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Some Exact Functors on the Category of *b*-Spaces (**)

Abstract. — We study the exactness of some functors defined by spaces of normal sequences, measurable functions and normal measurable functions on the category of *b*-spaces ***b***, and we define the integral of functions with values in a *b*-space.

Su alcuni funtori esatti della categoria dei *b*-spazi

Sunto. — Si studia la proprietà di esattezza per alcuni funtori definiti da spazi di successioni normali, di funzioni misurabili e di funzioni misurabili normali nella categoria dei *b*-spazi ***b***. Si definisce inoltre un integrale per funzioni a valori in *b*-spazi.

1. - Introduction and notations

Spaces of sequences in a Banach space and spaces of functions with values in a Banach space are well known. In this paper we shall define and study such spaces of sequences or of functions with values in *b*-spaces of L. Waelbroeck [7]. Functors will be defined first on the category of Banach spaces **Ban**, and are extended to the category **b**. We shall consider spaces of sequences or more generally of families of elements of a *b*-space. Our study will be devoted respectively to spaces of measurable and summable functions.

We shall consider spaces whose elements are families with an arbitrary index set, always denoted by X. At times, such spaces can be usufel. However most spaces used in the applications are true sequence spaces.

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In the first time, we shall consider b-subspaces of the space $\mathbb{C}^X$, and we will define normal b-subspaces of $\mathbb{C}^X$. If Λ is a normal b-subspace of $\mathbb{C}^X$, we define a functor $\Lambda : \mathbf{Ban} \to \mathbf{Ban} : E \mapsto \Lambda(E)$ and we will show that it is possible to extend it to a functor $\Lambda : \boldsymbol{b} \to \boldsymbol{b} : E \mapsto \Lambda(E)$.

As examples, we shall consider a complete finite or σ-finite measure space $(\Omega, \Re, \mu)$ and the space of Bochner measurable mappings from Ω into a b-space E, that we call $L^0(\Omega, E)$. If B is a completant bounded subset of the b-space E, the space $L^0(\Omega, E_B)$ is a completely metrizable topological vector space. We will prove that the functor $L^0(\Omega, .) : \boldsymbol{b} \to \boldsymbol{E.V.} : E \to L^0(\Omega, E)$ is exact. So if E is a b-space and F a bornologically closed subspace of E, $L^0(\Omega, E/F)$ is the vector space $L^0(\Omega, E)/L^0(\Omega, F)$.

The problem is that the topology of $L^0(\Omega)$ is not locally convex and convexity is important in Functional Analysis. Because of this situation, we will introduce normal b-subspaces of $L^0(\Omega)$, and for each normal b-subspace Ξ, we will define an exact functor $\Xi(.) : \mathbf{Ban} \to \mathbf{Ban}$ when Ξ is a Banach normal subspace of $L^0(\Omega)$. Hence it can be extended to $\boldsymbol{b}$. In particular, if E is a b-space and F a bornologically closed subspace of E, we will define the space $L^p(\Omega, E/F)$ and the Orlicz space $L_\varphi(\Omega, E/F)$, where φ is an Orlicz convex function.

The classical spaces $L^p(\Omega)$ are examples of normal Banach subspaces of $L^0(\Omega)$. The preceding results are valid for them. So we can speak of the functor $L^p(\Omega, .) : \boldsymbol{b} \to \boldsymbol{b} : E \to L^p(\Omega, E)$ and this functor is exact. The space $L^p(\Omega, E)$ is a Banach space, or a b-space, according to the nature of E.

If Ω and Ω' are two measure spaces, $\Omega \times \Omega'$ is a measure space, the Fubini theorem shows that if E is a Banach space and $f \in L^p(\Omega \times \Omega', E)$, then for almost all $x \in \Omega$, the function $f(x, .) : \Omega' \to E : y \mapsto f(x, y)$ is in $L^p(\Omega', E)$ and the function $f(., .) : \Omega \to L^p(\Omega', E) : x \mapsto f(x, .)$ is in $L^p(\Omega, L^p(\Omega', E))$. Thus for any Banach space E, $L^p(\Omega \times \Omega', E) \simeq L^p(\Omega, L^p(\Omega', E))$. This isomorphism extends immediately to the case of b-spaces.

When E is a Banach space, the integral is a bounded linear mapping $L^1(\Omega, E) \to \to E$. When E is a b-space, any function $f \in L^1(\Omega, E)$ belongs to some Banach space $L^1(\Omega, E_B)$, where B is a bounded completant subset of E. We can integrate f in the Banach space E_B. The result is independent of the bounded completant subset B.

Let us fix some notations and recall some definitions that will be used in this paper. Let **E.V.** denotes the category of vector spaces and linear mappings over the scalar field $\mathbb{R}$ or $\mathbb{C}$, and **Ban** the category of Banach spaces and bounded linear mappings. Let $(E, \| \|_E)$ be a Banach space. A Banach subspace F of E is a vector subspace endowed with a Banach norm $\| \|_F$ such that the inclusion $(F, \| \|_F) \to (E, \| \|_E)$ is continuous.

Let E be a real or complex vector space, and B be an absolutely convex set of E. Call E_B the vector space generated by B i.e. $E_B = \bigcup_{\lambda > 0} \lambda B$. The Minkowski functional of B, $\|x\|_B = \inf\{\lambda > 0 : x \in \lambda B\}$ is a semi-norm on E_B. It is a norm if and only if B does not contain any nonzero subspace of E. The set B is completant if its Minkowski functional is a Banach norm.

A bounded structure β on a vector space E is defined by a set of «bounded» subsets of E with the following properties:

1) Every finite subset of E is bounded; **2**) every union of two bounded subsets is bounded; **3**) every subset of a bounded subset is bounded; **4**) a set homothetic to a bounded subset is bounded; **5**) each bounded subset is contained in a completant bounded subset.

A b-espace (E, β) is a vector space E with a boundedness β. A subspace F of a b-space E is bornologically closed if the subspace $F \cap E_B$ is closed in the Banach space E_B for every completant bounded disk B of E.

Let (E, β_E) and (F, β_F) be two b-spaces. A linear mapping $u : E \to F$ is bounded, if it maps bounded subsets of E into bounded subsets of F. The mapping $u : E \to F$ is bornologically surjective if for every $B' \in \beta_F$, there exists $B \in \beta_E$ such that $u(B) = B'$. Let (E, β_E) be a b-space. A b-subspace of E is a subspace F with a boundedness β_F such that (F, β_F) is a b-espace and $\beta_F \subseteq \beta_E$. We design by $\boldsymbol{b}(E_1, E_2)$ the space of all bounded linear mappings $E_1 \to E_2$.

We denote by $\boldsymbol{b}$ the category of b-spaces and bounded linear mappings. For more information about b-spaces we refer the reader to [7] and [4].

2. - Normal sequence spaces

We shall consider spaces whose elements are families with an arbitrary index set, always denoted by X. Most of spaces used in the applications are true sequence spaces.

In this section, we consider b-subspaces of the space $\mathbb{C}^X$, and we define normal b-subspaces of $\mathbb{C}^X$. If Λ is a normal b-subspace of $\mathbb{C}^X$, we define a functor $\Lambda : \mathbf{Ban} \to \mathbf{Ban} : E \mapsto \Lambda(E)$ and we show that it is possible to extend it to a functor $\Lambda : \boldsymbol{b} \to \boldsymbol{b} : E \mapsto \Lambda(E)$.

Let X be a set. We recall that the product space $\mathbb{C}^X$ is a b-space for the following boundedness: A subset B is bounded in $\mathbb{C}^X$ if and only if, for all $x \in X$, the set $B(x) = \{f(x) : f \in B\}$ is bounded in $\mathbb{C}^X$.

Definition 2.1: *Let X be a set.*

i) *A subset B of $\mathbb{C}^X$ is normal if it is absolutely convex and $u . f \in B$ whenever $f \in B$ and u is an element of the unit ball of $l^\infty(X)$.*

ii) *A b-subspace Λ of $\mathbb{C}^X$ is said to be normal if every bounded subset of Λ is included in a normal bounded subset.*

iii) *A Banach subspace Λ of $\mathbb{C}^X$ is said to be normal if for all $f \in \Lambda$ and $u \in l^\infty(X)$ we have $u . f \in \Lambda$ and $\|u . f\|_\Lambda \leq \|u\|_\infty \|f\|_\Lambda$.*

The space $\mathbb{C}^X$ is a normal b-subspace of itself. The unit ball of a normal Banach sub-

space of $\mathbb{C}^X$ *is a completant normal bounded subset of* $\mathbb{C}^X$*. Every bounded subset of a normal b-subspace of* $\mathbb{C}^X$ *is included in a normal completant bounded subset.*

We define the space $\Lambda(E)$ *when* E *is a Banach space.*

DEFINITION 2.2: *Let* E *be a Banach space.*

1. *To each normal b-subspace* Λ *of* $\mathbb{C}^X$*, we associate the vector space* $\Lambda(E) = \{f: X \to E : \|f(.)\| \in \Lambda\}$*. A subset* B *of* $\Lambda(E)$ *is bounded iff the set* $\{\|f(.)\| : f \in B\}$ *is bounded in* Λ*.*

2. *If* Λ *is a Banach subspace of* $\mathbb{C}^X$*, the vector space* $\Lambda(E)$ *is normed by* $|||f|||_{\Lambda(E)} = \|(\|f(.)\|_E)\|_\Lambda$*. It is a Banach space.*

If Λ is a normal Banach subspace of $\mathbb{C}^X$, then $\Lambda(E)$ is a Banach space. If Λ is a normal b-subspace of $\mathbb{C}^X$ and E is a *b*-space, then $\Lambda(E)$ is a *b*-space. When E and F are two Banach spaces and $u : E \to F$ is a bounded linear mapping, we can also define $\Lambda(u) : \Lambda(E) \to \Lambda(F)$ as the mapping $f \mapsto u \circ f$. In this way, we have defined a functor $\Lambda(.)$ on the category **Ban** of Banach spaces, with values either in **Ban** or in ***b*** depending whether Λ is a b-subspace or a Banach subspace of $\mathbb{C}^X$.

We can extend the functor $\Lambda(.)$ to the category ***b***. If E is a *b*-space, the *b*-space $\Lambda(E)$ will be the inductive limit of the *b*-spaces or Banach spaces $\Lambda(E_B)$, where B ranges over the bounded completant subsets of E. It is clear that for $B \subset C$ the structural mapping $\Lambda(E_B) \to \Lambda(E_C)$ is injective, so that the inductive limit $\varinjlim_B \Lambda(E_B)$ can be viewed as an union of b-subspaces. So we let the following definition.

DEFINITION 2.3: *Let* Λ *be a normal b-subspace of* $\mathbb{C}^X$ *and* E *be a b-space. Then* $\Lambda(E)$ *is the union of the Banach spaces* $\Lambda(E_B)$*, where* B *ranges over the bounded completant subsets of* E*. If* E *and* F *are two b-spaces and* $u : E \to F$ *is a bounded linear mapping, then* $\Lambda(u)$ *is the bounded linear mapping* $\Lambda(E) \to \Lambda(F)$, $f \mapsto u \circ f$.

Again it is clear that we have defined a functor $\Lambda(.) : \boldsymbol{b} \to \boldsymbol{b}$.

Let Y be a set and E a *b*-space, we denote by $\beta(Y, E)$ the space of mappings $f: Y \to E$ such that $f(Y)$ is bounded in E. We endow $\beta(Y, E)$ with the equibounded boundedness (i.e. a subset B of $\beta(Y, E)$ is bounded if the set $\{f(x), f \in B, x \in Y\}$ is bounded in E).

In [2], we showed the following result:

PROPOSITION 2.4: *If* Y *is a set and* $u : E \to F$ *is a bornologically surjective bounded linear mapping between two b-spaces, then the mapping* $\beta(Y, u) : \beta(Y, E) \to \beta(Y, F)$, $f \mapsto u \circ f$ *is bornologically surjective.*

It follows from the preceding result that if Y is a set, E is a *b*-space and F is a bornologically closed subspace of E, then $\beta(Y, E/F) = \beta(Y, E)/\beta(Y, F)$. In fact, the functor $\beta(Y, .) : \boldsymbol{b} \to \boldsymbol{b}$ is exact, and then as the sequence $0 \to F \to E \to E/F \to 0$ is exact in the category ***b***, its image by the functor $\beta(Y, .)$, give the following exact sequence $0 \to \beta(Y, F) \to \beta(Y, E) \to \beta(Y, E/F) \to 0$, and the assertion follows.

Proposition 2.5: *Let E be a b-space, Λ be a normal b-subspace of $\mathbb{C}^X$ and $f: X \to E$ be a mapping. Then $(f(x))_{x \in X} \in \Lambda(E)$ iff there exist functions $\lambda \in \Lambda$ and $g \in \beta(X, E)$ such that $f = \lambda g$. A subset B of $\Lambda(E)$ is bounded iff there exist bounded subsets B_1 of Λ and B_2 of $\beta(X, E)$ such that $B \subset B_1 . B_2$.*

This is obvious. It is enough to consider the case where E is a Banach space. If $f \in \Lambda(E)$ we write $f(x) = \lambda(x)\, g(x)$ with $\lambda(x) = \|f(x)\|$ and $g(x) = \dfrac{f(x)}{\lambda(x)}$, if $f(x) \neq 0$, $g(x) = 0$ otherwise, if $f(x) = 0$, we use $\lambda(x) = 0$ and $g(x) = 0$. Thus $\lambda \in \Lambda$ and $\|g(x)\| \leqslant 1$ for all $x \in X$. The last part is proved in the same way.

If E, F and G are Banach spaces, then $(u, v): E \to F \to G$ is a complex in the category **Ban** if $v \circ u = 0$. The complex $(u, v): E \to F \to G$ is exact in **Ban** iff; the range of u is dense in the kernel of v.

We introduc left exact complexes in the category $\boldsymbol{b}$.

Definition 2.6: *A complex $(u, v): E \to F \to G$ of the category $\boldsymbol{b}$ is exact if v has a closed range (i.e. the b-space $v(F)$ is bornologically closed in G) and for all bounded subset C in F, $v(C) = 0$, there exists a bounded completant subset C_1 in F such that $C \subset C_1$, $v(C_1) = 0$, and there exists a bounded completant subset B in E such that $u(B) \subset C_1$ and $\bigcup_{M \in \mathbb{R}^+} M . u(B)$ is dense in the Banach space F_{C_1}.*

We begin by showing the following characterization of left exact complexes in the category $\boldsymbol{b}$.

Proposition 2.7: *Let E, F, G be b-spaces, and $(u, v): E \to F \to G$ be a complex of of the caetegory $\boldsymbol{b}$. Then (u, v) is exact iff for all triples, (B, C, D) of bounded subsets of E, F, G respectively, one can associate a triple of bounded completant subsets (B', C', D') of E, F, G respectively, such that $B \subset B'$, $C \subset C'$, $D \subset D'$; $u(B') \subset C'$, $v(C') \subset D'$, and the complex $(u_{E_{B'}}, v_{F_{C'}}): E_{B'} \to F_{C'} \to G_{D'}$ is exact in* **Ban**.

An exact complex of b-spaces is an inductive limit of exact complexes of Banach spaces. Begin with a bounded subset D in G. there exists a bounded completant subset D_1 of E such that $D \subset D_1$. Since the bounded linear mapping v has a closed range, the subset $D_1 \cap v(F)$ is completant and bounded in in the b-space $v(F)$, and then there exists a bounded completant subset C_1 in F such that $v(C_1) = D_1 \cap v(F)$. Since the set C_1 does not usually contain C, we choose a bounded completant subset C_2 in F such that $C \cup C_1 \subset C_2$. As the complex (u, v) is exact, there exist subsets B_3 and C_3, where B_3 is bounded and completant in E, and C_3 is bounded in F such that $B \subset B_3$, $C_2 \cap v^{-1}(0) \subset C_3$ and the subspace $u(E_{B_3})$ is dense in the Banach space F_{C_3}. We let $B' = B_3$, $C' = C_3$ and $D' = D_3$. The spaces $E_{B'}$, $F_{C'}$, and $G_{D'}$ are Banach spaces, $u_{E_{B'}}$

maps $E_{B'}$ into $F_{C'}$, $v_{F_{C'}}$ maps $F_{C'}$ into $G_{D'}$, and $v(C') = D' \cap v(F)$. This shows that the complex $(u_{E_{B'}}, v_{F_{C'}})$: $E_{B'} \to F_{C'} \to G_{D'}$ is exact in the category **Ban**.

Proposition 2.8: *The functor $\Lambda(.) : \boldsymbol{b} \to \boldsymbol{b}$ is exact.*

Let $G_1 \xrightarrow{v} G_2 \xrightarrow{w} G_3$ be an exact complex in the category $\boldsymbol{b}$, hence the bounded linear mapping v is bornologically surjective onto $Ker(w)$. Let B be a bounded subset in $Ker(\Lambda(w))$. According to the proposition 2.5, B is included in a product $B_1 . B_2$, where B_1 is a bounded subset of Λ and B_2 is a bounded subset of $\beta(X, G_2)$. Thus we can write each function $f \in B$ as $f = \lambda_f . g_f$, with $\lambda_f \in B_1$ and $g_f \in B_2$.

Consider the set $B_2' = \{g_f(x) : x \in X, f \in B\}$. Since $v \circ f = 0$ for all $f \in B$, the set B_2' is included in $Ker(w)$. Moreover it is bounded in the b-space G_2 as $B_2' \subset B_2(X) = \{f(x) : x \in X, f \in B_2\}$ and B_2 is bounded in $\beta(X, G_2)$. Thus there exists a bounded subset C_1 of G_1 such that $B_2' = u(C_1)$. For all $f \in B$ and $x \in X$, we choose (by the axiom of choise) a mapping h_f: $X \to C_1$ such that $g_f(x) = u(h_f(x))$. Then the function $\lambda_f . h_f$ is an element of $\Lambda(G_1)$ such that $f = \Lambda(u)(\lambda_f h_f)$. The set of functions $C = \{\lambda_f h_f : f \in B\}$ is bounded in $\Lambda(G_1)$ and $B = \Lambda(u)(C)$. It follows that the complex $\Lambda(G_1) \xrightarrow{\Lambda(u)} \Lambda(G_2) \xrightarrow{\Lambda(v)} \Lambda(G_3)$ is exact.

Corollary 2.9: *Let Λ be a normal b-subspace of $\mathbb{C}^X$, E be a b-space and F be a bornologically closed subspace of E, then $\Lambda(E/F) = \Lambda(E)/\Lambda(F)$.*

Now we give some examples.

Examples 2.10: **1.** The Banach space c_0 of all sequences of complex numbers which converge to 0, is a normal Banach subspace of $\mathbb{C}^{\mathbb{N}}$, and then if $\Lambda = c_0$, we obtain $c_0(E/F) = c_0(E)/c_0(F)$, when E is a b-space and F is a bornologically closed subspace of E.

2. If I is a set, the Banach spaces $l^p(I)$, $1 \leq p \leq \infty$ are normal Banach subspaces of $\mathbb{C}^I$, and then if E is a b-space and F is a bornologically closed subspace of E, we have $l^p(I, E/F) = l^p(I, E)/l^p(I, F)$.

3. Let φ be an Orlicz function (i.e. a convex continuous, non decreasing function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and $\varphi(x) > 0$ for all $x > 0$) and $(\Omega, \mathfrak{R}, \mu)$ be a σ-finite measure space. The Orlicz disk $D_\varphi(\Omega, \mu)$ is the set of μ-measurable functions f on Ω such that $\int_\Omega \varphi(|f(x)|)\, d\mu(x) \leq 1$. The set $D_\varphi(\Omega, \mu)$ is a completant subset of the space of measurable functions on $(\Omega, \mathfrak{R}, \mu)$. The Orlicz space $L_\varphi(\Omega, \mu)$ is the Banach space absorbed by $D_\varphi(\Omega, \mu)$, with the gauge of this set as norm. If $\Omega = \mathbb{N}$, with the measure which counts the points, the Banach space $L_\varphi(\Omega, \mu)$ is called l_φ.

The Banach space l_φ is a normal Banach subspace of $\mathbb{C}^{\mathbb{N}}$, and then if E is a b-space and F is a bornologically closed subspace of E, we have $l_\varphi(E/F) = l_\varphi(E)/l_\varphi(F)$.

By the previous results, the functors $l^\infty(I)(.)$ and $l_\varphi(.)$ are exacts on $\boldsymbol{b}$. We shall write $l^p(I, E)$ instead of $l^p(I)(E)$. Let us remark that $l^\infty(I, E) = \beta(I, E)$.

4. Let w_0 be the function $\mathbb{R}^n \to \mathbb{R} : s \mapsto w_0(s) = (1 + |s|)^{1/2}$, where $|s| = \left(\sum_{i=1}^{n} |s_i|^2\right)^{1/2}$. We define $\theta(s, w_0)$ as the space of function $u : \mathbb{R}^n \to \mathbb{C}$ such that the function $w_0(s)^N u(s)$ is bounded for some natural number $N \in \mathbb{N}$. A subset B of $\theta(s, w_0)$ is said to be bounded if there exists $N \in \mathbb{N}$ such that $\{w_0(s)^N u(s) : s \in \mathbb{R}^n, u \in B\}$ is bounded in $\mathbb{C}$. The space $\theta(s, w_0)$ is a normal b-subspace of $\mathbb{C}^{\mathbb{R}^n}$, and then if E is a b-space and F is a bornologically closed subspace of E, we have $\theta(s, w_0, E/F) = \theta(s, w_0, E)/\theta(s, w_0, F)$, where $\theta(s, w_0, E) = \{f : \mathbb{R} \to E$, there exists $N \in \mathbb{N}$ such that the set $\{w_0(t)^N f(t) : t \in \mathbb{R}\}$ is bounded in $E\}$, where $w_0(t) = (1+t^2)^{-1/2}$. A subset B of $\theta(\mathbb{R}, w_0, E)$ is bounded if there exists $N \in \mathbb{N}$ such that $\{w_o(t)^N f(t); t \in \mathbb{R}, f \in B\}$ is bounded in E.

REMARK 2.11: In [2], we obtained by different methods, that if E is a b-space and F is a bornologically closed subspace of E, then $c_0(E/F) = c_0(E)/c_0(F)$ and $\beta(Y, E/F) = \beta(Y, E)/\beta(Y, F)$.

3. - MEASURABLE FUNCTION SPACES

We suppose that the reader is familiar with the results about Bochner measurable functions with values in a Banach space which were treated in J. Diestel and J. UHL [3]. We consider a complete finite or σ-finite measure space $(\Omega, \mathfrak{R}, \mu)$ and the space $L^0(\Omega, E)$ of Bochner measurable mappings from Ω into a Banach space E. It is a completely metrizable topological vector space.

We prove that the functor $L^0(\Omega, .) : \boldsymbol{b} \to \mathbf{E.V.} : E \to L^0(\Omega, E)$ is exact. So if E is a b-space and F a bornologically closed subspace of E, $L^0(\Omega, E/F)$ is the vector space $L^0(\Omega, E)/L^0(\Omega, F)$.

If $(\Omega, \mathfrak{R}, \mu)$ is a complete finite or σ-finite measure space and E is a Banach space, $L^0(\Omega, E)$ is the space of (equivalence classes of) Bochner measurable functions $\Omega \to E$. So $L^0(\Omega, .)$ is a functor $\mathbf{Ban} \to \mathbf{E.V.}$. As the linear mapping $L^0(\Omega, u)$: $L^0(\Omega, E) \to L^0(\Omega, F)$, $f \mapsto u \circ f$ is injective when $u : E \to F$ is an injective bounded linear mapping, we extend the functor $L^0(\Omega, .)$ in a standard way to the category $\boldsymbol{b}$:

DEFINITION 3.1: *Let $(\Omega, \mathfrak{R}, \mu)$ be a complete measure space or σ-finite measure space, and let E be a b-space. Then the space $L^0(\Omega, E)$ is the inductive limite (i.e. union) of the vector spaces $L^0(\Omega, E_B)$, where B ranges over the bounded completant subsets of E. If $u : E \to F$ is a bounded linear mapping between b-spaces, then the mapping $L^0(\Omega, u)$: $L^0(\Omega, E) \to L^0(\Omega, F)$ is the inductive limit of the mappings $L^0(\Omega, u_{/E_B})$: $L^0(\Omega, E_B) \to L^0(\Omega, F_{u(B)})$.*

We shall prove:

THEOREM 3.2: *Let* $(\Omega, \Re, \mu)$ *be a complete measure space or σ-finite measure space and* $u: E \to F$ *be a bornologically surjective bounded linear mapping between b-spaces. Then the linear mapping* $L^0(\Omega, u)$: $L^0(\Omega, E) \to L^0(\Omega, F)$, $f \mapsto u \circ f$ *is surjective.*

As the inductive limit is an exact functor on the category $\boldsymbol{b}$ [5], we shall consider only the case where E and F are Banach spaces. To prove that the linear mapping $L^0(\Omega, u)$ is surjective, we try to lift up any function $g \in L^0(\Omega, F)$ to $f \in L^0(\Omega, E)$ such that $g = L^0(\Omega, u)(f) = u \circ f$. We shall use the fact that there exists a constant $A > 0$ such that for all $x \in F$, we have $\|x\| \leq A\|u(x)\|$.

The function g takes its values almost everywhere in a separable Banach subspace F_1 of F. Since F_1 is separable, for all $n \in \mathbb{N}$, we can construct a measurable partition of F_1 by sets $Y_{n,k}$ of diameter smaller than $\dfrac{1}{2^n}$ (we start with a countable measurable covering of F_1 by subsets $X_{n,k}$ of diameter smaller than $\dfrac{1}{2^n}$, and we let $Y_{n,1} = X_{n,1}$ and $Y_{n,k} = X_{n,k} \setminus X_{n,1} \cup \ldots \cup X_{n,k-1}$ for $k > 1$ (we drop the $Y_{n,k}$ which would be empty)).

The partition $(Y_{n,k})_k$ will now be used to construct, by induction, a series $\sum f_i$ of measurable functions from Ω to E which converges almost everywhere to a function f such that $g = u \circ f$.

First we construct the function f_0. For all k, we let $\Omega_{0,k} = g^{-1}(Y_{0,k})$ and we choose $x_{0,k} \in u^{-1}(Y_{0,k})$. The function $f_0 = \sum_k 1_{\Omega_{0,k}} x_{0,k}$ is a measurable function from Ω into E. For almost all $x \in \Omega_{0,k}$, we have the two relations $g(x) \in Y_{0,k}$ and $u(f_0(x)) = u(x_{0,k}) \in Y_{0,k}$. So $\|g(x) - u \circ f_0(x)\| \leq 1$.

Suppose that we have defined measurable functions $f_0, f_1, \ldots, f_n$ taking their values in E such that for all $i \in \{0, \ldots, n\}$ and for almost all $x \in \Omega$, we have

$$\|g(x) - u(f_0(x) + \ldots f_i(x))\| \frac{1}{2^i}.$$

Then, we consider the function $h = g - u \circ (f_0 + \ldots + f_n)$ and we let $\Omega_{n+1,k} = h^{-1}(Y_{n+1,k})$ (we keep only the values of k such that $h(\Omega) \cap Y_{n+1,k} \neq \phi$). For such a k, we choose also $x_{n+1,k} \in u^{-1}(Y_{n+1,k})$.

We notice that $\|x_{n+1,k}\| \leq A\|u(x_{n+1,k})\|$ and that

1) $u(x_{n+1,k}) \in Y_{n+1,k}$.
2) $Y_{n+1,k} \cap h(\Omega) \neq \emptyset$.
3) For almost all $x \in \Omega$, $\|h(x)\| \leq \dfrac{1}{2^n}$.
4) The diameter of $Y_{n+1,k}$ is less than $\dfrac{1}{2^{n+1}}$.

The conclusion is $\|u(x_{n+1,k})\| \leq \dfrac{1}{2^{n-1}}$, and $\|x_{n+1,k}\| \leq \dfrac{A}{2^{n-1}}$ for all k. Then we let $f_{n+1} = \sum_k 1_{\Omega_{n+1,k}} x_{n+1,k}$. Clearly, we have the following properties:

1. For all x, $\|f_{n+1}(x)\| \leq \dfrac{A}{2^{n+1}}$.

2. For almost all x, $\|h(x) - u \circ f_{n+1}(x)\| \leq \frac{1}{2^{n+1}}$.

From the first property, we deduce that the series $\sum_n f_n$ converges to a function $f \in L^0(\Omega, E)$, and from the second property, we deduce that for almost all x, we have $g(x) = u(f(x))$, i.e. the mapping $L^0(\Omega, u)$ is surjective.

COROLLARY 3.3: *The functor* $L^0(\Omega, .): \boldsymbol{b} \to \mathbf{E.V.}$ *is exact.*

By the proposition 2.7, it is enough to prove that the functor $L^0(\Omega, .): \mathbf{Ban} \to \to \mathbf{E.V.}$ is exact. Let

$$(0, v, w, 0): 0 \to E \to F \to G \to 0$$

be a short exact complex of **Ban**, we like to prove the exactness of the sequence

$$(0, L^0(\Omega, v), L^0(\Omega, w), 0): 0 \to L^0(\Omega, E) \to L^0(\Omega, F) \to L^0(\Omega, G) \to 0$$

in the category **E.V.**. The mapping $L^0(\Omega, v)$ is injective. Indeed, let $f \in L^0(\Omega, E)$ be such that $L^0(\Omega, v)(f) = 0$. Then for almost all $x \in \Omega$, we have $v(f(x)) = 0$. As the mapping v is injective, the function f vanishes almost everywhere.

It remains to show that the image of $L^0(\Omega, v)$ coincides with the kernel of $L^0(\Omega, w)$. This is clear, by what we have just proved in theorem 3.2, the image of $L^0(\Omega, v)$ is $L^0(\Omega, v(E))$. But this space coincides with $L^0(\Omega, w^{-1}(0))$ which is obviously the kernel of $L^0(\Omega, w)$.

COROLLARY 3.4: *Let* $(\Omega, \mathfrak{R}, \mu)$ *be a complete finite or σ-finite measure space. If* E *is a b-space and* F *a bornologically closed subspace of* E, *then* $L^0(\Omega, E/F) = = L^0(\Omega, E)/L^0(\Omega, F)$.

REMARK 3.5: If (X, d) is a metric space, μ is an inner regular finite measure on X and E is a Banach space. A mapping $f: X \to E$ is Luzin measurable if for all $\varepsilon > 0$, there exists a compact subset K_ε of X such that $\mu(X \backslash K_\varepsilon) < \varepsilon$ and $f_{|K_\varepsilon} \in C(K_\varepsilon, E)$. We denoted by $L^0_{Lus}(\Omega, E)$ the space of Luzin measurable mappings. It is a completely metrizable topological vector space for the topology of convergence in measure. If E is a b-space we define $L^0_{Lus}(\Omega, E)$ as the inductive limit of the inductive system $(L^0_{Lus}(\Omega, E_B))_B$, where B ranges over the bounded completant subsets of E. If we use the results of the paper [1] about continuous functions with values in b-space, we can show the same result as theorem 3.2 for Luzin measurable functions with values in b-spaces.

4. - NORMAL MEASURABLE FUNCTION SPACES

The problem studied in the preceding paragraph has a big flaw. The topology of the space $L^0(\Omega)$ is not locally convex. Now, we introduce normal b-subspaces of

$L^0(\Omega)$, and for each normal b-subspace Ξ, we define an exact functor $\Xi(.): \boldsymbol{b} \to \boldsymbol{b}$. When Ξ is a Banach normal subspace of $L^0(\Omega)$, the functor $\Xi(.)$ has an exact restriction to the category **Ban**. In particular, we define $L^p(\Omega, E/F)$ as the b-space $L^p(\Omega, E)/L^p(\Omega, F)$. If φ is an Orlicz convex function, we could in a similar way define $L_\varphi(\Omega, E/F)$.

Let E be a topological vector space, a subset B of E is bounded in the von Neumann boundedness of E if it is absorbed by all neighbourhoods of the origin of E.

The von Neumann boundedness is a vector boundedness, it is separated if and only if the topological vector space is separated. If E is locally convex, its von Neumann boundedness is convex, but there exist topological vector spaces E whose topologies are not locally convex but their von Neumann boundedness are convex (for example, take I a set not countable and $\mathbb{C}^{(I)}$ with the strongest vector topology).

If E is a locally convex space in which all bounded closed absolutely convex subsets are completants, then the space E endowed with the von Neumann boundedness is a b-space. For more information about the the von Neumann boundedness see [4] and [7].

DEFINITION 4.1: *A normal subset B of $L^0(\Omega)$ is an absolutely convex closed subset, which is bounded for the von Neumann boundedness and such that $u.f \in B$ whenever $f \in B$ and u is a function belonging to the unit ball of $L^\infty(\Omega)$.*

As the topological vector space $L^0(\Omega)$ is complete, its bounded, closed absolutely convex subsets are completants. So the normal subsets of $L^0(\Omega)$ are completants.

Let us recall that a Banach subspace F of a topological vector space E is a vector subspace of E with a norm for which F is complete and the embedding $F \to E$ is continuous.

DEFINITION 4.2: *A normal Banach subspace Ξ of $L^0(\Omega)$ is a Banach subspace of $L^0(\Omega)$ whose closed unit ball is a normal subset of $L^0(\Omega)$.*

In other words, the Banach subspace Ξ of $L^0(\Omega)$ is normal if and only if whenever $u \in L^\infty(\Omega)$ and $f \in \Xi$ then $u.f \in \Xi$ and $\|u.f\|_\Xi \leq \|u\|_\infty . \|.f\|_\Xi$.

EXAMPLE 4.3: The Banach spaces $L^p(\Omega)$ $(p \geq 1)$ are normal Banach subspaces of $L^0(\Omega)$.

DEFINITION 4.4: *A normal b-subspace Ξ of $L^0(\Omega)$ is a vector subspace of $L^0(\Omega)$ with a boundedness such that each bounded subset is contained in a bounded subset which is normal in $L^0(\Omega)$.*

Let us introduce now vector valued function spaces.

DEFINITION 4.5: *Let Ξ be a normal Banach subspace of $L^0(\Omega)$.*

1. *If E is a Banach space, then $\Xi(E)$ is the space of measurable mappings $f : \Omega \to E$ such that $\|f(.)\| \in \Xi$. The norm of a function $f \in \Xi(E)$ is defined as the norm in Ξ of the function $\|f(.)\|$. It is denoted by $\|f\|_{\Xi(E)} = \|\|f(.)\|_E\|_\Xi$. This defines a Banach space $\Xi(E)$.*

2. *If E is a b-space, then we define $\Xi(E) \simeq \bigcup_B \Xi(E_B)$, where B ranges over the bounded completant subsets of E.*

This definition can be extended to the case where Ξ is a normal b-subspace of $L^0(\Omega)$.

Definition 4.6: *Let Ξ be a normal b-subspace of $L^0(\Omega)$.*

1. *If E is a Banach space, then $\Xi(E)$ is the space of measurable mappings $f : \Omega \to E$ such that $\|f(.)\| \in \Xi$. A subset B of $\Xi(E)$ is bounded if $\{\|f(.)\| \mid f \in B\}$ is bounded in Ξ. This defines a b-space $\Xi(E)$.*

2. *If E is a b-space, then we define $\Xi(E) \simeq \bigcup_B \Xi(E_B)$, where B ranges over the bounded completant subsets of E.*

The next step is to show that the functor $\Xi(.)$: $\boldsymbol{b} \to \boldsymbol{b}$ is exact:

Theorem 4.7: *Let $u : E \to F$ be a bornologically surjective bounded linear mapping between b-spaces and Ξ be a normal b-subspace of $L^0(\Omega)$, then the bounded linear mapping $\Xi(u) : \Xi(E) \to \Xi(F)$, $f \mapsto u \circ f$ is bornologically surjective.*

Since the inductive limit is an exact functor on the category $\boldsymbol{b}$ [5], we shall consider only the case where E and F are Banach spaces. In the proof of the Theorem 3.2, for all $n \in \mathbb{N}$, we have constructed a countable measurable partition $\{Y_{n,k}\}$ of F_1 (defined in the theorem 3.2) such that all sets $Y_{n,k}$ have a diameter smaller than $\dfrac{1}{2^n}$. Here, we shall use a finer partition of F_1. We need a measurable and countable partition $(Y_{n,k})_k$ such that $Y_{n,0} = \{0\}$ and for $k > 0$, there exists $y_{n,k} \in F_1$ such that for all $y \in Y_{n,k}$ we have $\|y - y_{n,k}\| \leq \min\left\{\dfrac{1}{2^n}, \|y\|\right\}$. Let us show that such a partition exists.

Let $(z_k)_k$ be a dense sequence of elements of F_1. For all $y \in F_1$, with $y \neq 0$, there exists k such that $\|y - z_k\| \leq \min\left\{\dfrac{1}{2^n}, \|y\|\right\}$. So the union of the sets $X_{n,k} = \left\{y \in F_1 : \|y - z_k\| \leq \min\left\{\dfrac{1}{2^n}, \|y\|\right\}\right\}$ is $F_1 \setminus \{0\}$. The partition $(Y_{n,k})_k$ is then defined by the relations $Y_{n,0} = \{0\}$, $Y_{n,1} = X_{n,1}$ and $Y_{n,k} = X_{n,k} \setminus \bigcup_{j=1}^{k-1} X_{n,j}$.

If one of the sets $Y_{n,k}$ is empty, we drop it and renumber the sequence. Now each set $Y_{n,k}$ is included in one of the sets $X_{n,k}$ and so it is associated to one of the vectors z_k. That vector z_k will be the vector $y_{n,k}$. We have constructed the partition that we wanted.

Consider now $g \in \Xi$. For all k, we let $\Omega_{0,k} = g^{-1}(Y_{0,k})$ and we choose $x_{0,k} \in F$

such that $u(x_{0,k}) = y_{0,k}$. As in the proof of the theorem 3.2, we have $\|x_{0,k}\| \leqslant A\|y_{0,k}\|$ for some $A > 0$.

The function $f_0 = \sum_k 1_{\Omega_{0,k}} x_{0,k}$ belongs to $\Xi(E)$. Indeed, for almost all $x \in \Omega_{0,k}$, we have $f_0(x) = x_{0,k}$ and $g(x) \in Y_{0,k}$, so that

$$\|f_0(x)\| \leqslant A\|y_{0,k}\| \leqslant A(\|y_{0,k} - g(x)\| + \|g(x)\|) \leqslant$$
$$\leqslant A(\min\{1, \|g(x)\|\} + \|g(x)\|) \leqslant 2A\|g(x)\|.$$

As Ξ is a normal Banach subspace of $L^0(\Omega)$, we see that the function $\|f_0(.)\|$ belongs to Ξ, thus $f_0 \in \Xi(E)$. Moreover, for almost all x, we have $\|g(x) - u \circ f_0(x)\| \leqslant 1$.

We end the proof with an induction analogous to that of the theorem 3.2. Suppose we have defined measurable functions $f_0, f_1, \dots, f_n$ belonging to $\Xi(E)$ such that for all $i \in \{0, \dots, n\}$ and for almost all $x \in \Omega$, we have

$$\|g(x) - u(f_0(x) + \dots + f_i(x))\| \leqslant \frac{1}{2^i}.$$

Then, we consider the function $h = g - u \circ (f_0 + \dots + f_n)$ and we let $\Omega_{n+1,k} = h^{-1}(Y_{n+1,k})$ (we keep only the values of k such that $h(\Omega) \cap Y_{n+1,k} \neq \phi$). For such k, we choose also $x_{n+1,k}$ such that $u(x_{n+1,k}) = y_{n+1,k}$ and we let $f_{n+1} = \sum_k 1_{\Omega_{n+1,k}} x_{n+1,k}$. The function f_{n+1} belongs to $\Xi(E)$ because for almost all $x \in \Omega_{n+1,k}$, we have

$$\|f_{n+1}(x)\| = \|x_{n+1,k}\| \leqslant A\|y_{n+1,k}\| \leqslant$$
$$\leqslant A(\|y_{n+1,k} - h(x)\| + \|h(x)\|) \leqslant A\left(\min\left\{\frac{1}{2^{n+1}}, \|h(x)\|\right\} + \|h(x)\|\right) \leqslant 2A\|h(x)\|.$$

As Ξ is normal, $\|f_{n+1}(.)\|$ belongs to Ξ, f_{n+1} belongs to $\Xi(E)$ and $\|f_{n+1}\|_{\Xi(E)} \leqslant 2A\|h\|_{\Xi(E)} \leqslant \dfrac{A}{2^{n-1}}$.

On this way, we construct by induction a sequence $(f_i)_i$ of elements of $\Xi(E)$ such that the series $\sum_i f_i$ converges in $\Xi(E)$. Moreover, for almost all $x \in \Omega$ and for all n, $\|h(x) - u \circ f_{n+1}(x)\| \leqslant \dfrac{1}{2^{n+1}}$, in other words

$$\|g(x) - u \circ (f_0(x) + \dots + f_{n+1}(x))\| \leqslant \frac{1}{2^{n+1}}.$$

Thus in $\Xi(F)$, we have $g = u \circ (\sum_i f_i)$. This proves that the mapping $\Xi(u)$ is surjective.

COROLLARY 4.8: *The functor $\Xi(.)$: $\boldsymbol{b} \to \boldsymbol{b}$ is exact.*

Let $(0, v, w, 0): 0 \to E \to F \to G \to 0$ be a short exact complex of the category $\boldsymbol{b}$. First it is clear that the mapping $\Xi(v)$ is injective. By the Theorem 4.7, the mapping $\Xi(w)$ is bornologically surjective. It remains to prove that the kernel of the mapping $\Xi(w)$ coincides with the image of $\Xi(v)$ (as b-spaces). This is clear by what we have just proved, the image of $\Xi(v)$ is $\Xi(v(E))$. But this space coincides bornologically with the b-space $\Xi(w^{-1}(0))$ which is obviously the kernel of $\Xi(w)$.

COROLLARY 4.9: *Let Ξ be a normal b-subspace of $L^0(\Omega)$. If E is a b-space and F a bornologically closed subspace of E, then $\Xi(E/F) = \Xi(E)/\Xi(F)$.*

5. - SUMMABLE AND LOCALLY SUMMABLE FUNCTION SPACES

The classical spaces $L^p(\Omega)$ are examples of normal Banach subspaces of $L^0(\Omega)$. The results of the preceding paragraph are valid for them. So we can speak of the functor $L^p(\Omega, .): \boldsymbol{b} \to \boldsymbol{b}: E \to L^p(\Omega, E)$ and this functor is exact. The space $L^p(\Omega, E)$ is a Banach space or a b-space, according to the nature of E. Some complements can be added. We consider first the case $p \neq \infty$.

Let Ω and Ω' be two measure spaces, $\Omega \times \Omega'$ is a measure space. The Fubini theorem [3] shows that if E is a Banach space and $f \in L^p(\Omega \times \Omega', E)$ then for almost all $x \in \Omega$, the function $f(x, .): \Omega' \to E: y \mapsto f(x, y)$ is in $L^p(\Omega', E)$ and the function $f(., .): \Omega \to L^p(\Omega', E): x \mapsto f(x, .)$ is in $L^p(\Omega, L^p(\Omega', E))$. Thus for any Banach space E, $L^p(\Omega \times \Omega', E) \simeq L^p(\Omega, L^p(\Omega', E))$. This isomorphism extends immediately to the case of b-spaces. So we get.

PROPOSITION 5.1: *If Ω and Ω' are two measure spaces, the functors $L^p(\Omega \times \Omega', .)$: $\boldsymbol{b} \to \boldsymbol{b}$ and $L^p(\Omega, L^p(\Omega', .)): \boldsymbol{b} \to \boldsymbol{b}$ are isomorphic.*

EXAMPLE 5.2: The Banach space l^p is naturally isomorphic to the Banach space $L^p(\mathbb{N}, \mathfrak{R}(\mathbb{N}), \#)$, where $\mathfrak{R}(\mathbb{N})$ is the σ-algebra of all subsets of $\mathbb{N}$ and $\#$ is the measure on $\mathfrak{R}(\mathbb{N})$ which counts the number of elements. If $1 \leqslant p < \infty$, then $l^p(L^p(\Omega, E)) \simeq L^p(\Omega, l^p(E))$.

We can also consider spaces of locally p-summable functions. If we have to use only Banach spaces or Fréchet spaces, we could restrict ourselves to finite measure spaces. In the study of general b-spaces, σ-finite measure spaces are usefuls.

DEFINITION 5.3: *Let E be a b-space, and let $(\Omega, \mathfrak{R}, \mu)$ be a σ-finite measure space. If $(\Omega_n)_{n \in \mathbb{N}}$ is a partition of Ω such that $\mu(\Omega_n) < +\infty$ for all n, then $L^p_{\mathrm{loc}}(\Omega, E)$ is the direct product $\prod_{n=0}^{\infty} L^p(\Omega_n, E)$.*

In the case where E is a b-space, a consequence of that definition is that the set of values $f(\Omega)$ of the function $f \in L^p_{\mathrm{loc}}(\Omega, E)$ need not be contained in any Banach subspace E_B of E.

The proposition 5.1 extends immediately to the case of locally p-summable functions:

PROPOSITION 5.4: *If Ω and Ω' are two σ-finite measure spaces, the functors $L^p_{\mathrm{loc}}(\Omega \times \Omega', .) : \boldsymbol{b} \to \boldsymbol{b}$ and $L^p_{\mathrm{loc}}(\Omega, L^p_{\mathrm{loc}}(\Omega', .)) : \boldsymbol{b} \to \boldsymbol{b}$ are isomorphic.*

Consider next the case $p = \infty$. We have already defined the exact functor $\beta(X, .)$: $\boldsymbol{b} \to \boldsymbol{b}$. When X is a set and E is a b-space, we have $l^\infty(X, E) \simeq \beta(X, E)$.

If X is a set, we can consider the measure space $(X, \mathfrak{R}(X), \#)$, where $\#$ is the measure which counts the points. If E is a Banach space, the space $L^\infty(X, E)$ could still reasonably be called $l^\infty(X, E)$. However if a mapping $f : X \to E$ is an element of that space, its range is contained in a separable subspace of the Banach space E as the mapping has to be measurable. So this space $L^\infty(X, E)$ is different from what we have called $l^\infty(X, E)$ previously, except if the set X is countable. In that case the range $f(X)$ of f is clearly included in a separable subspace of E.

Thus, it is preferable to keep the old definition of $l^\infty(X, E)$ as the space of all bounded functions $X \to E$. With that definition, we can write $\beta(X, l^\infty(E)) \simeq \simeq l^\infty(\beta(X, E))$, whether X is countable or not.

The result remains valid if we consider a bornological space X whose boundedness has a countable basis $\beta(X, l^\infty(E)) \simeq l^\infty(\beta(X, E))$ for $f \in \beta(X, E)$ iff for all bounded subset B of X, $f_{|B} \in \beta(B, E)$.

6. - THE INTEGRAL

When E is a Banach space, the integral is a bounded linear mapping $L^1(\Omega, E) \to \to E$. When E is a b-space, any function $f \in L^1(\Omega, E)$ belongs to some Banach space $L^1(\Omega, E_B)$, where B is a bounded completant subset of E. We can integrate f in the Banach space E_B. The result is independent of the bounded completant subset B.

DEFINITION 6.1: *Let E be a b-space. The (Bochner) integral on E is the bounded linear mapping $L^1(\Omega, E) \to E$, which is the inductive limit of the bounded linear mappings $L^1(\Omega, E_B) \to E_B$: $f \mapsto \int_\Omega f(x)\, d\mu(x)$, where B ranges over the bounded completant subsets of E.*

We can go a step further and define the integral on $L^1(\Omega, E/F)$ when E is a b-space and F is a bornologically closed subspace of E.

DEFINITION 6.2: *If E is a b-space and F is a bornologically closed subspace of E, then the integral on $L^1(\Omega, E/F)$ is the morphism induced by the bounded linear mapping $L^1(\Omega, E) \to E : f \mapsto \int_\Omega f(x)\, d\mu(x)$.*

This definition is valid as the integral maps $L^1(\Omega, E)$ into E and $L^1(\Omega, F)$ into F.

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