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Semilinear Equations on Hausdorff Spaces with Symmetries (*$_*$*)

ABSTRACT. — Critical point problems involving Dirichlet forms on functions on a Hausdorff space, equipped with shifts by a noncompact invariance group, lack compactness due to the shifts. We use the functional-analytic version of cocentration-compactness to address variational semilinear problems in our general framework that applies, in particular, to semilinear elliptic problems on symmetric spaces, non-compact Lie groups, metric graphs, self-similar fractal tiles and point lattices.

Equazioni semilineari su spazi di hausdorff con simmetrie

SUNTO. — I problemi di punto critico relativi ad una forma di Dirichlet su uno spazio di Hausdorff con un gruppo di traslazioni mostrano una mancanza di compattezza dovuta alle traslazioni. Nel presente articolo si usa una versione funzionale analitica del principio di concentrazione-compattezza adatta a trattare problemi variazionali semilineari nel nostro quadro generale, che si applica, in particolare a problemi ellittici semilineari su spazi simmetrici, su gruppi di Lie non compatti, su grafi metrici, su frattali con struttura autosimilare e su reticoli di punti.

1. - INTRODUCTION

This paper studies existence problems for minimizers in Sobolev inequalities for Dirichlet forms on spaces of homogeneous type (cf. [4], [5]) and for related semilinear

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elliptic equations $Au = f(u)$ on the Hausdorff space. Existence in such problems is typically derived from compactness of Sobolev imbedding. The latter, in the general Dirichlet forms setting, has been proved for two cases. In one, [6], the problems are studied on a relatively compact space. In the other, [7], compactness in the Sobolev embedding is a result of adding a penalizing term term to the Dirichlet form, roughly speaking, a L^2-functional with a weight whose infimum over complements of compact sets grows to infinity with the set. Several existence results for semilinear elliptic problems based on the compactness results of [6] and [7], are proved in [16].

The present paper deals with elliptic problems associated with energy forms on Dirichlet spaces, where the Sobolev imbedding is not compact. It also addresses two instances of compact Sobolev embedding when the underlying Dirichlet space is not compact.

Many specific problems where the Sobolev embedding lacks compactness (e.g. Laplace operator on R^N, subelliptic operators on Lie groups, Laplace-Beltrami operators on symmetric Riemannian spaces, elliptic operators on metric graphs and infinite fractal tiles) possess a non-compact group acting on the underlying space (e.g. parallel translations on R^N, left Lie group shifts, isometry group of the symmetric space), which preserves both the energy form and the L^p-norms, but generates sequences of the form $u \circ \eta_k \rightharpoonup 0$. This phenomenon has been addressed by P.-L.Lions ([15] and subsequent papers) in the case of shifts on $\mathbb{R}^N$, by means of the concentration compactness principle. One can find in literature numerous adaptations of Lions' method to different problems that we don't quote here, following instead the unifying approach of [18], where the concentration compactness is studied for a general Hilbert space equipped with a dislocation group (cf. the definition in Section 3 below).

Invariance with respect to actions of a non-compact group is an intuitive attribute of an infinite homogeneous medium or an empty space: speaking metaphorically, the shift invariance means that under a transformation group that changes the location of observer without causing him destructive deformations and without altering the global energy, the medium should «still look the same». We assume that the group of shifts is robust enough, namely that there exists a compact neighborhood that any point can be brought into by an appropriate shift (condition (I)), and that the group is small enough so that compactness of a subset of the group (in the CO-topology) is subordinate to compactness of the subset's actions at a single point (condition (II)). From the technical angle, these conditions suffice to satisfy the definition of the dislocation group in the functional-analytic framework of concentration compactness.

Condition (III) can be understood as a requirement that the operator associated with the quadratic form will be in a way quasi-local or have a diagonal-heavy kernel. This paper postulates the local Sobolev inequality (condition IV), although condition (IV) is derived in the framework of [4], [5], [6] from general assumptions that include the scaled Poincaré inequality, an assumption on the space dimension in the Coifman-Weiss sense and the definition of the Dirichlet form. However, to require from the start that the quadratic form would be a Dirichlet form is to exclude applications with

non-local elliptic operators, for instance the discrete Laplace operator on $\mathbb{Z}^N$ addressed in Section 5.

In Section 2 we list the assumptions on the space, the shifts and the energy form. In Section 3 we formulate the concentration compactness results in the present settings. In Section 4 we prove compactness of Sobolev imbedding in two cases where the domain is not compact, in restriction to symmetric functions (generalization of 14], [10]) and for domains that are slim at infinity (generalization of [1]). In Section 5 we provide existence results and list applications for elliptic and sub-elliptic problems on Lie groups, Riemannian manifolds, lattices, fractal tiles and metric graphs.

2. - Assumptions and definitions

Let X be a locally compact Hausdorff space supplied with a group G acting continuously on X, and with a G-invariant Radon measure μ, $\operatorname{supp}(\mu) = X$. The group G is assumed to be a topological group in the compact-open (in what follows, CO-) topology.

We assume that

(I) there exists a relatively compact open set $V \subset X$, such that $\cup_{\eta \in G} \eta V = X$.

Let us define the set of shifts that keep two given sets intersected:

$$Q_M(A, B) := \{\eta \in M : \eta A \cap B \neq \emptyset\}, \quad M \subset G, \quad A, B \subset X, \tag{2.1}$$

(II) for any compact sets $K_1, K_2 \subset X$, the intersection set $Q_G(K_1, K_2)$ is compact in G.

This condition is fulfilled, in particular, by parallel translations on R^n and, more generally, by Lie group shifts and by isometries on Riemannian manifolds. Equivalently it may be formulated as a requirement that the map $X \times G \to X \times X$, $(x, \eta) \mapsto (x, \eta x)$ to be a proper map.

Let us introduce the following sequence of sets. Let $V_1 \subset X$. By induction we define

$$V_{n+1} := Q_G(V_n, V_n)\, V_n = \bigcup_{\eta \in G : \eta V_n \cap V_n \neq \emptyset} \eta V_n, \; n \in \mathbb{N}. \tag{2.2}$$

Note that the sequence V_n is monotone increasing. In what follows we will use the sequence V_n with the set $V_1 = V$, V as in (I).

The set of actions of G on $L^2(X, \mu)$

$$g_\eta u := u \circ \eta, \quad \eta \in G, \tag{2.3}$$

will be denoted as D.

Let H_0 be a D-invariant subspace of $C_0(X)$, dense in $L^2(X, \mu)$ (where $C_0(X)$ denotes the space of real valued continuous functions with compact support in X) and

equipped with a positive symmetric quadratic form $a : H_0 \times H_0 \to \mathbb{R}$, satisfying $a(u, u) \geqslant C \int_X u^2 \, d\mu$, $u \in H_0$, with some $C > 0$ and D invariant on H_0, i.e.

$$\forall u, v \in H_0, \quad \eta \in G, \quad a(u \circ \eta, v) = a(u, v \circ \eta^{-1}).$$

We define H as a Hilbert space by taking a completion of H_0 with respect to the norm $a(\cdot, \cdot)^{1/2}$. Obviously, H will be continuously imbedded into $L^2(X, \mu)$ and dense there. In what follows, all Hilbert space notations, unless specified otherwise, will refer to H. Note that density of H_0 in $L^2(X, \mu)$ assures the domain $D(A)$ of the operator associated with the form a is dense in $L^2(X, \mu)$ and that a is invariant with respect to the group actions on H:

$$\forall u, v \in H, \quad \eta \in G, \quad a(u \circ \eta, v) = a(u, v \circ \eta^{-1}). \tag{2.4}$$

The group D acts on H as a group of unitary operators.

We assume finally that there exist two open sets $U, U_0 \subset X$, such that $\overline{V_2} \subset U_0 \subset \overline{U_0} \subset U \subset V_3$ and a bounded non-negative real-valued function χ with support in U, such that $u \mapsto \chi u$ is a bounded operator on H, and that the following holds:

(III) If $J \subset G$ is such that $\{\eta U\}_{\eta \in J}$ covers X with uniformly finite multiplicity, then there is a $C > 0$ such that for every $u \in H$,

$$\sum_{\eta \in J} a(\chi \circ \eta u, \chi \circ \eta u) \leqslant C \quad a(u, u). \tag{2.5}$$

(IV) There is a number $2_X > 2$ such that for every $p \in [2, 2_X]$ there exist a $C > 0$ such for every $u \in H$,

$$\int_{U_0} |u|^p \, d\mu \leqslant C \quad a(\chi u, \chi u)^{p/2}. \tag{2.6}$$

Moreover, if $p < 2_X$, then the set of traces of u on $L^p(U_0, \mu)$, that satisfy inequality $a(\chi u, \chi u) < 1$ is relatively compact.

One may implement the conditions (I-IV) in the framework of Riemannian strongly local Dirichlet forms (see [2], [3], [4], [5]) as follows. We may assume that the action of G transforms intrinsic balls into intrinsic balls with the same radius, we may set without loss of generality $V = B(0, 1)$ and require that $\cup \{\eta B(0, 1)\}_{\eta \in J}$, $J \subset G$, covers X with uniformly finite multiplicity (the intrinsic ball $B(0, r)$ is the ball centered in 0 with radius r relative to the intrinsic distance on X, see [2], [3]. Condition (II) is to be posed. Let $U = B(0, 6)$, $U_0 = B(0, 3)$, the assumptions (III) and (IV) hold choosing $\chi u = \phi u$, where ϕ denotes the cut-off functions between the balls $B(0, 3)$ and $B(0, 6)$ and $2_X = \dfrac{2\nu}{\nu - 2}$ if $\nu > 2$ or 2_X any finite number greater than 2 if $\nu \leqslant 2$, where ν is the intrinsic dimension relative to the duplication property for μ, which is assumed to hold.

PROPOSITION 2.1: *If a sequence $\eta_k \in G$, is discrete (has no convergent subsequence), then $\forall u \in H$, $u \circ \eta_k \rightharpoonup 0$ in H.*

PROOF: By density, it suffices to take u with compact support and to consider weak convergence in $L^2(X, \mu)$ with test functions v having a compact support. From (II) it follows that for all k sufficiently large $\operatorname{supp}(u)$ will be disjoint from $\eta_k \operatorname{supp}(v)$ (or else η_k will have a convergent subsequence), so that $\operatorname{supp}(u) \circ \eta_k$ will be disjoint from $\operatorname{supp}(v)$ and $\int_X u \circ \eta_k v \, d\mu = 0$ for all k large. ■

PROPOSITION 2.2: *If $\eta_k \to \eta \in G$ in G and $u \in H$, then $u \circ \eta_k \to u \circ \eta$ in H.*

PROOF: Without loss of generality we can assume that $\eta = e$. By (2.4) the actions of group shifts on H are unitary operators, so it suffices to prove weak convergence. By the density assumptions, it suffices to verify that for $u, v \in C_0(X)$, $\int_X u \circ \eta_k v \, d\mu \to$ $\to \int_X u v \, d\mu$. The latter follows from the Lebesgue dominated convergence theorem, since the CO-convergence of η_k implies pointwise convergence for $u \circ \eta_k$. ■

3. - CONCENTRATION COMPACTNESS FOR GROUP SHIFTS ON HAUSDORFF SPACES.

DEFINITION 3.1: *Let H be a separable Hilbert space. We say that a group D of unitary operators on H is a group of dislocations if*

(*) *any sequence $g_k \in D$ that does not converge to zero weakly has a strongly convergent subsequence.*

DEFINITION 3.2: *Let $u, u_k \in H$. We will say that u_k converges to u weakly with concentration (under dislocations D), which we will denote as*

$$u_k \overset{(D)}{\rightharpoonup} u ,$$

if for all $\varphi \in H$,

$$\lim_{k \to \infty} \sup_{g \in D} (g(u_k - u), \varphi) = 0 . \tag{3.1}$$

THEOREM 3.3: ([18]) *Let $u_k \in H$ be a bounded sequence. Then there exists $w^{(n)} \in H$, $g_k^{(n)} \in D$, $k, n \in \mathbb{N}$ such that for a renumbered subsequence*

$$g_k^{(1)} = id, \quad g_k^{(n)^{-1}} g_k^{(m)} \rightharpoonup 0 \quad \text{for } n \neq m , \tag{3.2}$$

$$w^{(n)} = w \lim g_k^{(n)-1} u_k \tag{3.3}$$

$$\sum_{n \in \mathbb{N}} \|w^{(n)}\|^2 \leqslant \limsup \|u_k\|^2 \tag{3.4}$$

$$u_k - \sum_{n \in \mathbb{N}} g_k^{(n)} w^{(n)} \overset{(D)}{\rightharpoonup} 0 . \tag{3.5}$$

We recall that a sequence u_k is called a $(PS)_c$-sequence for a functional $\Phi \in C^1(H)$ if $\Phi(u_k) \to c$ and $\Phi'(u_k) \to 0$. The critical set of Φ will be denoted as $K := \{u \in H : \Phi'(u) = 0\}$.

PROPOSITION 3.4: *Let $\Phi \in C^1(H)$ satisfy the following conditions*:

(i) *Φ is invariant under D*: $\forall g \in D$, $\Phi \circ g = \Phi$,

(ii) *Φ' is continuous in the weak topology*: $u_k \rightharpoonup u \Rightarrow \Phi'(u_k) \rightharpoonup \Phi'(u)$. Then any bounded $(PS)_c$-sequence for Φ has a subsequence satisfying (3.5) with $w^{(n)} \in K$.

PROOF From (i) and the definition of Gateaux derivative follows immediately that for every $g \in D$, $\Phi' \circ g = g \circ \Phi'$. Let u_k be a bounded $(PS)_c$-sequence for Φ and let $w^{(n)}$, $g_k^{(n)}$ be as in Theorem 3.3. Then, by (ii),

$$\Phi'(w^{(n)}) = \operatorname{wlim} \Phi'(g_k^{(n)-1} u_k) = \operatorname{wlim} g_k^{(n)-1} \Phi'(u_k) = 0 . \qquad \blacksquare \tag{3.6}$$

Let now H, G and D be defined as in Section 2.

LEMMA 3.5: *The group D is a group of dislocations.*

PROOF: Since the operators g_η are unitary, it suffices to verify $(*)$. If $g_{\eta_k} \not\rightharpoonup 0$, then by Proposition 2.1, η_k has a subsequence convergent in the CO-topology to some $\eta \in G$. Then by Proposition 2.2 $g_{\eta_k} \to g_\eta$ in the strong operator sense. $\blacksquare$

LEMMA 3.6: *There is a subset J of G such that the sets in the collection $\{\eta V_1)\}_{\eta \in J}$ are mutually disjoint, while $\{\eta V_2\}_{\eta \in J}$ covers X. Moreover, the open cover $\{\eta V_3\}_{\eta \in J}$ has a uniformly finite multiplicity.*

PROOF: The second assertion of the lemma follows from the first one by the following argument. Without loss of generality consider multiplicity of the covering at a point in V_3. The multiplicity will not exceed the number of $\eta \in J$, such that $\eta V_3 \cap V_3 \neq \emptyset$. This number is not larger than the number of $\eta \in J$ such that $\eta V_1 \subset V_4$. Since these sets are disjoint, the multiplicity of the covering does not exceed $\dfrac{\mu(V_4)}{\mu(V_1)}$. Note that $\mu(V_1) > 0$ by assumption, and $\mu(V_4) < \infty$ since μ is Radon measure and V_4 is relatively compact due to (II).

Now let us construct the subset $J \subset G$: Since X is paracompact, we may first replace G with a subset $J_0 \subset G$, such that $X = \bigcup_{\eta \in J_0} \eta V_1$ is a locally finite cover. Namely, we find first a locally finite refinement $\eta V_\eta \subset \eta V_1$, $\eta \in J_0$. We can show then that finite multiplicity persists even if we replace every V_η with V_1. Since V_3 is relatively compact, the multiplicity of its covering by ηV_η's is uniformly finite. All ηV_η's that intersect V_2 lie in V_3. Therefore, by a finite measure argument, there are finitely many ηV_η's that intersect V_2. If we show that

$$V_2 \cap \eta V_\eta = \emptyset \Rightarrow V_1 \cap \eta V_1 = \emptyset, \tag{3.7}$$

then there will be finitely many $\eta \in J_0$ such that ηV_1 intersects V_1. Let us prove (3.7). If $\eta V_1 \cap V_1 \neq \emptyset$, then $\eta \in Q_G(V_1, V_1)$ and so $\eta V_1 \subset V_2$ and consequently, $\eta V_\eta \subset V_2$, which is a contradiction. It remains to note that argument above extends to covering of any shifted set ηV_1, $\eta \in J_0$ with the same upper bound for multiplicity.

By induction we define subsets $J_k = A_k \cup B_k \subset G$ such that the number of elements in A_k equals k and

$$X = \bigcup_{\eta \in A_k} \eta V_2 \cup \bigcup_{\zeta \in B_k} \eta V,$$

and $\eta V \cap \zeta V = \emptyset$ for any $\eta \in A_k$, $\zeta \in J_k$, $\eta \neq \zeta$. Furthermore $A_k \subset A_{k+1}$ for all k, while $B_k \supset B_{k+1}$ with $\bigcap_{k=0}^{\infty} B_k = \emptyset$. Since the cover $\{\eta V\}_{\eta \in J_0}$ was locally finite, the latter implies that any compact set $K \subset X$ is contained in $\bigcup_{\eta \in A_k} \eta V_2$ for sufficiently large k. Finally take $J := \bigcup_{k=0}^{\infty} A_k$. Begin with $A_0 := \emptyset$, $B_0 := J_0$. Write J_0 as a sequence $J_0 = \{\eta_1, \eta_2, \ldots\}$. So assume A_k, B_k have already been constructed. Let $m_k = \min\{m : \eta_m \in B_k\}$. Set $A_{k+1} := A_k \cup \{\eta_{m_k}\}$ and let $B_{k+1} := \{\eta \in B_k;\ \eta V \cap \eta_{m_k} V = \emptyset\}$. ■

LEMMA 3.7: *Let $r \in (2, 2_X)$ and let $u_k \in H$ be a bounded sequence. Then*

$$u_k \overset{(D)}{\rightharpoonup} 0 \Leftrightarrow u_k \to 0 \text{ in } L^r(X, \mu). \tag{3.8}$$

PROOF: *First, assume that $u_k \to 0$ in L^r. Then for every sequence $\eta_k \in G$, $u_k \circ \eta_k \rightharpoonup 0$ in L^r. However, since u_k is bounded in the Hilbert norm, $u_k \circ \eta_k \rightharpoonup 0$ in H and therefore, $u_k \overset{(D)}{\rightharpoonup} 0$.*

Assume now that $u_k \overset{(D)}{\rightharpoonup} 0$. By (IV), *there is a $C > 0$ such that*

$$\int_{\eta V_2} |u_k|^r d\mu \leqslant Ca(\chi \circ \eta u, \chi \circ \eta u) \left(\int_{\eta V_2} |u_k|^r d\mu \right)^{1-2/r}, \quad \eta \in G. \tag{3.9}$$

Due to Lemma 3.6, *there is a countable set $J \subset X$ such that for $i = 2, 3$ the sets ηV_i, $\eta \in J$, form a cover of finite multiplicity for X, so by adding terms in* (3.9) *over $\eta \in J$ and using* (III), *we obtain*

$$\int_X |u_k|^r d\mu \leqslant Ca(u_k, u_k) \sup_{\eta \in J} \left(\int_{V_2} |u_k \circ \eta^{-1}|^r d\mu \right)^{1-2/r} \leqslant \tag{3.10}$$

$$\leqslant 2C \left(\int_{V_2} |u_k \circ \eta_k^{-1}|^r d\mu \right)^{1-2/r}$$

for an appropriately chosen «near-supremum» sequence $\eta_k \in J$. It remains to note that by the compactness in (IV) *one has $u_k \circ \eta_k \to 0$ in $L^r(X, \mu)$, so that the assertion of the lemma follows from* (3.10). ■

We conclude the section with the proof of the global Sobolev inequality.

LEMMA 3.8: *For every* $p \in [2, 2_X]$ *there is a* $C > 0$ *such that for all* $u \in H$

$$\int_X |u|^p \, d\mu \leqslant Ca(u, u)^{p/2}. \tag{3.11}$$

PROOF: *We use the covering from Lemma* 3.6. *By* (IV),

$$\int_{\eta V_2} |u|^p \, d\mu \leqslant Ca(\chi \circ \eta u, \chi \circ \eta u)^{p/2}, \quad \eta \in J. \tag{3.12}$$

Adding the terms in (3.12) *over* J, *and taking into account that shifts of* V_2 *and of* V_3 *form a covering of* X *of finite multiplicity, we arrive, using* (III) *in the right hand side, at*

$$\int_X |u|^p \, d\mu \leqslant C \sup_{\eta \in J} a(\chi \circ \eta u, \chi \circ \eta u)^{p/2-1} a(u, u) \leqslant Ca(u, u)^{p/2}. \quad \blacksquare \tag{3.13}$$

4. - SOME COMPACT IMBEDDINGS

In this section we prove compactness of Sobolev imbeddings in two cases: compactness in presence of symmetries and compactness on non-compact domains thin at infinity.

Let us introduce the following condition on X, G:

(Γ) There is a subgroup Γ of G such that for every discrete sequence $\eta_k \in G$ there is an infinite subset Γ' of Γ such that for every $\gamma \in \Gamma'$, the sequence $\eta_k^{-1} \gamma \eta_k$ is discrete.

This condition is fulfilled, for example, when $X = \mathbb{R}^N$, $N > 2$, Γ is the group of rotations, and G is the product of Γ and the group of parallel shifts.

THEOREM 4.1: *Assume that* X, Γ *satisfy* (Γ). *Let* H_Γ *be a subspace of* H *consisting of functions satisfying* $u \circ \gamma = u$ *for every* $\gamma \in \Gamma$. *Then* H_Γ *is compactly imbedded into* L^p, $p \in (2, 2_X)$.

PROOF: Let $u_k \in H_\Gamma$ be a weakly convergent subsequence in H. Assume that for some discrete sequence η_k, the function $w := \operatorname{wlim} u_k \circ \eta_k^{-1} \neq 0$. Since for any $\gamma \in \Gamma$, $u = u \circ \gamma^{-1}$, we have also $w = \operatorname{wlim} u_k \circ \gamma^{-1} \circ \eta_k^{-1}$ and $w \circ \gamma = \operatorname{wlim} u_k \circ \gamma^{-1} \circ \eta_k^{-1} \circ \gamma = \operatorname{wlim} u_k \circ (\gamma^{-1} \eta_k^{-1} \gamma)$.

Let $\eta_k^\gamma = \gamma \eta_k \gamma^{-1}$. Then $(\eta_k^\gamma)^{-1} \eta_k^{\gamma'}$ is discrete if and only if $\eta_k^{-1} \gamma^{-1} \gamma' \eta_k$ is discrete, which due to (Γ) occurs for infinitely many pairs γ, γ'. Therefore, taking into account Proposition 2.1, we can choose a sequence $\gamma_n \in \Gamma'$ such that the assumptions

of Theorem 3.3 hold for the sequence $g_k^{(n)} u_k = u_k \circ \eta_k^{\gamma_n}$ The asymptotic expansion of u_k in Theorem 3.3 will so contain infinitely many terms $w^{(n)}$ corresponding to $\eta_k^{\gamma_n}$, each of them with the same norm as w. Then (3.4) implies that $w^{(n)} = 0$ for all $n > 1$. Thus, $u_k - \operatorname{wlim} u_k \xrightarrow{(D)} 0$ and by Lemma 3.7, $u_k \to \operatorname{wlim} u_k$ in L^p for any $p \in (2, 2_X)$. ■

Let us define, for an open set $\Omega \subset X$, the space $H(\Omega)$ as a closure in the H-norm of the set of elements of H whose supports lie in Ω. We note immediately that (3.11) yields a continuous imbedding of $H(\Omega)$ into $L^p(\Omega, \mu)$.

We now introduce the following class of domains.

Definition 4.2: *An open set $\Omega \subset X$ will be called asymptotically null if for every discrete sequence $\eta_k \in G$, $\mu(\liminf \eta_k \Omega) = 0$.*

(We recall that for a sequence of sets X_k, $\liminf X_k := \bigcup_n \bigcap_{k \geqslant n} X_k$ and $\limsup X_k := \bigcap_n \bigcup_{k \geqslant n} X_k$.)

For some open sets that are not relatively compact, the embedding into L^p can still be compact if there is not so much of the set left at infinity. The result is analogous to sufficient conditions of compactness in literature for the Euclidean case (e.g. [1]) and for the sub-laplacian on Lie groups ([8]).

Theorem 4.3: *Let $\Omega \subset X$ be an open asymptotically null set. Then for every $p \in (2, 2_X)$, $H(\Omega)$ is compactly embedded into $L^p(\Omega, \mu)$.*

Proof: We prove sequential compactness. Let u_k be a bounded sequence in $H(\Omega)$ regarded as a subspace of H. According to Theorem 3.3, all sequences $\eta_k^{(n)} = \eta_k^{\gamma_n}$ in the preceding theorem ($n > 1$) are discrete. Note that by the condition (IV), weak convergence in H implies convergence μ-a.e., so, up to a set of measure zero, the set in X where $w^{(n)}(x) \neq 0$ (because picking an infinite sequence of values outside Ω yields zero in the limit) is contained in the set of points that can be written as $x = \eta_k x_k$, $x_k \in \Omega$, for all $k \geqslant k_0(x)$, i.e. in $\liminf \eta_k \Omega$, which by the definition of asymptotically null set has measure zero. We conclude that for $n > 1$, $w^{(n)}(x) = 0$ a.e. Then Theorem 3.3 implies then that a subsequence of u_k converges weakly with concentration to $w^{(1)}$. By Lemma 3.7, $u_k \to w^{(1)}$ in $L^p(X, \mu)$. Since both u_k and $w^{(1)}$ are supported in $L^p(\Omega, \mu)$, the convergence in $L^p(\Omega, \mu)$ follows. ■

5. - Application to elliptic problems

5.1. - *Existence results*

Theorem 5.1: *Let $p \in (2, 2_X)$. Under assumptions of Section 2, there is a minimizer in the Sobolev inequality* (3.11).

PROOF: Let u_k be a minimizing sequence for the relation

$$c(p) = \inf_{\|u\|_{L^p(X,\mu)}=1} a(u, u). \tag{5.1}$$

We apply to u_k Theorem 3.3. Then

$$\sum \|w^{(n)}\|^2_{H(X)} \leqslant c(p). \tag{5.2}$$

At the same time it is easy to see that

$$\sum \|w^{(n)}\|^p_{L^p(X,\mu)} = \lim \|u_k\|^p_{L^p(X,\mu)} = 1. \tag{5.3}$$

From (5.1) and (5.2) follows that

$$\sum \|w^{(n)}\|^2_{H(X)} \geqslant c(p) \sum t_n^{2/p}, \tag{5.4}$$

where $t_n = \|w^{(n)}\|^p_{L^p(X,\mu)}$. Note now that (5.3) can be written now as $\sum t_n = 1$, so that with $p > 2$, $\sum t_n^{2/p} \leqslant 1$ only if all but one of t_n, say for $n = n_0$ equals zero. We conclude that $w^{(n_0)}$ is the minimizer. ■

Let $F \in C^1(\mathbb{R})$, $f(s) = F'(s)$, Assume that

$$\lim_{|s|\to\infty} |f(s)|/|s|^{2\chi-1} = 0 \tag{5.5}$$

and

$$\lim_{s\to 0} \frac{f(s)}{s} = 0. \tag{5.6}$$

Let

$$\varphi(u) := \int F(u)\, d\mu, \quad \Phi(u) = a(u, u) - 2\varphi(u), \quad u \in H. \tag{5.7}$$

Under (5.5) the functional Φ is of the class $C^1(H)$. Let

$$\sigma := \sup_{u\in H} \varphi(u)/a_1(u). \tag{5.8}$$

Let A denote the operator associated with the form a with respect to $L^2(X, \mu)$.

THEOREM 5.2: *Let the assumptions of Section* 2 *hold and* $\Omega = X$ *or let the assumptions of Section* 2,4 *hold and* Ω *be an asymptotically null open set, moreover let* (5.5) *and* (5.6) hold and

$$\sigma > 1, \tag{5.9}$$

then for every $\varepsilon > 0$ *there exists an* $\alpha \in [1-\varepsilon, 1]$ *and a* $u \in H(\Omega)\setminus\{0\}$ *satisfying*

$$Au = \alpha f(u) \tag{5.10}$$

in the variational and then semi-strong sense.

Note that condition (5.9) is satisfied if $F(s)/s^2 \to \infty$ when $s \to +\infty$.

THEOREM 5.3: *Assume in addition to the conditions of Theorem 5.2 that there exists a* $\mu > 2$ *such that*

$$f(s)s \geqslant \mu F(s), \quad s \in \mathbb{R}. \tag{5.11}$$

Then there exists a $u \in H(\Omega)\setminus\{0\}$ *satisfying*

$$Au = f(u)$$

in the variational and then semi-strong sense.

Note that in this case the condition (5.9) follows from (5.11).

Proofs of Theorem 5.2 and Theorem 5.11 are analogous to those in [19] for Theorem 4.1 and Theorem 4.2 respectively and can be omitted. We only sketch here a proof for having satisfied a weakened Palais-Smale condition.

For any bounded critical sequence u_k for the functional Φ with $\Phi(u_k) \to c \neq 0$, and any sequence $\eta_k \in G$, the sequence $u_k \circ \eta_k$ remains critical with the same critical value, and the *wlim* $u_k \circ \eta_k$ (on a renumbered subsequence) will be a critical point. If for any choice of η_k, $u_k \circ \eta_k \rightharpoonup 0$, then by Lemma 3.7, $u_k \to 0$ in $L^2(X, \mu)$, $\varphi'(u_k) = 0$ and so $u_k \to 0$ in H, which implies $\Phi(u_k) \to 0$, a contradiction. We conclude, that there is a sequence of shifts η_k such that weak limit of the critical sequence $u_k \circ \eta_k$ is non-zero.

In a similar manner one can prove existence of critical points for semilinear elliptic problems based on different saddle point geometries ([16]), with a reservation that, except for isoperimetric problems, it is much harder in non-compact problems to find critical sequences that converge to a point on a critical level and if $u \neq 0$ is a critical point and $\eta_k \in G$ is a discrete sequence, the sequence $u + u \circ \eta_k$ is critical and diverges. This complicates proofs of existence of multiple solutions that rely on distinction of critical points by their critical values.

5.2. - *Applications*

We list below examples where the conditions of Section 2 are satisfied.

1. Euclidean case ([15]).

Here $X = \mathbb{R}^N$, $N > 2$, $G = \mathbb{R}^N$ acting additively on X, μ is the Lebesgue measure, $H_0 = C_0^\infty(\mathbb{R}^N)$, $a(u, u) = \int_{\mathbb{R}^N} |\nabla u|^2 + u^2$, $2_X = 2N/(N-2)$. Concentrated weak convergence is equivalent to L^p-convergence, $2 < p < 2_X$.

2. Lattice-invariant problems on $\mathbb{R}^N$.

$X = \mathbb{R}^N$, $N > 2$, $G = \mathbb{Z}^N$, $a(u, u) = \int_{\mathbb{R}^N} p(x)\,|\nabla u|^2 + u^2$, where $p(x)$ is a continuous $\mathbb{Z}^N$-periodic function on $\mathbb{R}^N$, $2_X = 2N/(N-2)$. Concentrated weak convergence is equivalent to L^p-convergence, $2 < p < 2_X$. The case is fully analogous to $\mathbb{R}^N$.

3. Grid-symmetric Riemannian manifolds ([11]).

X is a complete $N-$ *dimensional* Riemannian manifold, $N > 2$. G is the group of isometries on X. Condition (I) (co-compactness, satisfied, in particular, by symmetric Riemannian spaces) is required. Condition (II) is a known property of the isometry group ([13]). μ is the Riemannian measure on X. The form a is the quadratic form of Laplace-Beltrami operator, with an added term $\int_X u^2\,d\mu$. Condition (III) follows from locality of the energy form, condition (IV) is the standard Sobolev inequality on a bounded subset, $2_X = 2^N/(N-2)$. Concentrated weak convergence is equivalent to L^p-convergence, $2 < p < 2_X$.

4. Subelliptic problems on Lie groups ([19]).

X is a connected Lie group, $G = X$ acts on X by left shifts, so that condition (I) is satisfied by any relatively compact set V. Condition (II) is implied by local compactness on the Lie group. μ is the left Haar measure on X. The form a is the quadratic form of a sub-elliptic operator (sum of squares of tangent vector fields satisfying Hörmander condition) with an added term $\int_X u^2\,d\mu$. Condition (III) follows from locality of the energy form, condition (IV) is the local Sobolev inequality for subelliptic forms (e.g.[22]); compactness of imbedding for bounded sets is due to [9], $2_X = 2Q/(Q-2)$, where Q is the homogeneous dimension of the group. Concentrated weak convergence is equivalent to L^p- convergence, $2 < p < 2_X$.

5. Metric graphs.

Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with a cusp-free piecewise C^1-boundary. Let $X = \Omega \times \mathbb{N}$ be a formal countable union of identical copies Ω_i, $i \in \mathbb{N}$ of Ω, and let $\Gamma_{ij} \subset \partial\Omega_i$, $j \in J_i \subset \mathbb{N}$, be a finite disjoint collection of C^1-hypersurfaces. Assume that some of Γ_{ij} are glued together in the sense that for every i, $i' \in IN$ there exists a set (possibly empty) $J_{ii'} \subset J_i$, a bijection $\pi_{ii'}: J_{ii'} \to J_{i'i}$, mappings $T^j_{ii'} \in O(N)$ and vectors $v^j_{ii'} \in \mathbb{R}^N$ with $j \in J_{ii'}$, such that $T^j_{ii'}\Gamma_{ij} + v^j_{ii'} = \Gamma_{i'\pi_{ii'}(j)}$. We assume that for every i there are only finitely many i' such that $J_{ii'} \neq \emptyset$. An elementary example is provided by a collection of faces of a unite cube translated by a unit cubic lattice, $\partial(0, 1)^{N+1} + \mathbb{Z}^{N+1}$, with $\Omega = (0, 1)^N$.

Let $H_0(X)$ be a subspace of $C^\infty(X)$ defined by conditions $u_i(x) = u_{i'}(T^j_{ii'}x + v^j_{ii'})$, $x \in \Gamma_{ij}$, $j \in J_{ii'}$. We may further restrict the space by assuming that $u_i(x) = 0$ for x in a neighborhood of $\partial\Omega_i \setminus \bigcup_{j \in J_{ii'},\, i' \in IN} \Gamma_{ij}$. Let $H(X)$ be the closure of $H_0(X)$ in the standard Sobolev norm given by the quadratic form $a(u, u) = \sum(\|\nabla u_i\|_2^2 + \|u_i\|_2^2)$.

Let G be a subgroup of permutations of $\mathbb{N}$ that preserves the structure

of the metric graph on X (i.e. the boundary identifications). The measure on X is the Lebesgue measure on Ω extended to X by countable additivity.

Condition (I) is generally not satisfied and has to be required. It is fulfilled in the example above with $G = \mathbb{Z}^{N+1}$ acting additively on $\mathbb{R}^{N+1}$.

Condition (II) is immediate. Condition (III) follows from the locality of the form and condition (IV) is the standard Sobolev inequality. Elliptic equations $Au = f$ hold as differential equations on the copies of Ω supplied with the boundary conditions for $\partial\Omega_i$: $\sum_{i'} \partial u_{ii'}(x)/\partial\nu(x) = 0$, $x \in \Gamma_{ij}$ where $u_{ii'}$ are the traces of relevant $u_{i'}$ on the identified above portions of $\partial\Omega_{i'}$. If the space $H_0(X)$ were restricted to functions vanishing near the portions of $\partial\Omega_i$ that are not glued to another $\partial\Omega_{i'}$, i.e., do not belong to any of the Γ_{ij}, then the Dirichlet condition holds on that portion of the boundary, otherwise the Neumann condition will hold.

Obviously, $2_X = 2N/(N-2)$. Concentrated weak convergence is equivalent to L^p-convergence, $2 < p < 2_X$. We remark that same symmetry groups play a role also in existence of band spectrum in problems with lattice symmetry on $\mathbb{R}^N$ as well as on metric graphs (cf. [20] and references therein).

6. Fractal tiles.

Let $p_i \in \mathbb{R}^N$, $i = 0, \dots, N$, $N \geqslant 2$, $p_0 = 0$, be vertices of a symmetric simplex in R^N and let Y be the Sierpinski gasket defined by these points p_i and maps $F_i(x) = \frac{1}{2}x + \frac{1}{2}p_i$. Let G be the lattice group in $\mathbb{R}^N$ generated by the points p_i and let $X = \bigcup_{g \in G} gY$. Let μ be the standard measure on Y extended by translations and countable additivity to the whole X (note that the copies of Y overlap only at the vertices that have measure zero). We remark that X is not a blow-up ([21]). Conditions (I) and (II) are immediate. Let $H_0 = C_0^\infty(X)$ and let a_0 be the quadratic form of the fractal Laplacian on Y appended with $\int_Y u^2\, d\mu$. Let χ be the characteristic function on Y. We define

$$a(u, u) = \sum_{g \in G} a_0(\chi u \circ g, \chi u \circ g) + \int_X u^2\, d\mu. \tag{5.13}$$

The form a is invariant by construction. Condition (III) is immediate from construction and the condition (IV) is the local Sobolev inequality (with compactness) with an arbitrarily large subcritical exponent p_0 $2 < p < 2_X = \infty$.

7. Discrete Laplacian.

Let $N > 2$, $X = \mathbb{Z}^N$, $G = \mathbb{Z}^N$ acting additively on X and let μ be the counting measure on X. Conditions (I) and (II) are immediate. We define the space H_0 here as the space of all functions on X with compact support. let

$$a(u, u) = \sum_{x \in \mathbb{Z}^N} \left[(N + \lambda)\, u(x)^2 - \sum_{i=1}^{N} u(x + e_i)\, u(x) \right], \quad \lambda > 0, \tag{5.14}$$

where $e_1 = (1, 0, \dots, 0), \dots, e_N = (0, \dots, 0, 1)$. The shift invariance of the form is obvious and the condition (III) follows immediately from the construction. Condition (IV) is the well-known discrete version of the local Sobolev inequality (the form (5.14)) with $\lambda = 0$ is, up to a scalar multiple, the quadratic form of the discrete Laplacian $\sum_i [u(x + e_i) + u(x - e_i) - 2u(x)]$, $2_X = 2N/(N-2)$. Weak concentrated compactness is equivalent to L^p-convergence on $\mathbb{Z}^N$, $2 < p < 2_X$.

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