m} \left| \left\{u \left(\frac{d}{du}\right)^2 \left(\frac{u}{p\left(\frac{1}{u}\right)}\right)\right\}_{u=\frac{1}{m}} \right|\right\}.$$

Therefore, to prove the theorem, we only require to show that

$$\Sigma_1 = \sum_{n=0}^{\infty} \Delta\left(\frac{1}{L_{n+1}}\right) \sum_{m=1}^{n+1} L_m \frac{y(m)}{m} \left| \left\{u \left(\frac{d}{du}\right)^2 \left(\frac{u}{p\left(\frac{1}{u}\right)}\right)\right\}_{u=\frac{1}{m}} \right| .$$

is convergent. And, uniformly in $0 < t < \pi$,

$$\Sigma_2 = \frac{t}{p\left(\frac{1}{t}\right)} \sum_{n=0}^{\infty} \Delta\left(\frac{1}{L_{n+1}}\right) \left| \sum_{m=1}^{n+1} L_m y(m) \cos m t \right| = O(1)$$

$$\Sigma_3 = \left| \frac{d}{dt} \frac{t}{p\left(\frac{1}{t}\right)} \right| \sum_{n=0}^{\infty} \Delta\left(\frac{1}{L_{n+1}}\right) \left| \sum_{m=1}^{n+1} L_m y(m) \frac{\sin m t}{m} \right| = O(1) .$$

By using (2.5), the convergence of Σ_1 follows from Lemma 1. Now, for $T = \left[\frac{1}{t}\right]$ we write

$$\Sigma_2 = \frac{t}{p\left(\frac{1}{t}\right)} \left(\sum_{n=0}^{T} + \sum_{T+1}^{\infty}\right) = \Sigma_{2.1} + \Sigma_{2.2} \text{, say.}$$

And

$$\Sigma_{2.1} = O\left\{\frac{t}{p\left(\frac{1}{t}\right)} \sum_{n=0}^{T} \Delta\left(\frac{1}{L_{n+1}}\right) \sum_{m=1}^{n+1} L_m y(m)\right\}$$

$$= O\left\{\frac{t}{p\left(\frac{1}{t}\right)} \sum_{m=1}^{T} L_m y(m) \sum_{n+1=m}^{T} \Delta\left(\frac{1}{L_{n+1}}\right)\right\}$$

(by changing the order of summation)

$$= O\left\{ \frac{t}{p\left(\frac{1}{t}\right)} \sum_{m=1}^{T} y(m) \right\}$$

$$= O(1) ,$$

by (2.1) and (2.2).

By using Abel's lemma, in view of (2.6), we have

$$\Sigma_{2.2} = O\left\{ \frac{t}{p\left(\frac{1}{t}\right)} \sum_{n=T+1}^{\infty} L_{n+1}\, y(n+1)\, \Delta\left(\frac{1}{L_{n+1}}\right) \cdot \max_{1 \leqslant m' \leqslant n+1} \left| \sum_{m=m'}^{n+1} \cos mt \right| \right.$$

$$= O\left\{ \frac{1}{p\left(\frac{1}{t}\right)} \sum_{n=T}^{\infty} \frac{|\Delta L_n|}{L_n}\, y(n) \right\}$$

$$= O(1) ,$$

by (2.7). Thus, on collecting the results, we follow that

$$\Sigma_2 = O(1) .$$

Now dividing the range of summation of Σ_3, just as in Σ_2, into $\Sigma_{3.1}$ and $\Sigma_{3.2}$, we follow, by using $\sin mt \leqslant mt$, (2.1) and (2.2), that $\Sigma_{3.1} = O(1)$. And by Abel's lemma, in view of (2.6), we have

$$\Sigma_{3.2} \leqslant \left| \frac{d}{dt}\left(\frac{t}{p\left(\frac{1}{t}\right)} \right) \right| \sum_{n=T+1}^{\infty} L_{n+1}\, y(n+1)\, \Delta\left(\frac{1}{L_{n+1}}\right) \cdot$$

$$\cdot \max_{1 \leqslant m' \leqslant n+1} \left| \sum_{m=m'}^{n+1} \frac{\sin mt}{m} \right|$$

$$= O\left\{ \left| \frac{d}{dt}\left(\frac{t}{p\left(\frac{1}{t}\right)} \right) \right| \sum_{n=T}^{\infty} \frac{|\Delta L_n|}{L_n}\, y(n) \right\}$$

$$= O(1) , \text{ by } (2.7).$$

This terminates the proof of Theorem 3.

5.2.1. — In this sub-section we consider some interesting corollaries of Theorem 3. Throughout k is suitable positive constant taken for the convenience in the analysis. Also we use the fact that $| R', L_n, 1 | \sim | R, L_n, 1 |$ and $| R, n, 1 | \sim | C, 1 |$, while stating the corollaries.

COROLLARY 1. — Let $\beta > 0$. Then $\emptyset_1(t)\left(\log \frac{k}{t}\right)^\beta \in BV(0, \pi)$ implies that $\sum_{n=1}^{\infty} A_n(x)\lambda(n) \in |R, \exp\{(\log n)^{1+\beta}\}, 1|$, whenever $\{\lambda(n)(\log n)^\beta\} \in (A)$.
This generalises the case $\alpha = 1$ of Dikshit [3].

COROLLARY 2. — Let $\beta > 0$. Then $\emptyset_1(t)\left(\log_2 \frac{k}{t}\right)^\beta \in BV(0, \pi)$ implies that $\sum_{n=3}^{\infty} A_n(x)\lambda(n) \in |R, \exp\{\log n(\log_2 n)^\beta\}, 1|$ whenever $\{\lambda(n)(\log_2 n)^\beta\} \in (A)$.
This generalises the following result due to Chandra [1]:

Theorem D. — $\emptyset_1(t)\log_2 \frac{k}{t} \in BV(0, \pi)$, then $\sum_{n=3}^{\infty} \frac{A_n(x)}{(\log n)^2 \log_2 n} \in |R, \exp\{\log n \log_2 n\}, 1|$.

COROLLARY 3. — If $\emptyset_1(t) \in BV(0, \pi)$, then $\sum_{n=1}^{\infty} \frac{A_n(x)}{(\log(n+1))^c} \in |R, \exp\{(\log n)^c(\log_2 n)^{-h}\}, 1|$, where $c > 0$ and $h > 1$.
This improves the case $\alpha = 1$ of Cheng [2], that is, the case $\alpha = 1$ of Theorem B of the present paper.

COROLLARY 4. — If $\emptyset_1(t) \in BV(0, \pi)$, then $\sum_{n=1}^{\infty} \frac{A_n(x)}{(\log(n+1)\log_2(n+2))^d} \in |R, \exp\{(\log n)^d\}, 1|$, where $d > 1$.
This generalises the following result due to Lal [5]:

Theorem E. — Let $c > 0$ and $d > 1$. Then $\emptyset_1(t) \in BV(0, \pi)$ implies that $\sum_{n=1}^{\infty} \frac{A_n(x)}{(\log(n+1))^{d+c}} \in |R, \exp\{(\log n)^d\}, 1|$.

The following corollary improves the case $\alpha = 1$ of Theorem B which is due to Cheng [2]. Remark 1 is also applicable for this corollary:

COROLLARY 5 — Let $\varepsilon > 0$. Then $\left\{\frac{\emptyset_1(t)}{\left(\log \frac{k}{t}\right)^\varepsilon}\right\} \in BV(0, \pi)$ implies that $\sum_{n=1}^{\infty} \frac{A_n(x)}{(\log(n+1))^{1+\varepsilon}} \in |C, 1|$.

Finally we give the following interesting result:

COROLLARY 6. — Let $\varepsilon > 0$. Then $\left\{\frac{\emptyset_1(t)}{\left(\log_2 \frac{k}{t}\right)^\varepsilon}\right\} \in BV(0, \pi)$ implies that $\sum_{n=1}^{\infty} \frac{A_n(x)}{(\log_2(n+2))^{1+\varepsilon}} \in |R, \log n, 1|$.

5.3. *Proof of Theorem* 2. — The case $\alpha = 1$ follows from Corollary 5 of Theorem 3. Thus we consider two cases: (i) $\alpha = 0$ and (ii) $0 < \alpha < 1$.

CASE (i) : $\alpha = 0$.

We have, by writing $g(t) = \emptyset(t)\left(\log\frac{k}{t}\right)^{-\varepsilon}$,

$$A_n(x) = \frac{2}{\pi}\int_0^\pi g(t)\left(\log\frac{k}{t}\right)^\varepsilon \cos nt\,dt$$

$$= \frac{2}{\pi} g(\pi)\int_0^\pi \left(\log\frac{k}{t}\right)^\varepsilon \cos nt\,dt$$

$$-\frac{2}{\pi}\int_0^\pi dg(t)\int_0^t \left(\log\frac{k}{u}\right)^{\varepsilon} \cos nu\,du ,$$

integrating by parts. Also, for $0 < t \leqslant \pi$, we have

$$\int_0^t \left(\log\frac{k}{u}\right)^\varepsilon \cos nu\,du = \frac{\sin nt}{n}\left(\log\frac{k}{t}\right)^\varepsilon + \frac{\varepsilon}{n}\int_0^t \frac{\sin nu}{u}\left(\log\frac{k}{u}\right)^{\varepsilon-1} du$$

$$= \frac{\sin nt}{n}\left(\log\frac{k}{t}\right)^\varepsilon + O\left\{\frac{1}{n(\log(n+1))^{1-\varepsilon}}\right\},$$

by (4.1) of Lemma 2. Therefore, by the hypothesis,

$$\sum_{n=1}^{\infty}\frac{|A_n(x)|}{(\log(n+1))^{1+\varepsilon}} \leqslant \frac{2}{\lambda}\int_0^\lambda |dg(t)| \left(\log\frac{k}{t}\right)^\varepsilon \sum_{n=1}^{\infty}\frac{|\sin nt|}{n(\log(n+1))^{1+\varepsilon}} +$$

$$+ O\left\{\sum_{m=1}^{\infty}\frac{1}{n(\log(n+1))^2}\right\} = \frac{2}{\lambda}\int_0^\lambda \|dg(t)\| \Sigma_1 + O(1), \text{ say}.$$

Now, for $0 < t < \pi$, we write

$$\Sigma_1 = \left(\log\frac{k}{t}\right)^\varepsilon \left\{\sum_{n=1}^{T}\frac{|\sin nt|}{n(\log(n+1))^{1+\varepsilon}} + \sum_{n=T+1}^{\infty}\frac{|\sin nt|}{n(\log(n+1))^{1+\varepsilon}}\right\} \leqslant$$

$$\leqslant t\left(\log\frac{k}{t}\right)^\varepsilon \sum_{n=1}^{T}(\log(n+1))^{-1-\varepsilon} + \left(\log\frac{k}{t}\right)^\varepsilon \sum_{n=T}^{\infty}\frac{1}{n(\log n)^{1+\varepsilon}}$$

$$= O\left\{t\left(\log\frac{k}{t}\right)^\varepsilon T(\log T)^{-1-\varepsilon}\right\} + O\left\{\frac{\left(\log\frac{k}{t}\right)^\varepsilon}{(\log T)^\varepsilon}\right\}$$

$$= O(1) .$$

This terminates the proof of Case (i).

CASE (ii) : $0 < \alpha < 1$.

The series $\sum_{n=1}^{\infty} \frac{A_n(x)}{(\log(n+1))^{1+\varepsilon}} \in |R, n, \alpha|$,

if

$$\int_1^{\infty} w^{-1-\alpha} \left| \int_0^{\pi} \varnothing(t)\, M(w,t)\, dt \right| dw < \infty .$$

Now

$$\int_0^{\pi} \varnothing(t)\, M(w,t)\, dt = \frac{1}{\Gamma(1-\alpha)} \int_0^{\pi} M(w,t) \left(\int_0^{t} (t-u)^{-\alpha}\, d\, \varnothing_\alpha(u) \right) dt$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^{\pi} d\, \varnothing_\alpha(u) \int_u^{\pi} (t-u)^{-\alpha} M(w,t)\, dt$$

$$= \int_0^{\pi} g(w,u)\, d\, \varnothing_\alpha(u)$$

$$= \Big[g(w,u)\, \varnothing_\alpha(u) \Big]_0^{\pi} - \int_0^{\pi} \varnothing_\alpha(u) \frac{\partial}{\partial u} g(w,u)\, du$$

$$= - \int_0^{\pi} \varnothing_\alpha(u) \frac{\partial}{\partial u} g(w,u)\, du + O \left\{ \frac{\omega^{\alpha}}{(\log \omega)} 1+\varepsilon \right\} \text{(by (3 . 4))}$$

$$= - \frac{1}{\Gamma(1+\alpha)} \int_0^{\pi} \varnothing_\alpha(u)\, u^{\alpha} \frac{\partial}{\partial u} g(w,u)\, du + O \left\{ \frac{\omega^{\alpha}}{(\log \omega)} 1+\varepsilon \right\}$$

$$= - \frac{1}{\Gamma(1+\alpha)} \frac{\varnothing_\alpha(\pi)}{\left(\log \frac{k}{\pi}\right)^{\varepsilon}} \int_0^{\pi} u^{\alpha} \left(\log \frac{k}{u}\right)^{\varepsilon} \frac{\partial}{\partial u} g(w,u)\, du + O \left\{ \frac{\omega^{\alpha}}{(\log \omega)} 1+\varepsilon \right\}$$

$$+ \frac{1}{\Gamma(1+\alpha)} \int_0^{\pi} d \left\{ \frac{\varnothing_\alpha(u)}{\left(\log \frac{k}{u}\right)^{\varepsilon}} \right\} \int_0^{u} x^{\alpha} \left(\log \frac{k}{x}\right)^{\varepsilon} \frac{\partial}{\partial x} g(w,x)\, dx .$$

Therefore for the proof of the theorem it is sufficient to show that

(5.3.1.) $$I_1 = \int_1^{\infty} w^{-1-\alpha} \,|\, G(w,\pi) \,|\, dw < \infty ,$$

and

(5.3.2.) $$I_2 = \int_1^{\infty} w^{-1-\alpha} \,|\, G(w,u) \,|\, dw = O(1) ,$$

uniformly in $0 < u < \pi$.

Proof of (5.3.1.). — On integrating by parts, we have

$$G(w,\pi) = -\int_0^{\pi} g(w,u)\, u^{\alpha-1}\left(\log\frac{k}{u}\right)^{\varepsilon}\left(\alpha - \frac{\varepsilon}{\log\frac{k}{u}}\right) du + O\{|g(\omega,\lambda)|\}$$

Therefore, by (1.7) and (1.5),

$$I_1 = \frac{1}{\Gamma(1-\alpha)}\int_1^{\infty} w^{-1-\alpha}\left|\sum_{n<w}(w-n)^{\alpha-1}\frac{n}{(\log(n+1))^{1+\varepsilon}}\right.\cdot$$

$$\cdot\int_0^{\pi} u^{\alpha-1}\left(\log\frac{k}{u}\right)^{\varepsilon}\left(\alpha-\frac{\varepsilon}{\log\frac{k}{u}}\right) du \int_u^{\pi}(t-u)^{-\alpha}\cos nt\, dt\,\Big|\, dw + O(1)\ (\text{by } (3\cdot 4))$$

$$= O\left\{\int_1^{\infty} w^{-1-\alpha}\left|\sum_{n<w}(w-n)^{\alpha-1} n\,\frac{1}{n(\log(n+1))^2}\right| dw\right\} + O(1)$$

(by (4.2), for $u = 0$, of Lemma 2)

$$= O(1),$$

by Lemma 1, since $\sum_{n=1}^{\infty}\frac{1}{n(\log(n+1))^2} \in |C, 0|$.

Proof of (5.3.2.). — From (3.4.1.) we have

$$g(w,u) = O\left\{\sum_{n<w}(w-n)^{\alpha-1}\frac{n^{\alpha}}{(\log(n+1))^{1+\varepsilon}}\right\}$$

(5.3.2.1.) $$= O\left\{\frac{w^{2\alpha}}{(\log w)^{1+\varepsilon}}\right\}.$$

And we write

$$I_2 = \int_1^{\frac{k}{u}} w^{-1-\alpha}|G(w,u)|\,dw + \int_{\frac{k}{u}}^{\infty} w^{-1-\alpha}|G(w,u)|\,du$$

$$\leqslant \int_1^{\frac{k}{u}} w^{-1-\alpha}|G(w,u)|\,du + \int_1^{\infty} w^{-1-\alpha}|G(w,\pi)|\,dw$$

$$+\int_{\frac{k}{u}}^{\infty} w^{-1-\alpha}|H(w,u)|\,dw$$

$$= I_{2,1} + I_{2,2} + I_{2,3},\ \text{say}.$$

The convergence of $I_{2,2}$ follows from (5.3.1.) and, by second mean value theorem,

$$I_{2,1} = u^{\alpha} \left(\log \frac{k}{u}\right)^{\varepsilon} \int_{1}^{\frac{k}{u}} w^{-1-\alpha} \left| \int_{\eta}^{u} \frac{\partial}{\partial x} g(w, x)\, dx \right| dw$$

$$(0 < \eta < u)$$

$$= O \left\{ u^{\alpha} \left(\log \frac{k}{u}\right)^{\varepsilon} \int_{1}^{\frac{k}{u}} w^{-1+\alpha} (\log w)^{-1-\varepsilon}\, dw \right.$$

(by (5.3.2.1.))

$$= O(1) .$$

Now, integrating by parts, we have

$$\Gamma(1+\alpha)\, H(w, u) = -g(w, u)\, u^{\alpha} \left(\log \frac{k}{u}\right)^{\varepsilon} + O\{|g(\omega, \lambda)|\}$$

$$- \int_{0}^{\pi} g(w, x)\, x^{\alpha-1} \left(\log \frac{k}{x}\right)^{\varepsilon} \left(\alpha - \frac{\varepsilon}{\log \frac{k}{x}}\right) dx$$

$$= O\left\{ \left(\log \frac{k}{u}\right)^{\varepsilon} w^{\alpha} (\log w)^{-1-\varepsilon} \right\}$$

(by (3.4))

$$+ O\left\{ \sum_{n<w} (w-n)^{\alpha-1}\, n\, \frac{1}{n(\log(n+1))^{2}} \right\}$$

$$+ O\left\{ \left(\log \frac{k}{u}\right)^{\varepsilon} \left| \int_{\frac{u}{\pi}}^{1} y^{\alpha-1} (1-y)^{-\alpha} \left(\sum_{n<w} (w-n)^{\alpha-1} \frac{\sin\left(\frac{n u}{y}\right)}{(\log(n+1))^{1+\varepsilon}} \right) dy \right| \right\}$$

(by 1.7) and (4.2) of Lemma 2)

$$= O\left\{ \left(\log \frac{k}{u}\right)^{\varepsilon} w^{\alpha} (\log w)^{-1-\varepsilon} \right\} + O\left\{ \sum_{n<w} (w-n)^{\alpha-1} (\log(n+1))^{-2} \right\},$$

by (3.2). Therefore

$$I_{2,3} = O\left\{ \left(\log \frac{k}{u}\right)^{\varepsilon} \int_{\frac{k}{u}}^{\infty} w^{-1} (\log w)^{-1-\varepsilon}\, dw \right\}$$

$$+ O\left\{ \int_{1}^{\infty} w^{-1-\alpha} \left| \sum_{n<w} (w-n)^{\alpha-1} \frac{1}{(\log(n+1))^{2}} \right| dw \right.$$

$$= O(1) \text{ , by Lemma 1.}$$

This terminates the proof of the theorem.

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