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# Classification of a Wide Set of Geometric Figures, Surfaces and Lines (Trajectories) (***) 

Dedicated to the Memory of Gaetano Fichera

Abstract. - Aim of this article is the analytical representation of a class of geometrical figures, surfaces and lines ("Surfaces of revolution" [7]. This class of surfaces includes the surfaces appearing in some problems of Shell Theory or problems of spreading of smoke-rings; furthermore, the lines of this class can be used for describing the complicated orbit of some celestial objects. In particular cases, this analytic representation gives back many classical objects (torus, helicoid, helix, ... etc.).

## Introduction

In this paper we give analytic representation of a large class of geometrical figures, surfaces, lines and trajectory. In previous articles [1-4] sets of Generalized Möbius Listing's bodies, which are a particular case of this class in static case, have been already defined.

Notations: In this article we use following notations:

- $X, Y, Z$, or $x, y, z$ - is the ordinary notation for coordinates;
- $t$-time value $-t \in[0,+\infty)$;
- $\tau, \psi, \theta$ - are space values (local coordinates or parameters in parallelogram):
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1. $\tau \in\left[\tau_{*}, \tau^{*}\right]$, where $\tau_{*}$ and $\tau^{*}$ are some non-negative constants;
2. $\psi \in[0,2 \pi]$;
3. $\theta \in[0,2 \pi h]$, where $b \in \mathbf{Z}$ (integer);

- $P R_{m} \equiv A_{1} A_{2} \ldots A_{m} A_{1}^{\prime} A_{2}^{\prime} \ldots A_{m}^{\prime}$ denotes an orthogonal prism, whose ends given by $A_{1} A_{2} \ldots A_{m}$ and $A_{1}^{\prime} A_{2}^{\prime} \ldots A_{m}^{\prime}$ are "regular polygons" $P_{m}$ and $m$ is the number of its angles or vertices (see fig. 1). In general case - edges of "regular polygons" are not always straight lines ( $A_{i} A_{i+1}$ may be, for example: edge of epicycloid or edge of hypocycloid);
- $P R_{0}$ - is a segment and $P_{0}$ is a point;
- $P R_{1}$ - is an orthogonal cylinder, which cross section is a a $P_{1}$ - plane figure without symmetry;
- $P R_{2} \equiv A_{1} A_{2} A_{1}^{\prime} A_{2}^{\prime}$ - is a rectangle, if $P_{2} \equiv A_{1} A_{2}$ is a segment of straight line;
- $P R_{\infty}$ - is an orthogonal cylinder, which cross section is a $P_{\infty}$-circle.

Let

$$
\begin{equation*}
x=p(\tau, \psi), \quad z=q(\tau, \psi) \tag{2}
\end{equation*}
$$

or

$$
x=p(\tau, \theta) \cos \psi
$$

$$
\begin{equation*}
z=p(\tau, \theta) \sin \psi \tag{*}
\end{equation*}
$$



Fig. 1.
are analytic representations of the "regular polygon" $P_{m}$, usually $p(0,0)=q(0,0)=0$ and the point $(0,0)$ is the center of symmetry of this polygon. In general the "shape" of Prism $P R_{m}$, may be changeable and his cross section, which is defined by $\left(2^{*}\right)$, depends on place of the cross section $(\theta)$.

- $D(p, q)$ or $D(p)$ - diameter of $P_{m}$, "regular polygon" or cross section.
- $O O^{\prime}$ - axis of symmetry of the prism $P R_{m}$;
- $L_{\rho}$ - Family of lines situated on the plane, which parametric representation are

$$
L_{\rho}=\left\{\begin{array}{l}
X=f_{1}(\rho, \theta)  \tag{3}\\
Y=f_{2}(\rho, \theta)
\end{array} \quad \rho \in\left[0, \rho^{*}\right), \quad \theta \in[0,2 \pi b], \quad b \in \mathbf{Z}\right.
$$

or

$$
L_{\rho}=\left\{\begin{array}{l}
X(\rho, \theta)=\rho_{1}(\theta) \cos \theta  \tag{*}\\
Y(\rho, \theta)=\rho_{2}(\theta) \sin \theta
\end{array}\right.
$$

We assume the following hypotheses
i) For any parameters $\rho_{1}, \rho_{2} \in\left[0, \rho^{*}\right], \rho_{1} \neq \rho_{2}$, the lines $L_{\rho_{1}}$ and $L_{\rho_{2}}$ have not common points.
ii) If $L_{\rho}$ is a closed curve, then for every fixed $\rho \in\left[0, \rho^{*}\right] f_{i}$ - are $2 p$-periodic functions $f_{i}(\rho, \theta+2 \pi)=f_{i}(\rho, \theta),(i=1,2)$.

- (4)

$$
g(\theta):[0,2 \pi] \longrightarrow[0,2 \pi]
$$

be an arbitrary function and for every $\Theta \in[0,2 \pi]$ there exists $\theta \in[0,2 \pi]$, such that $\Theta=g(\theta) ;$

- $\bmod _{m}(n)$ - natural number $<m$; for every two numbers $m \in \mathbf{N}$ (natural) and $n \in \mathbf{Z}$ (integer) there exists a unique representation $n=k m+j \equiv k m+\bmod _{m}(n)$, where $k \in \mathbf{Z}$ and $j \equiv \bmod _{m}(n) \in \mathbf{N} \cup\{0\}$;
- (5) $\mu \equiv\left\{\begin{array}{l}\frac{n}{m}, \quad \text { when } \quad m \in \mathbf{N} \quad \text { and } \quad n \in \mathbf{Z} \\ n\end{array} \quad\right.$ when $\quad m=\infty \quad$ and $\quad n \in \mathbf{Z} \quad($ or $n \in \mathbf{R}$ (Real))
- GML ${ }_{n}^{m}$ - Generalized Möbius Listing's body - is obtained by identifying of the opposite ends of the prism $P R_{m}$ in such a way that:
A) For any integer $n \in \mathbf{Z}$ and $i=1, \ldots, m$ each vertex $A_{i}$ coincides with $A_{i+n}^{\prime} \equiv A_{m_{0} d_{m}(i+n)}^{\prime}$, and each edge $A_{i} A_{i+1}$ coincides with the edge

$$
A_{i+n}^{\prime} A_{i+n+1}^{\prime} \equiv A_{\bmod _{m}(i+n)}^{\prime} A_{\text {mod }_{m}(i+n+1)}^{\prime}
$$

correspondingly;
B) The integer $n \in \mathbf{Z}$ is the number of rotations of the end of the prism with respect to the axis $O O^{\prime}$ before the identification. If $n>0$ the rotations are counter-clockwise, and if $n<0$ then rotations are clockwise.

Example 1: If $m=2$, then:
$G M L_{1}^{2}$ becomes a classical (or regular) Möbius band (see e.g. [1-6] and fig. 2 e.)); $G M L_{0}^{2}$ are surfaces:

- in regular cases: cylinder, ring, or frustrum of a cone (see also fig. 2 a.) b.) f.));
- in degenerated cases: disk, surface of cone, two cone (see also fig. 2 c.) d.)).

Example 2: If $m=4, P_{4} \equiv A_{1} A_{2} A_{3} A_{4}$ - is a rectangle and $P R_{4}$ is a parallelogram, then $G M L_{0}^{4}, G M L_{1}^{4}, G M L_{2}^{4}, G M L_{3}^{4}, G M L_{4}^{4}$ and $G M L_{14}^{4}$, correspondingly are shown in fig. 3 . Definition and some properties of $G M L_{n}^{m}$, can be found in [1-5].


Fig. 2.


Fig. 3.
I. - Static
I.1. In this part of the article we give parametric representations of a wide set of geometric figures, surfaces and lines under the following restrictions:

1) The $O O^{\prime}$-axis of symmetry (middle line) of the prism is transformed into some "basic line" (or "profile curve" [7]) $L_{\rho}$;
2) Rotation of the end of the prism is "semi-regular" along the middle line $O O^{\prime}$.
3) Every function does not depend on the time argument $t$.

Under the above restrictions the analytic representation of this class is given by the following formulas

$$
\begin{align*}
& X(\tau, \psi, \theta)=f_{1}([R+p(\tau, \psi) \cos (\mu g(\theta))-q(\tau, \psi) \sin (\mu g(\theta))], \theta) \\
& Y(\tau, \psi, \theta)=f_{2}([R+p(\tau, \psi) \cos (\mu g(\theta))-q(\tau, \psi) \sin (\mu g(\theta))], \theta)  \tag{6}\\
& Z(\tau, \psi, \theta)=K(\theta)+p(\tau, \psi) \sin (\mu g(\theta))+q(\tau, \psi) \cos (\mu g(\theta)),
\end{align*}
$$

or

$$
\begin{align*}
& X(\tau, \psi, \theta)=\left[\rho_{1}(\theta)+p(\tau, \theta) \cos (\psi+\mu g(\theta))\right] \cos (\theta) \\
& Y(\tau, \psi, \theta)=\left[\rho_{2}(\theta)+p(\tau, \theta) \cos (\psi+\mu g(\theta))\right] \sin (\theta)  \tag{*}\\
& Z(\tau, \psi, \theta)=K(\theta)+p(\tau, \theta) \sin (\psi+\mu g(\theta)),
\end{align*}
$$

where, respectively

- the arguments $(\tau, \psi, \theta)$ are defined in (1);
- the functions $f_{1}$ and $f_{2}$ or $\rho_{1}(\theta)$ and $\rho_{2}(\theta)$ are the "shape of plane basic line", defined by (3) or ( $3^{*}$ );
- the functions $p(\tau, \psi)$ and $q(\tau, \psi)$ or $p(\tau, \theta)$ denote the "shape of the radial cross section", defined by (2) or (2*);
- the function $g(\theta)$ is the "rule of twisting around basic line", defined by (4);
- $K(\theta)$ is the "Law of vertical stretching of figure" - an arbitrary smooth function;
- $\mu$ - is the "number of rotation", defined by (5);
- $R$ is the "radius" - some fixed real number.


## General remarks

- If the functions $\rho_{1}(\theta)=\rho_{2}(\theta)=K(\theta)=0$ and the numbers $R=\mu=0$, then (6) and $\left(6^{*}\right)$ define a set of "Rotation figures" (Spheres, Ellipsoids, Paraboloids, ...);
- If the "basic line" $L_{\rho}$ is a closed line, the "Law of vertical stretching of figure" $K(\theta)=0$ and $m$ is simultaneously: the number of angles or vertices of $P_{m}$ "regular polygon" ("shape of the radial cross section") and the denominator in (5) defining $\mu$, then equations (6) and $\left(6^{*}\right)$ define a set of "Toroids", in particular the Generalized Möbius Listing bodies $G M L_{n}^{m}$ are included;
- If the "basic line" $L_{\rho}$ is a closed line, the "Law of vertical stretching of figure" $K(\theta) \neq 0$ and $m$ in $P_{m}$ does not depend on $\mu$, then (6) and ( $6^{*}$ ) define set of "Helix bodies", in particular Helix, Helicoids with different cross sections are included;
- If the "basic line" $L_{\rho}$ is a spiral line, then $\left((6)\right.$ and $\left(6^{*}\right)$ define a set of "Cochlea bodies" (or "Seashells");
- If in (4) $g(\theta) \equiv \theta$, then restriction I.1. - 2) has the following formulation Rotation (twisting) of the end of the prism, (before the identification, when the "basic line" $L_{\rho}$ is a closed line), is called regular along the middle line $O O^{\prime}$. In this case (6) and ( $6^{*}$ ) define set of figures with "regular twisting" along the "basic line" $L_{p}$;
- If in (2) or $\left(2^{*}\right)$ the functions $p(\tau, \psi), q(\tau, \psi)$ or $p(\tau)$ also depend on $\theta$ (i.e. $p=p(\tau, \psi, \theta), q=q(\tau, \psi, \theta)$ or $p=p(\tau, \theta))$, then (6) and $\left(6^{*}\right)$ define a set of figure with "Variable shape of the radial cross section";
- If in (1) $\tau_{*}=0$, then (6) and $\left(6^{*}\right)$ give the analytic representation of a "Solid Body";
- If in (1) $\tau_{*}>0$, then (6) and $\left(6^{*}\right)$ give the analytic representation of a "Pipe" with thickness $\tau^{*}-\tau_{*}$ (or a "Shell", if $\tau^{*}-\tau_{*} \ll 1$ );
- If in (1) $\tau=$ const., then (6) and $\left(6^{*}\right)$ give the analytic representation of a Surface;
- If in (1) $\tau=$ const. and $\psi=$ const., then (6) and $\left(6^{*}\right)$ give the analytic representation of a Line, which is situated on the corresponding Surface;


## I.2. "Toroids"

In this part of the article we consider a set of Toroids.
In this case:

- The "basic line" $L_{\rho}$ is a closed line ( $3^{*}$ ) or in representation (3) the functions $f_{1}$ and $f_{2}$ satisfy the condition ii);
- The "Law of vertical stretching of figure" $K(\theta) \equiv 0$;
- $m$ is simultaneously: the number of angles or vertices of the "regular polygon" $P_{m}$ ("shape of the radial cross section") and the denominator of $\mu$, see definitions (5).


## General remarks for Generalized Möbius Listing's bodies GML $n_{n}^{m}$

- If $m$ - finite number of vertices of the "regular polygon" $P_{m}$ - in (2) is different from the denominator of $\mu$ in (5), then (6) and ( $6^{*}$ ) define a set of toroidal bodies with "Break" along the radial cross-section (see for example fig. 4);


Fig. 4.

- If $j$ (integral number) is the greatest common divisor of numbers $m$ and $\bmod _{m}(n)$ and $L_{p}$ is a sufficiently smooth curve, then the generalized Möbius-Listing body has a " $j$-colored surface" (i.e. it is possible to paint the surface of this figure in $j$ different colors without taking away the brush - when painting, it is prohibited to cross the edge of this figure. - For $m=2$, then the 1 or 2 -colored surfaces are 1 or 2 -sided surfaces respectively);
- If $j=0$, then we have a " $k$-colored" surface;
- If $j=1$, then the boundary edge of this body is a line;
- If $k=\infty$, then the generalized Möbius-Listing body is a toroidal body with circular cross section;
- If $D(p, q)$ or $D(p)$ is less then the diameter of "Basic line" $L_{R}$, then the figure, define by equations (6) or $\left(6^{*}\right)$ has not self-crossing points.

Case 1. Let $L_{R}$ be a circle with radius $R\left(\rho_{1}=\rho_{2}=R\right.$ in (3*)), $g(\theta) \equiv \theta$ in (4) and $m=2, P_{2} \equiv A_{1} A_{2}$ - is a segment of straight line - then in the parametric representation $\left(2^{*}\right) p(\tau)=\tau, \tau \in\left[-\tau^{*}, \tau^{*}\right]$, where $\tau^{*}$ is some constant, $\psi=\psi_{0}=$ const., $\psi_{0} \in[0,2 \pi]$ and $\theta \in[0,2 \pi b], b=1$ in (1), $\mu=n / 2$ in (5).

In this case $\left(6^{*}\right)$ is the analytic representation of:

- $G M L_{0}^{2}$, when $n=0$, this is a:
- Ring (when $R>\tau^{*}$ ) (see also fig. 2. b)) or Disk (when $R=\tau^{*}$ )(see also fig. 2. d)), if $\psi=\psi_{0}=0$ or $\pi$;
- Cylinder (for every $R>0$ ), if $\psi=\psi_{0}=\pi / 2$ or $3 \pi / 2$ (see also fig. 2. a));
- Cone (when $R=\tau^{*} \cos \psi_{0}$ )(see fig. 2 c .) , Two cone (when $R<\tau^{*} \cos \psi_{0}$ ) or Frustrum of a cone (see fig. 2 f.), (when $R>\tau^{*} \cos \psi_{0}$ ), if $\psi=\psi_{0}=$ const. $\neq 0, \pi / 2, \pi$ or $3 \pi / 2$.
- $G M L_{1}^{2}$, - regular Möbius band (or strip), when $n=1$, (see fig. 2 e.));
- $G M L_{n}^{2}$, - two sided surface, when $n=$ even number (examples for $n=2$ and $n=4$ in fig. 5);


Two-sided or Two-colored surfaces
Fig. 5.


Fig. 6.

- GML 2 , - one sided surface, when $n=$ odd number (examples for $n=-5$ and $n=3$ in fig. 6);

Remark 1: In this case:
a) $D(p, q)$ or $D(p)$ is a $2 \tau^{*}$-length of $A_{1} A_{2}$ or width of cross section of $G M L_{n}^{2}$;
b) If $\mu \neq n / 2$, then by formula (6) or $\left(6^{*}\right)$ define a non-closed surface;
c) Some properties of this figures are recalled in refs. [1-5];
d) If $\tau^{*}=\infty(\tau \in(-\infty, \infty))$ and (2) have following form:

$$
\begin{aligned}
& x(\tau, \psi)=a \sqrt{\pi} \int_{0}^{\tau} \cos (\pi t) d t \\
& z(\tau, \psi)=a \sqrt{\pi} \int_{0}^{\tau} \sin (\pi t) d t
\end{aligned}
$$

where $a$ is a constant and $R>a \pi$, then $G M L_{n}^{2}$ defined by (6) or ( $6^{*}$ ) for any $\mu=n / 2$, has not self-crossing points, but cross section has infinite width.

Case 2. Let $L_{R}$ be a circle with radius $R\left(\rho_{1}=\rho_{2}=R\right.$ in ( $\left.3^{*}\right)$ ), $g(\theta) \equiv \theta$ in (4) and $m=4, P_{4}$ is a rectangle, then in parametric representation (2)

$$
\begin{equation*}
x=p(\tau, \psi)=\frac{\tau \cos \psi}{|\cos \psi|+|\sin \psi|}, \quad z=q(\tau, \psi)=\frac{\tau \sin \psi}{|\cos \psi|+|\sin \psi|}, \tag{7}
\end{equation*}
$$

$\tau \in\left[\tau_{*}, \tau^{*}\right]$, where $\tau_{*}$ and $\tau^{*}$ are some constants; $\psi \in[0,2 \pi]$, and $\theta \in[0,2 \pi h], b=1$ in (1), $m=n / 4$ in (5).

Under these conditions $\left(6^{*}\right)$ is the analytic representation of:

- $G M L_{n}^{4}$ - solid body, if $\tau_{*}=0$;
- $G M L_{n}^{4}$ - pipe, with thickness $\tau^{*}-\tau_{*}$, if $\tau_{*}>0$ or Shell, if $0<\tau^{*}-\tau_{*} \ll 1$;
- $G M L_{n}^{4}$ - Surface, when $\tau=\tau^{*}$.

In fig. 3 are graphical representation of some examples of Generalized Möbius Listing's bodies $G M L_{n}^{4}$ :

1. $G M L_{0}^{4}, G M L_{4}^{4}$ - four-colored surfaces;
2. $G M L_{1}^{4}, G M L_{3}^{4}$ - one-colored surfaces;
3. $G M L_{2}^{4}, G M L_{14}^{4}$ - two-colored surfaces.

Case 3. Let $L_{R}$ be an ellipse with axes $\rho_{1}(\theta)=R_{1}, \rho_{2}(\theta)=R_{2}$ in $\left(3^{*}\right), g(\theta) \equiv \theta$ in (4) and $m=6$, then (2) is a parametric representation of $G M L_{n}^{6}$ epicycloid or hypocycloid $P_{6}$, with numbers $m=n / 6$ in (5); Then ( $6^{*}$ ) is the analytic representation of $G M L_{n}^{6}$ and in fig. 7 and 8 there are graphical representations of correspondingly:

1. $G M L_{0}^{6}, G M L_{6}^{6}$, and $G M L_{12}^{6}$ - six-colored surfaces with different cross sections (see fig. 7);
2. $G M L_{1}^{6}$ and $G M L_{5}^{6}$ - one-colored surfaces with cross section epicycloids, see fig. 8;
3. $G M L_{3}^{6}$ - two-colored surface, see fig. 8;
4. $G M L_{2}^{6}$ and $G M L_{4}^{6}$ - tree-colored surfaces, see fig. 8 .

CASE 4. Let $L_{R}$ be a circle with radius $R\left(\rho_{1}=\rho_{2}=R\right)$ in (3*), $g(\theta) \equiv \theta$ in (4) and $m=\infty, P R_{\infty}$ is an orthogonal Cylinder and $P_{\infty}$ is a circle, then in parametric representation $\left(2^{*}\right) p(\tau)=\tau, \tau \in\left[\tau_{*}, \tau^{*}\right]$, where $\tau_{*}$ and $\tau^{*}$ are some constants; $\psi \in[0,2 \pi], \theta \in[0,2 \pi h], h=1$ in (1), $\mu \in \mathbf{R}$.


Fig. 7.


Fig. 8.
Then $\left(6^{*}\right)$ define a set of toruses $G M L_{\mu}^{\infty}$. In particular $G M L_{0}^{\infty}$ is a classic:

- Spindle torus (with two radius when $R<\tau^{*}$ );
- Sphere (degenerate torus when $R=0$ ) and $\theta \in[0, \pi]$;
- Horn torus (with one radius if $R=\tau^{*}$ );
- Ring torus (with two radius if $R>\tau^{*}$ ) - (see e.g. [6], p. 1816);
- Toroidal Pipe or Shell with thickness $\tau^{*}-\tau_{*}$, if $\tau_{*}>0$;
- Toroidal Surface, if $\tau=\tau^{*}=$ const.

Remark 2: In the case of $G M L_{\mu}^{\infty}$, when functions $p(\tau)=\tau=\tau^{*}=$ const. $<R$ and


Fig. 9.
arguments $\psi=\psi_{0}=$ const. $\in[0,2 \pi]$ and $\theta \in[0,2 \pi h]$ in (1), then representation ( $6^{*}$ ) give a family of lines situated on the surface of a torus:

- If $\mu \in \mathbf{Z}$ (integer number) and $b=1$ in (1), then we have a closed curve, which is winding around the "little parts" (radial cross section is a disk, with radius $\tau^{*}$ ) of torus and $\mu$ - is the number of coils (examples when $\mu=15$ or $\mu=75$ are shown in fig. 9);
when $\mu=0$ we find a circle, which is situated on the surface of torus;
- If $\mu \in \mathbf{Q}$ (Rational number, $\mu=i / j$ ) and if $b=j$ in (1), then we have a closed curve with $i$ - coils around the "little parts" of the torus, which appear after $j$-circuits around the "big parts" of the torus (examples when $\mu=1 / 7,2 / 5$ or $\mu=5 / 2$ are shown in fig. 10);


Fig. 10.

If $\mu \in \mathbf{R} \backslash \mathbf{Q}$ (Irrational number) and $h=\infty$ in (1), then we have a non-closed curve, which makes infinite coils around the "little parts" of the torus after infinite circuits around the "big parts" of the torus, but this curve is not self crossing.

Case 5. Let $L_{R}$ be epicycloid, representation (3) has the following form

$$
X=(a+b) \rho \cos \theta-a \rho \cos \left[\frac{(a+b) \theta}{a}\right]
$$

$$
\begin{equation*}
\theta \in[0,2 \pi], \frac{b}{a} \in \mathbf{Z} \tag{8}
\end{equation*}
$$

$$
Y=(a+b) \rho \sin \theta-a \rho \sin \left[\frac{(a+b) \theta}{a}\right]
$$

$g(\theta) \equiv \theta$ in (4), $\mu=\infty$ and $P_{\infty}$ is a circle (see Case 4.); $\psi \in[0,2 \pi]$; and $\theta \in[0,2 \pi h], h=1$ in (1), $\mu \in \mathbf{Z}$. Then $\left(6^{*}\right)$ is an analytic representation of $G M L_{\mu}^{\infty}$. In particular, a graphical representation of $G M L_{0}^{\infty}$ is shown in fig. 11.


Fig. 11.

A different example of Toroidal surface is shown in fig. 12.


Fig. 12.

Case 6. Let $L_{R}$ be an ellipse and $P_{\infty}$ a circle. Assuming for $\left(2^{*}\right)$ the following form

$$
\begin{aligned}
& x=\tau(1+\varepsilon \sin \theta) \cos \psi \\
& z=\tau(1+\varepsilon \sin \theta) \sin \psi
\end{aligned} \quad \varepsilon=\text { const. }<1
$$



Fig. 13.
then equations (6) give the analytic representation of "Dupin's cyclide" (see fig. 13).

## I.3. "Helix bodies"

In this part of the article we consider set of "Helix bodies".

- "Basic line" $L_{\rho}$ is a closed line $\left(3^{*}\right)$ or in representation (3) the functions $f_{1}$ and $f_{2}$ satisfy the condition ii);
- "Law of vertical stretching of figure" $K(\theta) \neq 0$;
- The index $m$ in $P_{m}$ does not depend on $\mu$, see definitions (5);
- The number $b$ in (1) - number of coils of "Helix bodies" if $b \in \mathbf{N}$, (but in general $b$ is an arbitrary real number).


## General remarks

- If the function $K(\theta)>0(<0)$, then $(6)$ or $\left(6^{*}\right)$ define a set of "Helix bodies" which are situated in the upper (lower) parts of space $z>0(z<0)$;
- If the function $K(\theta)=K \cdot \theta \neq 0$, then (6) or $\left(6^{*}\right)$ define set of "Helix bodies" with constant step of vertical stretching. Number $K$ denotes the measure of this step;
- If $K(\theta)=K \cdot \theta \neq 0$, and number $|K|$ is greater than the diameter of the regular polygon $P_{m}$ divided by $2 \pi$ (in particular $|K|>\max (p(\theta)) / \pi$, - function $p(\theta)$ defined in $\left(2^{*}\right)$ ), then $(6)$ or $\left(6^{*}\right)$ define set of "Helix bodies", which have not self-crossing points;
- Functions in (2) or $\left(2^{*}\right)$ define the "shape of radial cross sections" of "Helix bodies" (for example when in $(2) p(\tau, \psi)=\tau$ and $q(\tau, \psi)=\sin \tau, L_{R}$ is a circle with radius $R>\tau^{*}$, function $K(\theta)=K \cdot \theta \neq 0, \mu=0, g(\theta) \equiv \theta, h=4$, then the graphical image of (6) is shown in fig. 14;


Fig. 14.

Case 7. Let $L_{R}$ be a circle with radius $R\left(\rho_{1}=\rho_{2}=R\right.$ in ( $\left.3^{*}\right)$ ), $g(\theta) \equiv \theta$ in (4) and $m=2, P_{2} \equiv A_{1} A_{2}$ is segment of straight line, then in parametric representation ( $2^{*}$ ) $p(\tau)=\tau, \tau \in\left[-\tau^{*}, \tau^{*}\right]$, where $\tau^{*}$ is some constant ( $2 \tau^{*}$ is the length of $A_{1} A_{2}$ ), $\psi=\psi_{0}=$ const., $\psi_{0} \in[0,2 \pi]$ and $\theta \in[0,2 \pi h], h \in \mathbf{R}$ in (1), $m=n / 2$ in (5) and function $K(\theta)=K \cdot \theta \neq 0$. Under these conditions equations ( $6^{*}$ ) define a set of "Straight Helicoids" - "Helix bodies with plane coils" (see fig. 15):

- "Straight Helicoid" - when $\mu=0$ in (5) and $R=\left|t^{*} \cos \psi\right|$ and if $\psi=\psi_{0}=$ $=$ const. $\neq \pi / 2$ or $3 \pi / 2$ ( $\psi_{0}$ - angle between the plane $O X Y$ and the "generatrix" line);
- "Straight Helicoid with Hole" - $\mu=0$ in (5) and $R>\left|\tau^{*} \cos \psi\right|$ and if $\psi=\psi_{0}=$ $=$ const. $\neq \pi / 2$ or $3 \pi / 2$;


Fig. 15.


Fig. 16.

- "Spring with plane coils" - for any $R>0, \mu=0$ in (5) and $K>\tau^{*} / \pi$ and argument $\psi=\pi / 2$ or $3 \pi / 2$ (axis $O Z$ and "generatrix" being parallel lines);
- Surface of Cylinder - for any $R>0, \mu=0$ in (5) and $K=\tau^{*} / \pi$ and argument $\psi=\pi / 2$ or $3 \pi / 2$;
- "Helix" line - if $\tau^{*}=0$.

CASE 8. Let $L_{R}$ be a rectangle and $f_{1}(\rho, \theta), f_{2}(\rho, \theta)$ in (3) have form (7), $g(\theta) \equiv \theta$ and $p(\tau, \psi), q(\tau, \psi)$ is the parametric representation of epicycloid, hypocycloid or ellipse, then (6) define a set of "Helix bodies" with rectangular "basic line" and different cross sections.

Some different examples of "Helix bodies" are shown in fig. 16 and 17.


Fig. 17.

## I.4. "Cochlea body"

In this part of the article we consider set of "Cochlea bodies", with following restrictions.

- The "basic line" $L_{\rho}$ is a spiral line ( $3^{*}$ ) or in representation (3) the functions $f_{1}$ and $f_{2}$ satisfy the condition $\mathbf{i}$ ) with the exception $\theta=0$;
- The index $m$ in $P_{m}$ does not depend on $\mu$, see definitions (5);
- The number $b$ - in (1) - denotes the number of coils of "Cochlea body" if $b \in \mathbf{N}$, but in general $b \in \mathbf{R}$ (Real number).
I.4.a. "Cochlea body" - with plane basic line ("Law of vertical stretch of figure" $K(\theta)=0)$.

Case 9. Let $L_{R}$ be a spiral $\left(\rho_{1}(\theta)=\rho_{2}(\theta)=R(1+\alpha \theta)\right.$ in ( $\left.\left.3^{*}\right)\right), g(\theta) \equiv \theta$ in (4) ("regular twisting") and $\mu=\infty, b$ in (1) - number of coils, then $\left(6^{*}\right)$ define the following set of figures (see fig. 18):


Fig. 18.

- "Spooling Bar" - when $|a|>\tau^{*} / \pi R$;
- "Spooling Bar with tangential Coils" - when $|a|=\tau^{*} / \pi R$;
- "Plane Cochlea body" - when $|a|<\tau^{*} / \pi R$;
"Spooling Bar" with "Radial cross section - rectangle" and with "regular twisting" is shown in fig. 19


Fig. 19.
I.4.b. "Cochlea body" - with space basic line ("Law of vertical stretch of figure" $K(\theta) \neq 0)$.

- $L_{R}$ and $K(\theta)$ are "Space Basic Line" of "Cochlea Body";
- The number $b$ in (1) - [ $h$ ] is a number of coils of "Cochlea Body";
- Functions in (2) or (2*) are "Shape of radial cross section" of "Cochlea Body";


Fig. 20.

Cochlea Bodies with Basic line - Rectangle


Fig. 21.

- The index $m$ in $P_{m}$ does not depend on $\mu$, see definitions (5);
- Function $g(\theta)$ in (4) - "Law of Twisting around Basic Line" of "Cochlea Body";

Case 10. Some examples of "Cochlea Bodies" are shown in fig. 20 and 21; Representation ( $6^{*}$ ) define, also, a line Winding around "Cochlea Bodies (see fig. 22), and Surfaces, which are Cross section of "Cochlea Bodies along the Basic Line (see fig. 23).

## Line winding around Cochlea body $-\mathbf{n} \cdot \theta=\mathbf{2 5} \cdot \theta$



Fig. 22.


Fig. 23.

## II. Dynamics

II.2. In this part of the article we give parametric representation of wide set of motion of geometric figures, surfaces, lines and point under following restrictions:

1) $O O^{\prime}$ - axis of symmetry (middle line) of the prism $P R_{m}$ transforms into the some line $L_{\rho}$ - "basic line";
2) Rotation of the end of the prism is semi-regular along the middle line $O O^{\prime}$.
3) Rotation of the end of the prism is semi-regular along the middle in time;
4) Rotation of the end of the cross section of prism around of basic line semi-regular in time - "Rotation in Orbit" ;

Under these conditions the analytic representation of this class given by the following formula

$$
\begin{aligned}
& X(\tau, \psi, \theta, t)=T_{1}(t)+ \\
& \quad+f_{1}([R+p(\tau, \psi, t) \cos (\mu g(\theta)+\xi \omega(t))-q(\tau, \psi, t) \sin (\mu g(\theta)+\xi \omega(t))], \theta+\zeta(t))
\end{aligned}
$$

$$
\begin{align*}
& Y(\tau, \psi, \theta, t)=T_{2}(t)+  \tag{9}\\
& \quad+f_{2}([R+p(\tau, \psi, t) \cos (\mu g(\theta)+\xi \omega(t))-q(\tau, \psi, t) \sin (\mu g(\theta)+\xi \omega(t))], \theta+\zeta(t))
\end{align*}
$$

$$
\begin{aligned}
& Z(\tau, \psi, \theta, t)=T_{3}(t)+ \\
& \quad+K(\theta)+p(\tau, \psi, t) \sin (\mu g(\theta)+\xi \omega(t))+q(\tau, \psi, t) \cos (\mu g(\theta)+\xi \omega(t))
\end{aligned}
$$

where:

- the arguments $(\tau, \psi, \theta, t)$ are defined by (1);
- $T_{i}(t)$ are arbitrary smooth functions for any $i=1,2,3$;
- the functions $f_{1}$ and $f_{2}$ - "Orbit" or "shape of plane basic line" - are defined by (3) or ( $3^{*}$ );
- the functions $p(\tau, \psi, t)$ and $q(\tau, \psi, t)$ - "shape of radial cross section" - are defined by (2);
- the function $g(\theta)$ - "rule of twisting around basic line" - is defined by (4);
- $\omega(t), \zeta(t), K(\theta, t)$ are arbitrary real smooth functions;
- $\mu, \xi$ - "number of rotation" - are defined by (5);
- $R$ - "radius" - is some fixed real number.

Also, in the previous notations the representation (9) may be rewritten in the following form

$$
\begin{array}{ll} 
& X(\tau, \psi, \theta, t)=T_{1}(t)+\left[\rho_{1}(\theta, t)+p(\tau, \theta, t) \cos (\psi+\mu g(\theta)+\xi \omega(t))\right] \cos (\theta+\zeta(t)) \\
\left(9^{*}\right) \quad & (\tau, \psi, \theta, t)=T_{2}(t)+\left[\rho_{2}(\theta, t)+p(\tau, \theta, t) \cos (\psi+\mu g(\theta)+\xi \omega(t))\right] \sin (\theta+\zeta(t))  \tag{*}\\
& Z(\tau, \psi, \theta, t)=T_{3}(t)+K(\theta, t)+p(\tau, \theta, t) \sin (\psi+\mu g(\theta)+\xi \omega(t)),
\end{array}
$$

where the functions $\rho_{1}(\theta, t)$ and $\rho_{2}(\theta, t)$ are defined by ( $3^{*}$ ); and the function $p(\tau, \theta, t)$ is defined by ( $2^{*}$ ) - including time $(t)$ and cross sections $(\theta)$;

General remarks

- Vector $T(t) \equiv\left(T_{1}(t), T_{2}(t), T_{3}(t)\right)$ - "trajectory of space displacement" of "Basic line" ("Orbit") of figure.
- If the arguments $t, \psi, \theta$ are fixed constants (i.e. functions are depending on time argument $t$ ) and vector $T(t)=0$, then equations (9) or ( $9^{*}$ ) define a line ("trajectory of displacement of point") situated on the boundary surfaces of the figure defined by equations (6) or ( $6^{*}$ ) (for example the lines in fig. 9,10 or 22 );
- If the arguments $\tau, \theta$ are fixed constants (functions are depending on arguments $t, \psi$ ) and vector $T(t)=0$, then representation (9) or (9*) define "trajectory of displacement of Radial Cross section" (2) or (2*) along of the Basic line of the figure defined by equations (6) or ( $6^{*}$ ).
- Function $K(\theta, t)$ - "Law of vertical stretch of figure" or "Law of vertical tension of figure".
a) If $K(\theta, t) \equiv$ const. and vector $T(t)=0$, then equations (9) or ( $9^{*}$ ) define a motion on a Toroid with plane Basic Line or a motion on a Generalized Möbius-Listing's Body (see Cases 1. - 5.);
b) If $\frac{\partial K(\theta, t)}{\partial t}=0, K(\theta, t)=K \cdot \theta \neq 0$ and vector $T(t)=0$, then equations (9) or (9*) define a motion on an "Helix body" with constant vertical steps (for example on the bodies shown in fig. 16, 17, 18 or 19);
c) If $\frac{\partial K(\theta, t)}{\partial t} \neq 0, T(t)=0$, then equations (9) or ( $9^{*}$ ) define a motion on an "Helix body" with vertical tension in time (examples in figures 20 or 21);
d) If $K(\theta, t)$ is a periodic function of time argument and vector $T(t)=0$, then equations (9) or (9*) define a "Springing figure" with respect to the $O Z$ axis (during the motion the vertical distances between coils of the figure are changing);
e) If $\frac{\partial K(\theta, t)}{\partial t}=0, K(\theta, t) \neq 0, T_{1}(t)=T_{2}(t)=0$ and $T_{3}(t)$ is aperiodic function, then equations (9) or (9*) define a motion on a "Jumped Helix body" with constant vertical step (vertical distances between coils of figure are constants, but there is a periodic parallel motion with respect to the $O Z$ axis);
$f$ ) Note that Motions "Springing figure" $d$ ) and "Jumped figure" $e$ ) are quite different (see previous points $d$ ) and $e$ )).
- "Orbit" or "shape of plane basic line" - $L_{\rho}$ functions are defined by (3) or ( $3^{*}$ ).
a) If $f_{1}$ and $f_{2}$ in (3) or $\rho_{1}(\theta, t)=\rho_{2}(\theta, t)=\rho(\theta, t)$ in ( $\left.3^{*}\right)$ are periodic functions of the argument $\theta(\rho(\theta, t)=\rho(\theta+2 \pi, t))$ and vector $T(t)=0$, then equations ( 9 ) or ( $9^{*}$ ) define a wide set of motions on Toroids or on "Helix bodies" ("shape of plane basic line" "Orbit" is a closed plane line) (see Cases 1. - 5.);
b) If $f_{1}$ and $f_{2}$ in (3) or $\rho_{1}(\theta, t)=\rho_{2}(\theta, t)=\rho(\theta, t)$ in $\left(3^{*}\right)$ are periodic functions of the argument $t$ (i.e. $\rho(\theta, t)=\rho(\theta, t+2 \pi)$ ) and vector $T(t)=0$, then equations (9) or ( $9^{*}$ ) define a wide set of motions on figures with "Pulsing Basic Line";
- "Rule of rotation in the orbit" around the $O Z$ axis, function $(\zeta(t)$ is an arbitrary real function).
a) If $\zeta(t)=$ const., then equations (9) or ( $9^{*}$ ) define a wide set of motions without rotation around the $O Z$ axis;
b) If $\zeta(t)$ is a periodic function, then equations (9) or ( $9^{*}$ ) define a wide set of motions with oscillation around the $O Z$ axis $(\zeta(t)$ is the "Law of Oscillation").
- "Rule of twisting around Orbit" ("Rule of twisting around Basic Line") $(\omega(t)$ is an arbitrary real function).
a) If $\omega(t)=$ const., then equations (9) or ( $9^{*}$ ) define a wide set of motions without twisting around the "Basic Line" of figure;
b) If $\omega(t)$ is a periodic function, then equations (9) or (9*) define a wide set of motions with "Rolling" (Periodical Twisting) around the Basic Line of figure;
- "Shape of radial cross section" functions $p(\tau, \psi, t)$ and $q(\tau, \psi, t)$ or $p(\tau, \theta, t)$ defined by (2) or (2*):
a) If functions in (2) or ( $2^{*}$ ) are equal to zero (for example $p_{1}(\tau, \theta, t)=p_{2}(\tau, \theta, t)=0$ ), then equations $(9)$ or $\left(9^{*}\right)$ define a wide set of motions on the "Orbit" (for example, in Case 1 - the "Orbit" is a Circle);
b) If these functions are depending only on $t$, then equations (9) or (9*) define a wide set of motions of point along the lines situated on the surface of specific figures (for example in surfaces: of "Generalised Mbius Listtings bodies" see fig. 2, "Helix body" see fig. 14 or "Straight Helicoids" see fig. 15) or in particular on the surface of $G M L_{n}^{2}$ (see fig. 2, 5 or 6);
c) If these functions in (2*) are equal to some constants, $p(\tau, \theta, t)=R$ and $K(q, t) \equiv$ const. and vector $T(t)=0$, then equations (9) or ( $9^{*}$ ) define a set of motions on "Toroidal Surfaces" (see Case 4);
d) If functions in (2*) and (3) does not depend on $t$ and $K(\theta, t) \equiv$ const. and vector $T(t)=0$, then equations ( $9^{*}$ ) define a set of motions on "Static Surfaces", which are defined by (6) or ( $6^{*}$ ) (see Section I);
$e)$ If functions in (2*) are periodic functions with respect to time argument $t$, then equations $\left(9^{*}\right)$ define a set of motions on bodies with periodically variable "Radial cross section - "Presstaltation";
$f)$ When $D(p, q)$ or $D(p)$ - diameter of Radial cross section is smaller then the radius of space Basic line, then equations (9) or ( $9^{*}$ ) define a set of motions on bodies without selfcrossing points.
- "Rule of Static Twisting around Basic Line" (function $g(\theta)$ is defined by (4)):
a) If $g(\theta)=0$ - then equations (9) or ( $9^{*}$ ) define a set of motions without twisting around the Basic Line;
b) If $g(\theta)=\theta$ - then equations (9) or ( $9^{*}$ ) define a set of motions on ta figures with "regular static (geometric) twisting" around the Basic Line.

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