Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
$124^{\circ}$ (2006), Vol. XXX, fasc. 1, pagg. 7-28

GREGORY M. CROSSWHITE (*) - STUART S. ANTMAN (**)

# A New Spin on Problems of Newton, the Bernoullis, and Abel 

In memoriam Gaetano Fichera

Abstract. - This paper treats some closely related conceptually simple problems associated with gravitational attraction and centrifugal force. They are more elaborate variants of problems treated by Newton, Jas. and Joh. Bernoulli, and Abel. They offer examples of qualitative methods in classical mechanics, of inverse problems, and of calculus of variations richer than those of the original versions of the problems.

## 1. - Introduction.

In his Principia [19] of 1687 Newton determined the gravitational attraction on a particle exerted by a body having a spherically symmetric mass distribution. (See Sec. XII of Book I.) The particle could be located anywhere, either within or without the body. Newton's Proposition XXXVIII, Theorem XII can immediately be used to determine the motion of a particle in a frictionless diametral tunnel through the earth, with the earth modelled as a ball of constant mass density. (This problem is posed and solved in the elementary physics text [22, pp. 351-352].) The motion is sinusoidal.

Newton's approach to studying such motions was modern in the sense of being qualitative, rather than analytic. He actually demonstrated that such motions are periodic. Our search through his Mathematical Papers [27], however, did not yield an explicit treatment of particle motion in a diametral tunnel.

In 1696 Joh. Bernoulli [3] posed the brachistochrone problem of determining the
(*) Indirizzo dell'Autore: Department of Physics, University of Washington, Seattle, WA 98195, email: gcross@washington.edu
(**) Indirizzo dell'Autore: Department of Mathematics, Institute for Physical Science and Technology, and Institute for Systems Research University of Maryland. College Park, MD 207424015, e-mail: ssa@math.umd.edu
shape of a frictionless wire joining two prescribed points along which a bead, under the constant gravitational attraction of a flat earth, descends in the least time. (See Bliss [4].) Within a year, its solution, a cycloid, was found by Newton, Leibniz, L'Hôpital, Joh. Bernoulli himself, and, to his dismay, his brother Jas. Bernoulli. Newton's solution, which took him a whole evening to construct, was published anonymously. But Joh. Bernoulli identified the author immediately "even as the lion by its paw print" ("tanquam ex ungue leonem"; see Westfall [26, p. 583].)

The related tautochrone problem is to find such a wire for which the time of descent of the bead is independent of the position of the starting point. The problem was posed by Huygens. In 1659 he used geometrical arguments to show that the solution of this problem is also a cycloid. (The work was published in [9].) Joh. Bernoulli observed that the tautochrone is the classical brachistochrone (see [13, p. 34]). More relevant for us is that in 1826 Abel [1] gave an analytic solution of a generalization of this problem formulated as an Abel integral equation.

We put a new spin on Newton's problem of a particle moving in a frictionless diametral tunnel through a stationary earth by replacing the stationary earth by a spherically symmetric planet spinning about an axis and by replacing the diametral tunnel with a tunnel of an arbitrary smooth shape. (In Book III of the Principia Newton considered spinning planets, and in discussing the oblateness of such planets he considered the behavior of tunnels filled with liquid.) We solve a related inverse problem of determining the density distribution of a planet from information about motions in tunnels. It leads to an Abel integral equation. We put a new spin on Bernoulli's problem by replacing the wire above a flat earth by an underground tunnel joining two prescribed points of our spinning planet. The presence of both spinning and gravity typically prevents the brachistochrone from being planar and prevents the Euler-Lagrange equations for its shape from being elementary. Our spin around these problems will take us through a beautiful and varied mathematical landscape.

Before attacking these problems, we first derive the gravitational attraction of spherically symmetric planet on a particle within it, and then formulate the equations of motion with great care. (This formulation affords an easy entrée into the mathematical structure of classical particle mechanics.)

Throughout this paper we denote both ordinary and partial derivatives by subscripts, and denote some ordinary derivatives by primes.

## 2. - The gravity of the situation

According to Newton's Law of Universal Gravitation, the force exerted on a particle at $\boldsymbol{x}$ per unit of its mass by a body occupying region $\mathcal{B}$ (given by the inverse-square law) is

$$
\begin{equation*}
G \int_{\mathcal{B}} \frac{\boldsymbol{y}-\boldsymbol{x}}{\boldsymbol{y}-\left.\boldsymbol{x}\right|^{3}} d m(\boldsymbol{y}) \tag{2.1}
\end{equation*}
$$

where $G$ is the universal gravitational constant and $\operatorname{dm}(\boldsymbol{y})$ is the differential mass (measure) at $\boldsymbol{y}$ in $\mathcal{B}$. We need to find the force exerted by a body with a spherically symmetric mass distribution on a particle within the body. Since the formula is not well known, we sketch its derivation, following Kellogg [15]. (For a modern version of Newton's derivation, see [7] or [22].)

Let $\left\{\boldsymbol{i}_{1}, \boldsymbol{i}_{2}, \boldsymbol{i}_{3} \equiv \boldsymbol{k}\right\}$ be a fixed right-handed orthonormal basis for Euclidean 3-space. Let

$$
\begin{equation*}
\boldsymbol{e}_{1}(\phi):=\cos \phi \boldsymbol{i}_{1}+\sin \phi \boldsymbol{i}_{2}, \boldsymbol{e}_{2}(\phi):=-\sin \phi \boldsymbol{i}_{1}+\cos \phi \boldsymbol{i}_{2} \equiv \boldsymbol{k} \times \boldsymbol{e}_{1}(\phi), \boldsymbol{e}_{3}:=\boldsymbol{k} \tag{2.2}
\end{equation*}
$$

We take the body to be a ball of radius $R$ centered at the origin. Then a typical point $\boldsymbol{y}$ of the ball can be located by the spherical coordinates $(r, \theta, \phi)$ by

$$
\begin{equation*}
\boldsymbol{y}=r\left[\sin \theta \boldsymbol{e}_{1}(\phi)+\cos \theta \boldsymbol{k}\right] . \tag{2.3}
\end{equation*}
$$

Without loss of generality, let the attracted particle occupy position $\boldsymbol{x}=z \boldsymbol{k}$ with $0 \leq z \leq R$. We assume that the ball has a mass density $\mu$ depending only on $r$, in consonance with radial symmetry, so that $d m(\boldsymbol{y})=\mu(r) r^{2} \sin \theta d r d \theta d \phi$. By virtue of the symmetry, the force on the particle at $z \boldsymbol{k}$ per unit of its mass, given by (2.1), only has a component in the $\boldsymbol{k}$-direction, given by

$$
\begin{align*}
-\Gamma^{\prime}(z): & =G \int_{0}^{2 \pi} \int_{0}^{\pi} \int_{0}^{R} \frac{r \cos \theta-z}{\left[r^{2}-2 r z \cos \theta+z^{2}\right]^{3 / 2}} \mu(r) r^{2} \sin \theta d r d \theta d \phi \\
& =2 \pi G \int_{0}^{R} d r \mu(r) r^{2} \int_{0}^{\pi} \frac{r \cos \theta-z}{\left[r^{2}-2 r z \cos \theta+z^{2}\right]^{3 / 2}} \sin \theta d \theta . \tag{2.4}
\end{align*}
$$

To evaluate the $\theta$-integral, we change the variable of integration from $\theta$ to $u:=\sqrt{r^{2}-2 r z \cos \theta+z^{2}}$, which leads to

$$
\begin{equation*}
\Gamma^{\prime}(z)=\frac{4 \pi G}{z^{2}} \int_{0}^{z} \mu(r) r^{2} d r=: \gamma(z) z \tag{2.5}
\end{equation*}
$$

The potential energy of the gravity force (per unit mass of the particle on which it acts) at radius $r$ is $\Gamma(r)$, defined from (2.5):

$$
\begin{equation*}
\Gamma(r):=\int_{0}^{r} \xi \gamma(\xi) d \xi \tag{2.6}
\end{equation*}
$$

(2.7) If $\mu$ is constant, then $\gamma=\frac{4 \pi}{3} G \mu$ (const) and $\Gamma(r)=\frac{2}{3} \pi \mu G r^{2}=\frac{1}{2} \gamma r^{2}$.

Thus the gravitational attraction acts like a sort of spring, nonlinear in $z$ when $\mu$ is not constant.

## 3. - Tunnelling

We suppose that a spherically symmetric planet spins about the axis $\boldsymbol{k}$ fixed in space with a constant angular speed $\omega$. Then the basis $\boldsymbol{e}_{1}(\omega t), \boldsymbol{e}_{2}(\omega t), \boldsymbol{k}$ is fixed in the planet. We identify a tunnel in the planet with a twice continuously differentiable curve fixed in the planet and lying entirely within it. Such a curve (at time $t$ ) has a parametrization of the form

$$
\begin{equation*}
s \mapsto \hat{\boldsymbol{r}}(s, t)=x_{1}(s) \boldsymbol{e}_{1}(\omega t)+x_{2}(s) \boldsymbol{e}_{2}(\omega t)+x_{3}(s) \boldsymbol{e}_{3} \tag{3.1}
\end{equation*}
$$

with $x_{1}, x_{2}, x_{3}$ independent of $t$ and with $|\boldsymbol{r}| \leq R$. (The functions $\boldsymbol{e}_{1}$ and $\boldsymbol{e}_{2}$ are defined in (2.2).) Let us set

$$
\begin{equation*}
\boldsymbol{r}(s)=x_{1}(s) \boldsymbol{i}_{1}+x_{2}(s) \boldsymbol{i}_{2}+x_{3}(s) \boldsymbol{i}_{3} \quad(\equiv \hat{\boldsymbol{r}}(s, 0)), \tag{3.2}
\end{equation*}
$$

and define the proper orthogonal transformation (rotation)

$$
\begin{equation*}
\boldsymbol{\Omega}(t)=\boldsymbol{e}_{1}(\omega t) \otimes \boldsymbol{i}_{1}+\boldsymbol{e}_{2}(\omega t) \otimes \boldsymbol{i}_{2}+\boldsymbol{k} \otimes \boldsymbol{k} \tag{3.3}
\end{equation*}
$$

as a sum of dyadic products. (A dyadic product $\boldsymbol{a} \otimes \boldsymbol{b}$ is a linear transformation with the defining property that $(\boldsymbol{a} \otimes \boldsymbol{b}) \boldsymbol{c}:=(\boldsymbol{b} \cdot \boldsymbol{c}) \boldsymbol{a}$.) Then

$$
\begin{equation*}
\hat{\boldsymbol{r}}(s, t)=\boldsymbol{\Omega}(t) \boldsymbol{r}(s) . \tag{3.4}
\end{equation*}
$$

The position $\hat{\boldsymbol{r}}(\hat{s}(t), t)$ of a particle in the tunnel at time $t$ is determined by its parameter $\hat{s}(t)$ (which could well be the arc length to the particle from some fixed point in the tunnel) by

$$
\begin{equation*}
\hat{\boldsymbol{r}}(\hat{s}(t), t)=x_{1}(\hat{s}(t)) \boldsymbol{e}_{1}(\omega t)+x_{2}(\hat{s}(t)) \boldsymbol{e}_{2}(\omega t)+x_{3}(\hat{s}(t)) \boldsymbol{e}_{3} \equiv \boldsymbol{\Omega}(t) \boldsymbol{r}(\hat{s}(t)) . \tag{3.5}
\end{equation*}
$$

Since $\frac{d}{d t} \boldsymbol{e}_{1}(\omega t)=\omega \boldsymbol{e}_{2}(\omega t)=\omega \boldsymbol{k} \times \boldsymbol{e}_{1}(\omega t)$, etc., the velocity and acceleration of the particle at time $t$ are thus

$$
\begin{align*}
\frac{d}{d t} \hat{\boldsymbol{r}}(\hat{s}(t), t)= & \hat{\boldsymbol{r}}_{s}(\hat{s}(t), t) \hat{s}_{t}(t)+\omega \boldsymbol{k} \times \hat{\boldsymbol{r}}(\hat{s}(t), t)  \tag{3.6}\\
\equiv & \boldsymbol{\Omega}(t)\left\{\boldsymbol{r}_{s}(\hat{s}(t)) \hat{s}_{t}(t)+\omega \boldsymbol{k} \times \boldsymbol{r}(\hat{s}(t))\right\} \\
\frac{d^{2}}{d t^{2}} \hat{\boldsymbol{r}}(\hat{s}(t), t)= & \hat{\boldsymbol{r}}_{s}(\hat{s}(t), t) \hat{s}_{t t}(t)+\hat{\boldsymbol{r}}_{s s}(\hat{s}(t), t) \hat{s}_{t}(t)^{2}  \tag{3.7}\\
& \quad+2 \omega \boldsymbol{k} \times \hat{\boldsymbol{r}}_{s}(\hat{s}(t), t) \hat{s}_{t}(t)+\omega^{2} \boldsymbol{k} \times[\boldsymbol{k} \times \hat{\boldsymbol{r}}(\hat{s}(t), t)] \\
\equiv & \boldsymbol{\Omega}(t)\left\{\boldsymbol{r}_{s}(\hat{s}(t)) \hat{s}_{t t}(t)+\boldsymbol{r}_{s s}\left(\hat{s}(t) \hat{s}_{t}(t)^{2}\right.\right. \\
& \left.+2 \omega \boldsymbol{k} \times \boldsymbol{r}_{s}(\hat{s}(t)) \hat{s}_{t}(t)+\omega^{2} \boldsymbol{k} \times[\boldsymbol{k} \times \boldsymbol{r}(\hat{s}(t))]\right\} .
\end{align*}
$$

(The last term in the first equation of (3.7), namely, $\omega^{2}\{\hat{\boldsymbol{r}}(\hat{s}(t), t)-[\hat{\boldsymbol{r}}(\hat{s}(t), t) \cdot \boldsymbol{k}] \boldsymbol{k}\}$, is the centrifugal acceleration. This is a misnomer for our problem because it is directed away from the axis $\boldsymbol{k}$ of spin rather than away from the center. A better term would be axifugal acceleration. The penultimate term in this equation is the Coriolis acceleration. The products of these accelerations with the negative of the mass of the particle are somewhat misleadingly called the centrifugal and Coriolis forces.)

Let the mass of the particle be denoted $m$. We assume that during its motion in a tunnel it is subject solely to the force of gravity and to the contact force exerted on it by the tunnel. This contact force has a component normal to the tunnel, which for algebraic convenience we scale by the mass and denote by $m \hat{\boldsymbol{n}}(t)$, with

$$
\begin{equation*}
\hat{\boldsymbol{n}}(t) \cdot \hat{\boldsymbol{r}}_{s}(\hat{s}(t), t)=0 \tag{3.8}
\end{equation*}
$$

We define $\boldsymbol{n}$ by

$$
\begin{equation*}
\hat{\boldsymbol{n}}(t)=: \boldsymbol{\Omega}(t) \boldsymbol{n}(t) \quad \text { so that } \quad \boldsymbol{n}(t) \cdot \boldsymbol{r}_{s}(\hat{s}(t))=0 \tag{3.9}
\end{equation*}
$$

We assume that the tangential component of the contact force is frictional, i.e., it opposes the motion. In particular, we assume that there is a function $f$ of $(s, \dot{s}, \boldsymbol{n})$ defined for $\dot{s} \neq 0$ such that the (tangential) friction force at time $t$ is

$$
\begin{equation*}
-m f\left(\hat{s}(t), \hat{s}_{t}(t), \boldsymbol{n}(t)\right) \hat{\boldsymbol{r}}_{s}(\hat{s}(t), t) \quad \text { with } \quad f(s, \dot{s}, \boldsymbol{n}) \dot{s} \geq 0 \quad \text { for } \quad \dot{s} \neq 0 \tag{3.10}
\end{equation*}
$$

(That this force should have this special dependence is a manifestation of a fundamental mechanical principle requiring that material properties, like frictional resistance, be invariant under rigid motions.)

Remark. The spinning generally prevents the velocity $\frac{d}{d t} \hat{\boldsymbol{r}}$ from being tangent to the tunnel. It is reasonable to assume that $f$ is continuous in $\dot{s}$ for $\dot{s} \neq 0$, and that $f$ is piecewise continuous in the other two variables. A body subject to dry friction can stay in equilibrium for a range of forces applied to it. Thus, for it (when $s$ and $\boldsymbol{n}$ are fixed), the friction force is not specified as a function of $\dot{s}$, but rather it is specified by a graph of $f$ vs. $\dot{s}$ in which there is a vertical segment at $\dot{s}=0$ containing the point $(\dot{s}, f)=(0,0)$. The dependence of $f$ on $s$ means that the frictional properties of the tunnel can vary with the location along it. We could have allowed $f$ to depend on $\boldsymbol{n}$ only through $|\boldsymbol{n}|$, but by eschewing this simplification we allow the frictional resistance to depend on which "side" of the tunnel the particle is pressed against. Although we devote some attention to the treatment of friction because the correct formulation of the equations accounting for it is a little tricky and because this formulation opens the way to new problems, we shall comment but briefly on the analysis of equations accounting for friction.

Newton's Second Law of Motion says that the total force on a particle equals its mass times its acceleration. Substituting the forces and acceleration into this law and cancelling the $m$ we obtain

$$
\begin{equation*}
\hat{\boldsymbol{r}}_{s} \hat{s}_{t t}+\hat{\boldsymbol{r}}_{s s} \hat{s}_{t}^{2}+2 \omega \boldsymbol{k} \times \hat{\boldsymbol{r}}_{s} \hat{s}_{t}+\omega^{2} \boldsymbol{k} \times(\boldsymbol{k} \times \hat{\boldsymbol{r}})=-\gamma(\hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}+\hat{\boldsymbol{n}}-f\left(\hat{s}, \hat{s}_{t}, \boldsymbol{n}\right) \hat{\boldsymbol{r}}_{s} . \tag{3.11}
\end{equation*}
$$

Here and below, the argumens of $\hat{\boldsymbol{r}}$ and its derivatives are $\hat{s}(t), t$. This vectorial equation, together with (3.8) corresponds to a system of four scalar equations for the unknown functions $\hat{\boldsymbol{s}}$ and $\hat{\boldsymbol{n}}$. In view of (3.4), (3.8), (3.9), we can replace $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{n}}$ in (3.11) with $\boldsymbol{r}$ and $\boldsymbol{n}$.

To get an equation for $\hat{s}$ alone, we project (3.11) onto the tangent $\hat{\boldsymbol{r}}_{s}(\hat{s}(t), t)$ to the rotating tunnel and onto its complement, or, equivalently, project the version of (3.11) without circumflexes onto $\boldsymbol{r}_{s}(\hat{s}(t))$ and onto its complement. The first projection is effected by taking the dot product of (3.11) with $\hat{\boldsymbol{r}}_{s}(\hat{s}(t), t) \equiv \boldsymbol{\Omega}(t) \boldsymbol{r}_{s}(\hat{s}(t))$ :

$$
\begin{equation*}
\left|\boldsymbol{r}_{s}\right|^{2} \hat{s}_{t t}+\boldsymbol{r}_{s} \cdot \boldsymbol{r}_{s s} \hat{s}_{t}^{2}=\omega^{2}\left[\boldsymbol{r} \cdot \boldsymbol{r}_{s}-(\boldsymbol{r} \cdot \boldsymbol{k})\left(\boldsymbol{r}_{s} \cdot \boldsymbol{k}\right)\right]-\gamma(|\boldsymbol{r}|) \boldsymbol{r} \cdot \boldsymbol{r}_{s}-f\left(\hat{s}, \hat{s}_{t}, \boldsymbol{n}\right)\left|\boldsymbol{r}_{s}\right|^{2} . \tag{3.12}
\end{equation*}
$$

Here and below, the argumen of $\boldsymbol{r}$ and its derivatives is $\hat{s}(t)$. (This projection is the fundamental step for transforming the Newtonian formulation of mechanics to the Lagrangian.) If $f$ is independent of $\boldsymbol{n}$, then this is an equation for $\hat{s}$ alone. Note that the first two terms on the right-hand side of (3.12) are autonomous, i.e., they depend on $t$ only through $\hat{s}$.

Next let

$$
\begin{equation*}
\boldsymbol{q}:=\frac{\boldsymbol{r}_{s}}{\left|\boldsymbol{r}_{s}\right|} \tag{3.13}
\end{equation*}
$$

We project (3.11) without circumflexes onto the complement of $\boldsymbol{r}_{s}$ by operating on (3.11) with $\boldsymbol{q} \times$ to get what we denote as $\boldsymbol{q} \times$ (3.11). Since $\boldsymbol{n} \cdot \boldsymbol{q}=0$, it follows that $\boldsymbol{n} \equiv \boldsymbol{q} \times(\boldsymbol{n} \times \boldsymbol{q})$, which we find from this projection, by operating on $\boldsymbol{q} \times(3.11)$ with $\boldsymbol{q} \times$ :

$$
\begin{align*}
\boldsymbol{n}= & \hat{s}_{t}^{2}\left[\boldsymbol{r}_{s s}-\left(\boldsymbol{r}_{s s} \cdot \boldsymbol{q}\right) \boldsymbol{q}\right]-2 \omega \hat{s}_{t} \boldsymbol{r}_{s} \times \boldsymbol{k} \\
& +\omega^{2}\{[\boldsymbol{r} \cdot \boldsymbol{q}-(\boldsymbol{r} \cdot \boldsymbol{k})(\boldsymbol{q} \cdot \boldsymbol{k})] \boldsymbol{q}-[\boldsymbol{r}-(\boldsymbol{r} \cdot \boldsymbol{k}) \boldsymbol{k}]\}  \tag{3.14}\\
& +\gamma(|\boldsymbol{r}|)[\boldsymbol{r}-(\boldsymbol{r} \cdot \boldsymbol{q}) \boldsymbol{q}] .
\end{align*}
$$

(This step is equivalent to operating on (3.11) with $\boldsymbol{I}-\boldsymbol{q} \otimes \boldsymbol{q}$.) We now substitute this $\boldsymbol{n}$ into (3.12) to get an equation for $\hat{s}$ alone, which is autonomous (because of the invariant form of $f$ ).

We could simplify the form (3.12) by taking $s$ to be the arc-length parameter of the curve $\boldsymbol{r}(s)$, so that $\boldsymbol{r}_{s} \cdot \boldsymbol{r}_{s}=1, \boldsymbol{r}_{s}=\boldsymbol{q}$, and $\boldsymbol{r}_{s s} \cdot \boldsymbol{r}_{s}=0$ :

$$
\begin{equation*}
\hat{s}_{t t}=\omega^{2}\left[\boldsymbol{r} \cdot \boldsymbol{r}_{s}-(\boldsymbol{r} \cdot \boldsymbol{k})\left(\boldsymbol{r}_{s} \cdot \boldsymbol{k}\right)\right]-\gamma(|\boldsymbol{r}|) \boldsymbol{r} \cdot \boldsymbol{r}_{s}-f\left(\hat{s}, \hat{s}_{t}, \boldsymbol{n}\right) . \tag{3.15}
\end{equation*}
$$

This parametrization removes $\boldsymbol{r}_{s s} \cdot \boldsymbol{q}$ from (3.14). For parts of our ensuing discussion, it is convenient to retain the general form (3.12).

## 4. - Qualitative behavior

Since (3.12) and (3.14) form an autonomous second-order equation for $\hat{s}$, we can determine its qualitative behavior from its phase portrait in the $(s, \dot{s})$-phase plane. For this purpose, we first obtain the energy inequality.

The potential energy of the gravity force (per unit mass of the particle on which it acts) at radius $r$ is $\Gamma(r)$, defined by (2.5).

We multiply (3.12) by $\hat{s}_{t}$ and use (3.10) to obtain the energy inequality

$$
\begin{equation*}
\frac{d}{d t}\left\{\frac{1}{2}\left|\boldsymbol{r}_{s}\right|^{2} \hat{s}_{t}^{2}-\frac{1}{2} \omega^{2}\left[|\boldsymbol{r}|^{2}-(\boldsymbol{r} \cdot \boldsymbol{k})^{2}\right]+\Gamma(|\boldsymbol{r}|)\right\}=-\left|\boldsymbol{r}_{s}\right|^{2} f\left(\hat{s}, \hat{s}_{t}, \boldsymbol{n}\right) \hat{s}_{t} \leq 0 . \tag{4.1}
\end{equation*}
$$

If $f=0$, i.e., if the tunnel is frictionless, then the trajectories in the $(s, \dot{s})$-phase plane are clockwise-oriented parametrized curves lying on the level curves of the energy, namely,

$$
\begin{equation*}
\frac{1}{2}\left|\boldsymbol{r}_{s}\right|^{2} \dot{s}^{2}-\frac{1}{2} \omega^{2}\left[|\boldsymbol{r}(s)|^{2}-(\boldsymbol{r}(s) \cdot \boldsymbol{k})^{2}\right]+\Gamma(|\boldsymbol{r}(s)|)=b \quad \text { (const). } \tag{4.2}
\end{equation*}
$$

If $f \neq 0$, then the essential properties of the phase portrait follow from the energy inequality (4.1), which implies that the trajectories in the phase plane pierce the level curves of energy in the direction of lower energy. (A special treatment, however, is typically required for $f$ 's correspond ing to dry friction.) Refinements of the phase portrait can be obtained by studying the singular points and by using LaSalle's invariance principle; see [23, Sec. VII.3].) Since this material is standard, we accordingly devote our attention to the case of a frictionless tunnel.

The structure of the terms in (4.2) confirms our intuition about the qualitative features of solutions, which we can read off from the phase portrait: The potential energy $\Gamma$ of the gravity force is a nowhere-negative convex function of its argument. For constant $\mu$, it is quadratic in $|\boldsymbol{r}|$, but $|\boldsymbol{r}|$ is an arbitrary function of $s$. Thus the gravity force tends to stabilize any motion. The potential energy $-\frac{1}{2} \omega^{2}\left[|\boldsymbol{r}(s)|^{2}-(\boldsymbol{r}(s) \cdot \boldsymbol{k})^{2}\right]$ of the centrifugal force is a nowhere-positive quadratic function of the projection of $\boldsymbol{r}$ onto the equatorial plane. It tends to destabilize any motion. Thus these two potentials compete for dominance. Their relative strengths are influenced by the latitudes through which the tunnel meanders, by the magnitude $\omega$ of the spin, and by the range of the gravitational function $\gamma$. To illustrate this competition, we take $\gamma=$ constant, and show how the qualitative behavior of the phase portrait changes markedly as the parameter $\omega^{2} / \gamma$ varies. In this case, (3.12) implies that its equilibrium points satisfy

$$
\begin{equation*}
\left(\omega^{2}-\gamma\right) \boldsymbol{r} \cdot \boldsymbol{r}_{s}=\omega^{2} x_{3} x_{3 s} \quad \Longleftrightarrow \quad\left(\omega^{2}-\gamma\right)\left(x_{1} x_{1 s}+x_{2} x_{2 s}\right)=\gamma x_{3} x_{3 s} . \tag{4.3}
\end{equation*}
$$

Note that if $x_{3} x_{3 s}=0$ everywhere, i.e., if the tunnel is confined to a horizontal plane, then the locations of the equilibrium points are independent of $\omega^{2} / \gamma$. But their types are not: As $\omega^{2} / \gamma$ crosses the value 1 , the generic equilibrium points switch from saddles to centers and vice versa. For the critical values $\omega^{2}=\gamma,(4.3)$ reduces to the degenerate $x_{3} x_{3 s}=0$, and (4.2) reduces to the degenerate

$$
\begin{equation*}
\left|\boldsymbol{r}_{s}\right|^{2} \dot{s}^{2}=2 h-\omega^{2}(\boldsymbol{r}(s) \cdot \boldsymbol{k})^{2} \tag{4.4}
\end{equation*}
$$

In Figure 1 we illustrate the dependence of the phase portrait on $\omega^{2} / \gamma$ for the specific horizontal quartic curve

$$
\begin{equation*}
x_{1}(s)=\left(s^{2}-1\right)\left(s^{2}-4\right)+3 \equiv\left(s^{2}-\frac{5}{2}\right)^{2}+\frac{3}{4}, \quad x_{2}(s)=s, \quad x_{3}(s)=1 \tag{4.5}
\end{equation*}
$$

so that

$$
\begin{equation*}
|\boldsymbol{r}|^{2}=\left[\left(s^{2}-\frac{5}{2}\right)^{2}+\frac{3}{4}\right]^{2}+s^{2}+1, \quad\left|\boldsymbol{r}_{s}\right|^{2}=16 s^{2}\left(s^{2}-\frac{5}{2}\right)^{2}+1 \tag{4.6}
\end{equation*}
$$

with $|\boldsymbol{r}|^{2} \leq R^{2}$ to keep the tunnel within the planet. (When (4.6) holds, the set of $s$ satisfying $R^{2}>50$, i.e., the set of $s$ for which $R^{2}$ exceeds the local maximum of $|\boldsymbol{r}|^{2}$ (at


FIg. 1. - $(s, \dot{s} / 4 \omega)$-phase portraits for (3.12) for $f=0$, corresponding to the level curves of (4.2) when (4.5) holds. The upper portrait is for $\frac{\gamma}{\omega^{2}}=\frac{3}{2}$ (so that gravity dominates) and the lower for $\frac{\gamma}{\omega^{2}}=\frac{1}{2}$ (so that centrifugal force dominates). The motion on these curves is to the right in the upper half plane and to the left in the lower half plane. The nonzero equilibrium points are at ( $\pm 1.4839,0$ ), their positions indicating the scaling. The separatrices are shown with thicker curves.
$s=0$ ), is an interval centered at the origin; if $|\boldsymbol{r}|^{2}>R^{2}$, this set is a pair of disjoint intervals symmetrically disposed about the origin.) Given any $R$, the phase portraits of Figure 1 are valid only where $|\boldsymbol{r}|^{2} \leq R^{2}$.

The interpretation of these portraits follows the discussion given above: The point corresponding to $s=0$, which is at a local maximum of the distance of the tunnel from the axis, is unstable when gravity dominates centrifugal force and becomes stable when the centrifugal force dominates. Note that the nonzero equilibrium points do not lie on the minimizers of $x_{1}$. The sharp bends in the phase portraits are due to the relatively large range of $\left|\boldsymbol{r}_{s}\right|^{2}$ as $s$ varies over $[-2,2]$.

## 5. - An inverse problem for the density

An inverse problem for a differential equation is to determine functions appearing in the equation from properties of a family of solutions of the equation. Newton solved one of the first inverse problems of mathematical physics when he determined that the only central force allowing planetary orbits to be conic sections is given by the inverse square law (see [26, p. 420]). Here we sketch the solution of another inverse problem, which he might have appreciated: Determine a radially symmetric mass density $\mu$ of a planet from data about the motion of a particle in a tunnel in the planet.

For simplicity, we limit our attention to a diametral frictionless tunnel through the planet:

$$
\begin{equation*}
\boldsymbol{r}(s)=\left[\sin \psi \boldsymbol{i}_{1}+\cos \psi \boldsymbol{k}\right]_{s} \tag{5.1}
\end{equation*}
$$

where $\psi$ is a constant. For it, (4.2) reduces to

$$
\begin{equation*}
\frac{1}{2} \dot{s}^{2}+V(s)=h, \quad V(s):=\Gamma(|s|)-\frac{1}{2} \omega^{2} s^{2} \sin ^{2} \psi \tag{5.2}
\end{equation*}
$$

The level curves $(5.2)_{1}$ in the $(s, \dot{s})$-phase plane are symmetric about the $s$ - and $\dot{s}$-axes.
For the unknown density $\mu$ we seek, let us provisionally assume that $\mu(r)$ is positive for $r$ in some interval ( $0, \bar{a}$ ]. Recall that (2.5) and (2.6) yield

$$
\begin{equation*}
\Gamma(r)=4 \pi G \int_{0}^{r} \frac{1}{\xi^{2}} \int_{0}^{\xi} \mu(\eta) \eta^{2} d \eta d \xi \tag{5.3}
\end{equation*}
$$

from which it follows that $\Gamma(0)=0=\Gamma^{\prime}(0)$ and that $\Gamma(r)>0$ and $\Gamma^{\prime}(r)>0$ for $r \in(0, \bar{a}]$. By taking two derivatives of (5.3), we immediately see that $\mu$ is determined from $\Gamma$. Let us provisionally take $\omega^{2} \sin ^{2} \psi$ so small that $V(0)=0=V^{\prime}(0)$ and that $V(s)>0$ and $V^{\prime}(s)>0$ for $s \in(0, \bar{a}]$. In this case, the origin of the phase portrait for (5.2) is a center, so that all nearby orbits correspond to periodic motions.

We now show that if we know how the period of motions about the planet's center depends on the amplitude (maximum displacement) $a$ for all amplitudes in $[0, \bar{a}]$, then we can determine $\Gamma$ on $[0, \bar{a}]$ and thence $\mu$ on $[0, \bar{a}]$. Let $b$ denote the speed of the particle at the center of the planet. Then (5.2) implies that

$$
\begin{equation*}
h=V(a) \equiv \Gamma(a)-\frac{1}{2} \omega^{2} a^{2} \sin ^{2} \psi=\frac{1}{2} b^{2} . \tag{5.4}
\end{equation*}
$$

These equations have unique positive-valued solutions for $a$ and $b$ in terms of $b$. When $\hat{s}$ moves in the open first quadrant of its phase portrait, its derivative $\hat{s}_{t}>0$, so that $\hat{s}$ has an inverse $s \mapsto \hat{t}(s)$, with $\hat{t}_{s}(s)=1 / \hat{s}_{t}(\hat{t}(s))=1 / \sqrt{2[h-V(s)]}$. In view of the symmetry of (5.2), the time lapse for $\hat{s}$ to make one pass from 0 to $a$ is a quarter of the period $\tau$, so that

$$
\begin{equation*}
\tau=2 \sqrt{2} \int_{0}^{a} \frac{d s}{\sqrt{b-V(s)}} \tag{5.5}
\end{equation*}
$$

We suppose that $\tau$ is a given function of $a$ (or $b$ ) and therefore of $b$, denoting the latter by $h \mapsto \hat{\tau}(h)$. We assume that $\tau$ is continuously differentiable and that $\tau(0)=0$. Let us make the invertible change of variables $v=V(s)$, denoting the solution of this equation by $s=\sigma(v)$. Then (5.5) yields the following Abel integral equation for $\sigma^{\prime}$ :

$$
\begin{equation*}
\hat{\tau}(h)=2 \sqrt{2} \int_{0}^{b} \frac{\sigma^{\prime}(v) d v}{\sqrt{b-v}} \tag{5.6}
\end{equation*}
$$

The solution of this equation (which uses elementary methods) [16, pp. 71 ff .] is given by

$$
\begin{equation*}
\sigma^{\prime}(v)=\frac{1}{\pi} \frac{d}{d v} \int_{0}^{v} \frac{\hat{\tau}(b) d b}{\sqrt{v-h}} . \tag{5.7}
\end{equation*}
$$

$V, \Gamma, \mu$ are thus immediately determined from (5.7), with $\mu$ enjoying the provisional restrictions imposed on it. If $\hat{\tau}$ lacks the mild restrictions imposed on it, then the density need not satisfy the provisional restrictions. (For a detailed generalization of the method of this section, see [25].)

## 6. - The below-ground brachistochrone

Given two points $\boldsymbol{a}$ and $\boldsymbol{b}$ fixed in a spinning planet, we seek a tunnel, if any, by which a particle can move from $\boldsymbol{a}$ to $\boldsymbol{b}$ in the shortest time solely under the action of the gravitational and centrifugal forces. (The tunnel's curve is the brachistochrone, from the Greek for "shortest time".) We assume that the tunnel is frictionless. (There are several papers cited in Section 9 that treat the classical above-ground brachistochrone with friction. Their methods could presumably be imported to handle our problem in the presence of friction.) We accordingly take $f=0$, and use the energy equation corresponding to (4.2):

$$
\begin{equation*}
\frac{1}{2}\left(\boldsymbol{r}_{s} \cdot \boldsymbol{r}_{s}\right) \hat{s}_{t}(t)^{2}-\frac{1}{2} \omega^{2}\left[|\boldsymbol{r}|^{2}-(\boldsymbol{r} \cdot \boldsymbol{k})^{2}\right]+\Gamma(|\boldsymbol{r}|)=b \quad(\text { const }) . \tag{6.1}
\end{equation*}
$$

We limit our attention to twice continuously differentiable curves parametrized so that $\boldsymbol{r}_{s}$ vanishes nowhere. We take $l>0$ so that

$$
\begin{equation*}
\boldsymbol{r}(0)=\boldsymbol{a}, \quad \boldsymbol{r}(l)=\boldsymbol{b} \tag{6.2}
\end{equation*}
$$

We prescribe the initial speed

$$
\begin{equation*}
\left|\hat{s}_{t}(0) \boldsymbol{r}_{s}(0)\right|=c \tag{6.3}
\end{equation*}
$$

relative to the planet. Then $b$ is a given function of $c$ and $\boldsymbol{a}$ :

$$
\begin{equation*}
h=\frac{1}{2} c^{2}-\frac{1}{2} \omega^{2}\left[|\boldsymbol{a}|^{2}-(\boldsymbol{a} \cdot \boldsymbol{k})^{2}\right]+\Gamma(|\boldsymbol{a}|) . \tag{6.4}
\end{equation*}
$$

On the closure of any open time interval on which $\hat{s}_{t}$ vanishes nowhere, $\hat{s}$ has an inverse $s \mapsto \hat{t}(s)$, with $\hat{t}_{s}(s)=1 / \hat{s}_{t}(\hat{t}(s))$. Thus the time needed for a particle to move along
the curve $\boldsymbol{r}$ from $\boldsymbol{a}$, starting with an initial speed $c$ relative to the planet, to $\boldsymbol{b}$ on such an interval is

$$
\begin{align*}
T[\boldsymbol{r}, l] & :=\int_{0}^{l} \frac{d s}{\hat{s}_{t}(\hat{t}(s))}=\int_{0}^{l} \frac{\sqrt{\boldsymbol{r}_{s} \cdot \boldsymbol{r}_{s}}}{\sqrt{u(\boldsymbol{r})}} d s,  \tag{6.5}\\
u(\boldsymbol{r}) & :=2 h+\omega^{2}\left[|\boldsymbol{r}|^{2}-(\boldsymbol{r} \cdot \boldsymbol{k})^{2}\right]-2 \Gamma(|\boldsymbol{r}|) .
\end{align*}
$$

( $T[\boldsymbol{r}, l]$ could be infinite.) We seek a twice continuously differentiable curve $\boldsymbol{r}$ satisfying the boundary conditions (6.2) with $|\boldsymbol{a}|,|\boldsymbol{b}| \leq R$ that minimizes $T$. Rather than requiring that $|\boldsymbol{r}| \leq R$ a priori, we verify this inequality a posteriori. (Alternatively, we could merely assume that $\boldsymbol{r}$ lies entirely within a larger planet. In either case, we avoid dealing with the change in the form of the gravitation attraction at the surface of the planet.)

It is important to note that we are fixing $l$, but not fixing the parametrization of $\boldsymbol{r}$. Finding a minimizing $\boldsymbol{r}$ gives its parametrization. If, on the other hand, we were to fix the parametrization, e.g., by using the arc-length parametrization so that $\left|\boldsymbol{r}_{s}\right|=1$, then we would have to handle this restriction as a constraint (preferably in the form $\boldsymbol{r}_{s} \cdot \boldsymbol{r}_{s}=1$ ) and introduce a suitable Lagrange multiplier. In this case $l$ would not be known a priori.

If $T$ has a minimizer $\overline{\boldsymbol{r}}$ over the class of twice continuously differentiable functions $\boldsymbol{r}$ satisfying (6.2) and satisfying $\left|\boldsymbol{r}_{s}(s)\right|>0$ for all $s \in[0,1]$, then $\overline{\boldsymbol{r}}$ must satisfy the EulerLagrange equation :

$$
\begin{equation*}
\frac{d}{d s}\left[\frac{\boldsymbol{r}_{s}}{\left|\boldsymbol{r}_{s}\right| \sqrt{u}}\right]=-\frac{\left|\boldsymbol{r}_{s}\right| u_{r}}{2 u^{3 / 2}}, \quad s \in(0, l) . \tag{6.6}
\end{equation*}
$$

We call a solution of (6.6) satisfying the boundary conditions (6.2) an extremal. We eschew a full treatment of the variational problem for (6.5), with its existence theory, regularity theory, second-variation tests, etc., commenting briefly on these matters in Section 9. We content ourselves with the study of (6.6), which will in fact yield nonexistence and nonuniqueness theorems.

## 7. - Analysis of the euler-lagrange equations

For computational simplicity and for visualization, it is convenient to take $s$ to be the arc-length parameter, in which case $\left|\boldsymbol{r}_{s}\right|=1$, and the resulting simplification of (6.6) is equivalent to the system

$$
\begin{equation*}
\boldsymbol{r}_{s}=\sqrt{u} \boldsymbol{p}, \quad \boldsymbol{p}_{s}=-\frac{u_{\boldsymbol{r}}}{2 u^{3 / 2}} . \tag{7.1}
\end{equation*}
$$

This system admits the integral

$$
\begin{equation*}
|\boldsymbol{p}|^{2}-\frac{1}{u}=\text { const }, \tag{7.2}
\end{equation*}
$$

which $(7.1)_{1}$ shows to be nothing more than a consequence of $\left|\boldsymbol{r}_{s}\right|=1$, which ensures that the constant should be 0 .

Remark. It is easy to see that (7.1) is not Hamiltonian. We can convert (6.6) into the very simple Hamiltonian form first by setting $\boldsymbol{p}:=\frac{\boldsymbol{r}_{s}}{\left|\boldsymbol{r}_{s}\right| \sqrt{u}}$ so that this definition and (6.6) yield the first-order system

$$
\begin{equation*}
\boldsymbol{r}_{s}=\left|\boldsymbol{r}_{s}\right| \sqrt{u} \boldsymbol{p}=: \lambda \boldsymbol{p}, \quad \boldsymbol{p}_{s}=-\frac{\left|\boldsymbol{r}_{s}\right| u_{r}}{2 u^{3 / 2}}=-\frac{\lambda u_{r}}{2 u^{2}} . \tag{7.3}
\end{equation*}
$$

Since $\lambda>0$, we introduce a new independent variable $\sigma=\tilde{\sigma}(s)$ as the integral of

$$
\begin{equation*}
\tilde{\sigma}_{s}=\lambda=\left|\boldsymbol{r}_{s}\right| \sqrt{u} . \tag{7.4}
\end{equation*}
$$

We denote $\boldsymbol{r}(\tilde{s}(\sigma))$ by $\tilde{\boldsymbol{r}}(\sigma)$, etc., and we introduce the Hamiltonian function

$$
\begin{equation*}
H(\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{p}})=\frac{1}{2}|\tilde{\boldsymbol{p}}|^{2}-\frac{1}{2 u} . \tag{7.5}
\end{equation*}
$$

Then (7.1) reduces to the Hamiltonian equations:

$$
\begin{equation*}
\tilde{\boldsymbol{r}}_{\sigma}=H_{\tilde{\boldsymbol{p}}} \equiv \tilde{\boldsymbol{p}}, \quad \tilde{\boldsymbol{p}}_{\sigma}=-H_{\tilde{\boldsymbol{r}}} \equiv-\frac{u_{\tilde{\boldsymbol{r}}}}{2 u^{2}} . \tag{7.6}
\end{equation*}
$$

We immediately see that if $\tilde{\boldsymbol{r}}, \tilde{\boldsymbol{p}}$ satisfy this first-order system, then

$$
\begin{equation*}
2 H=|\tilde{\boldsymbol{p}}|^{2}-\frac{1}{u}=\text { const } ; \tag{7.7}
\end{equation*}
$$

cf. (7.2). (The formulation leading to (7.6) is due to Carathéodory [5, Chap. 14].)
Remark. System (7.6) is more attractive than system (7.1) because its equations are simpler and because it has Hamiltonian structure. For our present discussion, there is no need for the Hamiltonian structure. Despite its slight complications, we shall use (7.1) because it simplifies the initial condition (6.3), it simplifies initial conditions complementary to it, which are used in the shooting method in the next section, and for the shooting method it allows the range of $s$ to be estimated in terms of the radius $R$.

Remark. The change of variables (7.4) could have been made in the functional (6.5), the form of whose integrand is invariant under any such change of variables. The conversion of (6.5) to an arc-length parametrization, however, cannot be carried out directly: Simply replacing the numerator of the integrand of the second integral in (6.5) produces a degeneracy. Instead, the condition that $\left|\boldsymbol{r}_{s}\right|^{2}=1$ must be appended as a constraint, and the Lagrange Multiplier Rule must be invoked to yield Euler-Lagrange equations, which involve a scalar Lagrange multiplier function of $s$. Its elimination yields (7.1). We omit the details.

Let us describe $\boldsymbol{r}$ and $\boldsymbol{p}$ by cylindrical coordinates:

$$
\begin{equation*}
\boldsymbol{r}(s)=\rho(s) \boldsymbol{e}_{1}(\phi(s))+z(s) \boldsymbol{k}, \quad \boldsymbol{p}(s)=p_{1}(s) \boldsymbol{e}_{1}(\phi(s))+p_{2}(s) \boldsymbol{e}_{2}(\phi(s))+p_{3}(s) \boldsymbol{k} . \tag{7.8}
\end{equation*}
$$

Then

$$
\begin{align*}
\boldsymbol{r}_{s} & =\rho_{s} \boldsymbol{e}_{1}+\rho \phi_{s} \boldsymbol{e}_{2}+z_{s} \boldsymbol{k}, \\
\boldsymbol{p}_{s} & =\left(p_{1 s}-p_{2} \phi_{s}\right) \boldsymbol{e}_{1}+\left(p_{2 s}+p_{1} \phi_{s}\right) \boldsymbol{e}_{2}+p_{3 s} \boldsymbol{k}, \\
u & =2 h+\omega^{2} \rho^{2}-2 \Gamma\left(\sqrt{\rho^{2}+z^{2}}\right),  \tag{7.9}\\
u_{r} & =2\left[\omega^{2} \rho \boldsymbol{e}_{1}-\gamma\left(\sqrt{\rho^{2}+z^{2}}\right)\left(\rho \boldsymbol{e}_{1}+z \boldsymbol{k}\right)\right],
\end{align*}
$$

and (7.1) yields

$$
\begin{gather*}
\rho_{s}=\sqrt{u} p_{1}, \quad \rho \phi_{s}=\sqrt{u} p_{2}, \quad z_{s}=\sqrt{u} p_{3},  \tag{7.10}\\
p_{1 s}-p_{2} \phi_{s}=\rho\left[\gamma-\omega^{2}\right] u^{-3 / 2}, \\
p_{2 s}+p_{1} \phi_{s}=0,  \tag{7.11}\\
p_{3 s}=\gamma z u^{-3 / 2} .
\end{gather*}
$$

The substitution of (7.10) into $(7.11)_{2}$ yields $\left(u^{-1 / 2} \rho^{2} \phi_{s}\right)_{s}=0$, whence

$$
\begin{equation*}
u^{-1 / 2} \rho^{2} \phi_{s}=a \quad \text { (const). } \tag{7.12}
\end{equation*}
$$

(If we were to express the integrand in (6.5) in cylindrical coordinates, then the absence of $\phi$ from this integrand (i.e., the ignorability of $\phi$ ) would cause the correspond ing EulerLagrange equation to be equivalent to (7.12). This observation is a special case of Noether's Theorem; cf. [12].)

Equation (7.12) ensures that $\phi_{s}$ has a fixed sign. Thus a minimizing tunnel (if it exists) cannot start out moving west and later go east, or vice-versa. In particular, if the initial and terminal points $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same longitude, i.e., if they have the same coordinate $\phi_{1}$, then a minimizing tunnel must either lie in the $\left(\boldsymbol{e}_{1}\left(\phi_{1}\right), \boldsymbol{k}\right)$-plane, or else (what seems unlikely) wind around the axis with the initial and final values of $\phi$ varying by a nonzero integral multiple of $2 \pi$. On the other hand, if $\boldsymbol{a}$ and $\boldsymbol{b}$ have longitudes differing by $\pi$, i.e., if $\phi_{2}=\phi_{1}+\pi$, say, then, as we shall soon see, the brachistochrone need not be confined to the vertical plane containing these termini. These considerations show that the independent variable $s$ could be replaced by $\phi$ when the initial and terminal points $\boldsymbol{a}$ and $\boldsymbol{b}$ do not have the same longitude.

Equations $(7.10)_{3}$ and $(7.11)_{3}$ imply that

$$
\begin{equation*}
\left[u^{-1 / 2} z_{s}\right]_{s}=\gamma z u^{-3 / 2} . \tag{7.13}
\end{equation*}
$$

Thus, if $\xi$ is a value of $s$ at which $z(\xi)>0$ and $z_{s}(\xi)=0$, then $z_{s s}(\xi)>0$. This means that for $z>0, z$ can have at most one interior minimum and can have no interior maximum. (The analog of $(7.13)$ for $\tilde{z}(\sigma):=z(\tilde{s}(\sigma))$ (cf. (7.4)) is that $\tilde{z}$ is convex.)

Equations (7.10),$(7.11)_{1}$, (7.12) imply that

$$
\begin{equation*}
\left[u^{-1 / 2} \rho_{s}\right]_{s}=a^{2} \sqrt{u} \rho^{-3}+u^{-3 / 2} \rho\left[\gamma-\omega^{2}\right] . \tag{7.14}
\end{equation*}
$$

This equation ensures that $\rho$ can have at most one interior minimum and can have no interior maximum when the gravitational force dominates the centrifugal force in the
sense that $\gamma \geq \omega^{2}$. A similar result holds for $r=\sqrt{\rho^{2}+z^{2}}$. When the gravitational force dominates the centrifugal force, we can ask whether $\rho$ must be monotone.

It is clear that the system (7.10) and (7.11) is compatible with brachistochrones lying in a vertical plane, for which $\phi_{s}=0$. But (7.13) shows that this system is not compatible with brachistochrones lying in a horizontal plane, for which $z_{s}=0$, except at the equator, where the resultant of the gravitational and centrifugal forces is central.

System (7.1) and its equivalent versions (7.9), (7.10), and (7.11) admit the integrals (7.2) and (7.12). These integrals do not suffice to make this system totally integrable, i.e., reducible to a phase-plane analysis, except for motion in the equatorial plane. Indeed, for motion in a vertical plane, for which $a=0$, under the assumption that $z$ is strictly monotone, we can replace the independent variable $s$ with $z$ and reduce this system to a second-order ordinary differential equation for $\rho$, which is not autonomous because of the presence of the independent variable $z$ in $u$. In other words the presence of both $|\boldsymbol{r}|$, corresponding to gravity, and $|\boldsymbol{r}-(\boldsymbol{r} \cdot \boldsymbol{k}) \boldsymbol{k}|$ corresponding to the centrifugal force, in the $u$ of (6.5) (cf. (7.9)) is the obstacle to integrability.

## 8. - You can't always get there from here

Now we illustrate some of the rich possibilities for brachistochrones with numerical and simple qualitative results. Our governing equations are (7.13) and (7.14), or their equivalent system of first-order equations coming from $(7.10)_{1,3},(7.11)_{1,3},(7.12)$. These equations admit the integral (7.2). When $\alpha=0$, (7.13) and (7.14) have a similar structure, at least when $\gamma>\omega^{2}$. We take the density $\mu$ of the planet to be constant, so that $u$ is a nonhomogeneous quadratic form in $\rho$ and $z$. We limit our attention to the case that $\boldsymbol{a}$ and $\boldsymbol{b}$ lie on the boundary of the spherical planet. Our basic dimensionless parameter is $\omega^{2} / \gamma$.

Remark. When $\alpha \neq 0$, we could suppress the nonlinear term $a^{2} \sqrt{u} \rho^{-3}$ in (7.14) by replacing the independent variable $s$ with $\phi$, denoting the inverse of $\phi$ by $\hat{s}$, and then defining $v(\phi):=1 / \rho(\hat{s}(\phi))$. This classical device from the mechanics of a particle in a central force field has no apparent utility because it cannot simplify the coupled nonlinear ities on the right-hand sides of (7.13) and (7.14).

The starting point of our analysis is the observation that a trajectory for the EulerLagrange ordinary differential equations entering the planet at a prescribed point $\boldsymbol{a}$ on its boundary with a prescribed initial speed and having its first exit at some point $\boldsymbol{b}$ on its boundary determines a solution of these equations satisfying the boundary conditions (6.2). In other words, given such a trajectory for an initial-value problem, we can trivially identify the boundary-value problem that it satisfies. This trajectory is an extremal, i.e, a candidate for a brachistochrone, for which we can readily compute the time lapse (6.5).


Fig. 2. - Trajectories for $\omega^{2} / \gamma=0$ and $\omega^{2} / \gamma=0.7$ of extremals in a vertical plane that start with zero speed at a latitude making angle $\frac{\pi}{4}$ with the north pole. The axis of spin is vertical. The white regions near the initial point are filled with trajectories that are not shown.

In Figure 2 for $\omega^{2} / \gamma=0$ and $\omega^{2} / \gamma=0.7$ we show two sets of extremals in a vertical plane that start with zero speed at a latitude making angle $\frac{\pi}{4}$ with the north pole. When there is no spin, the left figure indicates that every point on and within the great circle through the north pole and the starting point are accessible. (We have not exhibited very shallow trajectories.) In this case there is a diametral extremal. The right figure shows what happens when there is some spin. The two nearly horizontal curves form the loci of points at which the speed on the extremals drops to zero, and at which the phase portrait would show that the motion continues in the opposite direction. Thus there are segments of the great circle that are inaccessible to such trajectories. If there were a brachistochrone capable of reaching points on these segments, then the brachistochrone could not lie in a vertical plane, and by symmetry there would have to be two of them. Note that the trajectories that terminate on the nearly horizontal curves in the right side of Figure 2 have the special features that they terminate with zero speed and that their reversal represents solutions starting and ending with zero speed. There are exactly three such trajectories with termini on the great circle.

For the classical above-ground brachistochrone with zero initial speed, the accessible termini are precisely those lying at or below the height of the starting point. For our problem, the issue of accessibility is more complicated:

It is not fruitful to explain the inaccessibility of points in terms of the gravitational and centrifugal forces, because their effect on the particle is due to their projection onto the unknown tangent to the tunnel at the point occupied by the particle. Instead, we note that the energy equation (6.1), (6.4) with $\gamma$ constant implies that

$$
\begin{equation*}
\left(\gamma-\omega^{2}\right)\left[|\boldsymbol{r}|^{2}-(\boldsymbol{r} \cdot \boldsymbol{k})^{2}\right]+\gamma(\boldsymbol{r} \cdot \boldsymbol{k})^{2} \leq 2 h \equiv c^{2}+\left(\gamma-\omega^{2}\right)\left[|\boldsymbol{a}|^{2}-(\boldsymbol{a} \cdot \boldsymbol{k})^{2}\right]+\gamma(\boldsymbol{a} \cdot \boldsymbol{k})^{2} . \tag{8.1}
\end{equation*}
$$

Denote the set of $\boldsymbol{r}$ satisfying this inequality by $\mathcal{A}$. An accessible terminal point $\boldsymbol{b}$ must lie in $\mathcal{A}$. (In the next paragraph we give simple sufficient conditions in a special case ensuring that there actually are extremals going to certain termini $\boldsymbol{b}$ in $\mathcal{A}$.) The boundary $\partial \mathcal{A}$ of $\mathcal{A}$ consists of the locus of points having zero terminal speed. If $\gamma>\omega^{2}$, then $\partial \mathcal{A}$ is an ellipsoid. The version of (8.1) corresponding to the left side of Figure 2 is obtained by setting $\gamma=0, c=0$, in which case $\mathcal{A}$ is the ball identical to the region occupied by the planet. $\partial \mathcal{A}$ corresponding to the right side of Figure 2 is an oblate ellipsoid, its intersection with the region of the vertical plane occupied by the planet is the region filled with trajectories. If $\gamma=\omega^{2}$, then $\mathcal{A}$ is the slab bounded by the planes $x_{3}= \pm \sqrt{2 h / \gamma}$. If $\gamma<\omega^{2}$, then $\partial \mathcal{A}$ is a hyperboloid. If, furthermore, $b>0$, then $\partial \mathcal{A}$ is a hyperboloid of two sheets, with $\mathcal{A}$ containing an interval along the $\boldsymbol{k}$-axis centered at the origin. If $b=0$, then $\partial \mathcal{A}$ is a cone of two sheets symmetric about the $\boldsymbol{k}$-axis with $\mathcal{A}$ intersecting this axis only at the origin. If $b<0$ (by virtue of a sufficiently large spin), then $\partial \mathcal{A}$ is a hyperboloid of one sheet symmetric about the $\boldsymbol{k}$-axis, with $\mathcal{A}$ lying outside of it. In this case, there is a region centered about the axis of spin in which no brachistochrone can enter. This hyperboloid intersects the $\left\{\boldsymbol{i}_{1}, \boldsymbol{i}_{2}\right\}$-plane in a circle of radius $A$, with

$$
\begin{equation*}
\left(\gamma-\omega^{2}\right) A^{2}=2 b \equiv c^{2}+\left(\gamma-\omega^{2}\right)\left[R^{2}-(\boldsymbol{a} \cdot \boldsymbol{k})^{2}\right]+\gamma(\boldsymbol{a} \cdot \boldsymbol{k})^{2}, \tag{8.2}
\end{equation*}
$$

by (8.1), where $R$ is the radius of the planet and $\boldsymbol{a}$ is taken to lie on the surface of the planet. Thus $A \leq R$ with equality only when $c=0=\boldsymbol{a} \cdot \boldsymbol{k}$. Except when $c=0=\boldsymbol{a} \cdot \boldsymbol{k}$, there is a belt of accessible terminal points on the surface of the planet centered on the equator, and $\mathcal{A}$ penetrates into a ring-like region. This means that if $b<0$, then there is an arc of accessible points in the opposite hemisphere lying on the great circle through $\boldsymbol{a}$ and the poles that cannot be reached by brachistochrones lying in the plane of the great circle. By symmetry, there must be at least two brachistochrones reaching each such accessible point.

We now determine when extremals meet prescribed boundary conditions, an issue we have so far avoided. For generality, let us temporarily allow the density $\mu$ to be variable. Equations (7.14) and (7.12), and (7.9) 3 yield

$$
\begin{align*}
{\left[u^{-1 / 2} \rho_{s}\right]_{s} } & =a^{2} \sqrt{u} \rho^{-3}+u^{-3 / 2}\left[\gamma(\rho)-\omega^{2}\right] \rho, \quad \phi_{s}=\alpha u^{1 / 2} \rho^{-2}, \\
u & =2 h+\omega^{2} \rho^{2}-2 \Gamma(\rho), \quad 2 h=c^{2}-\omega^{2} R^{2}+2 \Gamma(R) \tag{8.3}
\end{align*}
$$

when the tunnel is taken to start on the equator with $s=0$. We take it to start here at $\phi=0$. We multiply $(8.3)_{1}$ by $u^{-1 / 2} \rho_{s}$ to obtain the integral

$$
\begin{equation*}
\rho_{s}^{2}+\rho^{2} \phi_{s}^{2}-1+\delta u \equiv \rho_{s}^{2}+a^{2} u \rho^{-2}-1+\delta u=0 \tag{8.4}
\end{equation*}
$$

where $\delta$ is a constant that we recognize as zero because $(7.9)_{1}$ and the assumption that $\left|\boldsymbol{r}_{s}\right|=1$ implies that $\rho_{s}{ }^{2}+\rho^{2} \phi_{s}{ }^{2}=1$. (In other words, the integral (8.4) with $\delta=0$ is just a statement that $s$ is an arc-length parameter; cf. the remark following (7.2)). Thus, when $\gamma$ is a constant, (8.4) yields

$$
\begin{equation*}
\rho_{s}^{2}+\rho^{2} \phi_{s}^{2}=1, \quad \rho^{2} \phi_{s}^{2}=a^{2} \rho^{-2}\left[c^{2}+\left(\omega^{2}-\gamma\right)\left(\rho^{2}-R^{2}\right)\right] . \tag{8.5}
\end{equation*}
$$

Note that $R^{2} \phi_{s}(0)=a c$ by $(8.3)_{2}$, so that $\alpha$ determines $\phi_{s}(0)$. But (8.5) implies that it also determines $\rho_{s}(0)$ (because $\rho_{s}(0)$ must be $\leq 0$ if the tunnel is to begin within the planet.) In light of the development surrounding (8.1) and (8.2), we henceforth limit our attention to the case that

$$
\begin{equation*}
\lambda:=\omega^{2}-\gamma>0 \quad \text { so that } \quad 2 b=c^{2}-\lambda R^{2}, \quad \rho^{2} \phi_{s}^{2}=a^{2} \rho^{-2}\left[2 h+\lambda \rho^{2}\right] \tag{8.6}
\end{equation*}
$$

For each fixed set of parameters $\alpha^{2}, c^{2}, \omega^{2}-\gamma,(8.5)$ gives the level curve in the $\left(\rho, \rho_{s}\right)$ phase plane on which lie the trajectories for (8.3). Any level curve that corresponds to a trajectory that is a candidate for a brachistochrone (or more generally for an extremal) (i) must intersect the line $\rho=R$ because the initial and terminal points of the brachistochrone must lie on the equator and (ii) must connect these intersection points with a segment lying to the left of this line. The inequality $R^{2} \phi_{s}(0)^{2} \leq 1$, coming from the evaluation of $(8.5)_{1}$ at $\rho=R$, yields

$$
\begin{equation*}
a^{2} c^{2} \leq R^{2} \tag{8.7}
\end{equation*}
$$

The discussion following (8.5) shows that this inequality is a restriction on the initial direction.

Let us first study the case that $h>0$. Then $\rho^{2} \phi_{s}^{2}$ approaches $\infty$ as $\rho \searrow 0$, and the phase portrait of (8.4) shows that it meets requirements (i) and (ii). In particular, $\rho$ must have a positive lower bound on the level curve (8.5), so that there is a core around the axis that cannot be penetrated by brachistochrones, as we have already seen. Since $b>0$, inequality (8.7) yields

$$
\begin{equation*}
B:=1-a^{2} \lambda>0 . \tag{8.8}
\end{equation*}
$$

(This inequality comes directly from (8.5) by replacing $\rho_{s}{ }^{2}$ with 0 .) Equation (8.6) 3 implies that $\rho^{2} \phi_{s}^{2}$ asymptotically approaches $a^{2} \lambda$ as $\rho \rightarrow \infty$. Thus (8.8) implies that (8.5) ${ }_{1}$ describes a U -shaped curve in the $\left(\rho, \rho_{s}\right)$-phase plane that is symmetric about the $\rho$-axis and opens to the right. Equation (8.5) and the positivity of $h$ give an explicit lower bound for $\rho$ :

$$
\begin{equation*}
\rho^{2} \geq \frac{2 b a^{2}}{B}>0 \tag{8.9}
\end{equation*}
$$

If $h<0$, i.e., if $c^{2}<\lambda R^{2}$, then $\rho^{2} \phi_{s}^{2}$ approaches $-\infty$ as $\rho \searrow 0$. In this case, the phase portrait of (8.5) consists of curves asymptotic to the $\pm \rho_{s}$ axis on which $\rho$ has a single maximum, at the $\rho$-axis. If such a curve intersects the line $\rho=R$, then the intersection points are joined by a curve on which $\rho>R$. Therefore, (8.5) cannot meet the requirements (i) and (ii). If $b=0$ so that $c^{2}=\lambda R^{2}$, then $\rho_{s}^{2}=1-a^{2} \lambda$. If $1-a^{2} \lambda>0$, then the level curves are two lines parallel to the $\rho$-axis in the phase plane, so there can be no continuously differentiable brachistochrone starting and ending on the equator. If $1-a^{2} \lambda<0$, then this equation has no real solutions. If $1-a^{2} \lambda=0$, then $\rho$ must be the constant $R$, and $(8.3)_{1}$ implies that $\phi_{s}= \pm 1 / R$. This degenerate case corresponds to an equatorial tunnel. Since this degenerate case is the only novel result that holds when $b \leq 0$, we henceforth limit our attention to the case that $b>0$.

To determine a brachistochrone that terminates at the equator at a prescribed point with angle $\phi=\psi$, we must choose the initial direction determined by $\alpha$ so that

$$
\begin{equation*}
\psi=a \int_{0}^{l} u^{1 / 2} \rho^{-2} d s \tag{8.10}
\end{equation*}
$$

where $\rho$ satisfies (8.5) with $\rho(l)=R$. To exploit (8.10) we observe that (8.5) implies that

$$
\begin{equation*}
\left\{B \rho^{2}-2 b a^{2}\right\}^{-1 / 2}\left(\rho^{2}\right)_{s}= \pm 2 \tag{8.11}
\end{equation*}
$$

Note that the discussion leading to (8.9) ensures that the argument of the square root in (8.11) is positive.

We take $\rho(0)=R=\rho(l)$. The symmetry of the loop (8.5) about the $\rho$-axis implies that $\rho$ has its minimum at $s=\frac{1}{2} l$. The elementary integration of (8.11) gives

$$
\rho^{2}= \begin{cases}R^{2}-2 \sqrt{R^{2}-a^{2} c^{2}} s+B s^{2} & \text { for } \quad 0 \leq s \leq \frac{1}{2} l  \tag{8.12}\\ R^{2}-2 \sqrt{R^{2}-a^{2} c^{2}}(l-s)+B(l-s)^{2} & \text { for } \quad \frac{1}{2} l \leq s \leq l\end{cases}
$$

Equation (8.12) implies that $\rho^{2} \leq R^{2}$, as required. Note that (8.11) implies that $\rho_{s}\left(\frac{1}{2} l\right)=0$ and that $\rho^{2}\left(\frac{1}{2} l\right)$ is exactly the second term in (8.9), so (8.9) is sharp. Equating (8.9) with (8.12) at $s=\frac{1}{2} l$ gives

$$
\begin{equation*}
\frac{l}{2}=\frac{\sqrt{R^{2}-a^{2} c^{2}}}{B} . \tag{8.13}
\end{equation*}
$$

To evaluate the integral in (8.10), we use the symmetry of the level curve of (8.5) about the $\rho$-axis to write the integral as twice that over $\left[\frac{1}{2} l, l\right]$. By making the change of variables $\zeta=\rho(s)^{2}$ we can write (8.10) as

$$
\begin{align*}
\psi= & 2 a \int_{l / 2}^{l} \frac{\sqrt{2 h+\lambda \rho^{2}}}{\rho^{2}} d s \\
= & a \int_{2 b a^{2} / B}^{R^{2}} \frac{\sqrt{\lambda B \zeta^{2}+2 h\left[1-2 a^{2} \lambda\right] \zeta-(2 b)^{2} a^{2}}}{\zeta} d \zeta  \tag{8.14}\\
= & a c \sqrt{R^{2}-a^{2} c^{2}}+2 h a|a|\left[\frac{\pi}{2}-\arcsin \left(1-\frac{2 c^{2} a^{2}}{R^{2}}\right)\right] \\
& +\frac{a b\left[1-2 a^{2} \lambda\right]}{\sqrt{\lambda B}} \ln \frac{\sqrt{\lambda B c^{2}\left(R-a^{2} c^{2}\right)}+\lambda B R^{2}+b\left[1-2 a^{2} \lambda\right]}{2 a^{2} \lambda b+\left[1-2 a^{2} \lambda\right] b} .
\end{align*}
$$

As $a$ increases from 0 to its upper limit $R / c$ allowed by (8.7), the parameter $B$ decreases from 1 to $2 h / c^{2}$, and the right-hand side of (8.14), which is continuous in $a$, varies from 0 to $2 \pi h R^{2} / c^{2}$. The Intermediate-Value Theorem then implies that if $|\psi|<2 \pi h R^{2} / c^{2}$, then (8.14) has at least one solution $a$, and there is an extremal in the equatorial plane joining the point with polar coordinates $(R, 0)$ with the point $(R, \psi)$. In particular if $2 h R^{2} / c^{2}>1$, then there is an extremal terminating at $(R, \pi)$, and by symmetry, there must be at least two. The lack of monotonicity of the function $\alpha \mapsto a c \sqrt{R^{2}-a^{2} c^{2}}$ suggests that for appropriate ranges of the parameters there can be multiple extremals reaching other given terminal points.

An overly simplified characterization of our findings is that the larger $\omega^{2} / \gamma$, the smaller the accessible region, and the larger $c$, the larger the accessible region. The momentum associated with a large $c$ enables the particle to overcome much of the effect of the gravitational and centrifugal force.

## 9. - Comments

The development of the concept of energy, which pervades our study, was attributed by Whittaker [28, p. 62] to Huygens, Newton, Joh. and D. Bernoulli, and Lagrange. Euler can certainly be added to this list. Whittaker's classical book, among many, can be consulted for the basic mechanics used in this paper, albeit presented with a cumbersome notation.

There have been several classical and modern variations on Joh. Bernoulli's brachistochrone problems. Euler [10] treated the brachistochrone for a velocitydependent frictional force; cf. [13, p. 78 ff .]. For other treatments of brachistochrone problems with friction, see [2, 14, 17]. For the brachistochrone problem for a bead on a wire located outside an attracting spherically symmetric planet, in which case the attraction would be governed by the inverse-square law, see [20]. For the brachistochrone inside a stationary planet, see [8, 24]. For a relativistic version of the brachistochrone problem see [11]. Among the novelties of our work (as in [18]) is the competition between gravitational and centrifugal forces, and its consequence that brachistochrones are typically space curves. These curves are governed by systems of ordinary differential equations whose solutions cannot be determined by a phase-plane analysis, unlike our variation of Newton's problem. (In a somewhat antiquated locution, problems reducible to a phase-plane analysis are termed integrable by quadratures. Now such problems are called totally integrable.)

In this connection, we note that the classical brachistochrone is traditionally described as the graph of a function, and then a change of variables gives it a parametric representation as a cycloid. If this brachistochrone is initially given a parametric representation, as we do for our generalization, then the parametric representation for the cycloid appears directly [5]. Our brachistochrone problem does not appear to admit a second-order equation for a graph in Cartesian or cylindrical or spherical coordinates.

Our brachistochrone problem is an example of a parametric problem in the Calculus of Variations. Such problems provide technical obstacles for their analysis not present for classical nonparametric problems. The proof that our brachistochrone problem has a minimum can be based on the existence theorems for parametric problems stated in Chapter 14 of Cesari's book [6]. Their full proofs were to have been presented in a sequel to [6], which Cesari never lived to complete. Many can no doubt be found in his published papers. The most comprehensive treatment for parametric problems of necessary conditions for minimization and of sufficient conditions for solutions of the Euler-Lagrange equations to be local minimizers is given by Bliss [4]. The techniques presented in [4] could be used to supplement the theory of Sections 7 and 8.

There are numerous other variants of the brachistochrone problem: The admissible curves could be required to have a prescribed length (which of course, must exceed the distance between the starting and finishing points). In this case the problem is isoperimetric. We could constrain the admissible curves to lie in a prescribed plane or even a prescribed surface.

A brachistochrone for a given initial speed is typically is not one for another initial speed. As in Section 4, we could study motions in a brachistochrone tunnel for a given initial speed for different initial data.

And finally, why not y? A few books spell "brachistochrone" as "brachystochrone" apparently because "brachy" comes from the Greek adjective $\beta \rho a \chi$ ט́s for "short", with upsilon $v$ the vowel following $\chi$. But the superlative form of this Greek adjective is $\beta \rho a ́ \chi \iota \sigma \tau o \varsigma$ with an iota $l$ following $\chi$. Hence the i is appropriate. (See the Oxford English Dictionary. Stefan Hildebrandt kindly pointed out to us why i should be used.)

Acknowledgments. The preparation of this work was supported in part by a regular grant and a VIGRE grant from the NSF. The work of Crosswhite was conducted during his undergraduate years at the University of Maryland.

## REFERENCES

[1] N. H. Abel, Auflösung einer mechanischen Aufgabe, Crelle (1936); Résolution d'un problème de mécanique, Oeuvres, Vol. 1 (1881), 97-101.
[2] N. Ashby - W. E. Britten - W. F. Love - W. Wyss, Brachistochrone with Coulomb friction, Amer. J. Phys., 43 (1975) 902-906.
[3] Joh. Bernoulli, Acta Eruditorum = Opera, 1, 161.
[4] G. A. Bliss, Calculus of Variations, Open Court, 1925.
[5] C. Carathéodory, Calculus of Variations and Partial Differential Equations of the First Order, 2nd edn., Chelsea, 1982.
[6] L. Cesari, Optimization-Theory and Applications, Springer, 1983.
[7] S. Chandrasekhar, Newton's Principia for the Common Reader, Oxford Univ. Pr., 1995.
[8] A. Dorsey, Brachistochrone inside the earth, Lecture notes, (http://www.phys.ufl.edu/~dorsey/ phys4222/ notes/brach_inside.pdf.)
[9] C. Huygens, Horologium Oscillatorium, 1673.
[10] L. Euler, Methodus Inveniendi Lineas Curvis Maximi Minimivi Proprietate Gaudentes sive Solutio Problematis Isoperimetrici Latissimo Sensu Accepti, Bousquent, Lausanne, 1744, in Opera Omnia, Ser. I, Vol. 24.
[11] F. Giannoni - P. Piccione, The arrival time brachistochrones in general relativity, J. Geom. Anal., 12 (2002), 375-423.
[12] M. Giaquinta - S. Hildebrandt, Calculus of Variations, Vols. I, II. Springer, 1996.
[13] H. H. Goldstine, A History of the Calculus of Variations from the 17th through the 19th Century, Springer, 1980.
[14] L. Hawn - T. Kiser, Exploring the brachistochrone problem, Amer. Math. Monthly, 102 (1995), 328-336.
[15] O. D. Kellogg, Foundations of Potential Theory, Springer, 1929.
[16] P. Linz, Analytical and Numerical Methods for Volterra Equations, SIAM, 1985.
[17] S. C. Lipp, Brachistochrone with Coulomb friction, SIAM J. Control Optim., 35 (1997), 562584.
[18] P. Maisser, Brachystochronen als zeitkürreste Fabrspuren von Bobschitten, Z. Angew. Math. Mech., 78 (1998), 311-319.
[19] I. Newton, Philosophiae Naturalis Principia Mathematica, 1st edn. 1687, 2nd edn. 1713, 3rd edn. 1726.
[20] A. Parnovsky, Some generalizations of brachistochrone problem, Acta Phys. Pol. A93 Supplement (1998) S-55. (http://info.ifpan.edu.pl/firstep/aw-works/fsV/parnovsky/parnovsky.pdf)
[21] B. N. Pshenichny - Yu. M. Danilin, Numerical Methods in Extremal Problems, Mir, 1978.
[22] R. Resnick - D. Halliday - K. S. Krane, Pbysics, 4th edn., Wiley, 1992.
[23] N. Rouche - P. Habets - M. Laloy, Stability Theory by Liapunov's Direct Method, Springer, 1977.
[24] D. R. Smith, Variational Methods in Optimization, Prentice-Hall, 1974.
[25] M. Urabe, Nonlinear Autonomous Oscillations, Academic Pr., 1967.
[26] R. S. Westrall, Never at Rest, A Biography of Isaac Newton, Cambridge Univ. Pr., 1980.
[27] D. T. Whiteside, ed., The Mathematical Papers of Isaac Newton, Cambridge Univ. Pr., 19671981.
[28] E. T. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, 4th edn., Cambridge Univ. Pr., 1937.

