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Spaces of ϕ-Subgaussian Random Variables (**)

ABSTRACT. — We give a new definition of $Sub_\phi(\Omega)$ random variable. This definition is wider than a previous one, studied by one of the Authors. Moreover we prove some inequalities concerning the Sub_ϕ-norms in various contexts.

Spazi di variabili aleatorie ϕ-subgaussiane

SUNTO. — Si dà una nuova definizione di variabile aleatoria appartenente allo spazio $Sub_\phi(\Omega)$. Tale definizione è più ampia di una precedente, studiata da uno degli Autori. Inoltre si provano, in vari contesti, alcune diseguaglianze riguardanti le norme in tale spazio.

1. - INTRODUCTION

The notion of $Sub_\phi(\Omega)$ random variable is a very natural generalization of that of sub-Gaussian random variable, introduced by Kahane in the paper [4] and developed in [5-9]. The spaces $Sub_\phi(\Omega)$ were firstly defined in [1, 2] and studied in the book [3] as well. In this paper we present a new definition of $Sub_\phi(\Omega)$ random variable. This definition is wider than the previous one, and reveals itself of easier use. Most inequalities for the $Sub_\phi(\Omega)$ random variables proved in this paper are new or improve known inequalities.

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2. - Orlicz N-functions

Definition 2.1 [10] Let $\phi = \{\phi(x), x \in \mathbb{R}\}$ be a continuous even convex function. ϕ is called an *Orlicz N-function* if $\phi(0) = 0$, $\phi(x) > 0$ as $x \neq 0$ and the following conditions hold

$$(A_0) \lim_{x \to 0} \frac{\phi(x)}{x} = 0, \quad (A_\infty) \lim_{x \to \infty} \frac{\phi(x)}{x} = \infty .$$

Example 2.1: The following functions are N-functions:

$$\phi(x) = C|x|^\alpha, \quad C > 0, \alpha > 1;$$
$$\phi(x) = \exp\{|x|\} - |x| - 1;$$
$$\phi(x) = \exp\{a|x|^\alpha\} - 1, \quad a > 0, \alpha > 1;$$

$$\phi(x) = \begin{cases} \left(\frac{e\alpha}{2}\right)^{2/\alpha} x^2, & \text{as } |x| \leq \left(\frac{2}{\alpha}\right)^{1/\alpha} \\ \exp\{|x|^\alpha\}, & \text{as } |x| > \left(\frac{2}{\alpha}\right)^{1/\alpha}, \ 0 < \alpha < 1. \end{cases}$$

Lemma 2.1 [3, 10]: *For any N-function ϕ the following statements hold*:

a) $\phi(\alpha x) \leq \alpha\phi(x)$ *as* $x \in \mathbb{R}$, $0 \leq \alpha \leq 1$;
b) $\phi(\alpha x) \geq \alpha\phi(x)$ *as* $x \in \mathbb{R}$, $\alpha > 1$;
c) $\phi(|x| + |y|) \geq \phi(x) + \phi(y)$ *as* $x, y \in \mathbb{R}$;
d) *there exists a constant* $c > 0$, *such that* $\phi(x) > c|x|$ *as* $|x| > 1$;
e) *the function* $\psi(x) = \frac{\phi(x)}{x}$ *is monotone non-decreasing as* $x > 0$;
f) $\phi(x) = \int_0^{|x|} p(t)\, dt$, *where the density* $p = \{p(t), t \geq 0\}$ *is right continuous not-decreasing,* $p(0) = 0$ *and* $p(t) \to \infty$ *as* $t \to \infty$.

Definition 2.2 [10]: Let $\phi = \{\phi(x), x \in \mathbb{R}\}$ be an N-function. The function ϕ^* defined by

$$\phi^*(x) = \sup_{y \in \mathbb{R}} (xy - \phi(y))$$

is called the *Young-Fenchel transform* of ϕ.

Remark 2.1: If $x > 0$, then $\phi^*(x) = \sup_{y > 0}(xy - \phi(x))$. Moreover we have, for any $x \in \mathbb{R}$, $\phi^*(-x) = \phi^*(x)$.

LEMMA 2.2 [10]: *The Young-Fenchel transform of an N-function is an N-function as well and the following inequality holds* (Young-Fenchel inequality)

$$xy \leq \phi(x) + \phi^*(y), \quad \text{as} \quad x > 0, y > 0. \tag{2.1}$$

EXAMPLE 2.2: If $\phi(x) = \dfrac{|x|^p}{p}$, $p > 1$, then $\phi^*(x) = \dfrac{|x|^q}{q}$ where q is such that $\dfrac{1}{q} + \dfrac{1}{p} = 1$.

If $\phi(x) = \exp\{|x|\} - |x| - 1$ then we have $\phi^*(x) = (|x| + 1)\ln(|x| + 1) - |x|$.

CONDITION Q: An N-function ϕ satisfies condition Q if

$$\liminf_{x \to 0} \frac{\phi(x)}{x^2} = c > 0. \tag{2.2}$$

REMARK 2.2: It may happen that $c = \infty$.

EXAMPLE 2.3: The N-function $\phi(x) = c|x|^\alpha$ as $c > 0$, $1 < \alpha \leq 2$, satisfies condition Q, while the N-function $c|x|^\alpha$, $c > 0$, $\alpha > 2$ doesn't; on the other hand, it is easy to see that condition Q holds for the function

$$\phi(x) = \begin{cases} |x|^2, & |x| \leq 1 \\ |x|^\alpha, & |x| > 1 \end{cases} \quad \text{as} \quad \alpha > 2.$$

DEFINITION 2.3 [10]: Let ϕ_1 and ϕ_2 be two N-functions. Then ϕ_1 is said to be subordinate to ϕ_2 ($\phi_1 \prec \phi_2$) if there exist two constants $c > 0$ and $x_0 > 0$ such that for $x > x_0$ the inequality $\phi_1(x) < \phi_2(cx)$ holds. The N-functions ϕ_1 and ϕ_2 are said to be equivalent if both relations $\phi_1 \prec \phi_2$ and $\phi_2 \prec \phi_1$ hold.

REMARK 2.3: Let $\phi_1 \prec \phi_2$. In this case it is easy to prove that for any $x_0 > 0$ there exist two constants x_0 and $c(x_0)$ such that $\phi_1(x) < \phi_2(c(x_0)x)$ as $|x| > x_0$.

THEOREM 2.1: *For any N-function ϕ_1 there exists an N-function ϕ_2 which satisfies condition* Q *and such that $\phi_1 \sim \phi_2$.*

PROOF: Let ϕ_1 be an N-function. We define ϕ_2 as follows. Let $x_0 > 0$ be any constant and put

$$\phi_2(x) = \begin{cases} cx^2, & \text{as } 0 \leq x \leq x_0 \\ \phi_1(x) - \phi_1(x_0) + cx_0^2, & \text{as } x > x_0, \end{cases}$$

where $c = \dfrac{p(x_0)}{2x_0}$ and $p(t)$ is the density of ϕ_1. Then it is not difficult to see that $\phi_1 \sim \phi_2$ and ϕ_2 satisfies condition Q. ■

LEMMA 2.3 [10]: *Let ϕ_1 and ϕ_2 be two N-functions. Then*

a) *if* $\phi_1 < \phi_2$ *then* $\phi_2^* < \phi_1^*$,

b) *if* $\phi_1 \sim \phi_2$ *then* $\phi_2^* \sim \phi_1^*$.

LEMMA 2.4 [10]: *Let ϕ be an N-function and $\phi^{(-1)} = \{\phi^{(-1)}(x),\ x \in \mathbb{R}\}$ be the inverse function of ϕ. The following assertions hold*

a) $\phi^{(-1)}(x)$ *is a monotone increasing, concave continuous function such that* $\phi(0) = 0$, $\phi(x) > 0$ *as* $x > 0$, $\phi(x) \to \infty$ *as* $x \to \infty$;

b) $\phi^{(-1)}(\alpha x) \leqslant \alpha \phi^{(-1)}(x)$, *as* $\alpha \geqslant 1$;

c) $\phi^{(-1)}(\alpha x) \geqslant \alpha \phi^{(-1)}(x)$, *as* $0 \leqslant \alpha < 1$;

d) $\phi^{(-1)}(x + y) \leqslant \phi^{(-1)}(x) + \phi^{(-1)}(y)$;

e) *there exists such constant* $c > 0$ *that* $\phi^{(-1)}(\alpha x) \leqslant cx$, *as* $x > 1$;

f) *the function* $\theta(x) = \dfrac{\phi^{(-1)}(x)}{x}$, $x > 0$, *is monotone decreasing.*

3. - SPACES $Sub_\phi(\Omega)$. DEFINITIONS AND GENERAL PROPERTIES

Let $(\Omega, \mathcal{B}, P)$ be a standard probability space, fixed throughout.

DEFINITION 3.1: Let ϕ be an N-function satisfying condition Q. The random variable ξ belongs to the space $Sub_\phi(\Omega)$ if $E\xi = 0$, $E\exp\{\lambda\xi\}$ exists for all $\lambda \in \mathbb{R}$ and there exists a constant $a > 0$ such that the following inequality holds for all $\lambda \in \mathbb{R}$

$$E\exp\{\lambda\xi\} \leqslant \exp\{\phi(\lambda a)\}. \tag{3.1}$$

REMARK 3.1: Conditions Q and $E\xi = 0$ are necessary. In fact,

$$\begin{aligned} E\exp\lambda\xi &= 1 + \lambda E\xi + \frac{\lambda^2}{2}E\xi^2 + o(\lambda^2), \quad \text{as } \lambda \to 0, \\ \exp\phi(\lambda a) &= 1 + \phi(\lambda a) + o(\phi(\lambda a)), \qquad \text{as } \lambda \to 0. \end{aligned} \tag{3.2}$$

Inequality (3.1) holds for $\lambda > 0$ if the following holds

$$E\xi + \frac{\lambda}{2}E\xi^2 + \frac{o(\lambda^2)}{\lambda} \leqslant \frac{\phi(\lambda a)}{\lambda} + \frac{o(\phi(\lambda a))}{\lambda}, \quad \text{as } \lambda > 0.$$

Since $\dfrac{\phi(\lambda a)}{\lambda} \to 0$ as $\lambda \to 0$ then $E\xi \geqslant 0$. For $\lambda < 0$ (3.1) holds if

$$E\xi + \frac{\lambda}{2}E\xi^2 + \frac{o(\lambda^2)}{\lambda} \geqslant \frac{\phi(\lambda a)}{\lambda} + \frac{o(\phi(\lambda a))}{\lambda};$$

hence $E\xi \geqslant 0$, so that $E\xi = 0$. Now from (3.2) it follows that for $\lambda \to 0$

$$E\xi^2 + \frac{o(\lambda^2)}{\lambda^2} \leqslant \frac{\phi(\lambda a)}{\lambda^2} + \frac{o(\phi(\lambda a))}{\lambda^2}.$$

If $\liminf_{\lambda \to 0} \frac{\phi(\lambda)}{\lambda^2} = 0$, that there exists a sequence $\lambda_n \to 0$ such that $\frac{\phi(\lambda_n)}{\lambda_n^2} \to 0$ as $n \to \infty$, that is $E\xi^2 = 0$ and $\xi = 0$ with probability one.

The condition $\lim_{x \to \infty} \frac{\phi(x)}{x} = \infty$ excludes from our considerations the space of random variables which are bounded with probability one. In fact, if for all $\lambda \in \mathbb{R}$ and some $a > 0$

$$E \exp\{\lambda\xi\} \leqslant \exp\{a|\lambda|\},$$

then, for all $\lambda > 0$ we get

$$E \exp\{\lambda|\xi|\} \leqslant 2 \exp\{a\lambda\}.$$

It follows from Chebyshev inequality that for any $\varepsilon > 0$, $\lambda > 0$

$$P\{|\xi| > \varepsilon\} \leqslant \frac{E \exp\{\lambda|\xi|\}}{\exp\{\lambda\xi\}} \leqslant 2 \exp\{(a - \varepsilon)\lambda\}.$$

The right part of this inequality tends to zero as $\lambda \to \infty$ and $\varepsilon > a$ so that $P\{|\xi| > \varepsilon\} = 0$ if $\varepsilon > a$.

Consider now the following functional, defined on the space Sub_ϕ as (Ω)

$$\tau_\phi(\xi) = \inf(a \geqslant 0 : E \exp \lambda\xi \leqslant \exp \phi(a\lambda),\ \lambda \in \mathbb{R}). \tag{3.3}$$

It is evident that for all $\lambda \in \mathbb{R}$ the following inequality holds

$$E \exp\{\lambda\xi\} \leqslant \exp \phi(\lambda\tau_\phi(\xi)); \tag{3.4}$$

moreover

$$\tau_\phi(\xi) = \sup_{\lambda \neq 0} \frac{\phi^{(-1)}(\ln(E \exp\{\lambda\xi\}))}{|\lambda|}. \tag{3.5}$$

LEMMA 3.1: *Let $\xi \in Sub_\phi(\Omega)$, $\tau_\phi(\xi) > 0$, $\varepsilon > 0$. The following inequalities hold*

$$P\{\xi > \varepsilon\} \leqslant \exp\left\{-\phi^*\left(\frac{\varepsilon}{\tau_\phi(\xi)}\right)\right\};$$

$$P\{\xi < -\varepsilon\} \leqslant \exp\left\{-\phi^*\left(\frac{\varepsilon}{\tau_\phi(\xi)}\right)\right\};$$

$$P\{|\xi| > \varepsilon\} \leqslant 2\exp\left\{-\phi^*\left(\frac{\varepsilon}{\tau_\phi(\xi)}\right)\right\}.$$

PROOF: It follows from Chebyshev inequality that for all $\lambda > 0,\ \varepsilon > 0$

$$P\{\xi > \varepsilon\} \leqslant \frac{E\exp\{\lambda\xi\}}{\exp\{\lambda\varepsilon\}} \leqslant \exp\{\phi(\lambda\tau_\phi(\xi)) - \lambda\varepsilon\}.$$

It follows from this inequality that

$$\begin{aligned} P\{\xi > \varepsilon\} &= \inf_{\lambda > 0} \exp\{\phi(\lambda\tau_\phi(\xi)) - \lambda\varepsilon\} = \exp\{-\sup_{\lambda>0}(\lambda\varepsilon - \phi(\lambda\tau_\phi(\xi)))\} \\ &= \exp\left\{-\sup_{\lambda>0}\left(\lambda\tau_\phi(\xi)\,\frac{\varepsilon}{\tau_\phi(\xi)} - \phi(\lambda\tau_\phi(\xi))\right)\right\} \\ &= \exp\left\{-\phi^*\left(\frac{\varepsilon}{\tau_\phi(\xi)}\right)\right\}. \end{aligned}$$

The first inequality of this lemma is proved. The second inequality can be proved in the same way. The third inequality follows from

$$P\{|\xi| > \varepsilon\} \leqslant P\{\xi > \varepsilon\} + P\{\xi < -\varepsilon\}, \quad \text{as } \varepsilon > 0. \quad ■$$

THEOREM 3.1: *The space $Sub_\phi(\Omega)$ is a Banach space with respect to the norm $\tau_\phi(\cdot)$.*

PROOF: We first prove that $Sub_\phi(\Omega)$ is a linear space with norm $\tau_\phi(\cdot)$.

If $\xi = 0$ with probability one then $\tau_\phi(\xi) = 0$. Conversely, if $\tau_\phi(\xi) = 0$ then $E\exp\{\lambda\xi\} \leqslant 1$ for all $\lambda > 0$ and for any $\varepsilon > 0,\ \lambda > 0$

$$\begin{aligned} P\{|\xi| > \varepsilon\} &\leqslant \frac{E\exp\{\lambda|\xi|\}}{\exp\{\lambda\varepsilon\}} \leqslant (E\exp\{\lambda\xi\} + E\exp\{-\lambda\xi\})\exp\{-\lambda\varepsilon\} \\ &\leqslant 2\exp\{-\lambda\varepsilon\}. \end{aligned}$$

Let now $\lambda \to \infty$. Then we obtain that for any ε $P\{|\xi| > \varepsilon\} = 0$, that is $\xi = 0$ if and only if $\tau_\phi(\xi) = 0$.

It follows from (3.5) that as $a \neq 0$

$$\begin{aligned} \tau_\phi(a\xi) &= \sup_{\lambda \neq 0} \frac{\phi^{(-1)}(\ln(E\exp\lambda a\xi))}{|\lambda|} \\ &= |a| \sup_{a\lambda \neq 0} \frac{\phi^{(-1)}(\ln(E\exp\lambda a\xi))}{|a\lambda|} = |a|\tau_\phi(\xi). \end{aligned}$$

Now we prove that for any $\xi,\ \eta \in Sub_\phi(\Omega)$

$$\tau_\phi(\xi + \eta) \leqslant \tau_\phi(\xi) + \tau_\phi(\eta).$$

If $\tau_\phi(\xi)=0$ or $\tau_\phi(\eta)=0$ the above inequality is obvious. Let $\tau_\phi(\xi)\neq 0$ and $\tau_\phi(\eta)\neq 0$. It follows from Hölder inequality that for all $\lambda\in\mathbb{R}$, $p>0$, $\dfrac{1}{p}+\dfrac{1}{q}=1$,

$$E\exp\{\lambda(\xi+\eta)\}\leqslant\left(E\exp\{p\lambda\xi\}\right)^{\frac{1}{p}}\left(E\exp\{q\lambda\xi\}\right)^{1/q} \tag{3.6}$$

$$\leqslant\exp\left\{\frac{1}{p}\phi(\lambda p\tau_\phi(\xi))+\frac{1}{q}\phi(\lambda q\tau_\phi(\xi))\right\}.$$

Put in (3.6)

$$p=\frac{\tau_\phi(\xi)+\tau_\phi(\eta)}{\tau_\phi(\xi)},\qquad q=\frac{\tau_\phi(\xi)+\tau_\phi(\eta)}{\tau_\phi(\eta)};$$

then we obtain

$$E\exp\{\lambda(\xi+\eta)\}\leqslant\exp\{\lambda(\tau_\phi(\xi)+\tau_\phi(\eta))\},$$

hence $\tau_\phi(\xi+\eta)\leqslant\tau_\phi(\xi)+\tau_\phi(\eta)$.

Now we prove that the space $Sub_\phi(\Omega)$ is complete with respect to the norm $\tau_\phi(\cdot)$. Let the random variables ξ_n, $n\geqslant 1$, belong to the space $Sub_\phi(\Omega)$ and

$$\lim_{n\to\infty}\sup_{m\geqslant n}\tau_\phi(\xi_n-\xi_m)=0. \tag{3.7}$$

Therefore

$$\lim_{n\to\infty}\sup_{m\geqslant n}|\tau_\phi(\xi_n)-\tau_\phi(\xi_m)|\leqslant\lim_{n\to\infty}\sup_{m\geqslant n}\tau_\phi(\xi_n-\xi_m)=0$$

and $\sup\limits_n\tau_\phi(\xi_n)=\tau<\infty$. It follows from (3.7) and lemma 3.1 that for any $\varepsilon>0$

$$P\{|\xi_n-\xi_m|>\varepsilon\}\leqslant 2\exp\left\{-\phi^*\left(\frac{\varepsilon}{\tau_\phi(\xi_n-\xi_m)}\right)\right\}\to 0\quad\text{as } n,m\to\infty,$$

so that $\xi_n-\xi_m\to 0$ in probability. Hence ξ_n converge in probability to some random variable ξ_∞. We have now

$$\sup_n E[\exp\{\lambda\xi_n\}]^{1+\varepsilon}=\sup_n E\exp\lambda(1+\varepsilon)\,\xi_n \tag{3.8}$$

$$\leqslant\sup_n\exp\{\phi(\lambda(1+\varepsilon)\,\tau_\phi(\xi_n))\}$$

$$\leqslant\exp\phi(\lambda(1+\varepsilon)\tau)<\infty.$$

From (3.8) and the theorem of uniform integrability it follows that

$$E\exp\{\lambda\xi_\infty\}=\lim_{n\to\infty}E\exp\{\lambda\xi_n\}\leqslant\exp\phi(\lambda\tau_\phi^\infty),$$

where $\tau_\phi^\infty = \limsup_{n\to\infty} \tau_\phi(\xi_n)$. Hence $\xi_\infty \in Sub_\phi(\Omega)$ and

$$\tau_\phi(\xi_\infty) \leqslant \limsup_{n\to\infty} \tau_\phi(\xi_n). \tag{3.9}$$

The random variables $\xi_\infty - \xi_n$ belong to $Sub_\phi(\Omega)$. Now the inequality

$$\tau_\phi(\xi_\infty - \xi_n) \leqslant \sup_{m \geqslant n} \tau_\phi(\xi_m - \xi_n) \tag{3.10}$$

can be proved as we proved (3.9). It follows from (3.10) and (3.7) that $\tau_\phi(\xi_\infty - \xi_n) \to 0$ as $n \to \infty$. ■

REMARK 3.2: If $\phi(x) = \dfrac{x^2}{2}$ the space $Sub_\phi(\Omega) = Sub(\Omega)$ is the space of sub-Gaussian random variables.

LEMMA 3.2: *Let ξ be a random variable such that $E\xi = 0$ and $E\exp\{\lambda\xi\} = a(\lambda)$ exists for all $\lambda \in \mathbb{R}$. Then*

(i) *we have*

$$E\exp\{\lambda\xi\} \geqslant 1; \tag{3.11}$$

(ii) *there exist all moments $E|\xi|^\alpha$, $\alpha > 0$, and the next inequality holds*

$$E\exp|\xi|^\alpha \leqslant \left(\frac{\alpha}{e}\right)^\alpha \inf_{\lambda>0} \frac{a(\lambda) + a(-\lambda)}{\lambda^\alpha}. \tag{3.12}$$

(iii) *The function $\psi(\lambda) = \ln(a(\lambda))$ is convex; moreover for any real number x_0 there exists a constant $T = T(x_0)$ such that*

$$E\exp\{\lambda\xi\} \leqslant \exp\{T\lambda^2\} \tag{3.13}$$

as $|\lambda| < x_0$; we have $T = \sup_{|\lambda|<x_0} \dfrac{\ln(E\exp\lambda\xi)}{\lambda^2} < \infty$.

PROOF: From Jensen inequality we get $E\exp\{\lambda\xi\} \geqslant \exp\{\lambda E\xi\} = 1$, and (3.11) is proved. From the relation

$$\max_{x\geqslant 0} x^\alpha \exp\{-x\} = \left(\frac{\alpha}{e}\right)^\alpha, \quad \text{as } \alpha > 0$$

we deduce that for all $x > 0$, $\alpha > 0$ we have

$$x^\alpha \leqslant \left(\frac{\alpha}{e}\right)^\alpha \exp\{x\}. \tag{3.14}$$

It follows from (3.14) that for all $\lambda > 0$

$$(3.15)\qquad E|\lambda\xi|^{\alpha} \leqslant \left(\frac{\alpha}{e}\right)^{\alpha} E\exp\{\lambda\xi\} \leqslant \left(\frac{\alpha}{e}\right)^{\alpha}(E\exp\{\lambda\xi\} + E\exp\{-\lambda\xi\})$$

$$= \left(\frac{\alpha}{e}\right)^{\alpha}(a(\lambda) + a(-\lambda)).$$

Now (3.12) follows from (3.15).

We have

$$(3.16)\qquad \psi''(\lambda) = \frac{a''(\lambda)\,a(\lambda) - (a'(\lambda))^2}{a^2(\lambda)}.$$

It follows from Hölder inequality, that

$$(a'(\lambda))^2 = (E\xi\exp\{\lambda\xi\})^2 \leqslant E\xi^2\exp\{\lambda\xi\}\cdot E\exp\{\lambda\xi\} = a''(\lambda)\,a(\lambda),$$

so that $\psi''(\lambda) \geqslant 0$ and the function $\psi(\lambda)$ is convex. If $\lambda \to 0$ we have $E\exp\{\lambda\xi\} = 1 + \frac{1}{2}E\xi^2\lambda^2 + o(\lambda^2)$, hence $\psi(\lambda) = \frac{1}{2}\lambda E\xi^2 + o(\lambda^2)$. Relation (3.13) now follows from the last inequality and the convexity of $\psi(\lambda)$. ■

THEOREM 3.2: *Let $\phi_1(\lambda)$ and $\phi_2(\lambda)$ be two N-functions such that $\phi_1 \prec \phi_2$. Assume that $\xi \in Sub_{\phi_1}(\Omega)$; then $\xi \in Sub_{\phi_2}(\Omega)$ and there exists a constant $c(\phi_1, \phi_2)$ such that $\tau_{\phi_2}(\xi) \leqslant c(\phi_1, \phi_2)\tau_{\phi_1}(\xi)$.*

PROOF: It follows from remark 2.3 that for any $x_0 > 0$ there exists a number $D = D(x_0) > 0$ such that $\phi_1(x) \leqslant \phi_2(Dx)$, as $|x| > x_0$. Let $\xi \in Sub_{\phi_1}(\Omega)$, $\tau_1 = \tau_{\phi_1}(\xi) > 0$; then for all $\lambda > 0$ such that $|\lambda|\tau_1 \geqslant x_0$

$$(3.17)\qquad \exp\{\lambda\xi\} \leqslant \exp\{\phi_1(\lambda\tau_1)\} \leqslant \exp\{\phi_2(\lambda D\tau_1)\}.$$

Let λ be such that $|\lambda| \leqslant \frac{x_0}{\tau_1}$; then it follows from Lemma 3.2 that there exists a number $B(x_0)$ such that for $|\lambda| \leqslant \frac{x_0}{\tau_1}$ we have

$$(3.18)\qquad \exp\{\lambda\xi\} \leqslant \exp\{B(x_0)\,\tau_1^2\lambda^2\},$$

$$B(x_0) = \sup_{|\lambda| \leqslant x_0/\tau_1} \frac{\ln(E\exp\{\lambda\xi\})}{\tau_1^2\lambda^2} < \infty.$$

Since $B(x_0)$ decreases when x_0 decreases, it follows from (2.2) that there exist two numbers $z_0 > 0$ and $c_1 > 0$ such that for all $|x| \leqslant z_0$ we have $\phi_2(x) \geqslant c_1x^2$. Let x_0 be a number such that $\frac{x_0(B(x_0))^{1/2}}{c_1} \leqslant z_0$. (Such a number exists since $x_0(B(x_0))^{1/2} \to 0$ as

$x_0 \to 0$.) Then from (3.18) we deduce that for $|\lambda| \leqslant \dfrac{x_0}{\tau_1}$

$$(3.19) \qquad E\exp\{\lambda\xi\} \leqslant \exp\left\{c_1 \frac{B(x_0)}{c_1}\tau_1^2\lambda^2\right\} \leqslant \exp\left\{\phi_2\left(\lambda\tau_1\left(\frac{B(x_0)}{c_1}\right)^{1/2}\right)\right\}$$

since

$$|\lambda|\tau_1\left(\frac{B(x_0)}{c_1}\right)^{1/2} \leqslant x_0\tau_1\left(\frac{B(x_0)}{c_1}\right)^{1/2} \leqslant z_0.$$

It follows from (3.17) and (3.19) that $E\exp\{\lambda\xi\} \leqslant \exp\{\phi_2(\tau_1 L\lambda)\}$, where

$$L = \max\left(\left(\frac{B(x_0)}{c_1}\right)^{1/2}, D\right)$$

that is $\xi \in Sub_{\phi_2}(\Omega)$ and $\tau_{\phi_2}(\xi) \leqslant L\tau_{\phi_1}(\xi)$. ■

EXAMPLE 3.1: Let ξ be any bounded random variable with $E\xi = 0$; then $\xi \in Sub_\phi(\Omega)$ for all N-functions ϕ.

In order to prove the above statement, let ϕ be an N-function satisfying condition Q, a a real number with $a > 0$ and ξ a random variable with $|\xi| \leqslant r$ with probability one. Then

$$(3.20) \qquad \phi(x) = \phi\left(\frac{x}{a}a\right) \geqslant \frac{|x|}{a}\phi(a) \quad \text{as } |x| \geqslant a.$$

Hence it follows from (3.20) that

$$(3.21) \qquad E\exp\{\lambda\xi\} \leqslant \exp\left\{\frac{\phi(a)}{a}\frac{a|\lambda|}{\phi(a)}r\right\} \leqslant \exp\left\{\phi\left(a\frac{|\lambda|}{\phi(a)}r\right)\right\} \quad \text{as } |\lambda| > \frac{\phi(a)}{r}.$$

Let $|\lambda| < \dfrac{\phi(a)}{r}$; then from lemma 3.2 we deduce that there exists a number $T(a, r)$ such that

$$(3.22) \qquad E\exp\{\lambda\xi\} \leqslant \exp\{T(a, r)\lambda^2\}.$$

It follows from (2.2) that there exist two constants $c_1 > 0$ and $z_0 > 0$ such that

$$(3.23) \qquad \phi(x) \geqslant c_1 x^2 \quad \text{as } |x| < z_0,$$

$T(a, r)$ decreases as a decrease so that we can choose a constant $a > 0$ such that

$$\frac{\phi(a)T^{1/2}(a, r)}{r(c_1)^{1/2}} \leqslant z_0.$$

hence from (3.22) we get

$$\left(\frac{T(a,r)}{c_1}\right)^{1/2}|\lambda| \leqslant \frac{\phi(a)}{r}\frac{T^{1/2}(a,r)}{(c_1)^{1/2}} \leqslant z_0$$

and

$$E\exp\{\lambda\xi\} \leqslant \exp\left\{c_1\frac{T(a,r)}{c_1}\lambda^2\right\} \leqslant \exp\left\{\phi\left(\lambda\left(\frac{T(a,r)}{c_1}\right)^{1/2}\right)\right\}. \tag{3.24}$$

It follows from (3.23) and (3.24) that $E\exp\{\lambda\xi\} \leqslant \exp\{\phi(\lambda K)\}$, where $K = \max\left(\left(\frac{T(a,r)}{c_1}\right)^{1/2}, \frac{ar}{\phi(a)}\right)$; hence $\xi \in Sub_\phi(\Omega)$.

In some particular cases we can find other (more precise) norms in the spaces $Sub_\phi(\Omega)$.

Example 3.2: Let ξ be a random variable uniformly distributed in the interval $[-1, 1]$. Then $\xi \in Sub_\phi(\Omega)$ for all N-functions ϕ and

$$\tau_\alpha(\xi) \leqslant 6^{\frac{1-\alpha}{\alpha}} \quad \text{as } 1 < \alpha \leqslant 2, \tag{3.25}$$

where $\tau_\alpha(\xi) = \tau_{\phi_\alpha}(\xi)$, $\phi_\alpha = |x|^\alpha$. In fact

$$\begin{aligned} E\exp\{\lambda\xi\} &= \frac{1}{2}\int_{-1}^{1}\exp\{\lambda u\}\,du = \frac{1}{2\lambda}(e^\lambda - e^{-\lambda}) \\ &= \sum_{k=0}^{\infty}\frac{\lambda^{2k}}{(2k+1)!} = \sum_{k=0}^{\infty}\left(\frac{\lambda^2}{6}\right)^k\frac{6^k k!}{(2k+1)!}\frac{1}{k!} \\ &\leqslant \sum_{k=0}^{\infty}\left(\frac{\lambda^2}{6}\right)^k\frac{1}{k!} \leqslant \exp\left\{\frac{\lambda^2}{6}\right\}. \end{aligned}$$

If $\dfrac{|\lambda|}{\sqrt{6}} \leqslant 1$ then

$$\exp\left\{\frac{\lambda^2}{6}\right\} \leqslant \exp\left\{\left(\frac{|\lambda|}{\sqrt{6}}\right)^\alpha\right\} \leqslant \exp\left\{\left(\frac{|\lambda|}{6^{1-1/\alpha}}\right)^\alpha\right\}$$

so that for $|\lambda| \leqslant \sqrt{6}$ we have

$$E\exp\{\lambda\xi\} \leqslant \exp\left\{\left(\frac{|\lambda|}{6^{1-1/\alpha}}\right)^\alpha\right\} \tag{3.26}$$

It is obvious that $E\exp\{\lambda\xi\} = E\exp\{|\lambda||\xi|\} \leqslant \exp\{|\lambda|\}$.

If $\dfrac{|\lambda|}{\sqrt{6}} > 1$ then

$$\exp\{|\lambda|\} \leqslant \exp\left\{|\lambda| \frac{|\lambda|^{\alpha-1}}{(\sqrt{6})^{\alpha-1}}\right\} \tag{3.27}$$

$$= \exp\left\{\frac{|\lambda|^{\alpha}}{(\sqrt{6})^{\alpha-1}}\right\} = \exp\left\{\left(\frac{|\lambda|}{(\sqrt{6})^{\frac{\alpha-1}{\alpha}}}\right)^{\alpha}\right\}.$$

(3.25) now follows from (3.26), (3.27).

EXAMPLE 3.3: If ξ is a Gaussian random variable, $E\xi = 0$, $E\xi^2 = \sigma^2 > 0$, then $E\exp\{\lambda\xi\} \leqslant E\exp\left\{\dfrac{\lambda^2\sigma^2}{2}\right\}$, that is $\xi \in Sub_\phi(\Omega)$, where $\phi(x) = x^2/2$ and $\tau_\phi(\xi) = \sigma$.

EXAMPLE 3.4: Let ξ be a Poisson random variable, with $E\xi = a$ and put $\eta = \xi - a$; then

$$E\exp\{\lambda\eta\} = \exp\{a(e^{\lambda} - \lambda - 1)\};$$

this means that $\eta \in Sub_\phi(\Omega)$, where $\phi(\lambda) = a(e^{|\lambda|} - |\lambda| - 1)$ and $\tau_\phi(\xi) = 1$. It follows from example 2.2 that

$$\phi^*(\lambda) = a\left(\left(\frac{|\lambda|}{a} + 1\right)\ln\left(\frac{|\lambda|}{a} + 1\right) - \frac{|\lambda|}{a}\right)$$

and from Lemma 3.1 that for $\varepsilon > 0$ we have

$$P\{\eta > \varepsilon\} \leqslant \exp\left\{-\left[(\varepsilon + a)\ln\left(\frac{\varepsilon}{a} + 1\right) - \varepsilon\right]\right\}. \tag{3.28}$$

4. - CHARACTERIZATION OF THE SPACE $Sub_\phi(\Omega)$ AND SOME INEQUALITIES

LEMMA 4.1: *Let $\xi \in Sub_\phi(\Omega)$. Then for all $\alpha > 0$ the following inequality holds*

$$E|\xi|^{\alpha} \leqslant 2\left(\frac{\alpha}{e}\right)^{\alpha}(\tau_\phi(\xi))^{\alpha} \inf_{t > a} \exp\{\phi(t) - \alpha\ln(t)\} \tag{4.1}$$

$$\leqslant 2(\tau_\phi(\xi))^{\alpha}\left(\frac{\alpha}{\phi^{(-1)}(\alpha)}\right)^{\alpha} \quad \text{as } a > 0.$$

PROOF: It follows from inequalities (3.12) and (3.4) that for $\lambda > 0$ one has

$$E|\xi|^\alpha \leqslant 2\left(\frac{\alpha}{e}\right)^\alpha \lambda^{-\alpha} \exp\{\phi(\lambda\tau_\phi(\xi))\} \tag{4.2}$$

$$= 2\left(\frac{\alpha}{e}\right)^\alpha (\tau_\phi(\xi))^\alpha \left(\frac{1}{\lambda\tau_\phi(\xi)}\right)^\alpha \exp\{\phi(\lambda\tau_\phi(\xi))\}$$

$$= 2\left(\frac{\alpha}{e}\right)^\alpha (\tau_\phi(\xi))^\alpha \exp\{\phi(\lambda\tau_\phi(\xi)) - \alpha\ln(\lambda\tau_\phi(\xi))\}.$$

By setting $\lambda = \dfrac{\phi^{(-1)}(\alpha)}{\tau_\phi(\xi)}$ we obtain the second inequality in (4.1). ■

COROLLARY 4.1: *Let* $\xi \in Sub_\phi(\Omega)$; *then the following inequality holds*

$$\tau_\phi(\xi) \geqslant \frac{1}{\sqrt{2}}\theta_\phi(\xi), \tag{4.3}$$

where

$$\theta_\phi(\xi) = \sup_{n \geqslant 2} (E|\xi|^n)^{1/n} \frac{\phi^{(-1)}(n)}{n}.$$

Moreover $\theta_\phi(\xi)$ *is a norm on* $Sub_\phi(\Omega)$.

LEMMA 4.2: *Let* $\xi \in Sub_\phi(\Omega)$. *Then for* $k = 1, 2, \ldots$ *the following inequality holds*

$$|E\xi^k| \leqslant E|\xi|^k \leqslant 2(\tau_\phi(\xi))^k \frac{e^k}{(\phi^{(-1)}(k))^k} k!. \tag{4.4}$$

PROOF: The relation $\exp\{x\} = \sum_{k=0}^{\infty} \dfrac{x^k}{k!}$ yields that for $x > 0$ we have

$$x^k \leqslant k! \exp\{x\}.$$

For $x = |\xi|\lambda$ $(\lambda > 0)$ we get

$$E|\xi|^k \leqslant k!\, E\exp\{\lambda|\xi|\}\lambda^{-k} \leqslant k!\, 2\exp\{\phi(\lambda\tau_\phi(\xi))\}\lambda^{-k}.$$

By setting $\lambda = \dfrac{\phi^{(-1)}(k)}{\tau_\phi(\xi)}$ in the latter inequality we obtain (4.4). ■

Corollary 4.2: *Let $\xi \in Sub_\phi(\Omega)$. Then the following inequality holds*

$$\tau_\phi(\xi) \geq \frac{1}{e\sqrt{2}} \nu_\phi(\xi), \tag{4.5}$$

where

$$\nu_\phi(\xi) = \sup_{n \geq 2} |E\xi^n|^{1/n} \frac{\phi^{(-1)}(n)}{(n!)^{1/n}}.$$

Corollary 4.3: *Let $\xi \in Sub_\phi(\Omega)$ be a random variable with a symmetrical distribution (or such that all moments $E\xi^{2n+1} = 0$ for $n = 0, 1, 2, \ldots$); then*

$$\tau_\phi(\xi) \geq \frac{1}{e\sqrt{2}} \nu_{\phi,2}(\xi), \tag{4.6}$$

where

$$\nu_{\phi,2}(\xi) = \sup_{l \geq 2} \left(E\xi^{2l}\right)^{1/2l} \frac{\phi^{(-1)}(2l)}{(2l!)^{1/2l}}.$$

Corollary 4.4: *Let $\xi \in Sub_\phi(\Omega)$. Then we have*

$$\nu_\phi(\xi) \geq \theta_\phi(\xi), \quad \nu_\phi(\xi) \leq \exp\left\{\frac{49}{48}\right\} \theta_\phi(\xi). \tag{4.7}$$

Proof: The first inequality is evident. The second one follows from Stirling's formula $n! = n^n e^{-n} (2\pi n)^{1/2} e^{\theta_n}$ where $|\theta_n| \leq \dfrac{1}{12n}$. Indeed

$$(n!)^{-\frac{1}{n}} = \frac{1}{n} \frac{e^{1 + \theta_n / n}}{(2\pi n)^{1/2n}} \leq \frac{1}{n} e^{49/48} \quad \text{as } n \geq 2.$$

Lemma 4.3: *Let ξ be a random variable such that $E\xi = 0$, ϕ an N-function satisfying condition Q. Let $\lambda_0 > 0$ be any number and $c_0 = \inf\limits_{0 < |\lambda| \leq \lambda_0} \dfrac{\phi(\lambda)}{\lambda^2}$. Assume that*

$$\nu_\phi(\xi) = \sup_{n \geq 2} |E\xi^n|^{1/n} \frac{\phi^{(-1)}(n)}{(n!)^{1/n}} < \infty.$$

Let γ_1 be the root of the equation

$$\gamma = \lambda_0 \sqrt{c_0 (1 - \gamma)}, \tag{4.8}$$

γ_2 the root of the equation $\gamma^3 - 2(1-\gamma) = 0$ and γ_3 the root of the equation $\gamma = \phi^{(-1)}(2)\sqrt{c_0(1-\gamma)}$.

Then $\xi \in Sub_\phi(\Omega)$ and the following inequality holds

$$\tau_\phi(\xi) \leqslant S_\phi \nu_\phi(\xi), \tag{4.9}$$

where $S_\phi = \max\limits_{i=\overline{1,3}} \gamma_i^{-1}$.

Proof: Let γ be any number such that $\gamma \in (0, \min(\gamma_1, \gamma_2))$. Then

$$\gamma \leqslant \gamma_1 = \lambda_0 \sqrt{c_0(1-\gamma_1)} \leqslant \lambda_0 \sqrt{c_0(1-\gamma)}. \tag{4.10}$$

Let $\nu_\phi(\xi) = \nu$ and $\lambda_1 = \dfrac{\gamma}{\nu}\phi^{(-1)}(2)$. Then from (4.10) we get

$$\lambda_1 \nu = \gamma\phi^{(-1)}(2) \leqslant \lambda_0 \sqrt{c_0(1-\gamma)}\phi^{(-1)}(2). \tag{4.11}$$

We now have easily

$$E\exp\{\lambda\xi\} = 1 + \sum_{n=2}^{\infty} \frac{\lambda^n E\xi^n}{n!} \leqslant 1 + \sum_{n=2}^{\infty} \frac{|\lambda|^n |E\xi^n|}{n!} = S(\lambda). \tag{4.12}$$

Relation (4.12) yields that

$$S(\lambda) = 1 + \sum_{n=2}^{\infty} \frac{|\lambda|^n}{(\phi^{(-1)}(n))^n} \frac{(\phi^{(-1)}(n))^n}{n!} |E\xi^n| \leqslant 1 + \sum_{n=2}^{\infty} \left(\frac{|\lambda|\nu}{\phi^{(-1)}(n)} \right)^n. \tag{4.13}$$

For any number λ such that $|\lambda| < \lambda_1$ we get

$$S(\lambda) \leqslant 1 + \sum_{n=2}^{\infty} \left(\frac{|\lambda|\nu}{\phi^{(-1)}(2)} \right)^n. \tag{4.14}$$

From the relation

$$\frac{|\lambda|\nu}{\phi^{(-1)}(2)} \leqslant \frac{\lambda_1\nu}{\phi^{(-1)}(2)} = \gamma < 1$$

we obtain

$$\begin{aligned} S(\lambda) &\leqslant 1 + \left(\frac{|\lambda|\nu}{\phi^{(-1)}(2)} \right)^2 \left(1 - \frac{|\lambda|\nu}{\phi^{(-1)}(2)} \right)^{-1} \leqslant 1 + \left(\frac{|\lambda|\nu}{\phi^{(-1)}(2)} \right)^2 \frac{1}{1-\gamma} \\ &= 1 + c_0 \left(\frac{|\lambda|\nu}{\sqrt{c_0}\sqrt{1-\gamma}\phi^{(-1)}(2)} \right)^2. \end{aligned} \tag{4.15}$$

Relation (4.10) yields that

$$\frac{|\lambda|\nu}{\sqrt{c_0}\sqrt{1-\gamma}\phi^{(-1)}(2)} \leqslant \frac{\lambda_1\nu}{\sqrt{c_0}\sqrt{1-\gamma}\phi^{(-1)}(2)} = \frac{\gamma}{\sqrt{c_0(1-\gamma)}} \leqslant \lambda_0,$$

hence

$$c_0\left(\frac{|\lambda|\nu}{\sqrt{c_0(1-\gamma)}\phi^{(-1)}(2)}\right)^2 \leqslant \phi\left(\frac{\lambda\nu}{\sqrt{c_0(1-\gamma)}\phi^{(-1)}(2)}\right).$$

From the above we deduce that the following inequality holds, as $|\lambda| < \lambda_1$

$$S(\lambda) \leqslant 1 + \phi\left(\frac{\lambda\nu}{\sqrt{c_0(1-\gamma)}\phi^{(-1)}(2)}\right) \leqslant \exp\left\{\phi\left(\frac{\lambda\nu}{\sqrt{c_0(1-\gamma)}\phi^{(-1)}(2)}\right)\right\}. \tag{4.16}$$

Let now $|\lambda| > \lambda_1$. Since

$$\gamma = \frac{\lambda_1\nu}{\phi^{(-1)}(2)} \leqslant \frac{|\lambda|\nu}{\phi^{(-1)}(2)},$$

there exists an integer $n_\lambda \geqslant 2$ such that

$$\frac{|\lambda|\nu}{\phi^{(-1)}(n_\lambda+1)} < \gamma \leqslant \frac{|\lambda|\nu}{\phi^{(-1)}(n_\lambda)}. \tag{4.17}$$

Put now

$$A_1(\lambda) = \sum_{n=2}^{n_\lambda}\left(\frac{|\lambda|\nu}{\phi^{(-1)}(n)}\right)^n, \quad A_2(\lambda) = \sum_{n=n_\lambda+1}\left(\frac{|\lambda|\nu}{\phi^{(-1)}(n)}\right)^n.$$

We first bound $A_1(\lambda)$. From the inequality $\phi^{(-1)}(n_\lambda) \leqslant \dfrac{|\lambda|\nu}{\gamma}$, for $n \leqslant n_\lambda$ we get $n \leqslant$ $\leqslant n_\lambda \leqslant \phi\left(\dfrac{|\lambda|\nu}{\gamma}\right)$, hence

$$\frac{1}{n}\phi\left(\frac{|\lambda|\nu}{\gamma}\right) \geqslant 1 \quad (\text{as } n \leqslant n_\lambda). \tag{4.18}$$

From Lemma 2.1 and Lemma 2.4 it follows, for every n with $2 \leqslant n \leqslant n_\lambda$

$$\frac{|\lambda|\nu}{\phi^{(-1)}(n)} = \frac{1}{\phi^{(-1)}(n)}\phi^{(-1)}\left(\frac{n\phi(\lambda\nu)}{\phi\left(\frac{\lambda\nu}{\gamma}\right)}\frac{\phi\left(\frac{\lambda\nu}{\gamma}\right)}{n}\right)$$

$$\leqslant \frac{1}{n}\phi\left(\frac{\lambda\nu}{\gamma}\right)\frac{1}{\phi^{(-1)}(n)}\phi^{(-1)}\left(n\frac{\phi(\lambda\nu)}{\phi\left(\frac{\lambda\nu}{\gamma}\right)}\right) \leqslant \frac{1}{n}\phi\left(\frac{|\lambda|\nu}{\gamma}\right).$$

Hence

$$A_1(\lambda) \leqslant \sum_{n=2}^{n_\lambda}\left(\frac{1}{n}\phi\left(\frac{\lambda\nu}{\gamma}\right)\right)^n \leqslant \sum_{n=2}^{n_\lambda}\frac{1}{n!}\left(\phi\left(\frac{\lambda\nu}{\gamma}\right)\right)^n \leqslant \sum_{n=2}^{\infty}\frac{1}{n!}\left(\phi\left(\frac{\lambda\nu}{\gamma}\right)\right)^n. \tag{4.19}$$

We now bound $A_2(\lambda)$. From (4.17) we get $\dfrac{|\lambda|\nu}{\phi^{(-1)}(n_\lambda+1)} < \gamma < 1$, so that

$$\begin{aligned} A_2(\lambda) &\leqslant \sum_{n=n_\lambda+1}^{\infty}\left(\frac{|\lambda|\nu}{\phi^{(-1)}(n_\lambda+1)}\right)^n \\ &= \left(\frac{|\lambda|\nu}{\phi^{(-1)}(n_\lambda+1)}\right)^{n_\lambda+1}\left(1-\frac{|\lambda|\nu}{\phi^{(-1)}(n_\lambda+1)}\right)^{-1} \\ &\leqslant \gamma^{n_\lambda+1}\frac{1}{1-\gamma} \leqslant \gamma^3(1-\gamma)^{-1} \leqslant \gamma_2^3(1-\gamma_2)^{-1} \\ &= 2 \leqslant n_\lambda \leqslant \phi\left(\frac{\lambda\nu}{\gamma}\right). \end{aligned} \tag{4.20}$$

It follows from (4.19) and (4.20) that

$$E\exp\{\lambda\xi\} \leqslant S(\lambda) \leqslant \sum_{n=0}^{\infty}\frac{1}{n!}\left(\phi\left(\frac{\lambda\nu}{\gamma}\right)\right)^n = \exp\left\{\phi\left(\frac{\lambda\nu}{\gamma}\right)\right\} \quad \text{as } |\lambda| > \lambda_1. \tag{4.21}$$

Relations (4.16) and (4.21) yield that, for all $\lambda \in \mathbb{R}$, we have

$$E\exp\{\lambda\xi\} \leqslant \exp\{\phi(S_\gamma\nu\lambda)\}, \tag{4.22}$$

where $S_\gamma = \max\left(\gamma^{-1}, ((c_0(1-\gamma))^{1/2}\phi^{(-1)}(2))^{-1}\right)$. It is now easy to prove that

$$\inf_{\gamma\in(0,\min(\gamma_1\gamma_2)]} S_\gamma = S_\phi = \max_{i=\overline{1,3}} \gamma_i^{-1},$$

and inequality (4.8) follows. ■

EXAMPLE 4.1: Let ϕ be the N-function $\phi(x) = |x|^\alpha$, $1 \leq \alpha \leq 2$ and $\lambda_0 > 0$ any number. In this case

$$c_0 = \inf_{0<|\lambda|\leq\lambda_0} \frac{\phi(\lambda)}{\lambda^2} = \lambda_0^{\alpha-2}, \qquad \phi^{(-1)}(2) = 2^{1/\alpha}.$$

Hence γ_1 is the root of the equation $\gamma = \lambda_0\sqrt{\lambda_0^{\alpha-2}(1-\gamma)}$, i. e. $\gamma_1 = \frac{1}{2}\left[\lambda_0^{\alpha/2}\sqrt{\lambda_0^\alpha 4} - \lambda_0^\alpha\right]$, γ_2 is the root of the equation $\gamma^3 - 2(1-\gamma) = 0$, $(\gamma_2 \sim 0.770917)$, γ_3 is the root of the equation $\gamma = 2^{1/\alpha}\sqrt{\lambda_0^{\alpha-2}(1-\gamma)}$, i. e.

$$\gamma_3 = \frac{1}{2}\left[2^{1/\alpha}\lambda_0^{(\alpha/2)-1}\sqrt{\lambda_0^{\alpha-2}2^{2/\alpha}4} - \lambda_0^{\alpha-2}2^{2/\alpha}\right].$$

Put $z_1 = \left(\frac{\gamma_2^2}{1-\gamma_2^2}\right)^{1/\alpha}$, $z_2 = \left(\frac{(1-\gamma_2^2)2^{2/\alpha}}{\gamma_2^2}\right)^{1/(2-\alpha)}$. Then it is not difficult to see that, if $\lambda_0 > \max(z_1, z_2)$, we have $\gamma_1 > \gamma_2$ and $\gamma_3 > \gamma_2$. Hence in this case we get $S_\phi = \gamma_2^{-1} \sim 1.2971565$. ■

From corollary 4.2 and Lemma 4.3 we get the following result:

THEOREM 4.1: *The random variable ξ belongs to $Sub_\phi(\Omega)$ if and only if $E\xi = 0$ and $\nu_\phi(\xi) < \infty$. The norms $\nu_\phi(\xi)$ and $\tau_\phi(\xi)$ are equivalent.*

5. - ORLICZ SPACES OF EXPONENTIAL TYPE

DEFINITION 5.1. [3]: Let ψ be an arbitrary N-function. The Orlicz space generated by the N-function

$$U(x) = \exp\{\psi(x)\} - 1, \qquad x \in \mathbb{R}$$

is called an *Orlicz space of exponential type*.

We shall be interested in the Orlicz space of exponential type generated by the Young-Fenchel transform ϕ^* of an N-function ϕ. We shall denote such a space by $\mathrm{Exp}_{\phi^*}(\Omega)$. The Luxemburg norm in $\mathrm{Exp}_{\phi^*}(\Omega)$ is denoted by σ_{ϕ^*}; for any random variable ξ we have

$$\sigma_{\phi^*}(\xi) = \inf\{a > 0 : E[\exp\phi^*(\xi/a)] \leq 2\}.$$

The space $\mathrm{Exp}_{\phi^*}(\Omega)$ is a Banach space with respect to the norm $\sigma_{\phi^*}(\cdot)$.

The following Lemma is a modification of a Lemma from [3].

LEMMA 5.1: *Let ϕ be an N-function and $\xi \in \mathrm{Exp}_{\phi^*}(\Omega)$. Then for any $p \geq 1$ the following inequality holds*:

$$\left(E|\xi|^p\right)^{1/p} \leq 2^{1/p} \frac{p}{\phi^{(-1)}(p)} \sigma_{\phi^*}(\xi). \tag{5.1}$$

PROOF: It will be enough to prove (5.1) if $\sigma_{\phi^*}(\xi) > 0$. In this case the following inequalities hold for $p > 0$, $x \in \mathbb{R}$:

$$\begin{aligned}
|x|^p \exp\{-\phi^*(x)\} &\leq \sup_{x\in\mathbb{R}} |x|^p \exp\{-\phi^*(x)\} \\
&= \sup_{x\in\mathbb{R}} |x|^p \exp\left\{-\sup_{\lambda>0}(\lambda|x| - \phi(\lambda))\right\} \\
&= \sup_{x\in\mathbb{R}} |x|^p \exp\left\{\inf_{\lambda>0}(\phi(\lambda) - \lambda|x|)\right\} \\
&= \sup_{x\in\mathbb{R}} \inf_{\lambda>0} |x|^p \exp\{\phi(\lambda) - \lambda|x|\} \\
&= \inf_{\lambda>0}\left[\exp\{(\phi(\lambda)\} \sup_{x\in\mathbb{R}} |x|^p \exp(-\lambda|x|)\right] \\
&= \left(\frac{p}{e}\right)^p \inf_{\lambda>0} \lambda^{-p} \exp\{\phi(\lambda)\}.
\end{aligned}$$

Then for all $x \in \mathbb{R}$

$$|x|^p \leq \left(\frac{p}{e}\right)^p \exp\{\phi^*(x)\} \inf_{\lambda>0} \lambda^{-p} \exp\{\phi(\lambda)\}.$$

Substituting $x = \dfrac{|\xi|}{\sigma_{\phi^*}(\xi)}$ gives

$$\begin{aligned}
E|\xi|^p &\leq (\sigma_{\phi^*}(\xi))^p E\exp\left\{\phi\left(\frac{\xi}{\sigma_{\phi^*}(\xi)}\right)\right\}\left(\frac{p}{e}\right)^p \inf_{\lambda>0} \lambda^{-p}\exp\{\phi(\lambda)\} \\
&\leq 2(\sigma_{\phi^*}(\xi))^p \left(\frac{p}{e}\right)^p \left(\frac{e}{\phi^{(-1)}(p)}\right)^p \\
&= 2(\sigma_{\phi^*}(\xi))^p \left(\frac{p}{\phi^{(-1)}(p)}\right)^p. \qquad \blacksquare
\end{aligned}$$

LEMMA 5.2: *Let ϕ be an N-function satisfying condition* Q. *Let ξ be a random variable such that $\xi \in \mathrm{Exp}_{\phi^*}(\Omega)$ and $E\xi = 0$. Then $\xi \in Sub_\phi(\Omega)$ and*

$$\tau_\phi(\xi) \leqslant S_\phi e^{\frac{49}{48}} \sigma_{\phi^*}(\xi), \tag{5.2}$$

where S_ϕ is defined below (4.8).

PROOF: It follows from Lemma 5.1 (inequality 5.1) and Stirling's formula that

$$\nu_\phi(\xi) = \sup_{n\geqslant 2} |E\xi^n|^{1/n} \frac{\phi^{(-1)}(n)}{(n!)^{1/n}} \leqslant \sup_{n\geqslant 2} 2^{1/n} \frac{n}{\phi^{(-1)}(n)} \sigma_{\phi^*}(\xi) \frac{\phi^{(-1)}(n)}{(n!)^{1/n}}$$

$$\leqslant \sup_{n\geqslant 2}\left(2^{1/n}\sigma_{\phi^*}(\xi)\frac{ne}{n2^{1/2n}\pi^{1/2n}n^{1/2n}e^{\theta_n/n}}\right)$$

$$\leqslant \sigma_{\phi^*}(\xi) \sup_{n\geqslant 2} \frac{2^{1/2n} e\, e^{1/12n^2}}{2^{1/2n}\pi^{1/2n}} \leqslant \sigma_{\phi^*}(\xi)\, e^{\frac{49}{48}} < \infty .$$

Now from Lemma 4.3 we get that $\xi \in Sub_\phi(\Omega)$ and

$$\tau_\phi(\xi) \leqslant S_\phi \nu_\phi(\xi) \leqslant S_\phi e^{\frac{49}{48}} \sigma_{\phi^*}(\xi). \qquad \blacksquare$$

LEMMA 5.3 [3]: *Let ξ be a random variable such that*

$$P\{|\xi| \geqslant x\} \leqslant C \exp\left\{-\psi\left(\frac{x}{p}\right)\right\},$$

where $\psi(x)$ is N-function; then $\xi \in \mathrm{Exp}_\psi(\Omega)$ and

$$\sigma_\psi(\xi) \leqslant (1+C)\, D. \tag{5.3}$$

LEMMA 5.4: *Let $\xi \in Sub_\phi(\Omega)$; then $\xi \in \mathrm{Exp}_{\phi^*}(\Omega)$ and the following inequality holds*:

$$\sigma_{\psi^*}(\xi) \leqslant 3\,\tau_\phi(\xi). \tag{5.4}$$

PROOF: If $\xi \in Sub_\phi(\Omega)$ then it follows from Lemma 3.1 that

$$P\{|\xi| > \varepsilon\} \leqslant 2 \exp\left\{-\phi^*\left(\frac{\xi}{\tau_\phi(\xi)}\right)\right\}.$$

From Lemma 5.3 we deduce that $\xi \in \mathrm{Exp}_{\phi^*}(\Omega)$ and inequality (5.4) holds. $\blacksquare$

COROLLARY 5.1: *The random variable $\xi \in Sub_\phi(\Omega)$ if and only if $\xi \in \mathrm{Exp}_{\phi^*}(\Omega)$ and the norms $\sigma_{\psi^*}(\xi)$ and $\tau_\phi(\xi)$ are equivalent.*

This corollary follows from Lemmas 5.2 and 5.3.

THEOREM 5.1: *The random variable* $\xi \in Sub_\phi(\Omega)$ *if and only if* $E\xi = 0$ *and there exist two constants* $C > 0$ *and* $D > 0$ *such that*

$$P\{|\xi| > x\} \leqslant C \exp\left\{-\phi^*\left(\frac{x}{D}\right)\right\} \tag{5.5}$$

for any $x > 0$. *If* (5.5) *holds then*

$$\tau_\phi(\xi) \leqslant S_\phi e^{\frac{49}{48}} (1 + C) D, \tag{5.6}$$

where S_ϕ *is defined below* (4.9).

PROOF: If $\xi \in Sub_\phi(\Omega)$ then it follows from Lemma 3.1 that (5.5) holds with $C = 2$, $D = \tau_\phi(\xi)$. Conversely, if (5.5) holds we get from Lemma 5.2 that $\xi \in \mathrm{Exp}_{\phi^*}(\Omega)$ and $\sigma_{\psi^*}(\xi) \leqslant (1 + C) D$. Now again from Lemma 5.2 we deduce that $\xi \in Sub_\phi(\Omega)$ and

$$\tau_\phi(\xi) \leqslant S_\phi e^{\frac{49}{48}} \sigma_{\psi^*}(\xi). \qquad \blacksquare$$

EXAMPLE 5.1: Let ξ be a random variable having centered Weibull distribution, i. e.

$$P\{\xi > x\} = \frac{1}{2} \exp\left\{-\frac{1}{\alpha} x^\alpha\right\} \quad \text{as } x > 0;$$

$$P\{\xi < x\} = \frac{1}{2} \exp\left\{-\frac{1}{\alpha} |x|^\alpha\right\} \quad \text{as } x < 0.$$

Let $\alpha > 2$. Since

$$P\{|\xi| > x\} = \exp\left\{-\frac{1}{\alpha} x^\alpha\right\} \quad \text{as } x > 0,$$

then it follows from Theorem 5.1 that $\xi \in Sub_\phi(\Omega)$, where

$$\phi_\beta(x) = \frac{1}{\beta} |x|^\beta, \ \frac{1}{\beta} + \frac{1}{\alpha} = 1 \quad \text{and} \quad \tau_{\phi_\beta}(\xi) \leqslant 2 S_{\phi_\beta} e^{\frac{49}{48}},$$

where S_{ϕ_β} is defined in (4.8).

Consider now the particular case $\phi_p(x) = \dfrac{|x|^p}{p}$ with $1 < p < 2$ or $\phi_p(x) = \dfrac{|x|^p}{p}$ if $|x| \geqslant 1$ and $\phi_p(x) = \dfrac{x^2}{p}$ if $|x| < 1$, $p > 2$. In this case we can improve inequality (5.4). Our result is the following

PROPOSITION 5.1: *We have the inequality*

$$\sigma_{\phi^*}(\xi) \leqslant L\tau_{\phi_p}(\xi),$$

where

$$L = \left(\frac{2e^{\frac{1}{12}}}{\sqrt{2\pi}} + 1 \right)^{1/q}$$

PROOF: Consider first the case $p > 2$. A simple calculation shows that

$$\inf_{t \geqslant 1} \exp\{\phi_p(t) - s \log t\} = \left(\frac{e}{s} \right)^{s/p}, \quad \text{as } s > 1.$$

Let $\tau_{\phi_p}(\xi) = \tau$. Then (see Lemma 4.1)

$$E|\xi|^s \leqslant 2 \left(\frac{s}{e} \right)^{s/q} \tau^s \tag{5.7}$$

Since $\phi_p^*(x) \leqslant \dfrac{|x|^q}{q}$ $(q < 2)$ we have by (5.7) $(a > 0)$

$$E\exp\{\phi^*(\xi/a)\} = E\exp\{|\xi|^q/qa^q\} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{E|\xi|^{kq}}{(qa^q)^k} \tag{5.8}$$

$$\leqslant 1 + 2 \sum_{k=1}^{\infty} \left(\frac{k^k}{k!e^k} \right) \left(\frac{\tau}{a} \right)^{qk}.$$

It follows from Stirling's formula that

$$\frac{k^k}{k!\,e^k} = \frac{1}{\sqrt{2\pi}\sqrt{k}e^{\theta_k}} \leqslant \frac{e^{\frac{1}{12}}}{\sqrt{2\pi}}.$$

Therefore

$$E\exp\left\{ \phi^*\left(\frac{\xi}{a} \right) \right\} \leqslant 1 + \frac{2e^{\frac{1}{12}}}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \left(\frac{\tau}{a} \right)^{kq}.$$

The above geometric series converges if $\tau/a < 1$ and in such case its sum is equal to

$$1 + \frac{2e^{\frac{1}{12}}}{\sqrt{2\pi}} \frac{(\tau/a)^q}{(1 - (\tau/a)^q)}.$$

Then the last quantity in (5.8) is not greater than 2 if $a/\tau > L > 1$ and this gives the statement of Proposition 5.1. If $1 < p < 2$ the proof is the same $\left(\phi^*(x) = \dfrac{|x|^q}{q}\right)$. ■

6. - A CHARACTERISATION OF THE Sub_ϕ-NORM τ_ϕ FOR SYMMETRIC RANDOM VARIABLES

In this section we shall consider only symmetric random variables, which we shall call, for the sake of brevity, simply random variables.

For every real number $t \neq 0$, let M_t be the convex function defined by

$$M_t(x) = \frac{\cosh(tx) - 1}{e^{\phi(t)} - 1}, \quad x \in \mathbb{R}.$$

Clearly, for every random variable ξ, the function $t \mapsto EM_t(\xi)$ is symmetric. Moreover

$$EM_t(\xi) = \frac{Ee^{t\xi} - 1}{e^{\phi(t)} - 1}.$$

Let now ξ be a fixed random variable. Put, for each t,

$$A_t = \{a > 0 : EM_t(\xi/a) \leqslant 1\}.$$

We shall assume that A_t is nonempty. We have

PROPOSITION 6.1: *$A_t = A_{-t}$. Moreover A_t is a closed, left bounded half-line.*

The proof of Proposition (6.1) is an easy consequence of two lemmas:

LEMMA 6.1: *The (symmetric) function $t \mapsto Ee^{t\xi}$ is increasing for $t > 0$ (hence decreasing for $t < 0$).*

For every t, put

$$\tau_t(\xi) = \inf A_t.$$

LEMMA 6.2.

$$EM_t\left(\frac{\xi}{\tau_t(\xi)}\right) \leqslant 1.$$

The proofs of Lemmas 6.1 and 6.2 are straightforward.

Set now

$$E_t = \{\xi : \tau_t(\xi) < \infty\}.$$

We are interested in analyzing the structure of E_t and the properties of τ_t on E_t. Since $A_t = A_{-t}$ we have

$$\tau_t(\xi) = \tau_{-t}(\xi); \quad E_t = E_{-t}.$$

Hence there is no loss of generality in confining ourselves to the case $t \geqslant 0$.

We shall prove the following result

THEOREM 6.1: *E_t is a vector space and τ_t is a norm on E_t.*

PROOF: It is easy to see that E_t is a vector space and τ_t is a seminorm on it (recall that M_t is convex). It remains to see that $\tau_t(\xi) = 0$ implies $\xi = 0$. The relation $\tau_t(\xi) = 0$ amounts to saying that, for every $a > 0$, we have

$$EM_t\left(\frac{\xi}{a}\right) \leqslant 1,$$

or, equivalently,

$$Ee^{t\xi/a} \leqslant e^{\phi(t)}.$$

By the exponential Chebicev inequality, we deduce that, for every $u > 0$

$$P\{\xi > u\} = P\{e^{t\xi/a} > e^{tu/a}\} \leqslant Ee^{t\xi/a}e^{-tu/a} \leqslant e^{\phi(t)}e^{-tu/a}.$$

By letting a go to zero, we get $P\{\xi > u\} = 0$ for every $u > 0$, hence $P\{\xi > 0\} = 0$ and also $P\{\xi \neq 0\} = 0$ because of the symmetry of ξ. ■

Now, for every random variable ξ, put

$$\hat{\tau}(\xi) = \sup_t \tau_t(\xi),$$

and consider the set

$$\mathcal{S}(\Omega) = \{\xi \in \cap_t E_t : \hat{\tau}(\xi) < \infty\}.$$

We are interested in the structure of the pair $(\mathcal{S}(\Omega), \hat{\tau})$. First of all, $\mathcal{S}(\Omega)$ is non-empty, since all symmetric variables in $Sub_\phi(\Omega)$ belong to it. Moreover, it is clear by its very construction that

PROPOSITION 6.2: *$\mathcal{S}(\Omega)$ is a vector space and $\hat{\tau}$ is a norm on it.*

As we have said just now, we have the inclusion

$$Sub_\phi(\Omega) \subseteq \mathcal{S}(\Omega).$$

As a matter of fact, the inclusion is a set-theoretic equality:

PROPOSITION 6.3: *$\mathcal{S}(\Omega)$ coincides with the subspace of $Sub_\phi(\Omega)$ consisting of the symmetric random variables. Moreover $\widehat{\tau} = \tau_\phi$ on $\mathcal{S}(\Omega)$.*

For the proof of proposition 6.3 we need a simple lemma:

LEMMA 6.3: *Let ξ be a random variable. Put*

$$A = \{a > 0 : EM_t(\xi/a) \leq 1,\ \forall t\} = \{a > 0 : Ee^{t\xi/a} \leq e^{\phi(t)}\};$$

$$B = \{b > 0 : Ee^{t\xi} \leq e^{\phi(bt)},\ \forall t.\}$$

Then we have $A = B$.

PROOF OF PROPOSITION 6.3: We have $A = \bigcap_t A_t$; since A_t is a left bounded half-line for each t, the same is true for A. Moreover, by the preceding lemma, for every random variable ξ we have

$$\widehat{\tau}(\xi) = \sup_t \tau_t(\xi) = \inf A = \inf B = \tau_\phi(\xi). \qquad \blacksquare$$

7. - COMPARISON OF THE NORMS τ_ϕ AND σ_{ϕ^*}. SECOND PART

Let q be the density of ϕ^* i.e. the function such that

$$\phi^*(x) = \int_0^{|x|} q(t)\, dt.$$

We assume that q is differentiable and

$$\inf_u (q'(u) + q^2(u)) = H > 0.$$

REMARK 7.1: The above assumption is verified for the functions

$$\phi(x) = \frac{|x|^p}{p};$$

$$\phi(x) = e^{|x|} - |x| - 1.$$

Put now

$$\delta = \phi^{(-1)}(\log 3)$$

$$L = \sup_{|t| \leq \delta} \frac{t^2}{\phi(t)}.$$

We remark that

$$L \geqslant \limsup_{t\to 0} \frac{t^2}{\phi(t)} = \frac{1}{c} \geqslant 0$$

and that $L < \infty$ since $c > 0$.

Last, we set

$$A = \max\{\sqrt{10L/3H},\, 1\}.$$

and

$$\delta = \phi^{-1}(\log 3).$$

We are going to prove the following

PROPOSITION 7.1: *For every symmetric random variable $\xi \in \mathcal{S}(\Omega)$ we have*

$$\sup_{|t|\leqslant\delta} \tau_t(\xi) \leqslant A\sigma_{\phi^*}(\xi).$$

We need a

LEMMA 7.1: *For $|t| \leqslant \delta$*

$$M_t(x) \leqslant e^{\phi^*(Ax)} - 1.$$

PROOF (OF THE LEMMA): For $|t| \leqslant \delta$ we have, by (2.1)

$$e^{tu} + e^{-tu} \leqslant e^{A\delta u} + e^{-A\delta u} = 2\sum_{k=0}^{\infty} \frac{(\delta(Au))^{2k}}{(2k)!} \leqslant 2\sum_{k=0}^{\infty} \frac{(\phi(\delta)\,\phi^*(Au))^{2k}}{(2k)!}$$

$$= 2\sum_{k=0}^{\infty} \frac{(\log 3\,\phi^*(Au))^{2k}}{(2k)!} \leqslant \frac{10}{3} e^{\phi^*(Au)}$$

$$\leqslant \frac{10}{3HA^2} e^{\phi^*(Au)} (A^2 q^2(Au) + A^2 q'(Au)).$$

By recalling the inequality $z \leqslant e^z - 1$ we obtain also

$$2\frac{\phi(t)}{e^{\phi(t)} - 1}(e^{tu} + e^{-tu}) \leqslant \frac{20}{3HA^2} e^{\phi^*(Au)} (A^2 q^2(Au) + A^2 q'(Au));$$

now, by an integration in u between 0 and y, with $|y| \leqslant \delta$ we get

$$2\frac{\phi(t)}{e^{\phi(t)} - 1} \frac{e^{ty} - e^{-ty}}{t} \leqslant \frac{20}{3HA^2} e^{\phi^*(Ay)} Aq(Ay);$$

by another integration in y between 0 and x, with $|x| \leqslant \delta$ we get finally

$$M_t(x) = \frac{\cosh tx - 1}{e^{\phi(t)} - 1} \leqslant \frac{10L}{3HA^2}(e^{\phi^*(Ax)} - 1) \leqslant (e^{\phi^*(Ax)} - 1). \qquad \blacksquare$$

We are now ready to conclude the proof of the proposition. Let $a > 0$ be such that

$$E[e^{\phi^*(\xi/a)} - 1] \leqslant 1;$$

by the preceding lemma this implies that

$$EM_t(\xi/(Aa)) \leqslant 1,$$

so that we have the inclusion

$$A \times \{a > 0 : EM(\xi/a) \leqslant 1\} \subseteq \{b > 0 : EM_t(\xi/b) \leqslant 1\}$$

and this amounts to saying that

$$\tau_t(\xi) \leqslant A\sigma_{\phi^*}(\xi),$$

hence the statement of Proposition 7.1 by taking the supremum in $|t| \leqslant \delta$.

The two following lemmas are straightforward

LEMMA 7.2: *For every pair of real numbers t, x we have*

$$e^{tx} + e^{-tx} \leqslant e^{\phi(t) + \phi^*(x)} + 1 \leqslant e^{\phi(t) + \phi^*(x)} + 2.$$

LEMMA 7.3: *For every pair of real numbers t, x we have*

$$M_t(x/2) \leqslant M_t(x)/2.$$

Put again $\delta = \phi^{(-1)}(\log 3)$. We have

PROPOSITION 7.2: *For every symmetric random variable X we have*

$$\sup_{|t| > \delta} \tau_t(\xi) \leqslant 2\sigma_{\phi^*}(\xi).$$

PROOF: From Lemma 7.2 we easily get the relation

$$M_t(x) \leqslant \frac{1}{2}\left(1 + \frac{1}{e^{\phi(t)} - 1}\right)\exp\{\phi^*(x)\}. \tag{7.1}$$

Since $|t| > \delta$ we have

$$\frac{1}{2}\left(1 + \frac{1}{e^{\phi(t)} - 1}\right) \leqslant \frac{3}{4}. \tag{7.2}$$

Let now $a > 0$ be such that

$$E[e^{\phi^*(\xi/a)} - 1] \leqslant 1;$$

from relations (7.1) and (7.2) we get

$$EM_t(\xi/a) \leqslant (3/4)\, E[e^{\phi^*(\xi/a)} - 1] + 3/4 \leqslant (3/2) < 2.$$

Hence, we deduce from Lemma (7.3) that

$$EM_t(\xi/(2a)) \leqslant (1/2)\, EM_t(\xi/a) \leqslant 1.$$

The above relation says that $2a \in A_t$, that is

$$2a \geqslant \tau_t(\xi). \tag{7.3}$$

On taking the infimum with respect to a in relation (7.3), we get

$$2\sigma_{\phi^*}(\xi) \geqslant \tau_t(\xi);$$

we now obtain the required relation by taking the supremum in t. ■

Proposition 7.1 and 7.2 together with Proposition 6.3 yield

PROPOSITION 7.3: *For every symmetric random variable in* $Sub_\phi(\Omega)$ *we have*

$$\tau_\phi(\xi) \leqslant \max\{A, 2\}\, \sigma_{\phi^*}(\xi),$$

where A is the number defined in Proposition 7.1. ■

We now drop the assumption of symmetry and use an argument of symmetrization: let ξ be any variable in $Sub_\phi(\Omega)$ and η an independent copy of ξ. Denote by C the number $\max\{A, 2\}$. By Jensen inequality and Proposition 7.3 we have

$$Ee^{t\xi} \leqslant Ee^{t(\xi - \eta)} \leqslant \phi(tC\sigma(\xi - \eta)) \leqslant \phi(2tC\sigma(\xi)),$$

since σ_{ϕ^*} is a norm and $\sigma_{\phi^*}(\xi) = \sigma_{\phi^*}(\eta)$.

Hence we deduce the

PROPOSITION 7.4: *For every random variable* ξ *in* $Sub_\phi(\Omega)$ *we have*

$$\tau_\phi(\xi) \leqslant 2C\sigma_{\phi^*}(\xi).$$

8. - INDEPENDENT RANDOM VARIABLES IN $Sub_\phi(\Omega)$

THEOREM 8.1 [3]: *Let* $\xi_1, \xi_2, \ldots, \xi_n \in Sub_\phi(\Omega)$ *be independent random variables. If the function* $\phi(|x|^{1/p})$, $x \in \mathbb{R}$ *is convex for some* $p \in [1, 2]$, *then*

$$\tau_\phi^p\left(\sum_{k=1}^n \xi_k\right) \leqslant \sum_{k=1}^n \tau_\phi^p(\xi_k). \tag{8.1}$$

Proof: Since the function $\phi(|x|^{1/p})$ is convex the statement follows from the relations:

$$E\exp\left\{\lambda\sum_{k=1}^{n}\xi_k\right\} = \prod_{k=1}^{n} E\exp\{\lambda\xi_k\} \leqslant \prod_{k=1}^{n}\exp\{\phi(|\lambda|\tau_\phi(\xi_k))\}$$

$$= \exp\left\{\sum_{k=1}^{n}\phi\left(\left((|\lambda|\tau_\phi(\xi_k))^p\right)^{1/p}\right)\right\}$$

$$\leqslant \exp\left\{\phi\left(\lambda\left(\sum_{k=1}^{n}\tau_\phi^p(\xi_k)\right)^{1/p}\right)\right\}. \qquad \blacksquare$$

Corollary 8.1: *Let* $\xi_k \in Sub_\phi(\Omega)$, $k=\overline{1,\infty}$ *and assume that the function* $\phi(|x|^{1/p})$, $x\in\mathbb{R}$, $p\in[1,2]$, *is convex. Then we have*

$$\tau_\phi^p\left(\sum_{k=1}^{\infty}\xi_k\right) \leqslant \sum_{k=1}^{\infty}\tau_\phi^p(\xi_k).$$

Example 8.1: Let η_k, $k=\overline{1,\infty}$ be independent random variables uniformly distributed in $[-1,1]$. Let $\theta=\sum_{k=1}^{\infty}a_k\eta_k$. It follows from example 3.2 that $\eta_k\in Sub_\phi(\Omega)$, where $\phi_\alpha(x)=|x|^\alpha$, $1<\alpha\leqslant 2$, and $\tau_{\phi_\alpha}(\xi)\leqslant 6^{(1-\alpha)/\alpha}$. If $\sum_{k=1}^{\infty}a_k^\alpha<\infty$ then

$$\tau_{\phi_\alpha}(\theta)\leqslant\sum_{k=1}^{\infty}a_k^\alpha\tau_{\phi_\alpha}(\eta_k)\leqslant 6^{1-\alpha}\sum_{k=1}^{\infty}a_k^\alpha=A_\alpha<\infty$$

that is $\eta\in Sub_\phi(\Omega)$. In this case the following inequality holds

$$P\{|\theta|>\varepsilon\}\leqslant 2\exp\left\{-\phi_\alpha^*\left(\frac{\varepsilon}{A_\alpha}\right)\right\}=2\exp\left\{-c_\alpha\left(\frac{\varepsilon}{A_\alpha}\right)^{\frac{\alpha}{\alpha-1}}\right\},$$

where $c_\alpha=\alpha^{-\frac{1}{\alpha-1}}-\alpha^{-\frac{\alpha}{\alpha-1}}$.

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