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Linkage coefficient and surface of given rectifiable boundary in 3-space (*)

The first part of the following exposition deals with the linkage coefficient of two disjoint, simple, closed, rectifiable curves C and $\varkappa$, in Euclidean 3-space. The second part has to do with the definition and some properties of a surface S, when its boundary is the curve C, given in advance. The basis of the latter is the linkage coefficient with C of testing curveo (**).

The linkage coefficient is defined directly as a *function of curves*, and will be denoted by $\nu(\varkappa, C)$. It turns out to be symmetric in $\varkappa$ and C, this being a property related to the setting in 3-space R^3 (***).

We imagine an electric current of unit strength on the curve C. It may be represented *almost everywhere* on C as a vector in terms of the direction cosines at the point Q of C. It generates a magnetic field, and the change of its magnetic potential one or more times around the closed path $\varkappa$ turns out to be an integral multiple of 4π. Thus this integer will yield a definition of $\nu(\varkappa, C)$. The change of magnetic potential of an arc of $\varkappa$ yields a definition of the change of solid angle as a point P travels along the arc. Sometimes it is convenient to take C as an arc as well as, or instead of, $\varkappa$. In the terms of Physics we introduce sources and sinks. This combination of values of current and field we denote by $I(\varkappa, C)$. To employ the combination of electric and magnetic fields as a lead to a mathematical idea is an old and natural device (*).

Our rectifiable curve in R^3 offers most of the complications of a simple continuous curve: knots, etc., and the complications obtained by means of the condensation of singularities. In general the surfaces that we shall deal with in Part II, bounded by such curves, will not be simple, although it may happen that a curve with a knot will be the boundary of a surface, even a ruled surface, which does not intersect itself.

Occasionally we may consider the case where C consists of a finite number of disjoint closed curves, and S may or may not be made of disjoint pieces (**).

(*) Memoria presentata dall'Accademico MAURO PICONE.

(**) (4a). References are to the bibliography.

(***) (4b), p. 614.

(*) (6), p. 409 ff. (7) a, b.

(**) (4a), (4b).

Part I

Linkage coefficient of two simple rectifiable curves

1. Properties of the vector H

Let ξ, η, ζ be the coordinates of a generic point Q of a given simple closed curve C, and τ be the parameter of a homeomorphic representation of C onto a circumference (or onto an interval whose end points are identified). In order that ξ, η, ζ may be taken as continuous functions of bounded variation in τ, it is necessary and sufficient that C be rectifiable (*).

It is convenient to take τ so that $\tau_2 - \tau_1 \geq \widehat{Q_1 Q_2}$ where $\widehat{Q_1 Q_2}$ is a positive length along C, and in particular, if nothing else is prescribed, to take $\tau_2 - \tau_1 = \widehat{Q_1 Q_2}$. The derivatives $d\xi/d\tau$, $d\eta/d\tau$, $d\zeta/d\tau$ will exist, and be in absolute value ≤ 1, on a certain set E of measure equal to the length of C; in fact, for any ΔC of C, $|\Delta\xi/\Delta\tau| \leq 1$, etc. (**). For $Q = (\xi, \eta, \zeta)$ in C and $P = (x, y, z)$ in $R^3 - C$, and $r = QP$, the function r^{-1} and its derivatives of all orders are continuous in all six coordinates, uniformly if $r \geq e > 0$. The x, y, z axes are taken respectively the same as the ξ, η, ζ axes. We have $r^2 = (\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2$, whence

$$(1.1) \qquad \frac{\partial r}{\partial x} = -\frac{\partial r}{\partial \xi} = \frac{x-\xi}{r}, \quad \frac{\partial \log r}{\partial x} = \frac{x-\xi}{r^2}, \quad \frac{\partial r^{-1}}{\partial x} = -\frac{x-\xi}{r^3}.$$

We imagine a unit electric current on C, considered as the value along C of the vector of components ξ, η, ζ, and denote by H a vector (L, M, N) to represent the intesity of the magnetic field at the point $P = (x, y, z)$. The well known properties of the magnetic field suggests a corresponding mathematical definition of the components of H in terms of simple Stieltjes integrals, defined as limits of sums (*):

$$(1.2) \qquad \begin{aligned} L(P) &= \int_C \frac{\partial r^{-1}}{\partial \zeta} d\eta(\tau) - \frac{\partial r^{-1}}{\partial \eta} d\zeta(\tau), \\ M(P) &= \int_C \frac{\partial r^{-1}}{\partial \xi} d\zeta(\tau) - \frac{\partial r^{-1}}{\partial \zeta} d\xi(\tau), \\ N(P) &= \int_C \frac{\partial r^{-1}}{\partial \eta} d\xi(\tau) - \frac{\partial r^{-1}}{\partial \xi} d\eta(\tau). \end{aligned}$$

On account of cur choice of τ so that $|\Delta\xi|$, $|\Delta\eta|$, $|\Delta\zeta|$ are $\leq |\Delta\tau|$, it follows that all of the derivatives numbers (left inferior, left superior, etc.) of each of the functions $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$ have numerical values ≤ 1. At any point in E all four derivative numbers are equal. The boundedness of these numbers insures the absolute continuity of the corresponding functions.

(*) ( ), pp. 62-67, p. 63.

(**) (³), pp. 76-80.

(*) (⁴) and Appendix I. Theorems quoted from (⁴) will be denoted as B 1, B 2, etc.

If a function $f(\tau)$ is absolutely continuous, that is to say, determines a function of point sets that is absolutely continuous, in an interval (a, b), then each of the derivative numbers $\Lambda(\tau)$ of $f(\tau)$ is summable in (a, b). For any interval (τ_1, τ_2) (*).

$$f(\tau_2) - f(\tau_1) = \int_{\tau_1}^{\tau_2} \Lambda(\tau)\, d\tau .$$

It is important to note that by construction:

Each of the derivative numbers Λ is measurable Borel on the point set C.

If P is a point (x, y, z) in $R^3 - C$, the derivatives of r^{-1} of arbitrary order are uniformly continuous in ξ, η, ζ for $Q = (\xi, \eta, \zeta)$ on C. Hence for $P \varepsilon C$, with regard to the nature of the derivatives of r^{-1}, as is formulae (1.1), we have

$$\frac{\partial L}{\partial x} = \int_C \left(\frac{\partial^2 r^{-1}}{\partial x \partial \zeta} \frac{d\eta}{d\tau} - \frac{\partial^2 r^{-1}}{\partial x \partial \eta} \frac{d\zeta}{d\tau} \right) d\tau = - \int_C \frac{\partial^2 r^{-1}}{\partial x \partial z} d\eta(\tau) - \frac{\partial^2 r^{-1}}{\partial x \partial y} d\zeta(\tau), \text{ etc.}$$

and similarly for further differentiation. By direct calculation one finds (**):

$$\text{div. } H = \frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} = 0 . \tag{1.3}$$

Let Q be a point of E, that is, a point where C has a unique tangent $\overrightarrow{dC}$, and let $\overrightarrow{dH}$ be the infinitesimal contribution to H at Q. Then

$$\overrightarrow{dH} \perp \overrightarrow{dC} \quad \text{at } Q. \tag{1.4}$$

In fact, at a point of E,

$$\frac{d\zeta}{d\tau} = \zeta \text{ component of } \frac{\overrightarrow{dC}}{d\tau}$$

$$\frac{\partial r^{-1}}{\partial \zeta} \frac{d\eta}{d\tau} - \frac{\partial r^{-1}}{\partial \eta} \frac{d\zeta}{d\tau} = \zeta \text{ component of } \frac{\overrightarrow{dH}}{d\tau}$$

whence (1.4) follows. In fact the scalar product of the two vectors vanishes.

If instead of a closed curve C, the vector H involves only an arc $\widehat{Q_1 Q_2}$ of C, that is,

$$L(P) = \int_{Q_1}^{Q_2} \left\{ \frac{\partial r^{-1}}{\partial \zeta} d\eta(\tau) - \frac{\partial r^{-1}}{\partial \eta} d\zeta(\tau) \right\} \tag{1.5}$$

with corresponding expressions for $M(P)$ and $N(P)$, for integration along C, the relations (1.3) and (1.4), of course, are still valid.

(*) (³), pp. 76-80.

(**) H is itself *rot* K, where K is the vector

$$K(P) = \int_C r^{-1} d\xi(\tau), \int_C r^{-1} d\eta(\tau), \int_C r^{-1} d\zeta(\tau) ,$$

whence again (1.3) follows.

The x component of rot H is $\partial N/\partial y - \partial M/\partial z$, where, from (1.5) and the corresponding terms for y and z components which are obtained from (1.2), we have

$$\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = -\int_{Q_1}^{Q_2} \left\{ \frac{\partial^2 r^{-1}}{\partial y^2} d\xi(\tau) - \frac{\partial^2 r^{-1}}{\partial y \partial x} d\eta(\tau) - \frac{\partial^2 r^{-1}}{\partial z \partial x} d\zeta(\tau) + \frac{\partial^2 r^{-1}}{\partial z^2} d\xi(\tau) \right\}.$$

But $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^{-1} = 0$. Therefore adding and substracting inside the integral the term $\frac{\partial^2 r^{-1}}{\partial x^2} d\xi(\tau)$, we obtain

$$(1.6) \qquad \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = \frac{\partial}{\partial x} \int_{Q_1}^{Q_2} \left\{ \frac{\partial r^{-1}}{\partial x} d\xi(\tau) + \frac{\partial r^{-1}}{\partial y} d\eta(\tau) + \frac{\partial r^{-1}}{\partial z} d\zeta(\tau) \right\}$$

$$= -\frac{\partial}{\partial x} \int_{Q_1}^{Q_2} \left\{ \frac{\partial r^{-1}}{\partial \xi} d\xi(\tau) + \frac{\partial r^{-1}}{\partial \eta} d\eta(\tau) + \frac{\partial r^{-1}}{\partial \zeta} d\zeta(\tau) \right\}.$$

We wish to prove that rot $H \equiv 0$ if C is a closed curve and $P = (x, y, z)$ is not on C. We shall first establish a lemma of general character, with hypothesis slightly less restrictive than that which might be used for r^{-1}.

Lemma 1.1. Let $f(\xi, \eta, \zeta)$ and its first derivatives be continuous on spherical neighborhoods $0(Q, \delta)$, Q being a generic point of C for an interval $Q' \leq Q \leq Q''$ (*). It generates a tubular neighborhood $0(C, \delta)$ of C. *Then, for integration along* C, *as measured by* τ,

$$(1.7) \qquad \int_{Q'}^{Q''} \frac{\partial f}{\partial \xi} d\xi(\tau) + \frac{\partial f}{\partial \eta} d\eta(\tau) + \frac{\partial f}{\partial \zeta} d\zeta(\tau) = f(Q'') - f(Q').$$

Incidentally, $0(C, \delta)$ is thus defined as a simply covered set of points. We denote the left member of (1.7) as I, and write it in the form

$$I = \sum_{i=1}^{n} \int_{\tau_i}^{\tau_{i+1}} \left\{ \frac{\partial f}{\partial \xi} d\xi(\tau) + \frac{\partial f}{\partial \eta} d\eta(\tau) + \frac{\partial f}{\partial \zeta} d\zeta(\tau) \right\}$$

$$\tau(Q') = \tau_1, \ \tau_1 < \tau_2 < \cdots < \tau_{n+1}, \ \tau_{n+1} = \tau(Q''), \ \tau_{i+1} - \tau_i \leq \delta_1 ,$$

where $\delta_1 \leq \delta$. We take δ_1 small enough so that the following expressions remain in $0(C, \delta)$. Consider then the expression

$$I' = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial \xi}\right)_{\tau=\tau_i} \int_{\tau_i}^{\tau_{i+1}} d\xi(\tau) \text{ etc., where}$$

$$\int_{\tau_i}^{\tau_{i+1}} d\xi(\tau) = \xi(\tau_{i+1}) - \xi(\tau_i)$$

(*) The $\leq$ sign denotes order on the curve, as a substitute for $\tau(Q') \leq \tau(Q'')$, etc.

by definition of the Stieltjes integral. We have

$$I' = \sum_{i=1}^{n} \left\{ \left(\frac{\partial f}{\partial \xi}\right)_{\tau_i} [\xi(\tau_{i+1}) - \xi(\tau_i)] + \left(\frac{\partial f}{\partial \eta}\right)_{\tau_i} [\eta(\tau_{i+1}) - \eta(\tau_i)] + \left(\frac{\partial f}{\partial \zeta}\right)_{\tau_i} [\zeta(\tau_{i+1}) - \zeta(\tau_i)] \right\}.$$

As to $I - I'$, a slight generalization of Theorem B 2 of (2) yield the following: Given $\varepsilon > 0$, $\delta_1 > 0$ may be chosen sufficiently small so that

$$(1.8) \qquad I - I' \leq O_1^\delta T_\xi + O_2^\delta T_\eta + O_3^\delta T_\zeta \leq \varepsilon,$$

in which T_ξ, T_η, T_ζ are the total variations of ξ, η, ζ respectively and O_1^δ, O_2^δ, O_3^δ are the maximum values of the oscillations of the continuous functions $\partial f/\partial \xi$, $\partial f/\partial \eta$, $\partial f/\partial \zeta$ in the intervals $\tau_i \leq \tau \leq \tau_{i+1}$ on C, $(\tau_{i+1} - \tau_i) \leq \delta_1$. Thus I' may be interpreted as belonging to a polygonal arc C' inscribed in C with vertices $Q_i = Q(\tau_i)$. For P on the chord $\overline{Q_i Q_{i+1}}$ we take τ as linear between $\tau(Q_i)$ and $\tau(Q_{i+1})$.

On the other hand, the right hand member of (1.7) may written in the form

$$\begin{aligned}
(1.9) \qquad I'' &= f(Q'') - f(Q') \\
&= \sum_{i=1}^{n} f(\xi_{i+1}, \eta_{i+1}, \zeta_{i+1}) - f(\xi_i, \eta_i, \zeta_i) \\
&= \sum_{i=1}^{n} \{ f(\xi_{i+1}, \eta_i, \zeta_i) - f(\xi_i, \eta_i, \zeta_i) \} + \{ f(\xi_{i+1}, \eta_{i+1}, \zeta_i) \\
&\quad - f(\xi_{i+1}, \eta_i, \zeta_i) \} + \{ f(\xi_{i+1}, \eta_{i+1}, \zeta_{i+1}) - f(\xi_{i+1}, \eta_{i+1}, \zeta_i) \}
\end{aligned}$$

where ξ_i means $\xi(\tau_i)$, etc. Therefore

$$(1.10) \qquad I'' = \sum_{i=1}^{n} \left\{ \left(\frac{\partial f}{\partial \xi}\right) \xi_i', \eta_i, \zeta_i (\xi_{i+1} - \xi_i) + \left(\frac{\partial f}{\partial \eta}\right) \xi_{i+1}, \eta_i', \zeta_i (\eta_{i+1} - \eta_i) + \left(\frac{\partial f}{\partial \zeta}\right) \xi_{i+1}, \eta_{i+1}, \zeta_i' (\zeta_{i+1} - \zeta_i) \right\}$$

where for example $\xi_{i+1}, \eta_i', \zeta_i$ refers to a point in a small neighborhood of (ξ_i, η_i, ζ_i), and $|\xi_{i+1} - \xi_i|$, $|\eta_{i+1} - \eta_i|$, $|\zeta_{i+1} - \zeta_i|$ are all $\leq \Delta\tau$. Therefore given $\varepsilon > 0$, there is $\delta_2 > 0$ so that from (1.8) and (1.10), again with referemce to B 2 of (2) $|I'' - I'| \leq \varepsilon$. Therefore with $\delta < \delta_1$, $\delta < \delta_2$, the difference between the two members of (1.7) may made arbitrariy small.

QED

As application, substituting $f(\xi, \eta, \zeta) = r^{-1}$, (1.7) becomes

$$\int_{Q_1}^{Q_2} \frac{\partial r^{-1}}{\partial \xi} d\xi(\tau) + \frac{\partial r^{-1}}{\partial \eta} d\eta(\tau) + \frac{\partial r^{-1}}{\partial \zeta} d\zeta(\tau) = \overline{Q_2 P}^{-1} - \overline{Q_1 P}^{-1},$$

whence

$$\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = \frac{\partial}{\partial x}\{\overline{Q_2 P^{-1}} - \overline{Q_1 P^{-1}}\}.$$

Therefore for rot H at the point P we have the vector value

$$\text{(1.11)} \qquad \text{rot } H(P) = \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\,(\overline{Q_1 P^{-1}} - \overline{Q_2 P^{-1}})$$

for $Q_1 Q_2$ an arc of C.

rot H (P) = 0 if $Q_1 Q_2$ is the closed curve C.

2. The function $I(\varkappa, C) = I(C, \varkappa)$.

Let $\varkappa$ be a second simple rectifiable arc, of points P, closed as a point set, which does not intersect C. Thus there is a tubular neighborhood $0(C, \rho)$ such that $\varkappa \,\Omega\, 0(C, \rho) = 0$.

We define integrals λ, μ, ν analogous to L, M, N. Let $r = \overline{PQ}$, $P = (x, y, z)$ on $\varkappa$ and $Q(\xi, \zeta, \zeta)$ on C. For additional clarity when convenient we write c_a^b for integration along C and $\int_c^d$ similarly for $\varkappa$. Accordingly we define

$$\text{(2.1)} \quad \begin{cases} \lambda(Q) = \lambda(\xi, \eta, \zeta) = \displaystyle\int_{P_1}^{P_2}\left\{\frac{\partial r^{-1}}{\partial z}\, d\,y(t) - \frac{\partial r^{-1}}{\partial y}\, d\,z(t)\right\} \\ \mu(Q) = \mu(\xi, \eta, \zeta) = \displaystyle\int_{P_1}^{P_2}\left\{\frac{\partial r^{-1}}{\partial x}\, d\,z(t) - \frac{\partial r^{-1}}{\partial z}\, d\,x(t)\right\} \\ \nu(Q) = \nu(\xi, \eta, \zeta) = \displaystyle\int_{P_1}^{P_2}\left\{\frac{\partial r^{-1}}{\partial y}\, d\,x(t) - \frac{\partial r^{-1}}{\partial x}\, d\,y(t)\right\}. \end{cases}$$

THEOREM I *With the above notation, we have the identity*

$$\text{(2.2)} \qquad c_{Q_1}^{Q_2}\, \lambda\, d\,\xi(\tau) + \mu\, d\,\eta(\tau) + \nu\, d\,\zeta(\tau) = \int_{P_1}^{P} L\, d\,x(t) + M\, d\,y(t) + N\, d\,z(t)\,,$$

$\varkappa$ and C being assumed to be disjoint as point sets, except when one is being deformed across the other.

This is the explicit form of the $I(C, \varkappa)$ of p. 2. The theorem states that $I(C,\varkappa) = I(\varkappa, C)$.

The first integral of the left member consists of the terms

$$\text{(I)} \quad \int_{Q_1}^{Q_2} d\,\xi(\tau) \int_{P_1}^{P_2} \frac{\partial r^{-1}}{\partial z}\, d\,y(t)\,, \qquad \text{(II)} - \int_{Q_1}^{Q_2} d\,\xi(\tau) \int_{P_1}^{P_2} \frac{\partial r^{-1}}{\partial y}\, d\,z(t)\,,$$

making use of (2.1). The rest of the left member is obtained by permuting these pairs cylically. The right member has a similar structure with corresponding terms

(I′) and (II′), but not corresponding pair's. To (I) corresponds

$$(\mathrm{I}') \qquad -\int_{P_1}^{P_2} d\,y(t)\int_{Q_1}^{Q_2}\frac{\partial r^{-1}}{\partial \xi}\,d\,\xi(\tau) \ \text{ or } \ \int_{P_1}^{P_2} d\,y(t)\int_{Q_1}^{Q_2}\frac{\partial r^{-1}}{\partial z}\,d\,\xi(\tau)\,,$$

since $-\partial r^{-1}/\partial\zeta = \partial r^{-1}/\partial z$. The theorem merely states that a change in the order of integration is legitimate. For convenience we take τ and t as lengths along the respective arcs.

The expression (I) may be rewritten in the form

$$(\mathrm{I}) \qquad \sum_{i=0}^{k}\int_{\tau_i}^{\tau_{i+1}}\left\{\sum_{j=0}^{m}\int_{t_j}^{t_{j+1}}\frac{\zeta - z}{r^3}\,d\,y(t)\right\}d\,\xi(\tau)\,, \qquad \begin{matrix} i = 0,\ 1,\dots,k\ , \\ j = 0,\ 1,\dots,m\ , \end{matrix}$$

where $Q_1 = Q(\tau_0)$, $Q_2 = Q(\tau_{k+1})$, $P_1 = P(t_0)$, $P_2 = P(t_{m+1})$, and k and m are determined by choosing $\delta > 0$ and taking $(\tau_{i+1} - \tau_i) < \delta$, $(t_{j+1} - t_j) < \delta$.

Let T denote the length of the longer of the arcs $\varkappa$ and C, thus greater than or equal to the total variations of $\xi(\tau)$ and $y(t)$.

As in Theorem B 2 of (²) denote by 0_δ the maximum of the oscillations of $(\zeta - z)/r^3$, $r = \overline{QP}$, for Q in any of the arcs of C defined by a $(\tau_{i+1} - \tau_i) \leq \delta$ and P in any arc given by a $(t_{j+1}) - t_j) \leq \delta$. Let r_j denote the distance when P is restricted to a P_j, and r_{ij} the distance between specific points Q_i and P_j.

With reference to the expression above for (I), we wish to evaluate the difference

$$\int_{\tau_i}^{\tau_{i+1}} d\,\xi(\tau)\left\{\int_{t_j}^{t_{j+1}}\frac{\zeta - z}{r^3}\,d\,y(t)\right\} - \int_{\tau_i}^{\tau_{i+1}}\frac{\zeta_i - z_j}{r_{ij}^{\,3}}\,[y(t_{j+1}) - y(t_j)]\,d\,\xi(\tau)\,,$$

where ideed the second part is merely

$$-\frac{\zeta_i - z_j}{r_{ij}^{\,3}}\,[y(t_{j+1}) - y(t_j)]\;[\xi(\tau_{i+1}) - \xi(\tau_i)]\,.$$

We have

$$\left|\frac{\zeta - z}{r^3} - \frac{\zeta - z}{r_j^3}\right| \leq 0_\delta\,, \text{ with reference to integration on}$$

$$\left|\frac{\zeta - z_j}{r_j^3} - \frac{\zeta_i - z_j}{r_{ij}^{\,3}}\right| \leq 0_\delta\,, \text{ with reference to subsequent integration on } C$$

$$\left|\frac{\zeta - z}{r^3} - \frac{\zeta_i - z_j}{r_{ji}^{\,3}}\right| \leq 2\,0_\delta$$

$$\left|\int_{t_j}^{t_{j+1}}\frac{\zeta - z}{r^3}\,d\,y(t) - \frac{\zeta_i - z_j}{r_{ij}^{\,3}}\,[y(t_{j+1}) - y(t_j)]\right| \leq 2\,0_\delta\,(t_{j+1} - t_j)\,,$$

by integration of the $(\zeta - z)/r^3 - (\zeta_i - z_j)/r_{ij}^{\,3}$ and noting that $|\,y(t_{j+1}) - y(t_j)\,| \leq t_{j+1} - t_j$.

The integral in the left member, by Theorem B 3 of ([2]), is a continous function of τ.

For the moment let F_j stand for the expression between absolute value signs above. We have

$$\left|\sum_{j=0}^{m} F_j\right| \leq \sum_{j=0}^{m} |F_j| \leq 20_\delta \sum_{j=0}^{m} (t_{j+1} - t_j) \leq 2\, 0_\delta\, T\,.$$

Therefore

$$\left|\sum_{i=0}^{k} \int_{\tau_i}^{\tau_{i+1}} \left(\sum_{j=0}^{m} F_j\, d\,\tau\right)\right| \leq \sum_{i=0}^{m} \int_{\tau_i}^{\tau_{i+1}} 20_\delta\, T\, d\,\tau \leq 2\; 0_\delta\, T^2\,,$$

$$\left|(I) - \sum_{i=0}^{k}\ \sum_{j=0}^{m} \frac{\zeta_j - z_j}{r_{ij}^{\,2}} [y(t_{j+1}) - y(t_j)]\, [\xi(\tau_{i+1}) - \xi(\tau_i)]\right| \leq 20_\delta\, T^2\,.$$

If now we analyze (I′) similarly, we find that it also differs from the same polynomixal by $\leq\ 20_\delta\ T^2$. Hence

$$|(I) - (I')| \leq 40_\delta\, T^2\,.$$

But T is a constant and $0_\delta \to 0$ with δ. Therefore (I) = (I′). In the same way we find that the change of order of integration is ligitimate for every pair of corresponding terms in (2.2).

QED

We may write the identity (2.2) in a different from, and obtain a somewhat more general result, by making use of a second difference method of C. De la Vallée Poussin (*).

Given a function $f(\tau, t)$, its second difference is written as

$$\text{(2.3)}\qquad \Delta^2 f = f(\tau + h, t + k) - f(\tau, t + k) - f(\tau + h, t) + f(\tau, t),\ h > 0,\ k > 0.$$

Denote by ω the rectangle of lower left vertex (τ, t) and lengths of sides h, k, parallel to the τ, t axes, thus to define a function $g(\omega) = \Delta^2 f$. Let Ω be the union of a finite number of such non-overlapping rectangles ω_i and write

$$g(\Omega) = \Sigma\, g(\omega_i)\,.$$

It is additive on rectangles. Hence if $\omega_i = \Sigma\, \omega_{i,j}$, the $\omega_{i,j}$, being non-overlapping rectangles, then

$$g(\omega_i) = \Sigma_j\, g(\omega_{ij})\ ,\ g(\Omega) = \Sigma_{ij}\, g(\omega_{ij})\,.$$

If $g(\Omega) \to 0$ when meas $\Omega \to 0$, $g(\Omega)$ is absolutely continuous on rectangles, and, as definition, $f(\tau, t)$ is absolutely continuous on rectangles. In this way f τ(, t)

(*) ([1]) Ch. V, pp. 80-82, with reference also to theorems of H. Lebesgue.

determines a unique function $g(\omega)$ which is additive on measurable sets and absolutely continuous, coinciding with $f(\tau, t)$ on rectangles.

In (2.3) at an arbitrary point $P(\tau, t)$ let $h \to 0$ and $k \to 0$ in a regular manner, that is, so that there are two positive constants H and K, independent of P, such that

$$0 < H < \frac{h}{k} < K,\ H < K\ .$$

Returning to the function of point sets $g(\omega)$, we define $Dg(\omega)$ at a point $P(\tau, t)$ as a two-dimensional derivative at the point (*):

$$Dg_P = \lim_{\substack{h \to 0 \\ k \to 0}} \frac{g(\omega)}{\text{meas}\ \omega} = \lim \frac{\Delta^2 f}{h\, k}\ ,$$

h and k approaching 0 as above. It exist in the τ, t plane except at a set of Borel measure zero, where to a certain extent it may be defined according to convenience for its application. As a consequence of the absolute continuity of $g(\omega)$, we have

(2.4) $$\Delta^2 f(\tau, t) = g(\omega) = \int_{\omega} Dg\ dP,\ P = (\tau, t)\ .$$

The theory may be extended by introducing a general Stieltjes type integral, although in the case of our applications the derivative numbers are all bounded. We limit ourselves therefore to the hypothesis that $g(\omega)$ is absolutely continuous. Our task is to define an integral

(2.5) $$\Phi = \int_R \varphi(P)\ dg\ (\omega_P) = \int_R \varphi(\tau, t)\ d\,\Delta^2 f(\tau, t)$$

where $\varphi(\tau, t)$ is bounded and continuous in R, R being the union of a finite number of given rectangles in the τ, t space.

Since however $g(\omega)$ is continuable to measurable sets ω, it may be written as difference of two positive set functions in (2.5); similarly also $\Delta^2 f(\tau, t)$ may be considered as the difference of two positive functions. Hence it is sufficient to consider only positive set functions and also obviously only positive continuous functions $\varphi(P)$. From this point, for the purposes of our applications, it is necessary to consider only rectangular subdivisions ω_i of our τ, t space.

We define

(2.6) $$\Sigma_i = \Sigma_i\, \varphi(P)\, g(\omega_i)\ ,\ P_i \varepsilon\ \omega_i$$

to be summed over R; we define Σ'_i similarly, taking $P_i = P'_i$ where $\varphi(P'_i)$ is an upper bound of $\varphi(P)$ in ω_i, and secondly P_i as P''_i P''_ibeing a point where $\varphi(P''_i)$ is similarly a lower bound in ω_i. Thus we obtain upper and lower sums.

In the definition of $\Delta^2 f$ in (2.3), we write $h = h_m = \tau_{m+1} - \tau_m$, $k = k_n = t_{n+1} - t_n$, to form $\Delta\tau_m$ and Δt_n respectively requiring that they conform to fixed values of H and K, as they are subdivided. As in the case of one dimension,

(*) Hence it is related to a mixed second partial derivative.

one finds that any upper sum is $\geq$ every lower sum, and any lower sum is $\leq$ every upper sum. At every stage the number of rectangles is finite. And as in the proof for one dimension, they frame a single value, and that value is defined as the Φ in (2.5) above (*).

The advantege of this method of treatment is that the integral is defined directly as a double integral in (τ, t). As an application we turn to the (I) or (I') of Therorem I, which are thus covered together. In (2.5) we put

$$\text{(2.7)} \qquad \varphi(P) = \frac{\partial r^{-1}}{\partial z} = \frac{\zeta - z}{r^3} = -\frac{\partial r^{-1}}{\partial \zeta}, \quad f_j(\tau, t) = \xi_i(\tau)\, y_j(t),$$

from which, with a slight change of notation,

$$\Delta^2 f_{ij}(\tau, t) = \frac{\{\xi(\tau_i + \Delta\tau_i) - \xi(\tau_i)\}}{\Delta\tau_i}\,\frac{\{y(t_j + \Delta t_j) - y(t_j)\}}{\Delta t_j}\,\Delta\tau_i\,\Delta t_j$$

The function $\varphi(P)$ is bounded and continuous because $\varkappa \cap C = 0$, and $\Sigma\Delta^2 f_{ij}$ remains bounded because $|\Delta\xi| \leq \Delta\tau$, $|\Delta y| \leq \Delta t$ (**). Therefore

$$\Sigma_{ij}\frac{\zeta_i - z_j}{r_{ij}^3}\,\frac{\Delta^2 f_{ij}}{\Delta\tau_i\Delta t_j} \longrightarrow \int_R \left(\frac{\zeta - z}{r^3}\frac{d\xi}{d\tau}\frac{dy}{dt}\right) d\tau\, dt$$

as a double integral. We can assume that where the derivatives are not defined uniquely, they are assigned the values of the lower (forward) derivative numbers, so that they are defined everywhere as functions of the first Baire category, integrable in the Borel sense, unique derivatives except on sets of measures 0 on C and $\varkappa$.

Similarly for (II) and (II') we find the value $-\int_R \left(\frac{\eta - y}{r^3}\frac{d\xi}{d\tau}\frac{dz}{dt}\right) d\tau\, dt$. Thus we obtain the following double integral over the region R:

Theorem II.

$$\text{(2.8)} \quad \int_R r^{-3}\Big[(\zeta - z)\frac{d\xi}{d\tau}\frac{dy}{dt} - (\eta - y)\frac{d\xi}{d\tau}\frac{dz}{dt} + (\xi - x)\frac{d\eta}{d\tau}\frac{dz}{dt} - (\zeta - z)\frac{d\eta}{d\tau}\frac{dx}{dt}$$
$$+ (\eta - y)\frac{d\zeta}{d\tau}\frac{dx}{dt} - (\xi - x)\frac{d\zeta}{d\tau}\frac{dy}{dt}\Big]\, d\tau\, dt = I(\varkappa, C) = I(C, \varkappa),$$

(*) Cf. Theorem B 1 in (⁷).
(**) For the meaning of $\Sigma\Delta^2 f_{ij}$ see (I) and (I') in Theorem I.

R *being defined as in* (2.5).

And to this we may apply the Fubini theorem to obtain the iterated integrals analogus to equation (2.2). For example

$$(2.9) \qquad I(\varkappa, C) = I(C, \varkappa) =$$

$$\int_C d\tau \frac{d\xi}{d\tau} \int_\varkappa r^{-3} \left[(\zeta - z) \frac{dy}{dt} - (\eta - y) \frac{dz}{dt} \right] dt + \int_C d\tau \frac{d\eta}{d\tau} \int_\varkappa r^{-3} \left[(\xi - x) \frac{dz}{dt} - \right.$$

$$\left. - (\zeta - z) \frac{dx}{dt} \right] dt + \int_C d\tau \frac{d\zeta}{d\tau} \int_\varkappa r^{-3} \left[(\eta - y) \frac{dx}{dt} - (\xi - x) \frac{dy}{dt} \right] dt \, .$$

In terms of Stieltjes integrals we use later the right member of (2.2) in the form

$$(2.10) \qquad I(\varkappa, C) = \int_\varkappa dx(t) \int_C \frac{\partial r^{-1}}{\partial \zeta} d\pi(\tau) - \frac{\partial r^{-1}}{\partial \eta} d\zeta(\tau) +$$

$$\int_\varkappa dy(t) \int_C \frac{\partial r^{-1}}{\partial \xi} d\zeta(\tau) - \frac{\partial r^{-1}}{\partial C} d\zeta(\tau) +$$

$$\int_\varkappa dz(t) \int_C \frac{\partial r^{-1}}{\partial \eta} d\xi(\tau) - \frac{\partial r^{-1}}{\partial \xi} d\eta(\tau) \, .$$

In particular we may take $\varkappa$ and C as arcs, as well as closed curves.

Comment. Consider the case where in formula (2.10) we deal with arcs C_1 and C_2, taking τ as measures along each arc. Denote by C the union of the two arcs. Given the arc $\varkappa$ we have the formula

$$(2.11) \qquad I(\varkappa, C) = I(\varkappa, C_1 \cup C_2) = I(\varkappa, C_1) + I(\varkappa, C_2) \, .$$

Here, if $\varkappa$ is given, $I(\varkappa, C)$ is additive in C, and similarly, from (2.9), $I(\varkappa, C)$ is additive in $\varkappa$ if C is given.

In cases where an arc is repeated, it is retained as doubled in formula (2.11). If, however, we form $I(\varkappa, C)$ *mod* 2, direction along the arc has no significance, and in the corresponding formula the doubled arc disappears.

3. The linkage coefficient

It will be proved in the theorems of this section that if $\varkappa$ and C are disjoint closed curves, the value of $I(\varkappa, C)/4\pi$ is an integer, positive, negative or zero. *This value is defined to be the linkage coefficient* $v(\varkappa, C)$ *of* $\varkappa$ *and* C. For the definition of the related concept of *solid angle*, see Art. 4.

We shall have use for the Theorem of Stokes. It is a statement that, with proper restrictions of regularity, the integral of the tangential component of avector (L, M, N) around a closed curve is equal to the integral of *rot* (L, M, N) over a one-sided surface of wich the curve is the *sole* boundary.

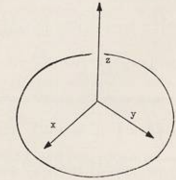

Fig. 1

As a concrete calculation of an $I(\varkappa, C)$ we take the following case:

Let C be the circumference $\xi^2 + \eta^2 = a^2$ *in the plane* $\zeta = 0$. *Let* $\varkappa$ *be the rectangle with vertices*

$$P_1 = (x_1, 0, -b), P_2 = (x_1, 0, b), P_3 = (x_2, 0, b), P_4 = (x_2, 0, -b)$$

all in the plane $y = 0$. *We assume* $b > 0, |x_1| < a, a < |x_2|$. Sufficiently, we may take $0 \leq x_1 < a$, $a < x_2$. Also for convenience, we take τ and t respectively as measure along C and measure along the segments of the rectangle x. We prove the following:

THEOREM A *Under the above specifications, we have*

$$I(\varkappa, C) = 4\pi.$$

Since r denotes the distance from a point $(\xi, \eta, 0)$ on C to a point $(x, 0, z)$ on the boundary of the rectangle, it follows that r is bounded uniformly away from 0, and $1/r$ is continuous on $\varkappa$ and C with all its derivatives. We have trigonometrically,

$$(3.1) \qquad r^2 = \alpha^2 + z^2, \quad \alpha^2 = a^2 + x^2 - 2x\xi, \quad \xi = a\cos\varphi.$$

From (2.9) we may eliminate elements of $\varkappa$ and C. Thus the pairs of integrals for $z = b$ and $z = -b$ reduce each to a single term from (2.9) (*):

$$I_{\overline{P_2 P_3}} = b \int_0^{2\pi} d\,t \frac{d\,\eta}{d\,\tau} \int_{x_1}^{x_2} \left(r^{-3} \frac{d\,x}{d\,t} \right) d\,t\,, \quad \text{for } z = b,$$

$$I_{\overline{P_4 P_1}} = -b \int_0^{2\pi} d\,\tau \frac{d\,\eta}{d\,\tau} \int_{x_2}^{x_1} \left(r^{-3} \frac{d\,x}{d\,t} \right) d\,t\,, \quad \text{for } z = -b,$$

where in both cases r^2 is given by (3.1). Therefore $I_{\overline{P_2 P_3}}$ and $I_{\overline{P_4 P_1}}$ cancel each other.

Before examining the contributions of the vertical sides of the figure, let us see what happens when $b \to \infty$. Consider a new rectangular boundary $\varkappa'$ with vertices:

$$P'_1 = (x_1, 0, -b-b),\ P'_2 = (x_1, 0, b+b),\ P'_3 = (x_2, 0, b+b),\ P'_4 = (x_2, 0, -b-b)$$

taking $b' > 0$. Since r does not vanish in the added rectangles or on their boundaries, the value of $I(\varkappa, C)$ [with reference to Stokes' theorem] is unchanged by their addition:

$$I(\varkappa, C) = I(\varkappa, C)\ .$$

In this operation the sides $P_2 P_3$ and $P_4 P_1$ have disappeared, and $I_{\overline{P'_2 P'_3}}$ and $I_{\overline{P'_4 P'_1}}$ cancel, as in the original figure. Incidentally, as $b' \to +\infty$, the contributions of $I_{\overline{P'_2 P'_3}}$ and $I_{\overline{P'_4 P'_1}}$ vanish separately, since each is of order r^{-2} as $r \to \infty$.

Accordingly the original rectangle may be considered variable, given by its vertices as b increases. Denote by $\varkappa_1$ and $\varkappa_2$ the boundaries, $\overline{P_1 P_2}$ and $\overline{P_3 P_4}$, and by $I(\varkappa_1, C)$ and $I(\varkappa_2, C)$ their contributions to $I(\varkappa, C)$. The value of the $I(\varkappa, C)$ of Theorem A is therefore

$$I(\varkappa, C) = \lim_{b \to \infty} I(\varkappa_1, C) + \lim_{b \to \infty} I(\varkappa_2, C)\ .$$

We shall prove that the first term has the value 4π, and the second, the value zero.

From (3.1) (*),

$$(3.2) \qquad I(\varkappa_1, C) = 2 \int_C d\,\tau \left[(\xi - x_1) \frac{d\,\eta}{d\,\tau} - \eta \frac{d\,\xi}{d\,\tau} \right] \int_0^b (z^2 + \alpha^2)^{-3/2}\, d\,z,$$

(*) We have $\frac{d\,y}{d\,t}$, $\frac{d\,z}{d\,t}$, ζ, $\frac{d\,\zeta}{d\,\tau} = 0$, $z = b$ and $z = -b$.

(**) In order to write $I(\varkappa^2, C)$ explicitly, we eliminate from (2.9) the elements y, d y/d t, d x at as well as ζ, d ζ/d τ.

with the corresponding expression for $I(\varkappa_2, C)$, except that $\varkappa_2$ is directd downward. We denote the respective limits above as $I_\infty(\varkappa_1, C)$ and $I_\infty(\varkappa_2, C)$.

As indefinite integral, $\int (z^2 + \alpha^2)^{-3/2}\, d\,z = \frac{z}{\alpha^2}(z^2 + \alpha^2)^{-1/2}$,

whence

$$\int_{-\infty}^{\infty} (z^2 + \alpha^2)^{-3/2}\, d\,z = 2\,\frac{z}{\alpha^2}(z^2 + \alpha^2)^{-1/2}\Big]_0^b \text{ as } b \to \infty = \frac{2}{\alpha^2}\,.$$

The difference between $\int_{-b}^{b}$ and $\int_{-\infty}^{\infty}$ is $\frac{2}{\alpha^2}\left(1 + \frac{\alpha^2}{b^2}\right)^{-\frac{1}{2}} - \frac{2}{\alpha^2} = \frac{2}{\alpha^2}\left\{\left(1 + \frac{\alpha^2}{b^2}\right)^{-\frac{1}{2}} - 1\right\}$, and this for large b, is of the order of $\frac{2}{\alpha^2}\left\{1 - \frac{1}{2}\,\frac{\alpha^2}{b^2} - 1\right\}$, thus uniformly, as $b \to \infty, < 1/b^2$.

Therefore we may substitute $\int_0^\infty$ for $\int_0^b$ in the interior integral and obtain the result:

$$(3.3) \qquad I_\infty(\varkappa_1, C) = \int_C \frac{2}{\alpha^2}\left\{(\xi - x_1)\frac{d\,\eta}{d\,\tau} - \eta\,\frac{d\,\xi}{d\,\tau}\right\} d\,\tau \text{ and from (3.1),}$$

$$= \int_0^{2\pi} \frac{2\,a^2 - 2\,ax_1\,\cos\varphi}{a^2 + x_1^2 - 2\,ax_1\,\cos\varphi}\, d\,\varphi = 2\,\pi + (a^2 - x_1^2)\int_0^{2\pi} \frac{d\,\varphi}{a^2 + x_1^2 - 2\,ax_1\,\cos\varphi}$$

where $\tau = a\,\varphi$, $d\,\tau = ad\,\varphi$, $\eta = a\sin\varphi$, $\frac{d\,\eta}{d\,\tau} = \cos\varphi$, etc. The expression $-I_\infty(\varkappa_2, C)$ is given by the same formula substituting x_2 for x_1.

An indefinite integral is again avaiable (*):

$$\int \frac{d\,\varphi}{m + n\cos\varphi} = \frac{-1}{\sqrt{m^2 - n^2}}\,\text{arc sin}\,\frac{n + m\cos\varphi}{m + n\cos\varphi}\,, \text{ if } |m| > |n|\,.$$

For $I_\infty(x_1, C)$, $m = a^2 + x_1^2$, $n = -2\,ax_1$, $\sqrt{m^2 - n^2} = \sqrt{a^4 - 2\,a^2 x_1^2 + x_1^4} = a^2 - x^2$

as the positive radical. Hence

$$(3.4) \quad I_\infty(\varkappa_1, C) = 2\pi + (a^2 - x_1^2)\left[-\frac{1}{a^2 - x_1^2}\,\text{arc sin}\,\frac{-2\,ax_1 + (a^2 + x_1^2\cos\varphi}{a^2 + x_1^2 - 2\,ax_1\cos\varphi}\right]_{\varphi=0}^{2\pi}$$

The integrand in the last integral in (3.3) is positive, because $a^2 + x_1^2 > 2\,ax_1 \cos\varphi$. Hence the expression in bnackets of (3.4) must be increasing continuously from $\varphi = 0$ to $\varphi = 2\,\pi$. In fact the angle

$$= \text{arc sin}\,\frac{-2\,ax_1 + (a^2 + x_1^2)\cos\varphi}{a^2 + x_1^2 - 2\,ax_1\cos\varphi}$$

(*) B. O. Peirce, Table of Integrals, Boston, 1910. Cf. formula 300, p. 41.

is decreasing continuously, say from $\pi/2$ to $-3\pi/2$; thus by 2π. Hence, from (3.4)

$$I_\infty(\varkappa_1, C) = 2\pi + 2\pi = 4\pi .$$

In order to obtain $I(x_2, C)$ we make use of the relation

$$-I(\varkappa_2, C) = 2\, I_{\overline{0}\, P_2} = 2\pi + (a^2 - x_2^2) \int_0^{2\pi} \frac{d\varphi}{a^2 + x_2^2 - 2\, a x_2 \cos\varphi} .$$

In this case $m^2 - n^2 = a^4 - 2\, a\, x_2^2 + x_2^4$, but because $|x_2| > a$ we have $\sqrt{m^2 - n^2} = x_2^2 - a^2$. Accordingly

$$-I(\varkappa_2, C) = 2\pi + \frac{a^2 - x_2^2}{x_2^2 - a^2}\, 2\pi = 0.$$

Therefore $I(\varkappa, C) = 4\pi$.

QED

Comment to Theorem A. Evidently the rectangle $\varkappa$ and the circumferece C may be deformed in turn into other simple figures, so that $I(\varkappa, C)$ remains equal to 4π, provided that each is deformed over an area in which the respective rot $\varkappa \equiv 0$ and rot $C \equiv 0$, — thus into orthogonal circles such that the circumference of each intersects the disc of the other in a single point, also when the circles are no longer orthogonal, and when they are deformed into similarly arranged triangles, etc.

THEOREM III. *If C and $\varkappa$ are simple dis,oint polygonal cycles, each of a finite number of vertices, then $I(\varkappa, C)/4\pi$ is an integer, positive, negative or zero.*

By $Q_1, Q_2, \ldots, Q_n, Q_{n+1} = Q_1$, designate the vertices of C in order of τ, τ being measure along C. Form the planes, as far as they are determined and distinct, chich contain the following points:

$$\begin{array}{llll}
(Q_1, Q_2, Q_3), & (Q_1, Q_3, Q_4), & (Q_1, Q_4, Q_5), \ldots, & (Q_1, Q_{n-1}, Q_n) \\
 & (Q_2, Q_3, Q_4), & (Q_2, Q_4, Q_6), \ldots, & (Q_2, Q_{n-1}, Q_n) \\
 & \quad \cdot \quad \cdot \quad \cdot \quad \cdot & & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 & & & (Q_{n-2}, Q_{n-1}, Q_n)
\end{array}$$

Let $\overline{\gamma_i} > 0$ be the maximum length of any segment (Q_i, Q_{i+1}). It is assumed that $\overline{\gamma_i}$ is sufficiently small so that some of the above planes exist. In particular, if Q_i, Q_j are vertices of C there must be some vertex Q_k, $k \neq i$, $k \neq j$, which forms a plane with them. Otherwise all the vertices of C would lie in one line.

Take Q_0, a point not in any of the above planes. Then Q_0 is not collinear with any pair of vertices of C. For if Q_0 is collinear with some Q_i, Q_j, it must lie, as above, in some plane (Q_i, Q_j, Q_k). For convenience we represent the triangles as corresponding plane surface elements A_i of which the closures $\overline{A}_i$ form a certain surface S.

It is possible that nonadjacent triangles of S intersect. They can do so only in rays from Q_0. Hence they may be split into subtriangles which do not have interior points in common but may have common edges belonging to nonadjacent triangles. There may be several pairs of such triangles with the same common edge.

We rename the subtriangles A'_1, A'_2, ..., in order of their edges in C. For a triangle A_i is substituted triangles A'_j, A'_{j+1}, thus to have their triangular boundaries C'_j ordered conformally with C. Finally if we make the aspects of the plane areas $\bar{A}'_j$ conform to the order of their edges $\overline{Q'_j Q'_{j+1}}$, we obtain what may be called the *orientable surface* $\bar{S}$. In fact, on account of all the rays having emanated from the point Q_0, the aspect of A'_j is retained after a simple traversing of the entire circuit C (*).

Lemma to Theorem III. If the polygonal cycle $\varkappa$ traverses one of the triangles $A'_j = (Q_0, Q'_j, Q'_{j+1})$, of boundary C'_j, in a single point, say P, and does not meet $\bar{S}$ otherwise, then

$$I(\varkappa, C) = \pm 4\pi .$$

We assume a sense along $\varkappa$, — for example that of the parameter t. By P′ we denote the point at which the ray $Q_0 P$ is prolonged to meet C.

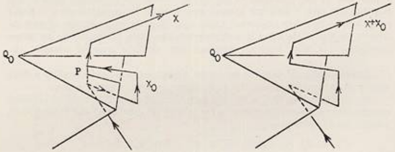

Fig. 2 Fig. 3

Let there be constructed a plane quadrilateral B_0 with boundary cycle $\varkappa_0$ of which one segment traverses $\bar{S}$ also at P, and in that segment coincides, but in the opposite sense, with a part or the whole, of a segment of $\varkappa$. The rest of $\varkappa_0$ completes the boundary of B_0, so that B_0 contains the point P′ and is bounded above and below PP′ by rays from Q_0. Thus (**)

$$I(\varkappa_0, C) = \mp 4\pi .$$

By $\varkappa'$ we denote the cycle

$$\varkappa' = \varkappa + \varkappa_0 .$$

(*) Cf Comment to Teorem A.
(**) Cf Comment to Theorem II.

We note that by taking a point P″ on $\varkappa$, distant from P, we may write

$$\varkappa' = \varkappa\,[P'' \text{ to } P] + \varkappa_0 + \varkappa\,[P \text{ to } P''] = \{\varkappa\,[P'' \text{ to } P] + \varkappa_0\} + \varkappa\,[P \text{ to } P''] \,.$$

Hence $\varkappa'$ does not cross $\bar{S}$. The corresponding terms in $I(\varkappa', C)$ are additive in the above polygonal arcs (**).

It still remains to be proved that $I(\varkappa', C) = 0$. For this purpose, we construct a sphere G, of surface Γ, of center Q_0 and of radius R large enough so that G contains the boundary $C = \Sigma C'_j$ and the whole of the polygonal cycle $\varkappa'$. We project the vertices of $\varkappa'$ radially outward until they lie in Γ, introducing, if necessary, additional vertices in the segments of $\varkappa'$, so that the altered segments shall not intersect. $\bar{S}$. Note also that the construction will coalesce the segments that are radial from Q_0.

In accordance with this construction let $P_1, \ldots, P_m$, $(P_{m+1} = P_1)$ be the reduced vertices on $\varkappa'$, and $P'_1, \ldots, P'_m$, $(P'_{m+1} = P'_1)$ the corresponding vertices on Γ. In this way we form successive quadrilaterals $B_i, B_{i+1}, \ldots$, of boundaries b_i, $b_{i+1}, \ldots$, and vertices $(P_i, P'_i, P'_{i+1}, P_{i+1})$, $(P_{i+1}, P'_{i+1}, P'_{i+2}, P_{i+2})$, ... Thus $\overline{P'_{i+1} P_{i+1}}$ of b_i cancels $\overline{P_{i+1} P'_{i+1}}$ of b_{i+1}.

Denote by $\varkappa'_R$ the resulting cycle with vertices in Γ. We have $I(\varkappa', C) - I(\varkappa'_R, C) = 0$, by Stokes's Theorem, because rot $H = 0$ at all points of B_i.

We note that for large R, with reference to formula (1.1), we have $I(\varkappa'_R, C) \to 0$ of order one as $R \to \infty$. Hence finally $I(\varkappa', C) = 0$ and $I(\varkappa' C) = \pm 4\pi$.
QED

The validity of Theorem III follows immediately. Since $\varkappa$ does not meet C, it must as a closed set fo points remain at a finite distance from C, and therefore, being of finite length, may cut $\bar{S}$ only a finite number of times. Hence it may be replaced by a finite number of polygonal cycles $\varkappa_k$. Therefore

$$I(\varkappa, C) = \Sigma\, I(\varkappa_k, C),$$

which is a finite multiple of 4π.
QED

THEOREM IV *Let* C *and* $\varkappa$ *be given as simple closed rectifiable curves which are disjoint. Then* $I(\varkappa, C)/4\pi$ *is an integer.*

The proof of this theorem can be made most simply when $I(\varkappa, C)$ is expressed by means of Stieltjies integrals, as in Theorem I. There is also implied reference to Theorem A. The theorem is proved in two analogous parts. In the first we assume that $\varkappa$ remains fixed as a polygonal cycle.

Let $0(C, \rho)$ be a tubular neighborhood of C, of radius ρ, and C' an arbitrary closed polygonal cycle inscribed in C, ρ being small enough so that $\varkappa \cap 0(C, \rho) = 0$. Denote by $Q_1, Q_2, \ldots, Q_n, Q_{n+1}, Q_{n+1} = Q_1$ the vertices of C', and by η the maximum length among arcs $Q_i Q_{i+1}$ of C, thus greater than any segment $\overline{Q_i Q_{i+1}}$ of C'. It may happen that C' can intersect itself. However C' may be ade a simple cycle by « infinitesimal » displacements of its vertices.

In an arc $Q_i Q_{i+1} = C_i$ of C the parameter τ is taken as measure along C. In the corresponding segment $\overline{Q_i Q_{i+1}} = C'_i$ of C' we introduce a parameter τ' as follows:

At any vertex Q_i, $\tau'_i = \tau_i$. Between Q_i and Q_{i+1}, τ' will be linear. Hence $\Delta \tau' \geq \Delta \tau \geq |\Delta \xi|$, atc. As in Theorem I denote $\theta_{\bar{\eta}}$ the maximum total oscillation among subintervals of C or C' of lengths $< \bar{\eta}$ and by T the total lenght of C, greater than that of any C'. Also take T as great as the length of $\varkappa$.

As in Theorem I, consider a single term of $I(\varkappa, C)$, for example

$$\int_{\varkappa} dx(t) \int_{C} \frac{\xi - z}{r^3} d\eta(\tau)$$

and the corresponding term of C'. We compare both of the interior integrals with the same finite sum. Thus

$$\int_{C} \frac{\zeta - z}{r^3} d\eta(\tau) \text{ differs from } \sum_{i=1}^{n} \frac{\zeta_i - z}{r_i^3} [\eta(\tau_{i+1}) - \eta(\tau_i)] \text{ by } \theta_{\bar{\eta}} T ,$$

where z is temporarily constant, and $r^2_i = (x - \xi_i)^2 + (y - \eta_i)^2 + (z - \zeta_i)^2$.

But

$$\int_{C'} \frac{\zeta' - z}{r'^3} d\eta(\tau') \text{ differs from the same sum also by } \leq \theta_{\bar{\eta}} T.$$

Therefore

$$\left| \int_{C'} \frac{\zeta' - z}{r'^3} d\eta(\tau') - \int_{C} \frac{\zeta - z}{r^3} d\eta(\tau) \right| \leq 2\, \theta_{\bar{\eta}}\, T.$$

Finally, by performing the linear operation $\int_{\varkappa} [\quad] dx(t)$ we obtain

$$\left| \int_{\varkappa} dx(t) \int_{C'} \frac{\zeta - z}{r'^3} d\eta(\tau') - \int_{\varkappa} dx(t) \int_{C} \frac{\zeta - z}{r^3} d\eta(\tau) \right|$$
$$\leq \int_{\varkappa} |dx(t)| \left\{ \left| \int_{C'} \frac{\zeta' - z}{r'^3} d\eta(\tau') - \int_{C} \frac{\zeta - z}{r^3} d\eta(\tau) \right| \right\} \leq 2\, \theta_{\bar{\eta}}\, T^2 .$$

There are six such comparisons in relating $I(\varkappa, C)$ with $I(\varkappa, C')$. Therefore

$$| I(\varkappa, C) - I(\varkappa, C') | \leq 12\, \theta_{\bar{\eta}}\, T^2 .$$

But T is fixed and $\theta_{\bar{\eta}}$ may be made arbitrarily small. Therefore given $\varepsilon > 0$, we can take $\bar{\eta} > 0$ small enough so that

$$\left| \frac{I(\varkappa, C)}{4\pi} - \frac{I(\varkappa, C')}{4\pi} \right| < \varepsilon .$$

By theorem III, $I(\varkappa, C')/4\pi$ is an integer. It follows that also $I(\varkappa, C)/4\pi$ is an integer because we can choose a sequence ε_i, $\varepsilon_i \to 0$. For the second part, the proof is similar: C is fixed and $\varkappa$ becomes a rectifiable curve.

COROLLARY *Given* C *and* $\varkappa$, *closed, simple and rectifiable, the the value of* $I(\varkappa, C)$ *remains constant during deformations of* $\varkappa$ *and of* C, *unless* C *and* *intersect.*

As preparation for the next theorem, take $\varkappa$ and C as disjoint simple cycles, and let Q be a point of the set E of C (that is, where all four derivative numbers exist and are equal). Construct a solid circular double cone K of axis the tangent to C at Q such that

a) the arc C_Q of C that passes through Q remains within K,
b) the cone K lies in an $0(C, \rho)$ which does not intersect $\varkappa$.

Let $\overline{\Gamma}$ be a sphere of center Q, closed as a point set, $\overline{\Gamma} \subset 0(C, \rho)$. It may happen that there are arcs of C that intersect $\overline{\Gamma}$, but there is none that is C_Q. Let then $\{\overline{\Gamma}_i\}$ be a sequence of spheres, closed as point sets, of center Q and radii ρ_i, $\rho_i > \rho_{i+1}$; and let C_i be an arc of C which intersects $\overline{\Gamma}_i$ but is not C_Q. It follows that C_i intersects $\overline{\Gamma}$, and in an arc of length l_i, which remains $\geq$ some constant $\bar{l}$, > 0.

If there is no last ρ_i, the total length of the portions of C in $\overline{\Gamma}$, $= \Sigma (C_i \cap \overline{\Gamma})$, would be infinite, in contradiction to the hypothesis that C is rectifiable. Therefore there is a $\overline{\Gamma}_0$, which is a least $\overline{\Gamma}_i$, and the arc C_0 of C_Q in $\overline{\Gamma}_0 \cap K$ is unique.

Denote by D_0 a disc of $\overline{\Gamma}_0$, which intersects K only in Q. Denote its boundary as $\varkappa_0$. From Theorem A, Comment, $I(\varkappa_0, C) = \pm 4\pi$.

THEOREM V «These things being so» (*), *take* P_0 *a point of* $\varkappa_0$, P *a point of the given* $\varkappa$ *and construct an arc* γ *from* P *to* P_0 *or from* P_0 *to* P. *Taking* $\gamma' = \gamma$, *andalso* γ'' *to* γ *construct the cycle.*

$$\varkappa' = \varkappa + \gamma' + \varkappa_0 - \gamma'' .$$

Then $I(\varkappa', C) - I(\varkappa, C) = \mp 4\pi$ *and* $v(\varkappa', C) - v(\varkappa, C) = \pm 1$, *depending on the sense of* $\gamma + \varkappa_0$.

THEOREM VI Gauss's Integral (**). *Given two closed dis*$_4$*oint curves, named* $\mathfrak{z}_1$, $\mathfrak{z}_2$, *where* $\mathfrak{z}_1 = [f(z_1)]$, $\mathfrak{z}_2 = [g(z_2)]$ *and* z_1 *and* z_2 *are vectors from the origin, referring to positions on the repective curves, then*

$$\text{(H9)} \qquad v(\mathfrak{z}_1, \mathfrak{z}_2) = -\frac{1}{4\pi} \int_{\mathfrak{z}_1} \int_{\mathfrak{z}_2} \frac{1}{|g - f|^3} D(g - f, dg, df) ,$$

(*) Quae cum ita sint.

(**) (1a) The book «ALEXANDROFF UND HOPF» discusses the «Integral von Gauss» in lower case priting on page 497 in the supplementary portion of Chapter XII. The account is not intended to be complete, and for details refers the reader elsewhere.

Professor S. S. Chern kindly looked over my ms. and called my attention to this portion of (1a).

The desiderability of using the Alexander Duality Theorem in the incomplete (1a) had led me over some years to a study of (1b).

(D standing for determinant) *is the linkage coefficient of $\mathfrak{z}_1$ and $\mathfrak{z}_2$.*

The « linkage coefficient » involves the « index of intersection ».

In cartesian coordinates, f is the vector (x, y, z) and g the vector (ξ, η, ζ) so that $g - f = r$, — the vector $(x - \xi, y - \eta, z - \zeta)$. Thus $|g - f| = r$.

The equation (H9) is the special case in R^3 of (H8) in R^n. In (H8) the determinant is written:

$$D\left(g - f, \frac{\partial f}{\partial a_1}, \dots, \frac{\partial f}{\partial a_r}, \frac{\partial g}{\partial b_1}, \dots, \frac{\partial g}{\partial b_s}\right) da_1, \dots, d\,b_s\,,$$

in which the derivatives are written explicitly. =ence in (H9) it should be

$$D\left(g - f, \frac{d\,f}{d\,a}, \frac{d\,g}{d\,b}\right) d\,a, d\,b\,.$$

We return however to an analog of (H9) except that we shall interpret the integral as Stieltjes integrals.

From the double integral (2.8) of Theorem II in which R denotes the domain ($\varkappa$, C), and from (2.10) we see that we can form the following equation, which is a generalization of (H9):

$$(2.8') \qquad \frac{I(\varkappa, C)}{4\pi} = \frac{I(C, \varkappa)}{4\pi} = \frac{1}{4\pi} \iint_{(\varkappa, C)} \frac{1}{r^3} \begin{vmatrix} x - \xi & d\,x\,(t) & d\,\xi\,(\tau) \\ y - \eta & d\,y\,(t) & d\,\eta\,(\tau) \\ z - \zeta & d\,z\,(t) & d\,\zeta\,(\tau) \end{vmatrix}\,.$$

The value of (2.8′) is $\pm\, v\,(\varkappa, C)$, the linkage coefficient, if $\varkappa$ and C are closed rectifiable curves.

As seen on our pages 3 and 4, the four derivative numbers with respect to t of x (t), y (t), z (t) each, are all numerically ≤ 1, and for each function are equal on a set that includes a common set $E_\varkappa$ of measure equal to that of $\varkappa_0$. Similarly there exists a corresponding set E_C. Therefore, for d x (t) and d ξ (τ) may be substituted (d x/d t) d t and (d ξ/d τ) d τ, etc., and the integrals interpreted as Lebesgue or Borel integrals.

The relation (2.8′) is valid from Theorem II, for arcs as well as cycles, and therefore applies to the treatment of solid angle, that being a function of arc as well as cycle C.

4. Solid angle

As mentioned in the Introduction, we consider the solid angle as a function of curves, suggested by the notion of magnetic potential.

DEFINITION 4.1. Let $\varkappa$ be a simple rectifiable arc, $\varkappa_{01}$ the portion of it from a point P_0 to a point P_1, and C a simple rectifiable cycle disjoint from $\varkappa$. With the notation of (2.2), Theorem I, the quantity

$$\text{(4.1)} \qquad I(\varkappa_{01}, C) = \int_{P_0}^{P_1} L \, d\, x(t) + M \, d\, y(t) + N \, d\, z(t) = V(P_1) - V(P_0)$$

is defined to be *the change in the solid angle subtended at* P *by* C *as* P *moves along* $\varkappa$ *from* P_0 *to* P_1.

If by extension P_0 is taken as a point at ∞, and $V(P_0)$ assigned the value 0, the value resulting from (4.1) is defined to be *the solid angle subtended by* C *at* P_1 *as* P *moves along* $\varkappa$ *from the chosen point at* ∞ *to* P_1.

With reference to (4.1) we prove that the expression $I(\varkappa_{10}, C)$, where $\varkappa_{10} = - \varkappa_{01}$, converges to a finite value as $P_0 \to +\infty$ along the z axis. We have

$$I(\varkappa_{10}, C) = \int_{z_1}^{z_0} N \, d\, z(t) , \quad x = y = 0 .$$

Let r be the distance from the point (ξ, η, ζ) on C to a point z on the z axis. We assume that the range of values of ζ is small in comparision with those of ξ and η, and represent by z_1 a point P on the positive z axis such that if $z > z_1$, $z - z_1$ is less than r and remains less than r as $z \to +\infty$. For a given point P on the z axis we represent by $r(z)$ the minimal value of r, for Q on C. Thus $z < r(z)$ for $z > z_1$.

From the definition (1.2) and the formula (1.1) we obtain the value

$$\text{(4.2)} \qquad I(\varkappa_{10}, C) = \int_{z_1}^{z_0} \left\{ \int_C \frac{1}{r^3} \left(- \eta \frac{d\,\xi}{d\,\tau} + \xi \frac{d\,\eta}{d\,\tau} \right) d\,\tau \right\} d\,z ,$$

since $x = y$ O. By definition of τ, $|d\,\xi / d\,\tau| \leq 1$, $|d\,\eta / d\,\tau| \leq 1$ and at every point on C

$$\left| - \eta \frac{d\,\xi}{d\,\tau} + \xi \frac{d\,\eta}{d\,\tau} \right| \leq |\eta| + |\xi| .$$

But $|\eta| + |\xi| <$ some constant k. Hence, since $r(z) > z - z_0$,

$$|I(\varkappa_{10}, C)| \leq k \int_{z_1}^{z_0} \left\{ \int_C \frac{d\,\tau}{[r(z)]^3} \right\} d\,z \leq k \text{ meas } C \int_{z_1}^{z_0} \frac{1}{z^3} \, d\,z .$$

Hence

$$\text{(4.3)} \qquad |I(\varkappa_{10}, C)| \to k \, z_1^{-2} \text{ meas } C, \text{ as } z_0 \to +\infty .$$

Lemma 4. Let C_0 be the boundary of a triangle A_0, and let $\varkappa_0$ be the entire z axis. Then, according as the zaxis meets A_0 once or $A_0 + C_0$ not at all,

$$I(\varkappa_0, C_0) = \pm 4\pi \text{ or } 0 .$$

The proof is established by the method of Lemma 3.1 (and Comment). The choice of the + or — sign above depends on the orientation of $\mathbf{x}$ with respect to C.

The analysis of Theorems III and IV shows how $\varkappa$ may be replaced by another path from $z = -\infty$ to $z = +\infty$. Also one point P_0 at ∞ may be moved to another by adding an arc on the sphere at ∞.

As the simplest example take

C as the circumference $\xi^2 + \eta^2 = a^2$, $\zeta = 0$

$\varkappa = \varkappa_{01}$ as the segment $x = 0$, $y = 0$, $z_0 \leq z \leq z_1$.

We have

$$I(\varkappa_{01}, C) = \int_{z_0}^{z_1} N\, d\,z(t) = \int_{z_0}^{z_1} d\,z(t) \left\{ \int_C \frac{\partial r^{-1}}{\partial \eta} d\,\xi(\tau) - \frac{\partial r^{-1}}{\partial \xi} d\,\eta(\tau) \right\},$$

where, since we must use derivatives, $r^2 = \xi^2 + \eta^2 + z^2$,

$$\frac{\partial r^{-1}}{\partial \eta} = -\frac{\partial r^{-1}}{\partial x} = -\frac{\eta - y}{(a^2 + z^2)^{3/2}}, \quad \frac{\partial r^{-1}}{\partial \xi} = \frac{-\xi}{(a^2 + z^2)^{3/2}}.$$

$$I(\varkappa_{01}, C) = \int_{z_0}^{z_1} d\,z(t) \int_C \frac{\xi\, d\,\eta(\tau) - \eta\, d\,\xi(\tau)}{(a^2 + z^2)^{3/2}}$$

$$= \int_{z_0}^{z_1} d\,z(t) \frac{a^2\, d\,\varphi(\tau)}{(a^2 + z^2)^{3/2}} = 2\pi a^2 \int_{z_0}^{z_1} \frac{d\,z}{(z^2 + a^2)^{3/2}}$$

$$= 2\pi a^2 \left. \frac{z}{a^2 \sqrt{z^2 + a^2}} \right]_{z_0}^{z_1} = 2\pi \left(\frac{z_1}{\sqrt{z_1^2 + a^2}} - \frac{z_0}{\sqrt{z_0^2 + a^2}} \right),$$

the indefinite integral being that of Art. 3.

If we let $z_0 \to -\infty$, we have

$$\lim_{z_0 \to -\infty} I(\varkappa_{01}, C) = V(P_1)$$

$$V(P_1) = 2\pi \left(\frac{z_1}{\sqrt{z_1^2 + a^2}} + 1 \right). \tag{4.4}$$

Thus

$$V(-\infty) = 0, \quad V(0) = 2\pi, \quad V(+\infty) = 4\pi.$$

5. On the gradient of the solid angle

The following is a rough translation of a quotation from the book of Alexandroff und Hopf.

« On the basis of potential-theoretic considerations not here to be discussed, the integral (9H) can be given another form, and therefore a different geo etric interpretation. We assume that for any point P not on C the solid angle W (P) subtended by C at P is determined by the integral (to use the notation of this ms.)

$$\text{(A)} \qquad \text{grad. } W = \int_C \frac{1}{r^3} [\vec{r} \times (d\,\xi(\tau), d\,\eta(\tau), d\,\zeta(\tau))]$$

the $\times$ meaning vector product and *grad* meaning vector gradient » (*). Here $\vec{r}$ is the vector $x-\xi, y-\eta, z-\zeta$.

Our purpose is to prove (A) from our own « potential-theoretic considerations ». From (4.1) the gradient of the solid angle $I(\varkappa_\beta, C)$ is the vector

$$(-L, -M, -N) = \text{grad } W = \left(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial W}{\partial z}\right) =$$

$$\left(-\int_C \frac{1}{r^3}\left[(z-\zeta)\frac{d\eta}{d\tau} - (y-\eta)\frac{d\zeta}{d\tau}\right] d\tau, -\int_C \frac{1}{r^3}\left[(x-\xi)\frac{d\zeta}{d\tau} - (z-\zeta)\frac{d\xi}{d\tau}\right] d\tau,\right.$$

$$\left. -\int_C \frac{1}{r^3}\left[(y-\eta)\frac{d\xi}{d\tau} - (x-\xi)\frac{d\eta}{d\tau}\right] d\tau\right).$$

But this can be expressed by using a vector product, viz. —

$$\text{grad } W = \int_C \frac{1}{r^3}\left[(x-\xi, y-\eta, z-\zeta) \times \left(\frac{d\xi}{d\tau}, \frac{d\eta}{d\tau}, \frac{d\zeta}{d\tau}\right)\right] d\tau$$

which confirms (A).

From this, if $\varkappa$ is also a cycle, an integration of $d\,W$ around $\varkappa$ ($\mathfrak{z}_2$ in the notation of ([1a])) brings us back, as we have seen already, to $1/4\pi$ times an integral multiple of $v(\varkappa, C)$ or to zero (**).

(*) ([1a]) p. 497, with reference to Picard, Traité d'Analyse, 1920, Vol. I. As will be seen in Part II, it goes to ([5]), 1916.
(**) Compare ([1a]) p. 498.

APPENDIX I

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APPENDIX II

Elementary properties of the Stieltjes integral

H. E. Bray

Def. 1. Consider two functions $\varphi(x)$ and $\alpha(x)$ defined at every point of the interval $a \leq x \leq b$. Let (a, b) be divided by the n points $x_1 < x_2 < \ldots < x_n$ into $n+1$ parts such that $x_{i+1} - x_i = \delta_i \leq \delta, i = 0, 1, 2, \ldots, n$, $[x_0 = a, x_{n+1} = b]$ and let the points ξ_i be chosen so that $x_i \leq \xi_i \leq x_{i+1}$. The quantity

$$\lim_{\delta \to 0} \sum_{i=0}^{n} \varphi(\xi_i) \{ \alpha(x_{i+1}) - \alpha(x_i) \}$$

if it exists, is called the *Stieltjes integral of* $\varphi(x)$, *whith regard to* $\alpha(x)$, *between the limits* a *and* b. *It is designated by*

$$\int_b^a \varphi(x)\, d\alpha(x) .$$

Theorem 1. If $\alpha(x)$ is a function of bounded variation, $a \leq x \leq b$, and if $\varphi(x)$ is continuous at every point of (a, b), then

$$\int_a^b \varphi(x)\, d\alpha(x)$$

exists.

Theorem 2. If $\alpha(s)$ is of bounded variation $a \leq s \leq b$ and if

$$\int_a^b \varphi\, d\alpha = \lim_{\delta \to 0} \sum_{i=0}^{n} \varphi(\xi_i) \{ \alpha(s_{i+1}) - \alpha(s_i) \}$$

exists, then

$$\int_a^b \varphi\, d\alpha - \sum_{i=0}^{n} \varphi(\xi_i) \{ \alpha(s_{i+1}) - \alpha(s_i) \} \leq 0_\delta\, T(b)$$

where T (b) is the total variation of α in the interval (a, b), 0_δ is the greatest oscillation of $\varphi(s)$ in the $n+1$ intervals (s_i, s_{i+1}) and $s_i \leq \xi_i \leq s_{i+1}$ (*).

(*) As an immediate corollary to Theorem 2, $\varphi(s)$ is continuos, $a \leq s \leq b$.

Def. 2. Consider a function $\alpha(x, s)$ which for every value of x, $c \leq x \leq d$ is of bounded variation in s, $a \leq s \leq b$. Let $T(x, s)$ be the corresponding total variation function in s, for a given value of x. Evidently $T(x, s) \leq T(x, b)$.

If $T(x, b)$ is a bounded function of x, i.e. if a positive constant K can be found such that

$$T(x, b) < K, \quad c \leq x \leq d,$$

then $\alpha(x, s)$ is said to be a function of uniformly bounded variation in s for all x in (c, d).

Theorem 3. If $\varphi(s)$ is continuous $a \leq s \leq b$, and if $\alpha(x, s)$ is a function of uniformly bounded variation in s for all values of x, $c \leq x \leq d$, and if at every point x in (c, d) a set of values of s (not necessarly independent of x) dense in (a, b) and including a and b, can be found such that for each s in the set of values of s, $\alpha(x, s)$ is continuous in x then

$$\Phi(x) = \int_a^b \varphi(s)\, d_s\, \alpha(x, s)$$

is continuous, $c \leq x \leq d$.

Theorem 4. If $\gamma(x)$ is a function of bounded variation, $c \leq x \leq d$, — and if $\alpha(x, s)$ is of uniformly bounded variation in s, $a \leq s \leq b$, for every value of x, $c \leq x \leq d$, and continuous in x for every value of s, then

$$\Psi(s) = \int_a^b \alpha(x, s)\, d\gamma(x)$$

is a function of bounded variation, $a \leq s \leq b$.

Theorem 5. If $\varphi(s)$ is continuous, $a \leq s \leq b$; if $\gamma(x)$ is of bounded variation, $c \leq x \leq d$; and if $\alpha(x, s)$ is continuous in x for all values of s in (a, b) and of uniformly bounded variation in s for all values of x in (c, d), then the integrals

$$\int_c^d \left[\int_a^b \varphi(s)\, d_s\, \alpha(x, s)\right] d\gamma(x), \quad \int_a^b \varphi(s)\, d_s \int_c^d \alpha(x, s)\, d\gamma(x)$$

exist and are equal.

Theorem 6. If $\varphi(x)$ is continuous $a \leq x \leq b$ and if $\alpha(x)$ is of bounded variation in the same interval, then the integral

$$\int_a^b \alpha(x)\, d\varphi(x)$$

exist and is equal to

$$\alpha(x)\varphi(x)\Big|_a^b - \int_a^b \varphi(x)\, d\alpha(x).$$

Part II (*)

Surfaces of given rectifiable boundary

1. Surfaces of boundary C.

We state immediately the following definition, in spite of the necessity of explaining later a couple of its terms (set in quotation marks).

Definition I (**). Let C be given as a simple closed rectifiable curve. The point set $\bar{S}$, of closure S, is a surface of boundary C if S is compact and connected (that is, closed and bounded as a set of points, and not the sum of two or more such sets that are disjoint) and

(i) Outside a tubular neighborhood of C, written $0(C, \rho)$, ρ being arbitrarily small, S is contained in a finite number of « regular » surface elements $\bar{A}$, which have two-by-two in common at most an edge (a « regular » arc), or else one or two vertices. Such elements are not prevented from reaching into $0(C, \rho)$.

(ii) If $\varkappa$ is a simple closed rectifiable curve for which $v(\varkappa, C)$ (***) is odd, i.e. $v(\varkappa, C) = 1$, *mod.* 2, then $\varkappa$ traverses S.

(iii) If any portion of S, open in S, is removed, then (ii) will no longer be satisfied.

A regular surface element $\bar{A}$ in R^3 is to be bounded as a point set. Hence by (i), $\bar{S}$ *is bounded as a point set.*

As in (4a), for the definition and properties of such surface elements we follow the treat ent given by the late Professor O. D. Kellogg, in his book on potential theory (****).

A regular surface element $\bar{A}$ then is a bounded closed set of points, which for some orientation of Cartesian coordinates (x, y, z) admits a representation (a « standard » representation)

$$z = f(x, y)$$

in a support (x, y) plane, with $f(x, y)$ continuously differentiable (K. p. 105) But the continuity of derivatives is not necessary across boundaries.

When convenient we shall speak of $\bar{A}$, the set A minus its boundary, also as a regular surface element.

(*) References to Part I will be preceded by I., e.g., I. Theorem III.
(**) See (4a), pp. 786-788; note the requirement in the introduction, p. 786.
(***) $v(\varkappa, C)$ stands for « linkage coefficient », as in Part. I.
(****) (3) pp. 97-112. A reference to (3) is preceded by the letter K.

A portion S_0, open in $\bar{S}$, is understood to contain a two dimensional portion of some $\bar{A}$.

In the support plane the image of $\bar{A}$ is defined as a simply connected region $\bar{A}'$, which we shall see is to be bounded by a finite number of « regular » arcs, arranged in order, and such that the terminal point of the last arc is the initial point of the next following. A regular arc in space is of course similarly constructed.

A « regular » arc in the plane is a closed set of points, which for some orientation of the x, y axes admits a representation $y = f(x)$, where $f(x)$ is continuously differentiable (K p. 97).

A regular arc (ab) in space is a set of points which for some orientation of the axes admits a representation

$$y = f(x),\ z = \varphi(x),\ a \leq x \leq b,$$

where $f(x)$ and $\varphi(x)$ are continuously differentiable (K pp. 97-99).

A « regular » curve in the plane or in space is a set of points consisting of a finite number of regular arcs, arranged in order (K p. 99).

The following theorems are important for us.

Theorem K 3. The projection of a regular arc on a plane to which it is nowhere perpendicular consists of a finite number of regular arcs (K p. 99) (*).

Theorem K 6. The boundary of a regular surface element is a regular curve (K p. 106).

Theorem K 7. Any regular surface element « E » (for us, $\bar{A}$) can be subdivided into a finite number of regular surface elements $\bar{e}$, each with the property that if any system of coordinate axes be taken, in which the z axis does not make an angle of more than 70 degrees with any normal to $\bar{e}$, $\bar{e}$ admits a « standard » representation with this system of axes (K p. 108).

A corollary to this theorem would be that it can be applied to subdivisions of two regular surface elements $\bar{A}_1$, $\bar{A}_2$ that are adjacent along one regular arc.

Note 1. Sometimes it is convenient to consider surfaces to which are assigned more than one boundary (**). In these instances the same general definitions and requirements apply, except that the surfaces may be either or not connected.

Note 2. The surfaces which are dealt with in (5) are described on K p. 112, conditions a), b), c), d). A surface consists of a *finite* number of regular surface elements that are more strongly connected than seems necessary for us. The complex suggesWs a « pseudovarieta » (***).

But Kellogg's problem is fitting function to surface rather than surface to boundary.

(*) For esample, the railing of a circular stairway.

(**) For example, the case where the boundaries are two parallel circles with centers on the z axis and equal radii.

(***) (1b) p. 351 f.

2. *Two basic theorems.* We take τ as in Part I, at least as great as measure along C, usually equal to it for convenience, and t similarly for $\varkappa$. By E is denoted *the set of points in* C *where all three derivatives* $d\xi/d\tau$, $d\eta/d\tau$, $d\zeta/d\tau$ *exist.* This set is Borel measurable, of total measure equal to C, thus dense in C. Each of the derivatives is in absolute value ≤ 1. In fact all of the derivative numbers have this ≤ 1 property everywhere on C.

Lemma 2.1. Given Q_1 a point of E, there exists a circular cycle $\varkappa$, of radius arbitrarily small and center Q_1, such that $v(\varkappa, C) = 1$.

Therefore there exists the point P_1, P_1 *in* $x_1 \cap S$.

Denote by s the tangent line to C at Q_1 by V a double circular cone of axis s, and by θ the angular magnitude of V about s. Let D_1 be a circular disc with center Q_1, orthogonal to s. We take its radius small enough so that D_1 intersects C only at Q_1. Denote its boundary by $\varkappa_1$.

Then $v(\varkappa_1, C) = \pm 1$. In fact if we take a point P_1 on $\varkappa_1$ and shrink $\varkappa_1$ down to vanishing about P_1, $I(\varkappa_1, C)$ will become 0 as soon as $\varkappa_1$ crosses C. See I. Theorem II equation (2.9). Since $r \neq 0$, the interior vector integral vanishes.

Therefore it must originally have had the value $\pm 4\pi$, and $v(\varkappa_1, C)$ the value ± 1 (*).

It follows then from (ii) of Art. 1, Definition, that $\varkappa_1$ must intersect S.

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(*) I. Theorem V.