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Second Order Transmission Problems Across a Fractal Surface (**)

ABSTRACT. — A second order transmission problem across either a fractal layer S or the corresponding prefractal layer S_h is studied. Existence, uniqueness and regularity results for the weak solution, in both cases, are established.

Problemi di trasmissione del secondo ordine attraverso una superficie frattale

SUNTO. — Si studia un problema di trasmissione del secondo ordine in cui lo strato è una superficie frattale S oppure la corrispondente superficie prefrattale S_h. Si provano risultati di esistenza, unicità e regolarità della soluzione variazionale sia nel caso frattale che prefrattale.

INTRODUCTION

There is a huge literature dealing with transmission problems. Transmission problems arise naturally in various fields (see [44]). For instance, in electrostatics and magnetostatics the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem (in this regard see the paper by Pham Huy and Sanchez Palencia [42] and the references listed in).

Another completely different field is that of «hydraulic fracturing» (see the paper of Cannon and Meyer [9]) used in order to increase the flow of oil from a reservoir into a producing oil well. We refer to [9] for more details. Further examples can be found in Dautray and Lions [11].

In all these applications, the mathematical model is an elliptic or parabolic bound-

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ary value problem involving a transmission condition on the interface (layer) either of order zero, one or two.

In this paper we will deal with a model problem considered in [42]. This is a second order transmission problem with a «flat» smooth layer, formally stated as

$$(P_0)\quad \begin{cases} -\Delta u = f & \text{in } \Omega^i,\ i=1,2 \qquad j) \\ -c_0 \Delta_t u = \left[\dfrac{\partial u}{\partial n}\right] & \text{on } S, \qquad jj) \\ u = 0 & \text{on } \partial\Omega, \qquad jjj) \\ u^1 = u^2 & \text{on } S, \qquad jv) \\ u = 0 & \text{on } \partial S \qquad v), \end{cases}$$

where Ω is a regular bounded open set in $\mathbb{R}^3$, say a regular cylinder and S is a cross section, say a disk. S divides Ω in two subsets Ω^1 and Ω^2, u^i denotes the restriction of u to Ω^i, $[u] = u^1 - u^2$ denotes the jump of u across S and $\left[\dfrac{\partial u}{\partial n}\right] = \dfrac{\partial u^1}{\partial n_1} + \dfrac{\partial u^2}{\partial n_2}$ the jump of the normal derivatives across S, n_i being the outward normal vector; Δ is the Laplace operator in $\mathbb{R}^3$ and Δ_t denotes the tangential Laplacian on S, f is a given function in $L^2(\Omega)$.

The existence and the uniqueness of the weak solution of problem P_0 are proved in [42]. No regularity results are proved. As far as we know, the question of the regularity of solutions of this kind of transmission problems has been firstly addressed in [32], where it has been proved that the solution u is continuous in $\overline{\Omega}$, the restrictions u^i are in $H^2(\Omega^i)$ and the transmission condition *jj*) holds in the L^2-sense.

Problems of this type – with a transmission condition of the *second order* as in (P_0) – come out naturally from electrostatics and magnetostatics or from problems of «hydraulic fracturing». In the context of electrostatics or magnetostatics condition *jj*) is typical of infinitely conductive layers and c_0 is a positive constant representing the dielectric constant or the magnetic permeability; in hydraulic fracturing c_0 depends on the permeability of the reservoir and on the fluid properties.

From the point of view of the applications, mentioned before, it would be interesting to study those model-problems in which the absorption of the tensions, electric conduction or flows is the relevant effect and hence the surface effects have to be enhanced. In this context fractal layers provide new interesting settings.

In the present paper we study problem (P_0) when the layer S is not flat but is a fractal surface. In this case, the presence of a fractal interface changes drastically the nature of the problem. While in the problems considered in [42], [9] and in [32] the layer has the same Hausdorff dimension of the boundary of Ω, now in the fractal case, the layer has Hausdorff dimension greater than that of $\partial\Omega$.

Problem (P_0) can be seen as the Euler conditions satisfied by the minimizer of a suitable energy functional. Variational principles of this kind have been stated in [39]

both in the case when S is a pre-fractal or a fractal surface of the Koch type. In particular in [39] the existence of a variational (weak) solution (minimizer) is proved. In this paper we will consider the case in which S is a fractal surface of the von Koch snowflake type.

We shall be mainly concerned with the «strong» problem (P) associated with this geometry. We will also study the approximating problems (P_h) – with layer the pre-fractal interface S_h approximating S.

The paper is organized as follows. In Section 1 we recall the definition of the von Koch snowflake F and we describe the geometry of both the fractal and pre-fractal layer $S = F \times I$ and $S_h = F_h \times I$, $I = [0, 1]$. In section 2 we describe the relevant functional spaces which will be used as well as the trace theorems on S and S_h respectively. In section 3 we consider the variational formulation for the fractal transmission problem (P); existence and uniqueness for the weak solution of problem (P) in a «convenient» space are proved (see Proposition 3.5) and the transmission condition is interpreted in a suitable duality sense (see Theorem 3.7). In section 4 we consider the variational formulation for the pre-fractal transmission problem (P_h); existence and uniqueness for the weak solution of problem (P_h) in a convenient space are proved (see Proposition 4.2), regularity results for the weak solution of problem (P_h) and a strong interpretation of the transmission condition in the L^2-sense are established in Theorem 4.3.

We think that Theorem 4.3 can be useful to build approximating numerical schemes for the solutions of transmission problems with fractal layers.

1. - Geometry of the fractal layers S and S_h

In the paper, by $|P-P_0|$ we denote the Euclidean distance in $\mathbb{R}^D$ and the Euclidean balls by $B(P_0, r) = \{P \in \mathbb{R}^D\colon |P - P_0| < r\}$, $P \in \mathbb{R}^D$, $r > 0$. By the Von Koch snowflake F, we will denote the union of three com-planar Von Koch curves (see [12]) K_1, K_2 and K_3 as shown in Figure 1.a. We assume that the junction points A_1, A_3 and A_5 are the vertices of a regular triangle with unit side length, i.e. $|A_1 - A_3| = |A_1 - A_5| = |A_3 - A_5| = 1$. Obviously, F can also be seen as the union of the three other standard von Koch curves K_4, K_5 and K_6 (with junction points A_2, A_4 and A_6), as shown in Figure 1.b. From now on we assume that a clockwise orientation is given on F.

The Hausdorff dimension of the von Koch snowflake is given by $d_f = \dfrac{\ln 4}{\ln 3}$. This fractal is no longer self–similar (and hence, not nested).

One can define, in a natural way, a finite Borel measure μ_F supported on F by

$$\mu_F := \mu_1 + \mu_2 + \mu_3, \tag{1.1}$$

where μ_i denotes the normalized d_f-dimensional Hausdorff measure, restricted

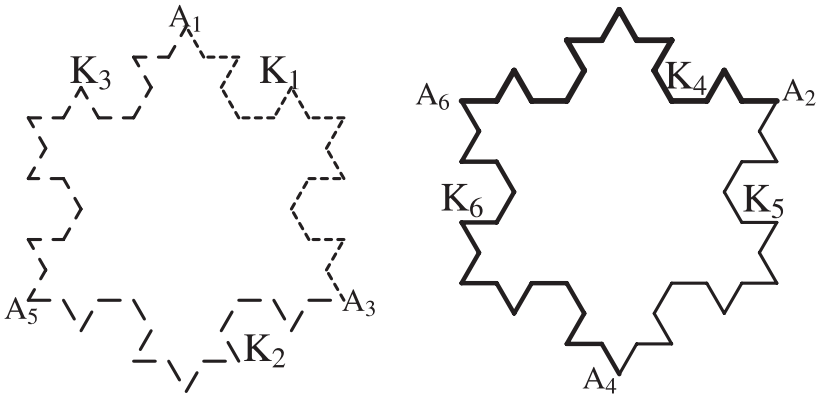

Fig. 1. – a: first decomposition; b: second decomposition.

to K_i, $i = 1, 2, 3$, it also holds that $\mu_F = \mu_4 + \mu_5 + \mu_6$, where μ_i is the normalized d_f-dimensional Hausdorff measure, restricted to K_i, $i = 4, 5, 6$.

K_1 is the uniquely determined self–similar set with respect to four suitable contractions $\psi^{(1)}, \dots, \psi^{(4)}$, with the same ratio $\frac{1}{3}$ (see Section 3.2 in [13]). We approximate K_1 by a sequence of finite sets of points. Let $V_0^{(1)} := \{A_1, A_3\}$, $V_{j_1 \dots j_n}^{(1)} := := \psi_{j_1}^{(1)} \circ \dots \circ \psi_{j_n}^{(1)}(V_0^{(1)})$ and

$$V_n^{(1)} := \bigcup_{j_1, \dots, j_n = 1}^{4} V_{j_1 \dots j_n}^{(1)}. \tag{1.2}$$

We set $V_*^{(1)} := \bigcup_{n \geqslant 0} V_n^{(1)}$. It holds that $K_1 = \overline{V_*^{(1)}}$. Let $K_1^{(0)}$ denote the unit segment whose endpoints are A_1 and A_3 and $K_{j_1 \dots j_n}^{(1)} := \psi_{j_1}^{(1)} \circ \dots \circ \psi_{j_n}^{(1)}(K_1^{(0)})$.

For $n > 0$, we denote

$$F_{(1)}^n = \{\psi_{j_1}^{(1)} \circ \dots \circ \psi_{j_n}^{(1)}(K_1^{(0)}),\ j_1, \dots, j_n = 1, \dots 4\}. \tag{1.3}$$

We set $K_1^{(1)} = \bigcup_{j=1}^{4} \psi_j^{(1)}(K_1^{(0)})$, $K_1^{(n+1)} = \bigcup_{M \in F_{(1)}^n} \bigcup_{j=1}^{4} \psi_j^{(1)}(M)$, here M denotes a segment of the $n+1$-th generation; $K_1^{(n+1)}$ the polygonal curve and $V_{n+1}^{(1)}$ the set of its vertices.

In a similar way, we approximate the von Koch curves $K_2, \dots K_6$ by the sequences $(V_n^{(2)})_{n \geqslant 0}, \dots, (V_n^{(6)})_{n \geqslant 0}$, and denote their limits by $V_*^{(2)}, \dots, V_*^{(6)}$, and the corresponding polygonal curves by $K_2^{(n+1)}, \dots, K_6^{(n+1)}$.

In order to approximate F, we define the increasing sequence of finite sets of points $\mathcal{V}_n := \bigcup_{i=1}^{3} V_n^{(i)} = \bigcup_{i=4}^{6} V_n^{(i)}$, $n \geqslant 1$, and $\mathcal{V}_* := \bigcup_{n \geqslant 1} \mathcal{V}_n$. It holds that $\mathcal{V}_* =$

$= \bigcup_{i=1}^{3} V_*^{(i)} = \bigcup_{i=4}^{6} V_*^{(i)}$ and $F = \overline{\mathcal{V}_*}$. In the following we denote by

$$F_{n+1} = \bigcup_{i=1}^{3} K_i^{(n+1)} = \bigcup_{i=4}^{6} K_i^{(n+1)} \tag{1.4}$$

the closed polygonal curve approximating F at the $(n+1)$-th step.

The measure μ_F has the property that there exist two positive constants c_1, c_2

$$c_1 r^d \leqslant \mu_F(B(P, r) \cap F) \leqslant c_2 r^d, \quad \forall P \in F \tag{1.5}$$

where $\mathrm{d} = d_f = \dfrac{\log 4}{\log 3}$ and where $B(P, r)$ denotes the Euclidean ball in $\mathbb{R}^2$. As μ_F is supported on F, it is not ambiguous to write in (1.5) $\mu_F(B(P, r))$ in place of $\mu_F(B(P, r) \cap F)$. In the terminology of the following section we say that F is a d-set with $\mathrm{d} = d_f$.

Remark 1.1: *The von Koch snowflake can be also regarded as a fractal manifold* (*see* [13] *Section* 3.2).

Let Q denote a bounded open set in $\mathbb{R}^3$; in our basic model Q denotes the parallelepiped $Q = (-1, 1)^2 \times (0, 1)$ and S denotes a «cylindrical» layer in Q of the type $S = F \times I$, where $I = [0, 1]$ and F is the von Koch snowflake. We assume that S is located in a median position inside Q and divides Q in two sub-domains Q^1 and Q^2 (see Figure 2).

We give a point $P \in S$ the cartesian coordinates $P = (x, y)$, where $x = (x_1, x_2)$ are the coordinates of the orthogonal projection of P on the plane containing F and y is the coordinate of the orthogonal projection of P on the y-line containing the interval I: $P = (x, y) \in S$, $x = (x_1, x_2) \in F$, $y \in I$.

One can define in a natural way, a finite Borel measure m supported on S as the

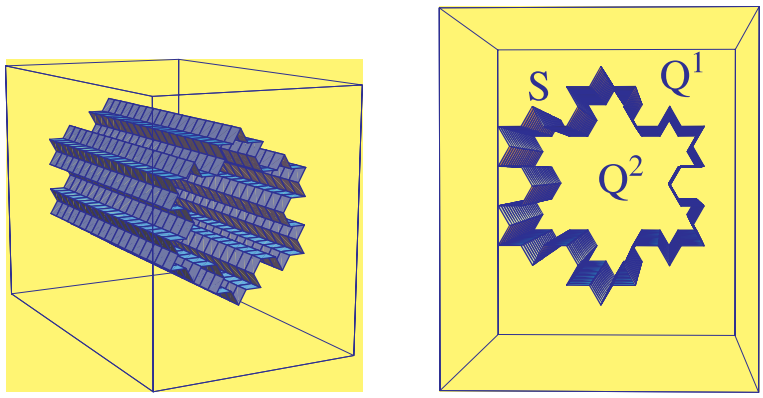

Fig. 2. – Two different viewpoints of the domain Q and the layer S.

product measure

$$\mathrm{d}m = \mathrm{d}\mu_F \times \mathrm{d}y \tag{1.6}$$

where $\mathrm{d}y$ denotes the one-dimensional Lebesgue measure on I. The measure m has the property that there exist two positive constants c_1, c_2

$$c_1 r^d \leqslant m(B(P, r) \cap S) \leqslant c_2 r^d, \quad \forall P \in S \tag{1.7}$$

where $\mathrm{d} = d_f + 1 = \dfrac{\log 12}{\log 3}$ and where $B(P, r)$ denotes the Euclidean ball in $\mathbb{R}^3$. As m is supported on S, it is not ambiguous to write in (1.7) $m(B(P, r))$ in place of $m(B(P, r) \cap S)$. In the terminology of the following section we say that S is a d-set with $\mathrm{d} = d_f + 1$.

By S_h we denote the pre-fractal layer of the type $S_h = F_h \times I$, $h = 1, 2, \dots$, F_h is the piecewise linear pre-fractal approximation of F at the step h see (1.4). S_h is a surface of polyhedral type. S_h divides Q in two sub-domains Q_h^i, $i = 1, 2$.

We give a point $P \in S_h$ the Cartesian coordinates $P = (x, y)$, where $x = (x_1, x_2)$ are the coordinates of the orthogonal projection of P on the plane containing F_h and y is the coordinate of the orthogonal projection P on the y-line containing the interval I.

2. - Functional spaces and traces

2.1. - *Sobolev spaces*

Let Q be a polyhedral domain: just to fix the ideas, the parallelepiped as in the previous section. For every integer $h \geqslant 1$ let S_h be the prefractal surface approximating the Koch-type surface S and let us denote every affine «face» of S_h by $S_h^{(j)}$; S_h divides Q into two subsets Q_h^1 and Q_h^2.

By $L^2(\cdot)$ we denote the Lebesgue space with respect to the Lebesgue measure on subsets of $\mathbb{R}^3$, which will be left to the context whenever that does not create ambiguity. Let T be a closed set of $\mathbb{R}^3$, by $C(T)$ we denote the space of continuous functions on T, by $C_0(T)$ we denote the space of continuous functions vanishing on ∂T. Let $\mathcal{G}$ be an open set of $\mathbb{R}^3$, by $H^s(\mathcal{G})$, where $s \in \mathbb{R}^+$ we denote the usual (possibly fractional) Sobolev spaces (see Necăs [41]); $H_0^s(\mathcal{G})$ is the closure of $\mathcal{D}(\mathcal{G})$, (the smooth functions with compact support on $\mathcal{G}$), with respect to the $\|\cdot\|_{H^s}$-norm. By $H^2_{\mathrm{loc}}(\mathcal{G})$ we denote the space of functions $u \in H^2(\mathcal{A})$ on every open set $\mathcal{A} \subset\subset \mathcal{G}$. In the following, we will make use of trace spaces on boundaries of polyhedral domains of $\mathbb{R}^3$.
By $H^r(S_h)$, $0 < r \leqslant 1$ we denote the Sobolev space on S_h, defined by local Lipschitz charts as in Necas [41].
By $H_0^1(S_h)$ we denote the closure in $H^1(S_h)$ of the set

$$\{v|_{\partial Q_h^2} : v \in C^\infty(Q_h^2),\ \text{v vanishes in a neighborhood of } S_h\}.$$

It is to be pointed out that the Sobolev space $H^r(S_h)$ (defined in [41])

coincides, with equivalent norms, with the trace space defined in Buffa e Ciarlet in [7] (see also [6] for the case of polygonal boundaries).

When $r > 1$ the trace spaces on non smooth boundaries can be defined in different ways; we now recall two trace theorems, specialized to our case, which we will use, referring to [16], [17] and [10] for a more general discussion.

For f in $H^s(\mathcal{G})$, we put

$$\gamma_0 f(P) = \lim_{r\to 0} \frac{1}{|B(P, r) \cap \mathcal{G}|} \int_{B(P, r)\cap \mathcal{G}} f(Q)\, \mathrm{d}Q \tag{2.1}$$

at every point $P \in \overline{\mathcal{G}}$ where the limit exists. It is known that the limit (2.1) exists at *quasi every* $P \in \overline{\mathcal{G}}$ with respect to the $(s, 2)$-capacity [1].

We now recall the results of Theorem 3.1 in [19] specialized to our case, referring to [17] and [10] for a more general discussion.

Proposition 2.1: *Let $\mathcal{G}$ denote respectively Q, Q_b^1, Q_b^2 and let Γ denote S_b, ∂Q_b^1, ∂Q_b^2, ∂Q. Let $\frac{1}{2} < s < \frac{3}{2}$. Then $H^{s-\frac{1}{2}}(\Gamma)$, is the trace space to Γ of $H^s(\mathcal{G})$ in the following sense*:

(i) *γ_0 is a continuous and linear operator from $H^s(\mathcal{G})$ to $H^{s-\frac{1}{2}}(\Gamma)$,*

(ii) *there is a continuous linear operator Ext from $H^{s-\frac{1}{2}}(\Gamma)$ to $H^s(\mathcal{G})$, such that $\gamma_0 \circ Ext$ is the identity operator in $H^{s-\frac{1}{2}}(\Gamma)$.*

In particular we use the Lions-Magenes space $H^{\frac{1}{2}}_{0,0}(S_b)$ defined as

$$H^{\frac{1}{2}}_{0,0}(S_b) = \{\theta \in L^2(S_b) : \exists v \in H_0^1(Q) : \gamma_0 v = \theta \text{ on } S_b\}, \tag{2.2}$$

equipped with the quotient norm

$$\|\theta\|_{H^{\frac{1}{2}}_{0,0}(S_b)} = = \inf\{\|v\|_{H^1(Q)} : v \in H_0^1(Q),\ \gamma_0 v = \theta \text{ on } S_b\}.$$

We note that $H^{\frac{1}{2}}_{0,0}(S_b)$ is the subspace of the functions $\theta \in H^{\frac{1}{2}}(S_b)$ for which the trivial extension $\tilde{\theta}_i$, $\tilde{\theta}_i = \theta$ on S_b and $\tilde{\theta}_i = 0$ on $\partial Q_b^i \setminus S_b$ belongs to the space $H^{\frac{1}{2}}(\partial Q_b^i)$; (see [7]).

Finally $(H^{\frac{1}{2}}_{0,0}(S_b))'$ denotes the dual space of $H^{\frac{1}{2}}_{0,0}(S_b)$.

For the present application, we will make use also of the Sobolev trace space $H^{\frac{3}{2}}(S_b)$; in the author's opinion, the definition of this Sobolev space (on polyhedral boundaries), which best fits to this problem, is that given in Section 2 of [7].

We set

$$H^{\frac{3}{2}}(S_b) = \{\varphi \in H^1(S_b) : \varphi|_{S_b^{(j)}} \in H^{\frac{3}{2}}(S_b^{(j)}),\ \forall S_b^{(j)} \in S_b\}.$$

Proposition 2.2: *$H^{\frac{3}{2}}(S_b)$, is the trace space to Q_b^i of $H^2(Q_b^i)$ in the following sense*:

(i) γ_0 *is a continuous and linear operator from* $H^2(Q_b^i)$ *to* $H^{\frac{3}{2}}(S_b)$,

(ii) *there is a continuous linear operator* Ext *from* $H^{\frac{3}{2}}(S_b)$ *to* $H^2(Q_b^i)$, *such that* $\gamma_0 \circ \mathrm{Ext}$ *is the identity operator in* $H^{\frac{3}{2}}(S_b)$,

(see Theorem 2.4 in [7] and Theorem 1 in [17]).

In the sequel we denote by the symbol $f|_{S_b}$ the trace $\gamma_0 f$ to S_b.

2.2. - *Besov spaces*

Definition 2.3: *Let* $T \subset \mathbb{R}^D$ *be a closed non-empty subset. It is a d-set* $(0 < d \leq D)$ *if there exists a Borel measure* μ *with* $\operatorname{supp} \mu = T$ *such that for some constants* $c_1 = c_1(T) > 0$ *and* $c_2 = c_2(T) > 0$

$$c_1 r^d \leq \mu(B(P, r)) \leq c_2 r^d \quad (P \in T,\ 0 < r \leq 1). \tag{2.3}$$

Such a μ *is called a d-measure on* T.

Proposition 2.4: *The set F is a d-set with* $d = d_f$. *The measure* μ_F *is a d-measure. The layer S is a d-set with* $d = d_f + 1$. *The measure m is a d-measure.*

See [13] and [39].

Throughout the paper c will denote different constants.

We now come to the definition of the Besov spaces. Actually there are many equivalent definitions of these spaces see for instance [43] and [24]. We recall here the one which best fits our aims and we will restrict ourselves to the case α positive and non-integer, $p = q = 2$; the general setting being much more involved see [24].

Let T be a d-set in $\mathbb{R}^D$.

Let $\alpha > 0$ non integer, $k = [\alpha]$ the integer part of α, j a D-dimensional multi-index of length $|j| \leq k$. If f and $\{f^{(j)}\}$ are functions defined μ-a.e. on T, we set

$$R_j(P, P') = f^{(j)}(P) - \sum_{|j+l| \leq k} \frac{f^{(j+l)}(P')}{l!} (P - P')^l,$$

where $f^{(0)} = f$ and l denotes a D-dimensional multi-index. We now define the Besov space $B_\alpha^{2,2}(T) \equiv B_\alpha^{2,2}(T, \mu)$.

Definition 2.5: *We say that* $f \in B_\alpha^{2,2}(T)$ *if there exists a family* $\{f^{(j)}\}$ *with* $|j| \leq k$, *as above, such that* $f^{(j)} \in L^2(T, \mu)$ *and* $\|\{a_n\}\|_{l_2} < \infty$ *where* a_n *is the smallest number such that*

$$\left(3^{nd} \int \int_{|P - P'| < 3^{-n}} |R_j(P, P')|^2 \, d\mu(P)\, d\mu(P')\right)^{1/2} \leq 3^{-n(\alpha - |j|)} a_n.$$

The norm of f in $B_\alpha^{2,2}(T)$ is

$$\|f\|_{B_\alpha^{2,2}(T)} = \|f\|_{2,\mu} + \|\{a_n\}\|_{l_2}.$$

The family $\{f^{(j)}\}$ in the previous definition is uniquely determined by f, as shown in [24], for d-sets with $d > D-1$.

Let us note that for $0 < \alpha < 1$ the norm $\|f\|_{B_\alpha^{2,2}(T)}$ can be written as

$$\|f\|_{2,\mu} + \left(\sum_{n=0}^{\infty} 3^{n(d+2\alpha)} \int \int_{|P-P'|<3^{-n}} |f(P)-f(P')|^2 \, d\mu(P) \, d\mu(P') \right)^{1/2}.$$

PROPOSITION 2.6: *Let T be a d-set, $T \subset \overline{Q}$. Let $s > \dfrac{(3-d)}{2}$, $\left(s - \dfrac{(3-d)}{2}\right) \notin \mathbb{N}$, then $B^{2,2}_{s-\frac{(3-d)}{2}}(T)$ is the trace space to T of $H^s(Q)$ in the following sense*:

(i) *γ_0 is a continuous linear operator from $H^s(Q)$ to $B^{2,2}_{s-\frac{(3-d)}{2}}(T)$,*

(ii) *there is a continuous linear operator* Ext *from $B^{2,2}_{s-\frac{(3-d)}{2}}(T)$ to $H^s(Q)$ such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator in $B^{2,2}_{s-\frac{(3-d)}{2}}(T)$.*

For the proof we refer to Theorem 1 of Chapter VII in [24], see also [43].

From Proposition 2.6 it follows that when $T = S$ and $s = 1$ the trace space of $H^1(Q)$ is $B^{2,2}_{\frac{d_f}{2}}(S)$.

Let $\beta = \dfrac{d_f}{2}$. The space $B^{2,2}_{\beta,0}(S)$ is a subspace of $B^{2,2}_{\beta}(S)$, more precisely

$$B^{2,2}_{\beta,0}(S) = \{u \in L^2(S, m) \mid \text{there exists } w \in H^1_0(Q) \text{ such that } \gamma_0 w = u \text{ on } S\}, \tag{2.4}$$

equipped with the norm

$$\|u\|_{B^{2,2}_{\beta,0}(S)} = \inf\{\|w\|_{H^1(Q)} : w \in H^1_0(Q),\ \gamma_0 w = u, \text{ on } S\}.$$

In the sequel we denote by the symbol $f|_S$ the trace $\gamma_0 f$ to S.

In the following, we also make use of the dual of Besov spaces on S. These spaces as shown in [25] coincide with a subspace of Schwartz distributions $D'(\mathbb{R}^3)$, which are supported in S. They are built by means of atomic decomposition. Actually, Jonsson and Wallin [25] proved this result in the general framework of d-sets. Here we do not give a detailed description of the duals of Besov spaces on d-sets and we refer to [25] for a complete discussion.

3. - VARIATIONAL FORMULATION FOR THE FRACTAL PROBLEM

3.1. - *The energy forms*

In Definition 4.5 of [13], a Lagrangian measure $\mathcal{L}_F$ on F and the corresponding energy form $\mathcal{E}_F$ as

$$\mathcal{E}_F(u, v) = \int_F \mathrm{d}\mathcal{L}_F(u, v) \tag{3.1}$$

with domain $D(F)$ have been introduced. The domain $D(F)$ – which is a Hilbert space

with norm $(\|u\|^2_{L^2(F,\,\mu_F)} + \mathcal{E}_F(u,\,u))^{\frac{1}{2}}$ – has been characterized in terms of the domains of the energy forms on K_i (see [13] Theorem 4.6).

In the following we will omit the subscript F, the Lagrangian measure will be simply denoted by $\mathcal{L}(u,\,v)$ and we will set $\mathcal{L}[u] = \mathcal{L}(u,\,u)$.

We define the energy forms E_S on the fractal layer $S = F \times I$ by setting

$$E_S[u] = \sigma^1 \int_I \int_F \mathcal{L}_x[u](\mathrm{d}x)\,\mathrm{d}y + \sigma^2 \int_F \int_I |D_y u|^2 \mathrm{d}y \mu_F(\mathrm{d}x) \tag{3.2}$$

where σ^1 and σ^2 are positive constants. Here $\mathcal{L}_x(\cdot,\,\cdot)(\mathrm{d}x)$ denotes the measure-valued Lagrangian (of the energy form $\mathcal{E}_F$ of F with domain $D(F)$) now acting on $u(x,\,y)$ and $v(x,\,y)$ as function of $x \in F$ for a.e. $y \in I$; $\mu_F(\mathrm{d}x)$ is the Hausdorff measure acting on each section F of S for a.e. $y \in I$ with $d_f = \dfrac{\log 4}{\log 3}$, $D_y(\cdot)$ denotes the derivative in the y direction.

The form E_S is defined for $u \in D(S)$ where $D(S)$ is the closure in the intrinsic norm

$$\|u\|_{D(S)} = (E_S[u] + \|u\|^2_{L^2(S,\,m)})^{\frac{1}{2}} \tag{3.3}$$

of the set

$$C_0(S) \cap L^2(0,\,1;\,D(F)) \cap H^1_0(0,\,1;\,L^2(F)) \tag{3.4}$$

where $L^2(F) = L^2(F,\,\mu_F(\mathrm{d}x))$.

In the following we shall also use the form $E_S(u,\,v)$ which is obtained from $E_S[u]$ by the polarization identity:

$$E_S(u,\,v) = \frac{1}{2}\{E_S[u+v] - E_S[u] - E_S[v]\}, \quad u,\,v \in D(S). \tag{3.5}$$

Proposition 3.1: *The space $D(S)$ is continuously embedded in $B^{2,2}_\alpha(S)$, $\alpha < 1$.*

Proof: According to [36] Section 2.2, we now introduce the following spaces:

$$W(0,\,1) = \{z : S \to \mathbb{R} : z \in L^2(0,\,1;\,D(F)) \cap H^1(0,\,1;\,L^2(F))\} \tag{3.6}$$

equipped with the norm

$$\|u\|_{W(0,\,1)} = (\|u\|^2_{L^2(0,\,1;\,D(F))} + \|D_y u\|^2_{L^2(0,\,1;\,L^2(F))})^{\frac{1}{2}}. \tag{3.7}$$

Obviously $D(S) \subset W(0,\,1)$.

From theorem 3.1 in [31] we deduce that $D(F)$ is embedded in $B^{2,2}_{d_f-\varepsilon}(F)$. Thus, in the notations of [36], we introduce the Hilbert space

$$B^{2,2}_{d_f-\varepsilon,\,1}(S) = \{L^2(0,\,1;\,B^{2,2}_{d_f-\varepsilon}(F)) \cap H^1(0,\,1;\,L^2(F))\} \tag{3.8}$$

with norm

$$\left(\int_0^1 \|u\|^2_{B^{2,2}_{d_f-\varepsilon}(F)} \, \mathrm{d}y + \|u\|^2_{H^1(0,\,1;\,L^2(F))} \right)^{\frac{1}{2}} . \tag{3.9}$$

From [36] page 8 it follows that the space $W(0,\,1) \subset B^{2,2}_{d_f-\varepsilon,\,1}(S)$, for every $\varepsilon > 0$. From the embedding Theorem 1 of [24] we deduce that the space $B^{2,2}_{d_f-\varepsilon,\,1}(S)$ is continuously embedded in $B^{2,2}_{\alpha}(S)$, $\alpha < 1$. ■

It can be proved, as in Proposition 3.1 of [39], that

Proposition 3.2: *In the previous notations and assumptions the form E_S with domain $D(S)$ is a regular Dirichlet form in $L^2(S,\,m)$ and the space $D(S)$ is a Hilbert space under the intrinsic norm* (3.3).

We now define the Laplace operator on S. As $(E_S,\,D(S))$ is a closed, bilinear form on $L^2(S,\,m)$, there exists (see Chap. 6, Theorem 2.1 in [18]) a unique self-adjoint, non positive operator Δ_S on $L^2(S,\,m)$ – with domain $\mathcal{D}(\Delta_S) \subseteq D(S)$ dense in $L^2(S,\,m)$ – such that

$$E_S(u,\,v) = -\int_S (\Delta_S u)\, v \, \mathrm{d}m, \quad u \in \mathcal{D}(\Delta_S),\ v \in D(S) . \tag{3.10}$$

Let $(D(S))'$ denote the dual of the space $D(S)$. We now introduce the Laplace operator on the fractal S as a variational operator from $D(S) \to (D(S))'$ by

$$E_S(z,\,w) = -\langle \Delta_S z,\, w \rangle_{(D(S))',\,D(S)} \tag{3.11}$$

for $z \in D(S)$ and for all $w \in D(S)$ where $\langle \cdot,\,\cdot \rangle_{(D(S))',\,D(S)}$ is the duality pairing between $(D(S))'$ and $D(S)$. We use the same symbol Δ_S to define the Laplace operator both as a self-adjoint operator in (3.10) and as a variational operator in (3.11). It will be clear from the context to which case we refer.

Consider now the space of functions $u : Q \to \mathbb{R}$

$$V(Q,\,S) = \{ u \in H^1_0(Q) : u|_S \in D(S) \} . \tag{3.12}$$

Proposition 3.3: *The space $V(Q,\,S) = \{ u \in H^1_0(Q) : u|_S \in D(S) \}$ is non trivial.*

Proof: We will prove that non trivial functions in $D(S)$ have a suitable extension in $H^1_0(Q)$.

Let I_1 denote the interval $\left[\frac{1}{4},\, \frac{3}{4} \right]$, and I_2 the interval $\left[\frac{3}{8},\, \frac{5}{8} \right]$, $S_1 = F \times I_1$ and $S_2 = F \times I_2$. Let ϕ be the capacity potential of S_2 with respect to S_1 (for its existence see Theorem 2.1.5 in [15]) the function $\tilde{\phi} = \phi$ on S_1 and $\tilde{\phi} = 0$ on $S \backslash S_1$ belongs to $D(S)$, its compact support is contained on S_1. From Proposition 3.1 and Proposition 2.6 Ext $\tilde{\phi} \in H^1(Q)$, then $w = \eta$ Ext $\tilde{\phi} \in H^1_0(Q)$, where η is a suitable cut-off function. ■

We now introduce the energy form

$$E[u] = \int_Q |\mathrm{D}u|^2 \mathrm{d}Q + c_0 E_S[u|_S] \tag{3.13}$$

defined on the domain $V(Q, S)$.
As c_0 is not relevant for our purposes we set $c_0 = 1$.
In the following, by $E(u, v)$, we will denote the corresponding bilinear form

$$E(u, v) = \int_Q \mathrm{D}u \cdot \mathrm{D}v \mathrm{d}Q + E_S(u|_S, v|_S) \tag{3.14}$$

defined on $V(Q, S) \times V(Q, S)$.
As in Theorem 3.2 of [39], it can be proved

Proposition 3.4: *The form $E[\cdot]$ defined in* (3.13) *is a regular Dirichlet form in $L^2(Q)$ and the space $V(Q, S)$ is a Hilbert space equipped with the scalar product*

$$(u, v)_{V(Q, S)} = \int_Q \mathrm{D}u\, \mathrm{D}v\, \mathrm{d}Q + E_S(u|_S, v|_S) + \int_S u|_S v|_S \mathrm{d}m \tag{3.15}$$

where $E_S(u|_S, v|_S)$ is the Dirichlet form defined in (3.5).

We denote by $|||u|||_{V(Q, S)}$ the norm in $V(Q, S)$ (associated with (3.15)) and we note that from the trace theorem (see Proposition 2.6) there exists a positive constant c, such that

$$\|u|_S\|_{L^2(S, m)} \leqslant c\|u\|_{H_0^1(Q)} \tag{3.16}$$

thus an equivalent norm in $V(Q, S)$ is

$$\|u\|_{V(Q, S)} = \left(E_S[u|_S] + \int_Q |\mathrm{D}u|^2 \mathrm{d}Q\right)^{\frac{1}{2}}. \tag{3.17}$$

Proposition 3.5: *Given $f \in L^2(Q)$, there exists a unique $u \in V(Q, S)$ such that*

$$E(u, v) = \int_Q fv \mathrm{d}Q \tag{3.18}$$

for every $v \in V(Q, S)$.

Moreover, u is obtained by

$$\min_{v \in V(Q, S)} \left\{ \frac{1}{2} E[v] - \int_Q fv\, \mathrm{d}Q \right\}. \tag{3.19}$$

Proof: The thesis follows by applying Lax-Milgram theorem to the bilinear form $E(u, v)$. ■

3.2. - The strong formulation of the transmission problem on the fractal layer S

We consider the problem (P) formally stated as

$$(3.20) \qquad (P) \quad \begin{cases} -\Delta u = f & \text{in } Q^i,\ i=1,2 \qquad j) \\ -\Delta_S u = \left[\dfrac{\partial u}{\partial n}\right] & \text{on } S, \qquad jj) \\ u = 0 & \text{on } \partial Q, \qquad jjj) \\ u^1 = u^2 & \text{on } S, \qquad jv) \\ u = 0 & \text{on } \partial S \qquad v), \end{cases}$$

where u^i is the restriction to Q^i, $\dfrac{\partial u^i}{\partial n_i}$, $i=1,2$ is the outward «normal derivative», to be defined in a suitable sense, $\left[\dfrac{\partial u}{\partial n}\right] = \dfrac{\partial u^1}{\partial n_1} + \dfrac{\partial u^2}{\partial n_2}$ is the jump of the normal derivative and Δ_S is the fractal Laplacian defined in Section 3.1.

We now prove that the variational solution of (3.18) satisfies problem (P) in a suitable strong sense. We state some preliminary results.

PROPOSITION 3.6: *The space $D(S)$ is embedded in $B^{2,2}_{\beta,0}(S)$, $\beta = \dfrac{d_f}{2}$.*

PROOF: The proof can be achieved by making use of a general extension theorem, proved by Jonsson in [22], for Besov spaces defined on general closed sets which are not possibly d-sets such as the set $T = \partial Q \cup S$. More precisely, for any given function $z \in D(S)$ consider the function $\tilde{z}$ defined as $\tilde{z} = z$ on S and $\tilde{z} = 0$ on $\partial Q \backslash S$. In order to extend $\tilde{z}$ to a function $w \in H^1_0(Q)$ we apply Theorem 1 in [22]. This theorem provides, in particular, a continuous linear operator $\widehat{E}xt$,

$$\widehat{E}xt : \tilde{B}^{2,2}_1(S \cup \partial Q) \to H^1(\mathbb{R}^3),$$

where $\tilde{B}^{2,2}_1(\cdot)$ is the Besov space defined in [22] page 356. We note that z belongs to $B^{2,2}_\alpha(S)$, $\alpha < 1$ (see Proposition 3.1) and $\tilde{z}$ belongs to $\tilde{B}^{2,2}_\beta(S \cup \partial Q)$, $\beta < \dfrac{2-d_f}{2}$, hence in particular to $\tilde{B}^{2,2}_1(S \cup \partial Q)$. Therefore $w = \widehat{E}xt\tilde{z}\,|_Q$ belongs to $H^1_0(Q)$ and $w|_S = z$ on S. ■

THEOREM 3.7: *Let u be the variational solution for problem* (3.18) *then we have that*

$$(3.21) \qquad u^i \in H^2_{\mathrm{loc}}(Q^i)$$

$$(3.22) \qquad \frac{\partial u^i}{\partial n_i} \in (B^{2,2}_{\beta,0}(S))', \quad \beta = \frac{d_f}{2}, \quad i=1,2$$

and the transmission condition jj) holds in $(D(S))'$ *that is*

$$\langle \Delta_S u|_S, z\rangle_{(D(S))', D(S)} = \left\langle \left[\frac{\partial u}{\partial n} \right]_S, z \right\rangle_{(D(S))', D(S)}$$

where $(B^{2,2}_{\beta,0}(S))'$ *is the dual of* $B^{2,2}_{\beta,0}(S)$ *defined in* (2.4); $(D(S))'$ *is the dual of* $D(S)$, Δ_S *is the variational operator from* $D(S) \to (D(S))'$ *defined in* (3.11) *and* $\langle \cdot, \cdot \rangle_{(D(S))', D(S)}$ *is the duality pairing between* $(D(S))'$ *and* $D(S)$.

PROOF: We recall that by u^i we denote the restriction to Q^i of the solution $u \in V(Q, S)$ of (3.18). We choose in (3.18) $v = \phi_i \in \mathcal{D}(Q^i)$ and we obtain

$$\int_{Q^i} \mathrm{D}u^i \mathrm{D}\phi_i \mathrm{d}Q = \int_{Q^i} f\phi_i \mathrm{d}Q \tag{3.23}$$

for every $\phi_i \in \mathcal{D}(Q^i)$ $(i = 1, 2)$. From the density of $L^2(Q^i)$ in $\mathcal{D}(Q^i)$ and from the fact that $f \in L^2(Q^i)$ we deduce that

$$-\Delta u^1 = f \quad \text{in } L^2(Q^1) \tag{3.24}$$

$$-\Delta u^2 = f \quad \text{in } L^2(Q^2) \tag{3.25}$$

This gives that $u \in V(Q^i) = \{u \in H^1(Q) \,|\, \Delta u^i \in L^2(Q^i)\}$ where the Laplacian is intended in the distributional sense. The classical theory on local regularity results (see [5]) gives also that $u^i \in H^2_{\mathrm{loc}}(Q^i)$.

Moreover, proceeding by duality (see Theorem 4.15 in [30] and [3] Appendix 4) we prove that the normal derivative $\dfrac{\partial u^i}{\partial n_i}$ is in the dual $(B^{2,2}_{\beta,0}(S))'$ of the space $B^{2,2}_{\beta,0}(S)$, where $\beta = \dfrac{d_f}{2}$ and

$$\left\langle \frac{\partial u^i}{\partial n_i}, v|_S \right\rangle_{(B^{2,2}_{\beta,0}(S))', B^{2,2}_{\beta,0}(S)} = \int_{Q^i} \mathrm{D}u^i \mathrm{D}v dQ + \int_{Q^i} v\Delta u^i dQ, \quad \text{for every } v \in H^1_0(Q). \tag{3.26}$$

From Proposition 3.6 and proceeding as in Section 6 of [30], it can be proved that the transmission condition

$$\Delta_S(u|_S) = \left[\frac{\partial u}{\partial n} \right] \quad \text{on } S$$

holds in $(D(S))'$ that is

$$\langle \Delta_S(u|_S), z\rangle_{(D(S))', D(S)} = \left\langle \left[\frac{\partial u}{\partial n} \right], z \right\rangle_{(D(S))', D(S)}. \qquad \blacksquare$$

As a consequence of Theorem 3.7, the (variational) solution of problem (3.18) is

the solution of problem (P) which can be rigorously stated as follows

$$(P)\quad \begin{cases} -\Delta u^i = f & \text{in } L^2(Q^i),\ i=1,2 \qquad j) \\ -\Delta_S u = \left[\dfrac{\partial u}{\partial n}\right] & \text{on } (D(S))' \qquad jj) \\ u = 0 & \text{on } H^{\frac{1}{2}}(\partial Q) \qquad jjj) \\ u^1 = u^2 & \text{on } B^{2,2}_{\frac{\alpha_j}{2}}(S) \qquad jv) \\ u = 0 & \text{on } B^{2,2}_{\frac{\alpha_j-1}{2}}(\partial S) \qquad v) \end{cases}$$

REMARK 3.8: *Actually from Proposition* 3.1 *one deduces that equalities jv) and v) respectively hold in* $B^{2,2}_{\alpha}(S)$ *and in* $B^{2,2}_{\alpha-\frac{1}{2}}(\partial S)$ *with* $\alpha < 1$.

4. - VARIATIONAL FORMULATION FOR THE PRE-FRACTAL LAYER PROBLEM

4.1. - *The energy forms*

By Q we denote the parallelepiped as defined in Section 2 and by S_h we denote the pre-fractal layer of the type $S_h = F_h \times I$, $h = 1, 2, \dots$, F_h is the piecewise linear pre-fractal approximation of F at the step h (see Section 1). S_h divides Q in two sub-domains Q^i_h, $i = 1, 2$.

We give a point $P \in S_h$ the Cartesian coordinates $P = (x, y)$, where $x = (x_1, x_2)$ are the coordinates of the orthogonal projection of P on the plane containing F_h and y is the coordinate of the orthogonal projection P on the y-line containing the interval I.

We first construct the energy forms E_S on the *pre-fractal* layers $S_h = F_h \times I$, $h \geqslant 1$. By ℓ we denote the natural arc-length coordinate on each edge of F_h and we introduce the coordinates $x_1 = x_1(\ell)$, $x_2 = x_2(\ell)$, $y = y$ on every affine «face» $S^{(j)}_h$ of S_h. By $\mathrm{d}\ell$ we denote the one-dimensional measure given by the arc-length ℓ and by $\mathrm{d}\sigma$ the surface measure on each face $S^{(j)}_h$ of S_h, that is $\mathrm{d}\sigma = \mathrm{d}\ell\,\mathrm{d}y$. We define E_{S_h} by setting

$$E_{S_h}[u] = \sum_j \left(\int_{S^{(j)}_h} (\sigma^1_h |D_\ell u|^2 + \sigma^2_h |D_y u|^2)\, \mathrm{d}\sigma \right) \tag{4.1}$$

where σ^1_h, σ^2_h are positive constants and $u \in H^1(S_h)$, the Sobolev space of functions on the piece-wise affine set S_h (see Section 2.1). By Fubini theorem, we can write this functional in the form

$$E_{S_h}[u] = \sigma^1_h \int_I \left(\int_{F_h} |D_\ell u|^2 \mathrm{d}\ell \right) \mathrm{d}y + \sigma^2_h \int_{F_h} \left(\int_I |D_y u|^2 \mathrm{d}y \right) \mathrm{d}\ell\,. \tag{4.2}$$

We denote the corresponding bilinear form by $E_{S_h}(u, v)$.

Consider now the space of functions $u : Q \to \mathbb{R}$

$$V(Q, S_h) = \{u \in H_0^1(Q) : u|_{S_h} \in H_0^1(S_h)\}, \tag{4.3}$$

it is not trivial as it contains $\mathcal{D}(Q)$.
Consider now the energy form

$$E^{(h)}[u] = \int_Q |\mathrm{D}u|^2 \mathrm{d}Q + E_{S_h}[u|_{S_h}] \tag{4.4}$$

defined on the domain $V(Q, S_h)$.
By $E^{(h)}(u, v)$ we will denote the corresponding bilinear form

$$E^{(h)}(u, v) = \int_Q \mathrm{D}u\, \mathrm{D}v\, \mathrm{d}Q + E_{S_h}(u|_{S_h}, v|_{S_h}) \tag{4.5}$$

defined on $V(Q, S_h) \times V(Q, S_h)$.

THEOREM 4.1: *The form $E^{(h)}[u]$, defined in* (4.4), *with domain $V(Q, S_h)$ is a regular Dirichlet form in $L^2(Q)$ and the space $V(Q, S_h)$ is a Hilbert equipped with the scalar product*

$$(u, v)_{V(Q, S_h)} = \int_Q \mathrm{D}u\, \mathrm{D}v\, \mathrm{d}Q + E_{S_h}(u|_{S_h}, v|_{S_h}) + \int_{S_h} u|_{S_h} v|_{S_h} \mathrm{d}\sigma.$$

PROOF: The completeness follows from the completeness of $H_0^1(Q)$ and $H_0^1(S_h)$ and from Proposition 2.1 (with $s = 1$, $\mathcal{G} = Q$ and $\Gamma = S_h$). The regularity of the form follows from Proposition 2.2 and from the density of $\mathcal{D}(Q)$ in $V(Q, S_h)$ (see also Proposition 4.1 in [33] where the two-dimensional case is studied). ■

We denote by $|||u|||_{V(Q, S_h)}$ the corresponding energy norm in $V(Q, S_h)$. By proceeding as in Section 4.1 one can prove, via the trace theorem that there exists a positive constant c

$$\|u\|_{L^2(S_h)} \leqslant c\|u\|_{H_0^1(Q)},$$

thus an equivalent norm in $V(Q, S_h)$ is

$$\|u\|^2_{V(Q, S_h)} = \int_Q \mathrm{D}u\, \mathrm{D}v\, \mathrm{d}Q + E_{S_h}[u|_{S_h}]. \tag{4.6}$$

PROPOSITION 4.2: *Given $f \in L^2(Q)$, for every $h \in \mathbb{N}$, there exists a unique $u_h \in V(Q, S_h)$ such that*

$$E^{(h)}(u_h, v) = \int_Q fv\, \mathrm{d}Q \tag{4.7}$$

for every $v \in V(Q, S_h)$.

Moreover u_h is obtained as the minimizer of the variational problem

$$E^{(h)}[u_h] = \min_{u \in V(Q, S_h)} \left\{ \frac{1}{2} E^{(h)}[u] - \int_Q fu \, \mathrm{d}Q \right\}. \tag{4.8}$$

PROOF: The thesis follows by applying Lax-Milgram theorem to the form $E^{(h)}(u, v)$. ■

4.2. - *The strong formulation of the transmission problem on the pre-fractal layer S_h*

We consider now the problems (P_h), formally stated as:

$$(P_h) \quad \begin{cases} -\Delta u = f & \text{in } Q_h^i,\ i = 1, 2 \qquad j) \\ -\Delta_{S_h} u = \left[\dfrac{\partial u}{\partial n} \right] & \text{on } S_h \qquad jj) \\ u = 0 & \text{on } \partial Q \qquad jjj) \\ u^1 = u^2 & \text{on } S_h \qquad jv) \\ u = 0 & \text{on } \partial S_h \qquad v) \end{cases} \tag{4.9}$$

where $u^i = u|_{Q_h^i}$, Δ_{S_h} is the «piecewise» tangential Laplacian on S_h associated to the Dirichlet form E_{S_h}, $\left[\dfrac{\partial u}{\partial n} \right] = \dfrac{\partial u^1}{\partial n_1} + \dfrac{\partial u^2}{\partial n_2}$ is the jump of the normal derivatives of u across S_h, n_i, $i = 1, 2$ being the outward normal to Q_h^i.

Let u_h^i denote the restriction of the variational solution u_h to Q_h^i. By usual duality arguments (see Appendix 4 in [3]) the normal derivatives $\dfrac{\partial u_h^i}{\partial n_i}$, $i = 1, 2$, belong to the dual space of $H^{\frac{1}{2}}_{0,0}(S_h)$ (see (2.2)).

Then, by the Green formula for Lipschitz domains, one can prove that the transmission condition *jj*) in (4.9) can be interpreted in the sense of the dual of $H^{\frac{1}{2}}_{0,0}(S_h)$ (see Proposition 2.2 in [32]).

THEOREM 4.3: *Let u_h be the variational solution for problem* (4.7) *then we have that*

$$u_h \in C(\overline{Q}) \tag{4.10}$$

$$u_h^1 \in H^{\frac{8}{5} - \varepsilon}(Q_h^1), \quad u_h^2 \in H^{\frac{7}{4} - \varepsilon}(Q_h^2) \tag{4.11}$$

$$\frac{\partial u_h^i}{\partial n_i} \in L^2(S_h), \quad i = 1, 2 \tag{4.12}$$

in particular conditions jjj), *jv*) *and v*) *are satisfied point-wise, j*) *and jj*) *almost every-*

where and

$$\Delta_{S_h} = \sigma_h^1 D_\ell^2 + \sigma_h^2 D_y^2. \tag{4.13}$$

Here D_ℓ^2 is the «piecewise» second order tangential derivative along the sides of F_h and D_y^2 the «usual» second order partial derivative in y.

In order to prove Theorem 4.3 we need some intermediate results.
Consider the weak solutions w_h^i, $\widehat{w}_h^i$ in $H^1(Q_h^i)$ of the following auxiliary problems

$$\begin{cases} \Delta \widehat{w}_h^i = 0 & \text{in} \quad Q_h^i \\ \widehat{w}_h^i = u_h & \text{on} \quad \partial Q_h^i \end{cases} \tag{4.14}$$

$$\begin{cases} -\Delta w_h^i = f & \text{in} \quad Q_h^i \\ w_h^i = 0 & \text{on} \quad \partial Q_h^i \end{cases} \tag{4.15}$$

As the link between u_h^i and the solutions of problems (4.14) and (4.15) is

$$u_h^i = w_h^i + \widehat{w}_h^i, \tag{4.16}$$

then the regularity of u_h^i follows from the regularity of w_h^i and $\widehat{w}_h^i$.

Proposition 4.4: *In the assumptions of Theorem* 4.3 *and notations* (4.14) *we have*

$$\begin{aligned} &\frac{\partial \widehat{w}_h^i}{\partial n_i} \in L^2(S_h) \\ &\left\| \frac{\partial \widehat{w}_h^i}{\partial n_i} \right\|_{L^2(S_h)} \leq c(h) \|\nabla_t u_h\|_{L^2(S_h)}, \quad i = 1, 2. \end{aligned} \tag{4.17}$$

Proof: The proof follows from an analogous result of Jerison and Kenig (see Theorem 3 and also the proof of Theorem 2 of [21]).

Note also that the right-hand side of (4.17) can be evaluated in terms of the L^2-norm of f in Q. ■

Proposition 4.5: *In the notations of* (4.15) *we have that*

$$w_h^i \in H^{s_i}(Q_h^i), \qquad \|w_h^i\|_{H^{s_i}(Q_h^i)} \leq c(\mu_i, h) \|f\|_{L^2(Q_h^i)}, \quad i = 1, 2 \tag{4.18}$$

where $s_i = 2 - \mu_i$ *with* $\frac{2}{5} < \mu_1 < 1$ *and* $\frac{1}{4} < \mu_2 < 1$; $c(\mu_i, h)$ *is a positive constant depending on* μ_i *and on* h.

Proof: The bounded domain Q_h^i has several intersecting edges on the boundary: let $\{P_\tau^i, \tau = 1, \dots, T\}$ be the set of the intersection points of the edges, near each intersection point P_τ^i the domain coincides with a cone K^τ cutting out a domain Ω_τ^i on the sphere $\mathcal{S}^2$. Let $r(P)$ be the distance from the point P

to the set M of the edges and $\varrho_\tau(P)$ be the distance from the point P to the point P_τ^i.

From Theorem 10.2.3 in [40] we deduce the following estimates for the second derivatives of w_b^i:

$$\sum_{|\alpha|=2} \int_{Q_b^i} r^{2\mu_i} \prod_{\tau=1}^{T} \varrho_\tau^{2(\sigma_\tau^i-\mu_i)} \, |\mathrm{D}^\alpha w_b^i|^2 \, \mathrm{d}x \, \mathrm{d}y \leqslant c(\mu_i, h) \|f\|_{L^2(Q)}^2, \quad i=1, 2 \tag{4.19}$$

the parameters σ_τ^i and μ_i satisfy the following condition

$$0 < -\mu_i + 1 < \frac{\pi}{\theta_i} \tag{4.20}$$

and

$$4(\sigma_\tau^i - 1)^2 < 1 + 4\Lambda_{1,\tau}^i \tag{4.21}$$

where $\Lambda_{1,\tau}^i = \Lambda_1(\Omega_\tau^i)$ is the first eigenvalue of the Dirichlet problem for the Laplace-Beltrami operator in Ω_τ^i, $\tau = 1, \dots, T$.

It is to be pointed out that for functions z supported in a small neighborhood of P_τ^i the product $\prod_{\tau=1}^{T} \varrho_\tau^{2(\sigma_\tau^i-\mu_i)}$ in (4.19) can be replaced by the single factor $\varrho_\tau^{2(\sigma_\tau^i-\mu_i)}$. Roughly speaking the loss of the smoothness depends only on the nearest singularity of the boundary ∂Q_b^i, namely either a vertex or an edge, the regularity being a local property.

Taking into account that the dihedral angles in Q_b^1 have opening equal to $\frac{2}{3}\pi$ or $\frac{5}{3}\pi$ it follows that μ_1 has to be chosen greater than $\frac{2}{5}$; on the other hand in Q_b^2, the dihedral angle have opening $\frac{\pi}{3}$ and $\frac{4}{3}\pi$ hence μ_2 has to be chosen greater than $\frac{1}{4}$. As to the choice of σ_τ^i, we firstly observe that any Ω_τ^i, $i = 1, 2$ is contained in the «lune»

$$\Omega = \left\{ \omega = (\omega_1, \omega_2) \in \mathcal{S}^2; \ \omega_1 \in (0, \pi), \ \omega_2 \in \left(0, \frac{5}{3}\pi\right) \right\};$$

taking into account the monotonicity properties of the first eigenvalue of Laplace-Beltrami operator (see *e.g.* [4] and [2]) we conclude that

$$\Lambda_{1,\tau}^i = \Lambda_1(\Omega_\tau^i) \geqslant \Lambda_1(\Omega).$$

Finally, Proposition 3.1 in [40] yields the explicit value of $\Lambda_1(\Omega) =$

$= \pi \cdot \frac{3}{5\pi}\left(1 + \pi \frac{3}{5\pi}\right) = \frac{3}{5}\left(1 + \frac{3}{5}\right)$, hence condition (4.21) is satisfied by any choice of $\sigma_\tau^i \geqslant 0$.

The choice in our case will be $\sigma_\tau^1 = \mu_1$, $\sigma_\tau^2 = \mu_2$, $\tau = 1, \dots, T$.

Denoting by δ the distance from the boundary, we have $\delta^{\mu_1} D^\alpha w_h^1 \in L^2(Q_h^1)$, $|\alpha| = 2$, $\mu_1 > \frac{2}{5}$, and

$$\|\delta^{\mu_1} D^\alpha w_h^1\|_{L^2(Q_h^1)} \leqslant c(\mu_1, h)\|f\|_{L^2(Q_h^1)}, \qquad |\alpha| = 2,\ \mu_1 > \frac{2}{5}. \tag{4.22}$$

Analogously, as in Q_h^2 the angles have opening equal to $\frac{\pi}{3}$ or $\frac{4\pi}{3}$, we prove that $\delta^{\mu_2} D^\alpha w_h^2 \in L^2(Q_h^2)$, $|\alpha| = 2$, $\mu_2 > \frac{1}{4}$, and

$$\|\delta^{\mu_2} D^\alpha w_h^2\|_{L^2(Q_h^2)} \leqslant c(\mu_2, h)\|f\|_{L^2(Q_h^2)}, \qquad |\alpha| = 2,\ \mu_2 > \frac{1}{4}. \tag{4.23}$$

The interpolation techniques allow us to reformulate in terms of the usual Sobolev spaces the results obtained in weighted Sobolev spaces. More precisely, by making use of Proposition 4.15 in [19], we have

$$\|w_h^i\|_{H^{2-\mu_i}(Q_h^i)} \leqslant c(\mu_i, h)\left\{\|\delta^{\mu_i} \sum_{|\alpha|=2} D^\alpha w_h^i\|^2_{L^2(Q_h^i)} + \|w_h^i\|^2_{H^1(Q_h^i)}\right\}^{\frac{1}{2}}. \tag{4.24}$$

We deduce the thesis from (4.22), (4.23) and (4.24): we have $w_h^1 \in H^{s_1}(Q_h^1)$ $\forall s_1 < \frac{8}{5}$ and $w_h^2 \in H^{s_2}(Q_h^2)$ $\forall s_2 < \frac{7}{4}$. ■

We are now in position to prove Theorem 4.3.

Proof: From Proposition 4.5 we deduce that

$$D^\alpha w_h^1 \in H^{\frac{3}{5}-\varepsilon}(Q_h^1), \quad D^\alpha w_h^2 \in H^{\frac{3}{4}-\varepsilon}(Q_h^2), \quad |\alpha| = 1 \tag{4.25}$$

then by trace results (see Proposition 2.1) we obtain for $i = 1, 2$

$$\frac{\partial w_i^h}{\partial n_i} \in L^2(S_h), \quad \left\| \frac{\partial w_i^h}{\partial n_i} \right\|_{L^2(S_h)} \leqslant c(\mu_i, h)\|f\|_{L^2(Q)}. \tag{4.26}$$

It follows from (4.26), (4.17) and (4.16) that $\frac{\partial u_i^h}{\partial n_i} \in L^2(S_h)$, $i = 1, 2$, hence the jump belongs to $L^2(S_h)$. As $H^{\frac{1}{2}}_{0,0}(S_h)$ is dense in $L^2(S_h)$ (see *e.g.* [7]), we deduce that the transmission condition *jj*) in (4.9) actually holds in the L^2-sense and in particular $\Delta_{S_h} u_h \in L^2(S_h)$. As $u_h \in H_0^1(S_h)$, from Theorem 8 in [8], we deduce that u_h is in particular in $H^{\frac{3}{2}}(S_h)$.

Denote by $\tilde{u}_h^i$ the trivial extension of u_h in ∂Q_h^i

$$\tilde{u}_h^i = \begin{cases} u_h & \text{on} \quad S_h \\ 0 & \text{on} \quad \partial Q_h^i \setminus S_h, \end{cases}$$

then $\tilde{u}_h^i|_{\partial Q_h^i}$ belongs in particular to $H^{\frac{3}{2}}(\partial Q_h^i)$ (see Proposition 2.11 in [7]). Let $\hat{u}_h^i = \tilde{E}xt\tilde{u}_h^i$ be a function in $H^2(Q_h^i)$ such that $\hat{u}_h^i|_{\partial Q_h^i} = \tilde{u}_h^i$ (see Proposition 2.2), then $\Delta \hat{u}_h^i \in L^2(Q_h^i)$ and $\|\Delta \hat{u}_h^i\|_{L^2(Q_h^i)} \leq c(h)\|f\|_{L^2(Q)}$.

We note that the restriction $u_h^i = u_h|_{Q_h^i}$ is the weak solution in $H^1(Q_h^i)$, of the problem

$$\begin{cases} -\Delta u_h^i = f & \text{in} \quad Q_h^i \\ u_h^i = 0 & \text{on} \quad \partial Q_h^i \setminus S_h \\ u_h^i = u_h & \text{on} \quad S_h. \end{cases}$$

Then the function $v_h^i := u_h^i - \hat{u}_h^i$ is the weak solution in $H_0^1(Q_h^i)$ of the Dirichlet problem

$$\begin{cases} -\Delta v_h^i = f + \Delta \hat{u}_h^i & \text{in} \quad Q_h^i \\ v_h^i = 0 & \text{on} \quad \partial Q_h^i. \end{cases}$$

By proceeding as in the proof of Proposition 4.5 we can obtain that $v_h^i \in H^{s_i}(Q_h^i)$ where $s_1 < \dfrac{8}{5}$ and $s_2 < \dfrac{7}{4}$. Finally u_h^i inherits the regularity of v_h^i as $\hat{u}_h^i$ is more regular, this yields (4.11).

We now prove (4.10), i.e. that $u \in C(\overline{Q})$. We note that from Morrey-Sobolev embedding it follows that $u_h \in C(S_h)$ because $H^{\frac{3}{2}}(S_h)$ is embedded in $C(S_h)$, and that $u_h^i \in C(\overline{Q_h^i})$ as $u_h^i \in H^{\frac{8}{5}-\varepsilon}(Q_h^i)$, $i = 1, 2$. We conclude the proof taking into account that $u_h^i|_{S_h} = u_h$, $i = 1, 2$. ■

From Theorem 4.3 it follows that the variational solution of problem (4.7) is the solution of problem (P_h) which can be rigorously stated as follows

$$(P) \quad \begin{cases} -\Delta u = f & \text{in } L^2(Q_h^i),\ i = 1, 2 & j) \\ -\Delta_S u = \left[\dfrac{\partial u}{\partial n} \right] & \text{on } L^2(S_h) & jj) \\ u = 0 & \text{on } H^{\frac{1}{2}}(\partial Q) \cap C(\partial Q) & jjj) \\ u^1 = u^2 & \text{on } H^1(S_h) \cap C(S_h) & jv) \\ u = 0 & \text{on } H^{\frac{1}{2}}(\partial S_h) \cap C(\partial S_h) & v). \end{cases}$$

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