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The Boundary Integral Method for Almost Incompressible Elastic Materials, a Revisit (***)

ABSTRACT. — Based on previous work by B. Reidinger and O. Steinbach [9, 10], we reconsider the relation between the solutions of the governing equations in linearised elasticity for the elastic and inelastic materials when the Poisson ratio tends to $\frac{1}{2}$. Whereas the boundary integral equations of linear elasticity model the material's elastic behaviour in a bounded region, the inelastic behaviour is modelled by the integral equations of the Stokes flow. The latter corresponds to the degenerate or reduced formulation of elasticity in terms of perturbation analysis. For the Dirichlet problem (the pure displacement problem), the perturbation turns out to be singular in the sense that in general the degenerate Stokes problem has no solution unless the boundary data satisfy appropriate compatibility conditions. In the latter case, the asymptotics corresponds to that of a light compressible Stokes flow and to the Cosserat spectrum at infinity as analyzed by R. Temam [11]. The Neumann problem (the pure traction problem), however, is a regular perturbation problem if the solvability conditions for the boundary tractions are satisfied. The mixed Dirichlet–Neumann problem may or may not be a singular perturbation problem depending on additional complementary conditions.

1. - THE RELATION BETWEEN THE LAMÉ AND THE STOKES SYSTEM

Let us consider the Lamé system

$$\mu\Delta\boldsymbol{v} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{v}) = \mathbf{0} \text{ in } \Omega \subset \mathbb{R}^3 \tag{1.1}$$

subject to

$$\text{Dirichlet:} \quad \boldsymbol{v}|_\Gamma = \boldsymbol{\varphi}\,, \text{or} \tag{1.2}$$

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(1.3) Neumann: $T\boldsymbol{v}|_\Gamma = \left(\lambda \operatorname{div}\boldsymbol{v} + 2\mu\dfrac{\partial \boldsymbol{v}}{\partial n} + \mu\boldsymbol{n} \times \operatorname{curl}\boldsymbol{v}\right)|_\Gamma = \boldsymbol{\psi}$, or

(1.4) mixed Dirichlet-Neumann: $\boldsymbol{v}|_\Gamma = \boldsymbol{\varphi}$ and $T\boldsymbol{v}|_{\Gamma_N} = \boldsymbol{\psi}$

boundary conditions. Ω is supposed to be a bounded strong Lipschitz domain whose boundary $\partial\Omega = \Gamma$ consists of L mutually separated closed Lipschitz surfaces Γ_j with $\Gamma = \bigcup_{j=1}^{L} \Gamma_j$ where Γ_1 denotes that component whose exterior domain is infinite.

We are interested in the elastic behaviour in Ω when the material becomes incompressible, i.e., The Poisson ratio υ tends to $\dfrac{1}{2}$,

$$\frac{1}{2} > \upsilon = \frac{1}{2}\frac{\lambda}{\lambda+\mu} \to \frac{1}{2}\,, \tag{1.5}$$

or $\lambda \to +\infty$.

Dividing (1.1) by μ and introducing the pressure

$$p := -\frac{1}{c}\operatorname{div}\boldsymbol{v}, c := 1 - 2\upsilon\,, \tag{1.6}$$

the Lamé system can be written in the mixed form

$$-\Delta\boldsymbol{v} + \nabla p = 0 \text{ and } \operatorname{div}\boldsymbol{v} = -cp \tag{1.7}$$

and if $c \to 0$ it degenerates to the Stokes system

$$-\Delta\boldsymbol{v}_0 + \nabla p_0 = \boldsymbol{0} \text{ and } \operatorname{div}\boldsymbol{v}_0 = 0 \tag{1.8}$$

which corresponds to the inelastic behaviour of a Bingham body [1, p. 14] or a slightly compressible flow [11, Chap. I, § 6]; and with $c^{-1} = \omega$ it also corresponds to the Cosserat spectrum near $\omega = +\infty$ [6, 11].

The solution's behaviour for $c \to 0$ can be understood by dealing with the corresponding elastic boundary potentials. As is well known, any solution to the Lamé equations can be represented in the form

$$\boldsymbol{v}(\boldsymbol{x}) = \int_\Gamma E_{e\ell}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\sigma}(\boldsymbol{y})ds_y - \int_\Gamma \left(T_y E_{e\ell}(\boldsymbol{x},\boldsymbol{y})\right)^\top \boldsymbol{\varphi}(\boldsymbol{y})ds_y \text{ for } \boldsymbol{x} \in \Omega \tag{1.9}$$

where the boundary charges are the boundary traction $\boldsymbol{\sigma} = T\boldsymbol{v}$ and $\boldsymbol{\varphi}$ the boundary displacement.

The fundamental tensor and elastic dipole kernel are given by

$$E_{e\ell}(\boldsymbol{x},\boldsymbol{y}) = \frac{\lambda+3\mu}{8\pi\mu(\lambda+2\mu)}\left\{\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\boldsymbol{I} + \frac{\lambda+\mu}{\lambda+3\mu}\,\frac{(\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})^\top}{|\boldsymbol{x}-\boldsymbol{y}|^2}\right\}, \tag{1.10}$$

$$\begin{aligned} T_y E_{e\ell}(\boldsymbol{x},\boldsymbol{y}) = \Bigg\{ &\left(\boldsymbol{I} + \frac{3(\lambda+\mu)}{\mu|\boldsymbol{x}-\boldsymbol{y}|^2}(\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})^\top\right)\left(\frac{\partial}{\partial n_y}\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right) \\ &+ \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^3}\left((\boldsymbol{x}-\boldsymbol{y})\boldsymbol{n}(\boldsymbol{y})^\top - \boldsymbol{n}(\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})^\top\right)\Bigg\}, \end{aligned} \tag{1.11}$$

respectively [7]. For $\lambda \to +\infty$, one obtains the fundamental tensor $E_{st}(\boldsymbol{x},\boldsymbol{y})$ and $K_{st}(\boldsymbol{x},\boldsymbol{y})$ of the Stokes system,

$$E_{st}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{8\pi\mu}\left\{\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\boldsymbol{I} + \frac{(\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})^\top}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right\}, \tag{1.12}$$

$$K_{st}(\boldsymbol{x},\boldsymbol{y}) = \frac{3}{4\pi\mu}\frac{(\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y})^\top}{|\boldsymbol{x}-\boldsymbol{y}|^2}\frac{\partial}{\partial n_y}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right), \tag{1.13}$$

respectively , and $\boldsymbol{v}_0$, the velocity can be represented by

$$\boldsymbol{v}_0(\boldsymbol{x}) = \int_\Gamma E_{st}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\sigma}(\boldsymbol{y})ds_y - \int_\Gamma K_{st}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})ds_y \tag{1.14}$$

with $\boldsymbol{\sigma}$ the hydrodynamic boundary traction and $\boldsymbol{\varphi}$ the boundary velocity, whereas the pressure field $\boldsymbol{p}_0(\boldsymbol{x})$ is given by the potentials

$$p_0(x) = \frac{1}{4\pi}\int_\Gamma\left(\nabla_y\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right)^\top\boldsymbol{\sigma}(\boldsymbol{y}) - \frac{\mu}{2\pi}\int_\Gamma\left\{\frac{\partial}{\partial n_y}\left(\nabla_y\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right)^\top\right\}\boldsymbol{\varphi}(\boldsymbol{y})ds_y\,. \tag{1.15}$$

For both systems, the solution of the corresponding boundary value problems is based on the boundary integral relations in the form of Calderon's projections obtained from taking the trace and the boundary traction of the representation formulae as

$$V_{e\ell}\boldsymbol{\sigma} = \left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi} \text{ and } \left(\frac{1}{2}I - K'_{e\ell}\right)\boldsymbol{\sigma} = D_{e\ell}\boldsymbol{\varphi} \text{ on } \Gamma\,, \tag{1.16}$$

$$V_{st}\boldsymbol{\sigma} = \left(\frac{1}{2}I + K_{st}\right)\boldsymbol{\varphi} \text{ and } \left(\frac{1}{2}I - K'_{st}\right)\boldsymbol{\sigma} = D_{st}\boldsymbol{\varphi} \text{ on } \Gamma\,, \tag{1.17}$$

respectively [3]. The boundary integral operators in (1.16) and (1.17) are well known and define continuous mappings on the indicated Sobolev spaces on the Lipschitz surface Γ with $|\varrho| \le \frac{1}{2}$ [2]:

$$K_{e\ell}\boldsymbol{\varphi}(x) = \int_{\Gamma\setminus\{x\}}\left(T_yE_{e\ell}(\boldsymbol{x},\boldsymbol{y})\right)^\top\boldsymbol{\varphi}(y)ds_y\,,\ H^{\frac{1}{2}+\varrho}(\Gamma) \to H^{\frac{1}{2}+\varrho}(\Gamma)\,, \tag{1.18}$$

$$K'_{e\ell}\boldsymbol{\sigma}(x) = \int_{\Gamma\setminus\{x\}}\left(T_xE_{e\ell}(\boldsymbol{x},\boldsymbol{y})\right)\boldsymbol{\sigma}(\boldsymbol{y})ds_y\,,\quad H^{-\frac{1}{2}+\varrho}(\Gamma) \to H^{-\frac{1}{2}+\varrho}(\Gamma)\,, \tag{1.19}$$

$$V_{e\ell}\boldsymbol{\sigma}(x) = \int_{\Gamma\setminus\{x\}}\left(E_{e\ell}(\boldsymbol{x},\boldsymbol{y})\right)\boldsymbol{\sigma}(\boldsymbol{y})ds_y\,,\qquad H^{-\frac{1}{2}+\varrho}(\Gamma) \to H^{\frac{1}{2}+\varrho}(\Gamma)\,, \tag{1.20}$$

$$(1.21)\qquad D_{e\ell}\boldsymbol{\varphi}(x) = -\frac{\mu}{4\pi}\big(\boldsymbol{n}(\boldsymbol{x})\times\nabla_x\big)^\top \int\limits_{\Gamma\setminus\{x\}} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\big(\boldsymbol{n}(\boldsymbol{y})\times\nabla_y\big)\boldsymbol{\varphi}(\boldsymbol{y})ds_y$$

$$-\mathcal{M}_x \int\limits_{\Gamma\setminus\{x\}} \left(4\mu^2 E_{e\ell}(\boldsymbol{x},\boldsymbol{y}) - \frac{\mu}{2\pi}\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\boldsymbol{I}\right)(\mathcal{M}_y\boldsymbol{\varphi})(y)ds_y$$

$$+\frac{\mu}{4\pi}\left(\sum_{\ell,k}^{3} m_{\ell,k}(\partial_x,\boldsymbol{x}) \int\limits_{\Gamma\setminus\{x\}} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} m_{k,j}(\partial_y,\boldsymbol{y})\varphi_\ell(\boldsymbol{y})ds_y\right)_{j=1,2,3},$$

$$H^{\frac{1}{2}+\varrho}(\Gamma)\to H^{-\frac{1}{2}+\varrho}(\Gamma)\,,$$

where

$$\mathcal{M}_y := \big((m_{j,k}(\partial_y,\boldsymbol{y}))\big) := \left(\left(n_k(\boldsymbol{y})\frac{\partial}{\partial y_j} - n_j(\boldsymbol{y})\frac{\partial}{\partial y_k}\right)\right)_{3\times 3}.$$

The corresponding operators of the Stokes system have the same mapping properties [5] and, in view of (1.10)–(1.13), are related to each other by

$$(1.22)\qquad V_{e\ell} = \frac{1}{1+c}V_{st} + \frac{2c}{1+c}\,\frac{1}{\mu}V_\Delta\,,$$

$$(1.23)\qquad K_{e\ell} = \frac{1}{1+c}K_{st} + \frac{c}{1+c}(K_\Delta + L_1)\,,$$

$$(1.24)\qquad K'_{e\ell} = \frac{1}{1+c}K'_{st} + \frac{c}{1+c}(K'_\Delta + L'_1)\,,$$

$$(1.25)\qquad D_{e\ell} = D_{st} + cL_2$$

where

$$(1.26)\qquad L_1\boldsymbol{\varphi} = \frac{1}{4\pi}\int\limits_{\Gamma\setminus\{x\}} \left(\frac{\boldsymbol{n}(y)\cdot\boldsymbol{\varphi}(\boldsymbol{y})(\boldsymbol{x}-\boldsymbol{y}) - (\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{\varphi}(\boldsymbol{y})\boldsymbol{n}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right)ds_y\,,$$

$$(1.27)\qquad L_2\boldsymbol{\varphi}(\boldsymbol{x}) = \mathcal{M}_x \int\limits_{\Gamma\setminus\{x\}} 4\mu^2\,\frac{1}{1+c}\left(E_{st}(\boldsymbol{x},\boldsymbol{y}) - \frac{1}{4\mu\pi}\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right)\mathcal{M}_y\boldsymbol{\varphi}(\boldsymbol{y})ds_y$$

and

$$(1.28)\qquad V_\Delta\boldsymbol{\sigma} = \int\limits_{\Gamma\setminus\{x\}} E_\Delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\sigma}(\boldsymbol{y})ds_y \text{ where } E_\Delta(\boldsymbol{x},\boldsymbol{y}) := \frac{1}{4\pi}\,\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,,$$

$$(1.29)\qquad K_\Delta\boldsymbol{\varphi} = \frac{1}{4\pi}\int\limits_{\Gamma\setminus\{x\}} \frac{\partial}{\partial n_y}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right)\boldsymbol{\varphi}(\boldsymbol{y})ds_y\,.$$

These integrals are either weakly singular or Tricomi–Mikhlin principal value integrals.

2. - The Dirichlet problem

Here, the boundary trace $\boldsymbol{\varphi} \in H^{\frac{1}{2}}(\Gamma)$ is given and the boundary traction $\boldsymbol{\sigma} \in H^{-\frac{1}{2}}(\Gamma)$ in (1.16) is unknown. So, we write (1.16) by using (1.20) in the form

$$V_{st}\boldsymbol{\sigma} = (1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi} - c\frac{2}{\mu}V_{\Delta}\boldsymbol{\sigma} \tag{2.1}$$

which becomes (1.16) for $c \to 0$. Whereas $V_{e\ell}$ is $H^{-\frac{1}{2}}(\Gamma)$- elliptic [2], the operator V_{st} has a non-trivial L-dimensional kernel (see also [9]), namely $\{\boldsymbol{n}_\ell = \boldsymbol{n}(\boldsymbol{x})\delta_{\ell k}$ for $\boldsymbol{x} \in \Gamma_k\,,\ \ell, k = 1, \ldots, L\}$ where $\boldsymbol{n}(\boldsymbol{x})$ is the unit normal vector field along $\bigcup_{\ell=1}^{L} \Gamma_\ell$ directed into the exterior of Ω,

$$\text{Ker } V_{st} = \text{Ker } \left(\frac{1}{2}I - K'_{st}\right) = \left\{\sum_{\ell=1}^{L} a_\ell \boldsymbol{n}_\ell \,|\, a_\ell \in \mathbb{R}\right\}. \tag{2.2}$$

Lemma 2.1: *Any eigenfunction* $\boldsymbol{\tau}_0 := \sum_{\ell=1}^{L} \gamma_\ell \boldsymbol{n}_\ell$ *of* V_{st} *on* Γ *generates a solution*

$$\boldsymbol{v}_0 = \boldsymbol{0} = \sum_{\ell=1}^{L} \gamma_\ell V_{st}\boldsymbol{n}_\ell\,,\ p_0 = \gamma_1$$

of the Stokes system (1.8) *in* Ω.

Proof: Since the pressure field at $\boldsymbol{x} \in \Omega$ for the simple layer potential $\boldsymbol{v}_0 = \sum_{\ell=1}^{L} \gamma_\ell V_{st}\boldsymbol{n}_\ell$ is given by

$$\begin{aligned} p_0 &= \sum_{\ell=1}^{L} \gamma_\ell \int_{\Gamma} \frac{1}{4\pi}\left(\nabla_y \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right)^{\top} \boldsymbol{n}_\ell(\boldsymbol{y})ds_y \\ &= \sum_{\ell=2}^{N} \gamma_\ell \int_{\Gamma_\ell} \left(\frac{\partial}{\partial n_y} E_\Delta(\boldsymbol{x},\boldsymbol{y})\right)^{\top} ds_y + \gamma_1 \int_{\Gamma_1} \left(\frac{\partial}{\partial n_y}\right)^{\top} E_\Delta(\boldsymbol{x},\boldsymbol{y})ds_y \\ &= 0 + \gamma_1\,, \end{aligned}$$

since $\boldsymbol{x} \in \Omega$ is exterior to Γ_ℓ for $\ell = 2, \ldots, N$ but interior for Γ_1.

With the constant pressure field we have $\Delta\boldsymbol{v}_0 = \boldsymbol{0}$ in Ω with $\boldsymbol{v}_0|_\Gamma = \boldsymbol{v}_0|_{\partial\Omega} = \boldsymbol{0}$; hence, $\boldsymbol{v}_0(\boldsymbol{x}) = \boldsymbol{0}$ for all $\boldsymbol{x} \in \overline{\Omega}$ which completes the proof.

On the other hand, $\boldsymbol{\sigma} \in H^{-\frac{1}{2}}(\Gamma)$ is uniquely determined by (1.16) since $(V_{e\ell})^{-1} : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ exists. Therefore the right-hand side of (2.1) is orthogonal to Ker V_{st}, i.e.,

$$\left\langle \left((1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\right)\boldsymbol{\varphi} - c\frac{2}{\mu}V_{\Delta}\boldsymbol{\sigma}, \boldsymbol{n}_\ell \right\rangle = 0 \ \text{ for } \ \ell = 1, \ldots, L\,. \tag{2.3}$$

Accordingly, we split $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \sum_{\ell=1}^{L} a_\ell \boldsymbol{n}_\ell \ \text{ where } \langle \boldsymbol{\sigma}_0, V_\Delta \boldsymbol{n}_k \rangle = 0 \ \text{ for } \ k = 1, \ldots, L\,. \tag{2.4}$$

Hence, (2.3) gives with the positive definite symmetric matrix

$$\underset{\sim}{\boldsymbol{\beta}}_\Delta := (\!(\langle V_\Delta \boldsymbol{n}_\ell, \boldsymbol{n}_k \rangle)\!)_{L\times L} = (\!(\beta_{\ell k})\!)_{L\times L} \tag{2.5}$$

the relations

$$a_\ell = \frac{\mu}{2c} \sum_{k=1}^{L} (\underset{\sim}{\boldsymbol{\beta}}_\Delta^{-1})_{\ell k} \left\langle (1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi}, \boldsymbol{n}_k \right\rangle, \quad \ell = 1, \ldots, L\,. \tag{2.6}$$

With (1.23), $(1+c)\left(\frac{1}{2}I + K_{e\ell}\right) = \frac{1}{2}cI + \left(\frac{1}{2}I + K_{st}\right) + c(K_\Delta + L_1)$ and

$$\left\langle (1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi}, \boldsymbol{n}_k \right\rangle = \langle \boldsymbol{\varphi}, \boldsymbol{n}_k \rangle + c\left\langle \left(\frac{1}{2}I + K_\Delta + L_1\right)\boldsymbol{\varphi}, \boldsymbol{n}_k \right\rangle$$

we find

$$a_\ell = \frac{\mu}{2c} \sum_{k=1}^{L} (\underset{\sim}{\boldsymbol{\beta}}_\Delta^{-1})_{\ell k} \langle \boldsymbol{\varphi}, \boldsymbol{n}_k \rangle + \frac{\mu}{2} \sum_{k=1}^{L} (\underset{\sim}{\boldsymbol{\beta}}_\Delta^{-1})_{\ell k} \left\langle \left(\frac{1}{2}I + K_\Delta + L_1\right)\boldsymbol{\varphi}, \boldsymbol{n}_k \right\rangle \tag{2.7}$$

So, Equation (2.1) becomes

$$V_{st}\boldsymbol{\sigma}_0 + c\frac{2}{\mu} V_\Delta \boldsymbol{\sigma}_0 = (1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi} - \sum_{k,\ell=1}^{L} (\underset{\sim}{\boldsymbol{\beta}}_\Delta^{-1})_{\ell k} \left\langle (1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi}, \boldsymbol{n}_k \right\rangle \boldsymbol{n}_\ell$$

which is uniquely solvable for $\boldsymbol{\sigma}_0$ satisfying (2.4), and which can be stabilized [9, 10] as

$$\begin{aligned} V_{st}\boldsymbol{\sigma}_0 + \sum_{j=1}^{L} \langle \boldsymbol{\sigma}_0, V_\Delta \boldsymbol{n}_j \rangle V_\Delta \boldsymbol{n}_j + c\frac{2}{\mu} V_\Delta \boldsymbol{\sigma}_0 &= (1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi} \\ &\quad - \sum_{k,\ell=1}^{L} (\underset{\sim}{\boldsymbol{\beta}}_\Delta^{-1})_{k\ell} \left\langle (1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi}, \boldsymbol{n}_k \right\rangle V_\Delta \boldsymbol{n}_\ell\,. \end{aligned}$$

For $0 \le c$ sufficiently small, this equation can be extended to the **whole space** $H^{-\frac{1}{2}}(\Gamma)$,

$$\begin{aligned} V_{st}\boldsymbol{\sigma}_0 + \sum_{j=1}^{L} \langle \boldsymbol{\sigma}_0, V_\Delta \boldsymbol{n}_j \rangle V_\Delta \boldsymbol{n}_j + \frac{2c}{\mu} V_\Delta \boldsymbol{\sigma}_0 &= (1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi} \\ &\quad - \sum_{k,\ell=1}^{L} (\underset{\sim}{\boldsymbol{\beta}}_\Delta^{-1})_{k\ell} \left\langle (1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi}, \boldsymbol{n}_k \right\rangle V_\Delta \boldsymbol{n}_\ell\,, \end{aligned} \tag{2.8}$$

and is still uniquely solvable there. To see this multiply with $\boldsymbol{n}_m$, integrate over Γ and

insert (2.4). Then one obtains for $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \sum_{\ell=1}^{L} a'_\ell \boldsymbol{n}_\ell$ the equation

$$\sum_{j=1}^{L} \gamma_j \beta_{jm} + \frac{2c}{\mu}\gamma_m = 0 \text{ with } \gamma_m = \sum_{\ell=1}^{L} a'_\ell \beta_{\ell m}$$

which implies $a'_\ell = 0$ for $\ell = 1, \dots, L$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0$.

The boundary traction $\boldsymbol{\sigma}$ is then given by

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \sum_{\ell=1}^{L} a_\ell \boldsymbol{n}_\ell = \boldsymbol{\sigma}_0 + \frac{\mu}{2c}\sum_{\ell=1}^{L}\sum_{k=1}^{L} (\underset{\sim}{\boldsymbol{\beta}}_\Delta^{-1})_{\ell k}\langle \boldsymbol{\varphi}, \boldsymbol{n}_k\rangle \boldsymbol{n}_\ell + \frac{\mu}{2}(\underset{\sim}{\boldsymbol{\beta}}_\Delta^{-1})_{\ell k}\left\langle \left(\frac{1}{2}I + K_\Delta + L_1\right)\boldsymbol{\varphi}, \boldsymbol{n}_k\right\rangle \boldsymbol{n}_\ell \tag{2.9}$$

where $\boldsymbol{\sigma}_0$ is the solution of (2.8).

Collecting these results we obtain the following theorem.

THEOREM 2.2: *For $c \geq 0$, the elastic field of the Dirichlet problem* (1.1), (1.2) *has the form*

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{x}) = {} & \frac{1}{1+c}\left\{\int_\Gamma E_{st}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\sigma}_0(\boldsymbol{y})ds_y - \int_\Gamma K_{st}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})ds_y\right\} \\ & + \frac{1}{1+c}\sum_{\ell,k=1}^{L} (\underset{\sim}{\boldsymbol{\beta}}_\Delta^{-1})_{\ell k}\langle \boldsymbol{\varphi}, \boldsymbol{n}_\ell\rangle V_\Delta \boldsymbol{n}_k(\boldsymbol{x}) \\ & + \frac{c}{1+c}\left\{\frac{2}{\mu}V_\Delta \boldsymbol{\sigma}_0(\boldsymbol{x}) - \int_\Gamma \big(K_\Delta(\boldsymbol{x},\boldsymbol{y}) + L_1(\boldsymbol{x},\boldsymbol{y})\big)\boldsymbol{\varphi}(\boldsymbol{y})ds_y \right. \\ & \left. + \sum_{\ell,k=1}^{L} (\underset{\sim}{\boldsymbol{\beta}}_\Delta^{-1})_{\ell k}\left\langle \left(\frac{1}{2}I + K_\Delta + L_1\right)\boldsymbol{\varphi}, \boldsymbol{n}_\ell\right\rangle V_\Delta \boldsymbol{n}_k(\boldsymbol{x})\right\} \\ = {} & \frac{1}{1+c}\boldsymbol{u}_{st} + \frac{1}{1+c}\sum_{\ell,k=1}^{L} (\underset{\sim}{\boldsymbol{\beta}}_\Delta^{-1})_{\ell k}\langle \boldsymbol{\varphi}, \boldsymbol{n}_\ell\rangle\{(I + K_{st})V_\Delta \boldsymbol{n}_k\}(\boldsymbol{x}) + \frac{c}{1+c}\widetilde{\boldsymbol{u}}(c;\boldsymbol{x}). \end{aligned} \tag{2.10}$$

The function $\boldsymbol{\sigma}_0(\boldsymbol{x})$ is the unique solution of (2.8). *The constants a_j can then be explicitly computed from* (2.7). *For $c > 0$, the function $\boldsymbol{u}_{st}$ is the solution of the Stokes system to the projected Dirichlet data*

$$\boldsymbol{\varphi}_0 := \boldsymbol{\varphi} - \sum_{k,\ell=1}^{L} (\underset{\sim}{\boldsymbol{\beta}}_\Delta^{-1})_{k\ell}\langle \boldsymbol{\varphi}, \boldsymbol{n}_k\rangle V_\Delta \boldsymbol{n}_\ell .$$

The boundary traction is given by

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{\sigma}_0(\boldsymbol{x}) + \frac{1}{c}\,\frac{\mu}{2}\sum_{k,\ell=1}^{L} (\underset{\sim}{\boldsymbol{\beta}}_\Delta^{-1})_{k\ell}\langle \boldsymbol{\varphi}, \boldsymbol{n}_k\rangle \boldsymbol{n}_\ell(\boldsymbol{x}) + \sum_{\ell,k=1}^{L} (\underset{\sim}{\boldsymbol{\beta}}_\Delta^{-1})_{\ell k}\frac{\mu}{2}\left\langle \left(\frac{1}{2}I + K_\Delta + L_1\right)\boldsymbol{\varphi}, \boldsymbol{n}_k\right\rangle \boldsymbol{n}_\ell(\boldsymbol{x}). \tag{2.11}$$

Note that in (2.9), due to Lemma 2.1, the contributions from $\boldsymbol{n}_\ell$ for $\ell = 2, \ldots, L$ will be mapped to zero.

COROLLARY 2.3: *In general, the given boundary displacements will also depend on the Poisson ratio, i.e., on c. So let us assume that*

$$\langle \boldsymbol{\varphi}, \boldsymbol{n}_k \rangle = a_k + c b_k + \mathcal{O}(c^2) \tag{2.12}$$

as $0 \le c \to 0$. *Then* (2.10) *becomes*

$$\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{\sigma}_0(\boldsymbol{x}) &+ \frac{\mu}{2c} \sum_{\ell=1}^{L} \sum_{k=1}^{L} (\underset{\sim}{\beta}_\Delta^{-1})_{\ell k} a_k \boldsymbol{n}_\ell(\boldsymbol{x}) \\ &+ \frac{\mu}{2} \sum_{\ell=1}^{L} \sum_{k=1}^{L} (\underset{\sim}{\beta}_\Delta^{-1})_{\ell k} \left\{ \left\langle \left(\frac{1}{2} I + K_\Delta + L_1 \right) \boldsymbol{\varphi}, \boldsymbol{n}_k \right\rangle b_k \right\} \boldsymbol{n}_\ell(\boldsymbol{x}) + \mathcal{O}(c) \end{aligned} \tag{2.13}$$

as $0 < c \to 0$.

REMARK: As we can see from (2.9), for $c \to 0$, i.e. $\upsilon \to \frac{1}{2}$, the elastic displacement field $\boldsymbol{v}(\boldsymbol{x})$ does **not** tend to the Stokes solution $\boldsymbol{u}_{st}$ if $a_k \neq 0$ for some $k = 1, \ldots, L$. In this case, the second expression in (2.9) will remain, which defines a boundary layer all over Ω. Correspondingly, the boundary stress $\boldsymbol{\sigma}(\boldsymbol{x})$ blows up for $c \to 0$ and the relation

$$\frac{1}{c} \int_\Gamma \boldsymbol{\varphi} \cdot \boldsymbol{n} ds = \sum_{\ell=1}^{L} \langle \boldsymbol{\varphi}, \boldsymbol{n}_\ell \rangle \frac{1}{c} = -\int_\Omega p dx \tag{2.14}$$

shows that the total pressure in Ω will then become infinite.

Hence, the elastic Dirichlet problem is a **singular perturbation problem**. The elastic displacement field will converge to the Stokes field if and only if $\int_\Gamma \boldsymbol{\varphi} \cdot \boldsymbol{n} ds = \mathcal{O}(c)$ for $c \to 0$; otherwise a remaining stress field will arise.

Even if $a_k = 0$ but $b_k \neq 0$, the boundary traction will not converge to the hydrodynamic boundary traction of the Stokes flow and a residual stress generated by the last term in (2.10b) will remain. Moreover, since the right-hand side in (2.9) depends analytically on c, the displacement fields $\boldsymbol{v}(\boldsymbol{x})$ as well as the boundary traction $\boldsymbol{\sigma}(\boldsymbol{x})$ can be expressed in terms of asymptotic expansions in powers of c and we recover Temam's result [11, Chap. I § 6] for given Dirichlet data on Γ and no volume forces.

Of course, the boundary integral equation (2.8) can also be formulated in terms of a variational problem, i.e., to find $\boldsymbol{\sigma}_0 \in H^{-\frac{1}{2}}(\Gamma)$ as the solution of

$$\begin{aligned} a_D(\boldsymbol{\sigma}_0, \boldsymbol{\chi}) := \langle V_{st} \boldsymbol{\sigma}_0, \boldsymbol{\chi} \rangle &+ \sum_{\ell=1}^{L} \langle \boldsymbol{\sigma}_0, V_\Delta \boldsymbol{n}_\ell \rangle \langle V_\Delta \boldsymbol{n}_\ell, \boldsymbol{\chi} \rangle \\ &+ c \frac{2}{\mu} \langle V_\Delta \boldsymbol{\sigma}_0, \boldsymbol{\chi} \rangle = \langle \boldsymbol{\Phi}, \boldsymbol{\chi} \rangle \text{ for all } \boldsymbol{\chi} \in H^{-\frac{1}{2}}(\Gamma) \end{aligned} \tag{2.15}$$

where

$$\boldsymbol{\Phi} = (1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi} - \sum_{k,\ell=1}^{L}\left\langle (1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi}, \boldsymbol{n}_k\right\rangle V_\Delta \boldsymbol{n}_\ell \tag{2.16}$$

and where $a_D(\boldsymbol{\sigma}, \boldsymbol{\chi})$ is $H^{-\frac{1}{2}}(\Gamma)$-elliptic:

$$a_D(\boldsymbol{\sigma}, \boldsymbol{\sigma}) \geq \gamma_0 \|\boldsymbol{\sigma}\|^2_{H^{-\frac{1}{2}}(\Gamma)} \tag{2.17}$$

with $\gamma_0 > 0$ for $0 \leq c$.

Regularity: In view of the coerciveness property (2.14) and the mapping properties of K'_{st}, K'_Δ and L'_1; K_{st}, K_Δ and L_1 as in (1.18)-(1.21), respectively, we have for $\boldsymbol{\varphi} \in H^{\frac{1}{2}}(\Gamma)$ that $\boldsymbol{\sigma} \in H^{-\frac{1}{2}}(\Gamma)$ and $\boldsymbol{v} \in H^1(\Omega)$. If $\boldsymbol{\varphi}$ is given as $\boldsymbol{\varphi} \in H^{\frac{1}{2}+\varrho}(\Gamma)$ with $0 < \varrho < \frac{1}{2}$ on the Lipschitz boundary Γ then one obtains with the arguments as in [2] that $\boldsymbol{\sigma} \in H^{-\frac{1}{2}+\varrho}(\Gamma)$ and $\boldsymbol{v} \in H^{1+\varrho}(\Omega)$. Moreover, all the terms in the asymptotic expansions belong to the corresponding spaces.

3. - The Neuman problem

Now let us consider the problem (1.1), (1.3) where the boundary traction

$$T\boldsymbol{v}|_\Gamma = \boldsymbol{\psi} \in H^{-\frac{1}{2}}(\Gamma) \tag{3.1}$$

is given satisfying the compatibility conditions

$$\langle \boldsymbol{\omega}, \boldsymbol{\psi}\rangle = 0 \text{ for all rigid motions } \quad \boldsymbol{\omega} = \sum_{j=1}^{6} \omega_j \boldsymbol{m}_j(\boldsymbol{x}) \text{ with } \omega_j \in \mathbb{R} \tag{3.2}$$

and $\boldsymbol{m}_j(\boldsymbol{x})$ is the j-th column vector of

$$\begin{pmatrix} 1 & 0 & 0 & 0 & x_3 & -x_2 \\ 0 & 1 & 0 & x_2 & 0 & x_1 \\ 0 & 0 & 1 & -x_3 & -x_1 & 0 \end{pmatrix}. \tag{3.3}$$

Now the boundary traction $\boldsymbol{\sigma} = \boldsymbol{\psi} \in H^{-\frac{1}{2}}(\Gamma)$ is unknown and with (1.23) and (1.25), the equation (1.16) can be written as

$$D_{st}\boldsymbol{\varphi} + cL_2\boldsymbol{\varphi} = \left(\frac{1}{2}I - K'_{e\ell}\right)\boldsymbol{\psi}. \tag{3.4}$$

Here the $6L$-dimensional kernels of the hypersingular operators of $D_{e\ell}$, the Stokes- and the Lamé system, coincide and are given by

$$\text{Ker } D_{e\ell}|_{\Gamma_\ell} = \text{Ker } D_{st}|_{\Gamma_\ell} = \mathcal{L}\big\{(\boldsymbol{m}_{j\ell} = \boldsymbol{m}_j(\boldsymbol{x})\delta_{\ell k} \text{ for } \boldsymbol{x} \in \Gamma_k, . \\ \ell, k = 1, \ldots, L; j = 1, \ldots, 6\big\}, \tag{3.5}$$

where $\delta_{\ell k}$ denotes the Kronecker symbol.

Since (1.23) holds for any $c > 0$ we conclude that also

$$L_2 \boldsymbol{m}_{j\ell} = 0 \text{ for all } \ell = 1, \dots, L \text{ and } j = 1, \dots, 6\,. \tag{3.6}$$

Since the solution $\boldsymbol{\varphi}$ of the Lamé–Neumann problem exists and, hence, also solves (3.4) for $c > 0$, and $D_{st} + cL_2$ is selfadjoint, the right-hand side $\left(\frac{1}{2}I - K'_{e\ell}\right)\boldsymbol{\psi}$ satisfies the $6L$ orthogonality conditions

$$\left\langle \left(\frac{1}{2}I - K'_{e\ell}\right)\boldsymbol{\psi}, \boldsymbol{m}_{j\ell} \right\rangle = 0 \text{ for } \ell = 1, \dots, L\,, j = 1, \dots, 6\,. \tag{3.7}$$

LEMMA 3.1: *Any of the eigenfunctions $\boldsymbol{m}_{j\ell}$ on Γ generates a solution $\boldsymbol{u}_{0j\ell}$ of the Stokes system with the properties*

$$\boldsymbol{u}_{0j\ell}(\boldsymbol{x}) := -\int_{\Gamma_\ell} K_{st}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{m}_j(\boldsymbol{y})ds_y = \begin{cases} \boldsymbol{0} & \text{for } \boldsymbol{x} \in \overline{\Omega} \text{ if } \ell = 2, \dots, L\,, \\ \boldsymbol{m}_j(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \Omega \text{ if } \ell = 1\,,\ j = 1, \dots, 6\,; \end{cases} \tag{3.8}$$

$$p_{0\ell j}(\boldsymbol{x}) = \begin{cases} 0 \text{ for } \boldsymbol{x} \in \overline{\Omega} & \text{if } \ell = 2, \dots, L\,, \\ \operatorname{div}_x 2\mu \int_{\Gamma_1} K_\Delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{m}_j(\boldsymbol{y})ds_y & \text{if } \ell = 1\,,\ j = 1, \dots, 6\,. \end{cases} \tag{3.9}$$

PROOF: The hydrodynamic traction of $\boldsymbol{u}_{0j\ell}$ satisfies

$$T\boldsymbol{u}_{0j\ell}|_{\Gamma_\ell} = D_{st}|_{\Gamma_\ell}\boldsymbol{m}_j = \boldsymbol{0} \text{ on } \Gamma_\ell\,.$$

If $\ell = 2, \dots, L$ then $\boldsymbol{x} \in \Omega$ lies in the exterior of Ω_ℓ with the closed boundary surface Γ_ℓ. There, in $\mathbb{R}^3 \setminus \overline{\Omega}_\ell$ the pair $(\boldsymbol{u}_{0j\ell}, p_{0j\ell})$ is a solution of the homogeneous Neumann problem for the homogeneous Stokes system and $\boldsymbol{u}_{0j\ell} = \mathcal{O}(|\boldsymbol{x}|^{-2})$, $p_{0j\ell} = \mathcal{O}(|\boldsymbol{x}|^{-3})$ for $|\boldsymbol{x}| \to \infty$. Hence, $\boldsymbol{u}_{0j\ell} = \boldsymbol{0}$ and $p_{0j\ell} = 0$ for all $\boldsymbol{x} \in \mathbb{R}^3 \setminus \Omega_\ell$ (see [4, p. 16]).

For $j = 1$, $\boldsymbol{u}_{0j\ell}(\boldsymbol{x})$ is the solution of the homogeneous Neumann problem in the interior domain $\Omega_1 = \Omega \bigcup_{\ell=2}^{N} \overline{\Omega}_\ell$ and, therefore,

$$\boldsymbol{u}_{0j\ell}(\boldsymbol{x}) = \sum_{k=1}^{6} \varrho_k \boldsymbol{m}_k(\boldsymbol{x}) = -\int_{\Gamma_1} K_{st}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{m}_j(\boldsymbol{y})ds_y\,, \quad \varrho_k \in \mathbb{R}\,.$$

If $\boldsymbol{x} \to \Gamma_1$ we get with the jump relation

$$\sum_{k=1}^{6} \varrho_k \boldsymbol{m}_k(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{m}_j(\boldsymbol{x}) - K_{st}\boldsymbol{m}_j(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \Gamma_1\,.$$

Now we use the relation $\boldsymbol{m}_j = V_{st}|_{\Gamma_1}\boldsymbol{\tau}_j$ on Γ_1 with $\{\boldsymbol{\tau}_j\}_{j=1}^6$ a basis of the 6-dimensional eigenspace to $\frac{1}{2}I + K'_{st}$ on Γ_1 (see [3, Theorem 2.3.2]), together with $K_{st}V_{st} = V_{st}K'_{st}$ due

to the Calderon projection, and obtain

$$\sum_{k=1}^{6} \varrho_k \boldsymbol{m}_k(\boldsymbol{x}) = \boldsymbol{m}_j(\boldsymbol{x}) - \left(\frac{1}{2}I + K_{st}\right) V_{st}\boldsymbol{\tau}_j$$
$$= \boldsymbol{m}_j(\boldsymbol{x}) - V_{st}\left(\frac{1}{2}I + K'_{st}\right)\boldsymbol{\tau}_j = \boldsymbol{m}_j(\boldsymbol{x}),$$

with $\left(\frac{1}{2}I + K'_{st}\right)\boldsymbol{\tau}_j = \mathbf{0}$. Hence, $\varrho_k = \delta_{jk}$. This completes the proof of Lemma 3.1. □

Again, we split

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 + \sum_{\ell=1}^{L} \sum_{j=1}^{6} \omega_{j\ell} \boldsymbol{m}_{j\ell} \tag{3.10}$$

where $\boldsymbol{\varphi}_0$ satisfies the $6L$ orthogonality conditions

$$\langle \boldsymbol{\varphi}_0, \boldsymbol{m}_{j\ell} \rangle = 0 \text{ for } \ell = 1, \ldots, L,\ j = 1, \ldots, 6; \tag{3.11}$$

and write equation (3.4) as an equation on the **whole space** $H^{\frac{1}{2}}(\Gamma)$ in stabilized form

$$D_{st}\boldsymbol{\varphi}_0 + \sum_{\ell=1}^{L} \sum_{j=1}^{6} \langle \boldsymbol{m}_{j\ell}, \boldsymbol{\varphi}_0 \rangle \boldsymbol{m}_{j\ell} + cL_2\boldsymbol{\varphi}_0 = \left(\frac{1}{2}I - K'_{e\ell}\right)\boldsymbol{\psi}. \tag{3.12}$$

This equation is uniquely solvable and, because of (3.7), has the unique solution $\boldsymbol{\varphi}_0$. Collecting these properties and invoking Lemma 3.1 we have the following theorem.

THEOREM 3.2: *For $c \geq 0$ the elastic field of the Neumann problem* (1.1), (1.3) *with* (3.2) *has the form*

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{x}) &= \frac{1}{1+c}\left\{\int_\Gamma E_{st}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\psi}(\boldsymbol{y})ds_y - \int_\Gamma K_{st}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\varphi}_0(\boldsymbol{y})ds_y\right\} \\ &\quad + \frac{c}{1+c}\left\{\frac{2}{\mu}\int_\Gamma E_\Delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\psi}(\boldsymbol{y})ds_y - \int_\Gamma \Big(K_\Delta(\boldsymbol{x},\boldsymbol{y}) + L_1(\boldsymbol{x},\boldsymbol{y})\Big)\boldsymbol{\varphi}_0(\boldsymbol{y})ds_y\right\} \\ &\quad + \sum_{j=1}^{6} a_j \boldsymbol{m}_j(\boldsymbol{x}) \text{ for } \boldsymbol{x} \in \Omega \\ &= \frac{1}{1+c}\boldsymbol{u}_{st}(\boldsymbol{x}) + \sum_{j=1}^{6} a_j \boldsymbol{m}_j(\boldsymbol{x}) + \frac{c}{1+c}\,\widetilde{\boldsymbol{u}}(\boldsymbol{x}) \end{aligned} \tag{3.13}$$

where $a_\ell \in \mathbb{R}$ are arbitrary constants and $\boldsymbol{\varphi}_0(\boldsymbol{x})$ is the unique solution of the stabilized equation (3.12).

Note that the contributions $\omega_{j\ell}\boldsymbol{m}_{j\ell}$ for $\ell = 2, \ldots, L$ will be mapped to zero in (3.13) due to Lemma 3.1.

REMARK 3.3: As we can see from Theorem 3.2, here the elastic field $\boldsymbol{v}(\boldsymbol{x})$ tends for $c \to 0$ to the Stokes field. In the contrary to the Dirichlet problem, the traction problem is a regular perturbation problem for $c \to 0$ to the Stokes traction problem. Here, since the operator in (3.12) and also the right-hand side in (3.13) depend analytically on c, the displacement fields $\boldsymbol{\varphi}_0$ on Γ and $\boldsymbol{v}$ in Ω can be expressed in the form of regular asymptotic equations in powers of c.

Again, the boundary integral equation (3.12) can be formulated as a variational problem, i.e., to find $\boldsymbol{\varphi}_0 \in H^{\frac{1}{2}}(\Gamma)$ as the solution of

$$(3.14) \qquad a_N(\boldsymbol{\varphi}_0, \boldsymbol{\lambda}) = \langle D_{st}\boldsymbol{\varphi}_0, \boldsymbol{\lambda}\rangle + \sum_{\ell=1}^{L} \sum_{j=1}^{6} \langle \boldsymbol{m}_{j\ell}, \boldsymbol{\varphi}_0\rangle \langle \boldsymbol{m}_{j\ell}, \boldsymbol{\lambda}\rangle$$
$$+ c\langle L_2\boldsymbol{\varphi}_0, \boldsymbol{\lambda}\rangle = \left\langle \left(\frac{1}{2}I - K'_{e\ell}\right)\boldsymbol{\psi}, \boldsymbol{\lambda}\right\rangle \text{ for all } \boldsymbol{\lambda} \in H^{\frac{1}{2}}(\Gamma)\,.$$

The bilinear form a_N is $H^{\frac{1}{2}}(\Gamma)$-elliptic, i.e., there holds

$$(3.15) \qquad a_N(\boldsymbol{\lambda}, \boldsymbol{\lambda}) \geq \gamma_1 \|\boldsymbol{\lambda}\|^2_{H^{\frac{1}{2}}(\Gamma)} \text{ for all } \boldsymbol{\lambda} \in H^{\frac{1}{2}}(\Gamma)$$

with some constant $\gamma_1 > 0$.

Note that integration by parts reduces $\langle D_{st}\varphi_0, \boldsymbol{\lambda}\rangle$ to a bilinear form with weakly singular kernel operating on the surface derivatives of $\boldsymbol{\varphi}_0$ and $\boldsymbol{\lambda}$ (see [5]).

REGULARITY: In view of (3.15), as for the Dirichlet problem, we find $\boldsymbol{\varphi}_0 \in H^{\frac{1}{2}}(\Gamma)$ and $\boldsymbol{v} \in H^1(\Omega)$ for given $\boldsymbol{\psi} \in H^{-\frac{1}{2}}(\Gamma$ satisfying (3.2), and for $\boldsymbol{\psi} \in H^{-\frac{1}{2}+\varrho}(\Gamma)$ with $0 \leq \varrho < \dfrac{1}{2}$ on the Lipschitz boundary we get $\boldsymbol{\varphi}_0 \in H^{\frac{1}{2}+\varrho}(\Gamma)$ and $\boldsymbol{v} \in H^{1+\varrho}(\Omega)$.

4. - THE MIXED DIRICHLET–NEUMANN PROBLEM

In case of the mixed problem (1.1), (1.4) where $\Gamma = \overline{\Gamma_D \cup \Gamma_N}$ and $\Gamma_D \cap \Gamma_N = \emptyset$, ($\Gamma_D$ and Γ_N are open parts of Γ), we have

$$(4.1) \qquad \boldsymbol{v}|_{\Gamma_D} = \boldsymbol{\varphi} \in H^{\frac{1}{2}}(\Gamma_D) \text{ and } T\boldsymbol{v}|_{\Gamma_N} = \boldsymbol{\psi} \in H^{-\frac{1}{2}}(\Gamma_N) = (\widetilde{H}^{\frac{1}{2}}(\Gamma_N))'\,.$$

Without loss of generality we assume that $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are extended to $\boldsymbol{\varphi}_g$ and $\boldsymbol{\psi}_g$, respectively, to all of Γ; hence,

$$(4.2) \qquad \boldsymbol{v}|_{\Gamma_D} = \boldsymbol{\varphi}_g|_{\Gamma_D} \text{ and } T\boldsymbol{v}|_{\Gamma_N} = \boldsymbol{\psi}_g|_{\Gamma_N}$$

where now $\boldsymbol{\varphi}_g \in H^{\frac{1}{2}}(\Gamma)$ and $\boldsymbol{\psi}_g \in H^{-\frac{1}{2}}(\Gamma)$. Then

$$(4.3) \qquad \begin{aligned} \boldsymbol{v}|_\Gamma &= \boldsymbol{\varphi}_g + \widetilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi}_0 + \sum_{\ell=1}^{L} \sum_{j=1}^{6} \omega_{j\ell}\boldsymbol{m}_{j\ell} \in H^{\frac{1}{2}}(\Gamma) \text{ and} \\ T\boldsymbol{v}|_\Gamma &= \boldsymbol{\psi}_g + \widetilde{\boldsymbol{\psi}} = \boldsymbol{\sigma}_0 + \sum_{\ell=1}^{L} a_\ell \boldsymbol{n}_\ell \in H^{-\frac{1}{2}}(\Gamma) \end{aligned}$$

where $\boldsymbol{\varphi}_0$ satisfies (3.4) and $\boldsymbol{\sigma}_0$ satisfies (2.4) and $\widetilde{\boldsymbol{\varphi}} \in \widetilde{H}^{\frac{1}{2}}(\Gamma_N)$, $\widetilde{\boldsymbol{\psi}} \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_D)$.

The spaces $\widetilde{H}^{\frac{1}{2}}(\Gamma_N)$ and $\widetilde{H}^{-\frac{1}{2}}(\Gamma_D)$ are defined as subspaces of $H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma)$,

respectively, by

$$\widetilde{H}^{\frac{1}{2}}(\Gamma_N) := \text{closure of } C_0^\infty(\Gamma_N) \text{ in } H^{\frac{1}{2}}(\Gamma)\,, \text{ and}$$
$$\widetilde{H}^{-\frac{1}{2}}(\Gamma_D) := \text{closure of } C_0^\infty(\Gamma_D) \text{ in } H^{-\frac{1}{2}}(\Gamma)\,.$$

For the elastic displacement field with $c > 0$, the Calderon projector implies the system in mixed form on Γ, namely

$$\begin{aligned} V_{e\ell}(\widetilde{\boldsymbol{\psi}} + \boldsymbol{\psi}_g) - \left(\frac{1}{2}I + K_{e\ell}\right)(\widetilde{\boldsymbol{\varphi}} + \boldsymbol{\varphi}_g) &= 0\,, \\ -\left(\frac{1}{2}I - K'_{e\ell}\right)(\widetilde{\boldsymbol{\psi}} + \boldsymbol{\psi}_g) + D_{e\ell}(\widetilde{\boldsymbol{\varphi}} + \boldsymbol{\varphi}_g) &= 0\,. \end{aligned} \tag{4.4}$$

If we restrict the first equation to Γ_N and the second one to Γ_D then we get a **system in stabilized saddle point form**:

$$\begin{aligned} V_{e\ell}\widetilde{\boldsymbol{\psi}} - K_{e\ell}\widetilde{\boldsymbol{\varphi}} &= \left\{-V_{e\ell}\boldsymbol{\psi}_g + \left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi}_g\right\}|_{\Gamma_N} \text{ on } \Gamma_N\,, \\ K'_{e\ell}\widetilde{\boldsymbol{\psi}} + D_{e\ell}\widetilde{\boldsymbol{\varphi}} &= \left(\frac{1}{2}I + K'_{e\ell}\right)\boldsymbol{\psi}_g - D_{e\ell}\boldsymbol{\varphi}_g\Big\}|_{\Gamma_D} \text{ on } \Gamma_D \end{aligned} \tag{4.5}$$

which is uniquely solvable for $(\widetilde{\boldsymbol{\varphi}}, \widetilde{\boldsymbol{\psi}}) \in \widetilde{H}^{\frac{1}{2}}(\Gamma_N) \times \widetilde{H}^{-\frac{1}{2}}(\Gamma_D)$ and whose variational bilinear form on the product space $\widetilde{H}^{\frac{1}{2}}(\Gamma_N) \times \widetilde{H}^{-\frac{1}{2}}(\Gamma_D) \overset{\bullet}{\subset} H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ reads:

$$a_{DN}\big((\widetilde{\boldsymbol{\psi}}, \widetilde{\boldsymbol{\varphi}})\,;\,(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\chi}})\big) := \langle V_{e\ell}\widetilde{\boldsymbol{\psi}}, \widetilde{\boldsymbol{\sigma}}\rangle - \langle K_{e\ell}\widetilde{\boldsymbol{\varphi}}, \widetilde{\boldsymbol{\sigma}}\rangle + \langle \widetilde{\boldsymbol{\psi}}, K_{e\ell}\widetilde{\boldsymbol{\chi}}\rangle + \langle D_{e\ell}\widetilde{\boldsymbol{\varphi}}, \boldsymbol{\chi}\rangle\,, \tag{4.6}$$

and is $\big(\widetilde{H}^{\frac{1}{2}}(\Gamma_N)\big) \times \widetilde{H}^{-\frac{1}{2}}(\Gamma_D)\big)$ elliptic:

$$a_{DN}\big((\widetilde{\boldsymbol{\psi}}, \widetilde{\boldsymbol{\varphi}})\,;\,(\widetilde{\boldsymbol{\psi}}, \widetilde{\boldsymbol{\varphi}})\big) \geq \gamma_0\big(\|\widetilde{\boldsymbol{\varphi}}\|^2_{H^{\frac{1}{2}}(\Gamma)} + \big(\|\widetilde{\boldsymbol{\psi}}\|^2_{H^{-\frac{1}{2}}(\Gamma)}\big) \quad \text{for all } \widetilde{\boldsymbol{\varphi}} \in \widetilde{H}^{\frac{1}{2}}(\Gamma_N)\,,\ \widetilde{\boldsymbol{\psi}} \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_D)\,. \tag{4.7}$$

The constant $\gamma_0 > 0$ depends on Γ_D and Γ_N.

Inserting (1.22)–(1.29) leads to a system which is a perturbation of the degenerate Stokes system . For the first equations in the Stokes system we insert into (2.8) the relation (2.8a) and obtain

$$\begin{aligned} V_{st}\boldsymbol{\sigma} &+ \sum_{\ell=1}^{L}\langle \boldsymbol{\sigma}, V_\Delta \boldsymbol{n}_\ell\rangle V_\Delta \boldsymbol{n}_\ell + \frac{2c}{\mu}V_\Delta\boldsymbol{\sigma} \\ &= (1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi} - \sum_{\ell,k=1}^{L}(\underset{\sim}{\beta}_\Delta^{-1})_{\ell k}\left\langle (1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi}, \boldsymbol{n}_k\right\rangle V_\Delta \boldsymbol{n}_\ell \\ &\quad + \sum_{k=1}^{L}\frac{\mu}{2c}\langle \boldsymbol{\varphi}, \boldsymbol{n}_k\rangle V_\Delta \boldsymbol{n}_k + \sum_{\ell,k=1}^{L}(\underset{\sim}{\beta}_\Delta^{-1})_{\ell k}\langle \boldsymbol{\varphi}, \boldsymbol{n}_k\rangle V_\Delta \boldsymbol{n}_\ell \\ &\quad + \sum_{k=1}^{L}\frac{\mu}{2}\left\langle \left(\frac{1}{2}I + K_\Delta + L_1\right)\boldsymbol{\varphi}, \boldsymbol{n}_k\right\rangle V_\Delta \boldsymbol{n}_k \\ &\quad + c\sum_{\ell,k=1}^{L}(\underset{\sim}{\beta}_\Delta^{-1})_{\ell k}\left\langle \left(\frac{1}{2}I + K_\Delta + L_1\right)\boldsymbol{\varphi}, \boldsymbol{n}_k\right\rangle V_\Delta \boldsymbol{n}_\ell\,. \end{aligned} \tag{4.8}$$

For the second equations, the hypersingular equation in stabilized form reads

$$D_{st}\boldsymbol{\varphi} + \sum_{\ell=1}^{L}\sum_{j=1}^{6}\langle\boldsymbol{\varphi},\boldsymbol{m}_{j\ell}\rangle\boldsymbol{m}_{j\ell} + cL_2\boldsymbol{\varphi} = \left(\frac{1}{2}I - K'_{e\ell}\right)\boldsymbol{\psi}\,. \tag{4.9}$$

Now we insert $\boldsymbol{\sigma} = \boldsymbol{\sigma}_g + \widetilde{\boldsymbol{\sigma}}$ and $\boldsymbol{\varphi} = \boldsymbol{\varphi}_g + \widetilde{\boldsymbol{\varphi}}$ and recollect to obtain the equations in variational form:

$$\begin{aligned}
&\langle V_{st}\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\chi}}\rangle + \sum_{\ell=1}^{L}\langle\widetilde{\boldsymbol{\sigma}}, V_\Delta\boldsymbol{n}_\ell\rangle\langle V_\Delta\boldsymbol{n}_\ell,\widetilde{\boldsymbol{\chi}}\rangle + \frac{2c}{\mu}\langle V_\Delta\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\chi}}\rangle \\
&\quad - (1+c)\left\langle\left(\frac{1}{2}I + K_{e\ell}\right)\widetilde{\boldsymbol{\varphi}},\widetilde{\boldsymbol{\chi}}\right\rangle + \sum_{\ell,k=1}^{L}(\underset{\sim}{\beta}_\Delta^{-1})_{\ell k}\left\langle(1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\widetilde{\boldsymbol{\varphi}},\boldsymbol{n}_k\right\rangle\langle V_\Delta\boldsymbol{n}_\ell,\widetilde{\boldsymbol{\chi}}\rangle \\
&\qquad - \sum_{\ell,k=1}^{L}\frac{\mu}{2}\left\langle\left(\frac{1}{2}I + K_\Delta + L_1\right)\widetilde{\boldsymbol{\varphi}},\boldsymbol{n}_k\right\rangle\langle V_\Delta\boldsymbol{n}_\ell,\widetilde{\boldsymbol{\chi}}\rangle \\
&\qquad - c\sum_{\ell,k=1}^{L}(\underset{\sim}{\beta}_\Delta^{-1})_{\ell k}\left\langle\left(\frac{1}{2}I + K_\Delta + L_1\right)\widetilde{\boldsymbol{\varphi}},\boldsymbol{n}_k\right\rangle\langle V_\Delta\boldsymbol{n}_\ell,\widetilde{\boldsymbol{\chi}}\rangle \\
&\quad = \langle\boldsymbol{\Phi},\widetilde{\boldsymbol{\chi}}\rangle \qquad \text{for all } \widetilde{\boldsymbol{\chi}} \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_D)
\end{aligned} \tag{4.10}$$

and

$$\langle D_{st}\widetilde{\boldsymbol{\varphi}},\widetilde{\boldsymbol{\lambda}}\rangle + \sum_{\ell=1}^{L}\sum_{j=1}^{6}\langle\widetilde{\boldsymbol{\varphi}},\boldsymbol{m}_{j\ell}\rangle\langle\boldsymbol{m}_{j\ell},\widetilde{\boldsymbol{\lambda}}\rangle + c\langle L_2\widetilde{\boldsymbol{\varphi}},\widetilde{\boldsymbol{\lambda}}\rangle - \left\langle\left(\frac{1}{2}I - K'_{e\ell}\right)\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\lambda}}\right\rangle = \langle\boldsymbol{\Psi},\widetilde{\boldsymbol{\lambda}}\rangle\,. \tag{4.11}$$

Here, the right-hand sides are given by

$$\begin{aligned}
\boldsymbol{\Phi} &= -V_{st}\boldsymbol{\sigma}_g - \sum_{\ell=1}^{L}\langle V\boldsymbol{\sigma}_g,\boldsymbol{n}_\ell\rangle V_\Delta\boldsymbol{n}_\ell - \frac{2c}{\mu}V_\Delta\boldsymbol{\sigma}_g \\
&\quad + (1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi}_g - \sum_{\ell,k=1}^{L}(\underset{\sim}{\beta}_\Delta^{-1})_{\ell k}\left\langle(1+c)\left(\frac{1}{2}I + K_{e\ell}\right)\boldsymbol{\varphi}_g,\boldsymbol{n}_k\right\rangle V_\Delta\boldsymbol{n}_\ell \\
&\quad + \sum_{k=1}^{L}\frac{\mu}{2}\left\langle\left(\frac{1}{2}I + K_\Delta + L_1\right)\boldsymbol{\varphi}_g,\boldsymbol{n}_k\right\rangle \\
&\quad + c\sum_{\ell,k=1}^{L}(\underset{\sim}{\beta}_\Delta^{-1})_{\ell k}\left\langle\left(\frac{1}{2}I + K_\Delta + L_1\right)\boldsymbol{\varphi}_g,\boldsymbol{n}_k\right\rangle V_\Delta\boldsymbol{n}_\ell \\
&\quad + \frac{\mu}{2c}\sum_{k=1}^{L}a_k V_\Delta\boldsymbol{n}_k + \sum_{\ell,k=1}^{L}\left\{(\underset{\sim}{\beta}_\Delta^{-1})_{\ell k}a_k + \frac{\mu}{2}\delta_{\ell k}b_k\right\}V_\Delta\boldsymbol{n}_\ell + c\sum_{\ell,k=1}^{L}(\underset{\sim}{\beta}_\Delta^{-1})_{\ell k}b_k V_\Delta\boldsymbol{n}_\ell
\end{aligned} \tag{4.12}$$

and

$$\boldsymbol{\Psi} = -D_{st}\boldsymbol{\varphi}_g - \sum_{\ell=1}^{L}\sum_{j=1}^{6}\langle\boldsymbol{\varphi}_g,\boldsymbol{m}_{j\ell}\rangle\boldsymbol{m}_{j\ell} - cL_2\boldsymbol{\varphi}_g + \left(\frac{1}{2}I - K'_{e\ell}\right)\boldsymbol{\sigma}_g \tag{4.13}$$

together with the compatibility conditions

$$\langle \widetilde{\boldsymbol{\varphi}}, \boldsymbol{n}_k \rangle = a_k + cb_k - \langle \boldsymbol{\varphi}_g, \boldsymbol{n}_k \rangle \qquad \text{for } k = 1, \ldots, L, \tag{4.14}$$

where a_k and b_k are given constants.

Collecting the above relations we have the following theorem.

THEOREM 4.1: *For $c > 0$ the elastic field of the mixed Dirichlet–Neumann problem* (1.1), (1.4) *has the form*

$$\begin{aligned} \boldsymbol{v}(x) = \frac{1}{1+c} \Bigg\{ & \int_\Gamma E_{st}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\sigma}(\boldsymbol{y}) - \int_\Gamma K_{st}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\varphi}(\boldsymbol{y})ds_y \\ & + c\left(\frac{2}{\mu}\int_\Gamma E_\Delta(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\sigma}(\boldsymbol{y})ds_y - \int_\Gamma \big(K_\Delta(\boldsymbol{x},\boldsymbol{y}) + L_1(\boldsymbol{x},\boldsymbol{y})\big)\boldsymbol{\varphi}(\boldsymbol{y})ds_y\right)\Bigg\} \end{aligned} \tag{4.15}$$

where $\widetilde{\boldsymbol{\sigma}}$ and $\widetilde{\boldsymbol{\varphi}}$ are the solutions of the equations (4.10), (4.11) *and* (4.14) *where the constants a_k, b_k are given.*

REMARK: If we choose $a_k = b_k = 0$ in (4.14) then the mixed boundary value problem becomes a regular perturbation problem of the Stokes equations whereas for non vanishing a_k, b_k, the problem becomes singular if $c \to 0$.

Since the operators in (4.8) and (4.9) and also $\boldsymbol{\Psi}$ in (4.13) depend analytically on c, and $\boldsymbol{\Phi}$ in (4.12) is meromorphic, again the charges $\boldsymbol{\varphi}$ and $\boldsymbol{\sigma}$ on Γ and the displacement $\boldsymbol{v}$ in Ω admit again asymptotic developments in powers of c.

REGULARITY: The coerciveness estimate (4.12) implies that $\boldsymbol{\varphi} \in H^{\frac{1}{2}}(\Gamma)$, $\boldsymbol{\sigma} \in H^{-\frac{1}{2}}(\Gamma)$ and $\boldsymbol{v} \in H^1(\Omega)$ if Γ is Lipschitz. For higher regularity, however, i.e., if $\boldsymbol{\varphi} \in H^{\frac{1}{2}+\varrho}(\Gamma_D)$ and $\boldsymbol{\psi} \in H^{-\frac{1}{2}+\varrho}(\Gamma_N)$ are given with $0 < \varrho < \dfrac{1}{2}$ and Γ is just Lipschitz, higher regularity has not been proven yet. But if the Lipschitz surface contains a vicinity of the collision curve $\gamma = \overline{\Gamma}_N \cap \overline{\Gamma}_D$, that is, a piece of a Lyapounov surface, and if γ is a C^2-curve, then it can be shown that $\boldsymbol{v} \in H^{1+\varrho}(\Omega)$ [8].

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REFERENCES

[1] H.-D. ALBER, *Materials with Memory.* Lecture Notes in Mathematics **1682**, Springer-Verlag Berlin 1998.

[2] M. COSTABEL, *Boundary integral operators on Lipschitz domains: elementary results.* SIAM J. Math. Anal. **19** (1988) 613-626.

[3] G.C. Hsiao - W.L. Wendland, *Boundary Integral Equations.* In preparation.
[4] M. Kohr - I. Pop, *Viscous Incompressible Flow.* WIT Press, Southampton 2004.
[5] M. Kohr - W.L. Wendland, *Variational boundary integral equations for the Stokes system.* Appl. Anal. **85** (2006) 1343-1372.
[6] A. Koshevnikov, *A history of the Cosserat spectrum.* In: J. Roßmann, P. Takáč, G. Wildenhain (Eds.) The Maz'ya Anniversary Collection Vol. I. Birkhäuser-Verlag, Basel (1999), 225-236.
[7] V.D. Kupradze - T.G. Gegelia - M.O. Basheleishvili - T.U. Burchuladse, *Three-Dimenional Problems of the Mathematical Theory of Elasticity and Thermoelasticity.* North Holland, Amsterdam 1979.
[8] V.G. Maz'ya, *Boundary Integral Equations.* In: V.G. Maz'ya and S.M. Nikolski (Eds.) Encyclopaedia of Mathematical Sciences, Vol. 27, Analysis IV, Springer-Verlag Berlin (1991) 130-222.
[9] B. Reidinger - O. Steinbach, *A symmetric boundary element method for the Stokes problem in multiple connected domains.* Math. Methods Appl. Sci. **26** (2003) 77-93.
[10] O. Steinbach, *A robust boundary element method for nearly incompressible linear elasticity.* Numer. Math. **95** (2003) 553-562.
[11] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis.* North Holland Publ. Amsterdam 1979.

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