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Asymptotic Behavior of Some Nonlinear Subelliptic Relaxed Dirichlet Problems (* * *)

ABSTRACT. — We consider the asymptotic behavior of the solutions of a relaxed Dirichlet problem in a bounded open set Ω associated with the p-Lapacian relative to the vector fields $X = (X_1, \ldots, X_m)$ satisfying an Hörmander condition and to measures μ_ε, that do not charge sets of zero p-capacity (with respect to X). We prove that there exists a subsequence of μ_ε that Γ-converges to a measure μ of the same type and we give also correctors for the convergence of the solutions in $H_0^{1,p}(\Omega, X)$.

Comportamento asintotico di certi problemi di Dirichlet rilassati non lineari e sottoellittici

SUNTO. — Si considera il comportamento asintotico delle soluzioni di un problema di Dirichlet rilassato relativo al p-Laplaciano associato con dei campi vettori $X = (X_1, \ldots, X_m)$ soddisfacenti una condizione di Hörmander e a misure μ_ε, che non caricano insiemi di p-capacità zero (rispetto a X). Si prova che esiste una sottosuccessione di μ_ε che Γ-converge ad una misura μ dello stesso tipo e si danno correttori relativi alla convergenza delle soluzioni in $H_0^{1,p}(\Omega, X)$.

1. - INTRODUCTION

In this paper we study the asymptotic behavior of solutions of some subellitic nonlinear relaxed Dirichlet problems of monotone type. In the case of the Laplace opera-

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tor the notion of relaxed Dirichlet problems is defined in [15] and their asymptotic behavior is studied in terms of the Γ-convergence of the functional associated to the relaxed Dirichlet problem, [16]. For the extension of those results to the case of uniformly elliptic symmetric operators we refer to [7] and [11]. For the case of uniformly elliptic operators (also non symmetric) see [14]. The case of relaxed Dirichlet problems relative to a subdifferential of an integral convex functional defined on $H_0^{1,p}(\Omega)$, with $1 < p < +\infty$, is studied in [13]. The case relative to a partial differential operator on $H_0^{1,p}(\Omega)$, with a degree p homogeneity, has been studied in [17] which is the main reference of our paper, we also recall the more recent paper [19] where more general nonlinear elliptic problems in varying domains are studied. Concerning the general case of a symmetric Dirichlet form the notion of Γ-convergence has been introduced and studied in [28]; the asymptotic behavior of relaxed Dirichlet problems has been studied in the strongly local symmetric case in [4][12] and in [27] in some strongly local non symmetric case. Here we will study the asymptotic behavior of relaxed Dirichlet problems in the case of subelliptic operators generated by Hörmander's vector fields with a p-homogeneity in the fields and in particular of the subelliptic p-Laplacian. We use methods, which are an adaptation to the subelliptic framework of the one in [17]; we prefer this type of methods since they allow us to exhibit correctors. Finally we recall that the importance of the class of relaxed Dirichlet is that this class contains the class of Dirichlet problems in varying domains and that results concerning the asymptotic behavior of the Dirichlet problem for the Heisenberg p-Laplacian in periodically varying domains have been given in [3].

We now precise our framework.

Let $X_i = \sum_{j=1}^{N} a_{ij} \frac{\partial}{\partial x_j}$, $i = 1, \ldots, m$, be C^∞ vector fields on R^N satisfying an Hörmander condition, i.e. the vector X_i and their commutators up to the order k span R^N at every point. e denote by $X_i^* = -\sum_{j=1}^{N} \frac{\partial}{\partial x_j}(a_{ij}\cdot)$ the formal adjoint of the vector field X_i, moreover we denote by X the gradient with respect to the vector fields X_i.

We recall that there is a distance $d(x, y)$ connected with the vector fields, which may be defined as

$$d(x, y) = \sup\{\phi(x) - \phi(y);\ \phi \in C_0^1,\ |X\phi| \leq 1\} \tag{1.1}$$

[20, 21, 25, 26, 29, 30]. The distance $d(x, y)$ defines a topology on R^N which is equivalent to the Euclidean one, [21, 29]; moreover for every compact set $K \subset R^N$ there exists $\varepsilon > 0$ and a constant c_K such that

$$|x - y| \leq d(x, y) \leq c_K |x - y|^\varepsilon .$$

We denote by $B(x, r)$ the ball relative to the distance d with center in x and radius r. We fix now an open bounded set $\Omega \subset B$, where B is a ball with center in Ω and radius 4 *diam*(Ω).

We recall, [21, 25, 29, 30], that for the balls with center in B and radius $r \leq \overline{R}_0$ we have a duplication property

$$m(B(x, r)) \geq C\left(\frac{r}{R}\right)^{\nu} m(B(x, R)) \tag{1.2}$$

$r \leq \frac{R}{2}$, $R \leq \overline{R}_0$, where $\nu = N + k$, we define ν as the «intrinsic dimension» (or an estimate of) of our problem (where m denotes the Lebesgue measure).

We recall that in our case we have a Poincaré inequality on balls, i.e. there exists a constant $\overline{R}_0$ such that for $x \in B$, $r \leq \overline{R}_0$ and $p \geq 1$

$$\int_{B(x,\, r)} |u - u_r|^p \, dx \leq C r^p \int_{B(x,\, r)} |Xu|^p \, dx \tag{1.3}$$

where C is a constant independent of x and r and u_r denotes the average of u on $B(x, r)$ and $u \in C^1(\Omega)$, [22, 25, 26]. Using Poincaré inequality (1.3) we can prove that also Sobolev-Morrey-Campanato type inequality (relative to ν) holds, [1, 2, 22].

We denote by $H^{1,p}(\Omega, X)$, $1 < p < +\infty$ the completion of the functions in $C^{\infty}(\Omega)$ such that (1.4) is finite for the norm

$$\|u\|_{H^{1,p}(\Omega,\, X)} = \left(\int_{\Omega} |u|^p \, dx + \int_{\Omega} |Xu|^p \, dx\right)^{1/p} \tag{1.4}$$

REMARK 1.1: The space $H^{1,p}(\Omega, X)$ coincides with the space of all functions $u \in L^p(\Omega)$ such that the gradient Xu (in distribution sense) belongs to $L^p(\Omega)$, [23].

The space $H_0^{1,p}(\Omega, X)$ will be the completion of $C_0^{\infty}(\Omega)$ for the norm (1.4). We observe that the inequality (1.3) and the Sobolev-Morrey-Campanato type inequalities hold again for functions in $H^{1,p}(B(x, r), X)$.

LEMMA 1.2: *Let* $u \in H_0^{1,p}(B(x, r), X)$ *then*

$$\int_{B(x,\, r)} |u|^p \, dx \leq C r^p \int_{B(x,\, r)} |Xu|^p \, dx$$

where $x \in B$, $r \leq \frac{\overline{R}_0}{3}$ *and* C *is a constant independent of* x, r.

PROOF: We observe that the extension of u by 0 to $B(x, 2r)$ is in $H_0^{1,p}(B(x, 2r), X)$, then from (1.3) we have

$$\int_{B(x,\, 2r)} |u - u_{2r}|^p \, dx \leq C_1 r^p \int_{B(x,\, 2r)} |Xu|^p \, dx. \tag{1.5}$$

From (1.5) we have

$$|u_{2r}|^p \leq C_1 \frac{r^p}{m(\{x \in B(x, 2r);\ u = 0\})} \int_{B(x,\, r)} |Xu|^p\, dx .$$

We observe that from (1.2) there exists a ball $B_1 \subset\subset B(x, 2r) - B(x, r)$ such that $m(B_1) \geq C_2\, m(B(x, r))$. Then

$$|u_{2r}|^p \leq \frac{C_1}{C_2} \frac{r^p}{m(B(x, r))} \int_{B(x,\, r)} |Xu|^p\, dx .$$

From (1.5) we obtain

$$\int_{B(x,\, r)} |u|^p\, dx \leq C_3\, |u_{2r}|^p\, m(B(x, 2r) + C_4 r^p \int_{B(x,\, r)} |Xu|^p\, dx \leq C_5\, r^p \int_{B(x,\, r)} |Xu|^p\, dx$$

and the result follows. ■

A consequence of the Lemma 1.2. is that

$$\|u\|_{H_0^{1,p}(\Omega,\, X)} = \left(\int_\Omega |Xu|^p\, dx \right)^{1/p}$$

is a norm on $H_0^{1,p}(\Omega, X)$ equivalent to the norm $\|u\|_{H^{1,p}(\Omega,\, X)}$. Finally we observe that $H^{1,p}(\Omega, X)$ and $H_0^{1,p}(\Omega, X)$ are uniformly convex Banach spaces, [4]. We denote by $H^{-1,q}(\Omega, X)$ the dual of the space $H_0^{1,p}(\Omega, X)$, $\frac{1}{p} + \frac{1}{q} = 1$; again $H^{-1,q}(\Omega, X)$ is a reflexive Banach space. We have easily that $H_0^{1,p}(\Omega, X)$ is dense and compactly embedded in $L^p(\Omega)$, [4], then $L^q(\Omega)$ is dense in and compactly embedded $H^{-1,q}(\Omega, X)$.

Lemma 1.3: *Let $u \in H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$; there exists an uniformly bounded sequence $u_n \in C_0^\infty(\Omega)$ such that u_n converges to u in $H_0^{1,p}(\Omega, X)$.*

Proof: By definition there exists a sequence v_n in $C_0^\infty(\Omega)$ such that v_n converges to u in $H_0^{1,p}(\Omega, X)$.

Consider a non decreasing function $\beta_M \in C^\infty(R)$, $M = \sup_\Omega u$, such that

$$\beta_M(t) = t, \quad |t| \leq M; \qquad \beta_M(t) = M + 1, \quad |t| \geq (M+1)$$

$$\beta_M(0) = 0; \qquad \beta_M'(t) \leq 1; \qquad \beta_M'(t) = 1, \quad |t| < M + \frac{1}{2}\, .$$

let $u_n = \beta_M(v_n)$.

The sequence u_n is uniformly bounded and in $C_0^\infty(\Omega)$.

We have also $u_n \to u$ a.e. in Ω and $|Xu_n| = \beta_M'(v_n)\, |Xv_n|$, then $|Xu_n| \leq |Xv_n|$

and $Xu_n \to Xu$ a.e. in Ω. We end the proof by the dominated convergence theorem. ■

We give now the notion of p-capacity associated with the fields X_i.

Let O be a bounded open set and $E \subset\subset O$ we define

$$\mathrm{cap}_p(E, O; X) = \inf\left\{ \int_\Omega |Xv|^p\, dx;\ v \in C_0^\infty(O),\ v \geq 1 \text{ in a neighborhood of } E \right\}$$

LEMMA 1.4: *Let* $E \subset\subset O$; *then* $\mathrm{cap}_p(E, O; X) = 0$ *if and only if* $\mathrm{cap}_p(E, O'; X) = 0$, *where* O' *is a bounded open set with* $E \subset\subset O'$.

PROOF: It is enough to prove the result in the case $O \subset O'$.

From the definition we have

$$\mathrm{cap}_p(E, O'; X) \leq \mathrm{cap}_p(E, O; X)$$

then $\mathrm{cap}_p(E, O; X) = 0$ implies $\mathrm{cap}_p(E, O'; X) = 0$.

Let now $\mathrm{cap}_p(E, O'; X) = 0$. Let ϕ be a function in $C_0^\infty(O)$ with $\phi = 1$ on a neighborhood of E; there exists a sequence v_n in $C_0^\infty(O')$ such that $v_n = 1$ on a neighborhood of E and $\int_{O'} |Xv_n|^p\, dx \to 0$.

Let $w_n = \phi v_n$; we have $w_n \in C_0^\infty(O)$, $w_n = 1$ on a neighborhood of E and $Xw_n = \phi X v_n + v_n X\phi$. Since v_n converges to 0 in $H_0^{1,p}(O', X)$ we have $\int_O |Xw_n|^p\, dx \to 0$, then $\mathrm{cap}_p(E, O; X) = 0$. ■

We say that a property holds p-q.e. in Ω if holds up to a set of null p-capacity (with respect to Ω or to every bounded open set containing Ω).

We say that a function u is p-quasi-continuous in Ω if for every $\varepsilon > 0$ there exists a set $A_\varepsilon \subset \Omega$ with $\mathrm{cap}_p(A_\varepsilon, \Omega; X) < \varepsilon$ such that the restriction of u to $\Omega - A_\varepsilon$ is continuous.

We say that a sequence u_n converges p-quasi-uniformly to u in Ω if for every $\varepsilon > 0$ there exists a set $A_\varepsilon \subset \Omega$ with $\mathrm{cap}_p(A_\varepsilon, \Omega; X) < \varepsilon$ such that u_n converges uniformly to u in $\Omega - A_\varepsilon$.

It is easily proved that if $v_n \in C_0^\infty(\Omega)$ is a sequence converging in $H_0^{1,p}(\Omega, X)$, then v_n (at least after extraction of subsequences) converges p-quasi-uniformly in Ω, [10](see also [24] for the case $p = 2$).

Denote by $\tilde{v}$ the q.e. limit of the v_n and by v the limit of the v_n in $H_0^{1,p}(\Omega, X)$, then $\tilde{v}$ is a p-quasi-continuous representative (q.e.) of v.

We observe that from Proposition 6.1. and Corollary 6.7. of [10] two p-quasi-continuous representatives are equal q.e.. In the following we identify v with its p-quasi-continuous representative and we consider v as defined up to set of null p-capacity (with respect to Ω).

We observe that from the above results every function in $H^{1,p}(\Omega, X)$ has a p-quasi-continuous representative and that the convergence in $H^{1,p}(\Omega, X)$ implies, at least after extraction of subsequence the convergence q.e. in Ω.

Finally we say that a subset U of Ω is p-quasi-open if for every $\varepsilon > 0$ there exists a subset V of Ω with $\operatorname{cap}_p(V, \Omega; X) < \varepsilon$ and $U \cup V$ open.

Let O be an open bounded set and $E \subset\subset O$; we have

$$\operatorname{cap}_p(E, O; X) = \inf\left\{ \int_\Omega |Xv|^p\, dx;\ v \in H_0^{1,p}(O, X),\ v \geqslant 1 \text{ q.e. on } E \right\}$$

The infimum is really a minimum that is achieved by a function u_E called the *potential* of the set E with respect to O and we have $u_E = 1$ q.e. on E. Moreover we observe that u_E can also be defined as the solution of the following variational inequality

$$u_E \in H_0^{1,p}(O, X), \qquad u_E \geqslant 1 \text{ q.e. on } E$$

$$\int_O |Xu_E|^{p-2} Xu_E X(v - u_E)\, dx \geqslant 0$$

$$\forall v \in H_0^{1,p}(O, X), \qquad v \geqslant 1 \text{ q.e. on } E$$

We recall the following result:

LEMMA 1.5: *Let E be a closed subset of Ω. If $u \in H_0^{1,p}(\Omega, X)$ and $u = 0$ q.e. on E, then $u \in H_0^{1,p}(\Omega - E, X)$*

PROOF: We can assume without loss of generality $u \geqslant 0$ and $u \in L^\infty(\Omega)$.

We consider a sequence $v_n \geqslant 0$ in $C_0^\infty(\Omega)$ uniformly bounded and converging to $u \geqslant 0$ in $H_0^{1,p}(\Omega, X)$. Up to extraction of subsequences we have that v_n converges to u p-quasi-uniformly and that we can choose v_n such that

$$\operatorname{cap}_p\left(\left\{ |v_n - u| > \frac{1}{n} \right\}, \Omega; X\right) < \frac{1}{n}$$

Let E_n be the set $E \cap \left\{ |v_n - u| > \frac{1}{n} \right\}$ and let $w_n = \left(v_n - \frac{1}{n}\right)^+$, we have again that w_n converges to u in $H_0^{1,p}(\Omega, X)$. We observe that w_n are Lipschitz functions and $\operatorname{supp}(w_n) \subset (\Omega - E) \cup E_n$ moreover we have $\operatorname{cap}_p(E_n, \Omega; X) \leqslant \frac{1}{n}$.

There exists a sequence $u_n \in H_0^{1,p}(\Omega, X) \cap C(\Omega)$ with $u_n = 1$ on E_n, $0 \leqslant u_n \leqslant 1$ on Ω and such that u_n converges to 0 in $H_0^{1,p}(\Omega, X)$.

Consider the functions $\widetilde{w}_n = (1 - u_n)\, w_n$; we have that the support of $\widetilde{w}_n$ is contained in $\Omega - E$, so $w_n \in H_0^{1,p}(\Omega - E, X)$. Moreover it is easily proved that $\widetilde{w}_n$ converges to u in $H_0^{1,p}(\Omega, X)$; then, since the supports of $\widetilde{w}_n$ are contained in $\Omega - E$, $\widetilde{w}_n$ converges to u in $H_0^{1,p}(\Omega - E, X)$. ■

REMARK 1.6: Let Ω and Ω' be bounded open sets with $\Omega \subset \Omega'$, let u be in $H_0^{1,p}(\Omega, X)$. We denote again by u the extension of u by 0 to Ω'; then $u \in H_0^{1,p}(\Omega', X)$.

We shall frequently use the following Lemma about the approximation of the characteristic function of a p-quasi-open set.

We recall that the characteristic function $\mathbf{1}_E$ of a set E in Ω is defined as $\mathbf{1}_E = 1$ if $x \in E$ and $1_E = 0$ if $x \in \Omega - E$.

LEMMA 1.7: *For every p-quasi-open set U of Ω, there exists an increasing sequence v_n of functions in $H_0^{1,p}(\Omega, X)$ which converges to $\mathbf{1}_U$ q.e. in Ω.*

PROOF: Let U be p-quasi-open in Ω. Then there exists a sequence U_k of open sets of Ω with $\mathrm{cap}_p(U_k, \Omega; X) \leq \frac{1}{k}$ such that the sets $A_k = U \cup U_k$ are open. Therefore for every k there exists an increasing sequence of non-negative functions ϕ_h^k in $L^\infty(\Omega) \cap H_0^{1,p}(\Omega, X)$ and with $|X\phi_h^k| \leq M_h^k$ converging to 1_{A_k} pointwise q.e. in Ω. Since $\mathrm{cap}_p(U_k, \Omega; X) \leq \frac{1}{k}$ for every k there exists $u_k \in H_0^{1,p}(\Omega, X)$ such that $u_k \geq 1$ q.e. in U_k, $u_k \geq 0$ in Ω and $\int_\Omega |Xu_k|^p \, dx \leq \frac{1}{k}$. This implies that a subsequence of u_k converges to 0 q.e.. Moreover as $\phi_h^k \leq 1_{A_k}$, we have $(\phi_h^k - u_k)^+ \leq 1_U$ q.e.. Let us define

$$v_h = \max_{1 \leq k \leq h} (\phi_h^k - u_k)^+, \qquad \psi = \sup_h v_h.$$

Then $v_h \in H_0^{1,p}(\Omega, X)$, $v_h \geq 0$ q.e. in Ω, moreover the sequence v_h is increasing and $\psi \leq \mathbf{1}_U$ q.e. in Ω. For every $h \geq k$ we have $v_h \geq (\phi_h^k - u_k)$. As $U \subset A_k$ we get $\psi \geq (1 - - u_k)$ q.e. in U.

Taking the limit as $k \to +\infty$ along a suitable subsequence We obtain $\psi \geq 1$ q.e. in U. This shows $\psi = \mathbf{1}_U$ which concludes the proof. ■

2. - THE SPACE OF MEASURES $\mathcal{M}_0^p(\Omega, X)$ AND THE OPERATOR

MEASURES. A Radon measure on Ω is a continuous linear functional on $C_0(\Omega)$ the space of all continuous functions with compact support in Ω, Ω as in section 1. It is well known that for every Radon measure λ on Ω there is a countably additive set function μ, defined on the family of all relatively compact Borel subsets of Ω, such that $\lambda(u) = \int_\Omega u \, d\mu$ for every $u \in C_0(\Omega)$. In the following we identify λ with the set function μ.

A non-negative Borel measure will be a non-negative countably additive set function defined on the Borel σ-field of Ω with values in $[0, +\infty]$. It is well known that every non-negative Borel measure which is finite on compact subsets of Ω is a non-negative Radon measure (Hamos, Measure Theory, section 13). Let μ be a non-negative Borel measure, we denote by $L^r_\mu(\Omega)$, $1 \leq r \leq +\infty$, the usual Lebesgue space with respect to the measure μ.

We denote by $\mathcal{M}^p_0(\Omega, X)$ the sets of all non-negative Borel measures such that

(i) $\mu(B) = 0$ for every Borel set $B \subset \Omega$ with $\mathrm{cap}_p(B, \Omega; X) = 0$

(ii) $\mu(B) = \inf\{\mu(U),\ U$ quasi-open, $B \subset U\}$.

Property (ii) is a weak regularity property of the measure μ. Since any quasi-open set differs from a Borel set by a set of p-capacity 0, every quasi-open set is μ-measurable for every non-negative Borel measure μ which satisfies (i). Therefore $\mu(U)$ is well defined when U is quasi-open and condition (ii) make sense.

REMARK 2.1: The condition (ii) appears in [17] but does not appear in some previous definitions but will be essential in the proof of the uniqueness of the γ^A-limit (Remark 7.4).

For every open set $U \subset \Omega$ we consider the Borel measure μ_U defined as

$$\mu_U(B) = 0 \text{ if } \mathrm{cap}_p(B \backslash U, \Omega; X) = 0, \qquad \mu_U(B) = +\infty \text{ otherwise.} \tag{2.1}$$

As U is open it is easy to see that this measure belongs to $\mathcal{M}^p_0(\Omega, X)$. The measure μ_U will be useful in the study of the asymptotic behavior of sequences of Dirichlet problems in varying domains (see Remark 3.4 and Theorem 7.6).

If $\mu \in \mathcal{M}^p_0(\Omega, X)$ the space $H^{1,p}_0(\Omega, X) \cap L^p_\mu(\Omega)$ is well defined since the functions in $H^{1,p}_0(\Omega, X)$ are defined μ-almost everywhere in Ω.

It is easy to see that $H^{1,p}_0(\Omega, X) \cap L^p_\mu(\Omega)$ is a Banach space for the norm $\|u\|^p_{H^{1,p}_0(\Omega, X) \cap L^p_\mu(\Omega)} = \|u\|^p_{H^{1,p}_0(\Omega, X)} + \|u\|^p_{L^p_\mu(\Omega)}$.

Finally we say that a Radon measure σ belongs to $H^{-1,q}(\Omega, X)$ if there exists $f \in H^{-1,q}(\Omega, X)$ such that

$$\langle f, \phi \rangle = \int_\Omega \phi \, d\sigma \qquad \forall \phi \in C^\infty_0(\Omega) \tag{2.2}$$

where $\langle .,. \rangle$ denotes the pairing between $H^{-1,q}(\Omega, X)$ and $H^{1,p}_0(\Omega, X)$, $\frac{1}{p} + \frac{1}{q} = 1$. We identify f and σ. We observe that for every non-negative $f \in H^{-1,q}(\Omega, X)$ there exists a non-negative Radon measure σ such that (2.2) holds. Moreover every non-negative Radon measure in $H^{-1,q}(\Omega, X)$ belongs to $\mathcal{M}^p_0(\Omega, X)$.

The Monotone operator.

We will describe here the more generals operators to which our results apply; the proofs will be developed in the case of the subelliptic p-Laplace operator and are almost the same in the general case. Let $X_i = \sum_{j=1}^{N} a_{ij} \frac{\partial}{\partial x_j}$, $i = 1, \ldots, m$, be vector fields on R^N with C^∞ coefficients. We assume that the vector fields satisfy an Hörmander condition (i.e. the vector fields and their commutators up to the order k span R^N at every point).

We denote $X_i^\star = -\sum_{j=1}^{N} \left(\frac{\partial}{\partial x_j} a_{ij} . \right)$ the formal adjoint of the field X_i. We denote by X the vector $(X_1, \ldots, X_m)$.

Let $a : \Omega \times R^m \to R^m$ be a Borel function satisfying the following homogeneity condition

$$a(x, t\xi) = |t|^{p-2} t \quad a(x, \xi) \tag{2.3}$$

for every $x \in \Omega$, $t \in R$, $\xi \in R^m$, $1 < p < +\infty$, with the convention $|t|^{p-2} t = 0$ for $t = 0$ and $1 < p < 2$.

We list now some algebraic inequalities that we assume and that are different in the two cases $1 < p < 2$ and $p \geqslant 2$.

In the case $p \geqslant 2$ we assume that there exists constants C_0, $C_1 > 0$ such that

$$(a(x, \xi_1) - a(x, \xi_2),\ \xi_1 - \xi_2) \geqslant C_0 \, |\xi_1 - \xi_2|^p \tag{2.4}$$

$$|a(x, \xi_1) - a(x, \xi_2)| \leqslant C_1 (|\xi_1| + |\xi_2|)^{p-2} \, |\xi_1 - \xi_2| \tag{2.5}$$

for every $x \in \Omega$ and for every $\xi_1, \xi_2 \in R^m$, where $(.,.)$ denotes the scalar product in R^m.

In the case $1 < p < 2$ we assume that here exists constants C_0, $C_1 > 0$ such that

$$(a(x, \xi_1) - a(x, \xi_2), \xi_1 - \xi_2) \geqslant C_0 (|\xi_1| + |\xi_2|)^{p-2} \, |\xi_1 - \xi_2|^2 \tag{2.6}$$

$$|a(x, \xi_1) - a(x, \xi_2)| \leqslant C_1 \, |\xi_1 - \xi_2|^{p-1} \tag{2.7}$$

for every $x \in \Omega$ and for every $\xi_1, \xi_2 \in R^m$, $\xi_1 \neq \xi_2$.

We observe that (2.3) implies that

$$a(x, -\xi) = -a(x, \xi) \tag{2.8}$$

for every $x \in \Omega$ and for every $\xi \in R^m$, hence

$$a(x, 0) = 0 \tag{2.9}$$

for every $x \in \Omega$ while (2.4)-(2.7) and (2.9) imply that

$$(a(x, \xi), \xi) \geqslant C_0 \, |\xi|^p \tag{2.10}$$

$$|a(x, \xi)| \leqslant C_1 \, |\xi|^{p-1} \tag{2.11}$$

for every $x \in \Omega$ and for every $\xi \in R^m$.

We now define $A : H^{1,p}(\Omega, X) \to H^{-1,q}(\Omega, X)$, $\frac{1}{p} + \frac{1}{q} = 1$, by $Au =$

$= \sum_{ij=1}^{m} X_j^\star(a(x, Xu))$, i.e.

$$\langle Au, v\rangle = \int_\Omega (a(x, Xu), Xv)\, dx$$

for every $u \in H^{1,p}(\Omega, X)$, $v \in H_0^{1,p}(\Omega, X)$. This operator is strongly monotone on $H_0^{1,p}(\Omega, X)$.

The model case is the subelliptic p-Laplacian $\sum_i X_i^\star(|Xu|^{p-2} Xu)$, which corresponds to the choice $a(x, \xi) = |\xi|^{p-2}\xi$. We observe that in the case $p \geq 2$ the conditions (2.4) (2.5) are satisfied with $C_0 = 2^{2-p}$, $C_1 = (p-1)$ and for $1 < p < 2$ the conditions (2.6) (2.7) are satisfied with $C_0 = 1$, $C_1 = 2^{2-p}$. In the following we develop the proofs mainly in the case of subelliptic p-Laplacian, but easy modifications gives also the result under the above assumptions.

We say that u is a superharmonic (subharmonic) relative to the operator A in Ω if $u \in H^{1,p}(\Omega, X)$ and

$$\langle Au, v\rangle \geq (\leq) 0$$

$\forall v \in H_0^{1,p}(\Omega, X)$, $v \geq 0$.

We recall here some properties of sub- or superharmonics:

Proposition 2.2: *Let $u, v \in H^{1,p}(\Omega, X)$ be two superharmonic relative to A in Ω; then* $\min(u, v)$ *is again a superharmonic relative to A in Ω*, [10].

Proposition 2.3: *Let v be a nonnegative subharmonic relative to A in the ball $B \subset\subset \Omega$, then*

$$\sup_{\lambda B} v \leq C_\lambda \left(\frac{1}{m(B)} \int_B v\, dx \right)$$

for $\lambda \in (0, 1)$ fixed and C_λ constant dependent on λ, [8, 10].

Proposition 2.4: *Let v be a nonnegative superharmonic relative to A in Ω. There is a positive number r_0 such that for $0 < r < r_0$, $0 < s < \chi(p-1)$ ($\chi = \frac{\nu}{p-\nu}$ if $\nu < p$, where ν is the intrinsic dimension relative to Ω, $\chi > 1$ if $p \geq \nu$), then*

$$\left(\frac{1}{m(B)} \int_B v^s\, dx \right)^{1/s} \leq C \inf_B v$$

where $B = B(x_0, r)$ is such that $4B \subset \Omega$, [8, 10].

We say that $u \in H^{1,p}(\Omega, X)$ is a local solution of

$$Au = 0 \tag{2.13}$$

in Ω if u is both a super- and subsolution relative to A in Ω.

PROPOSITION 2.5: *Let u be a positive local solution of* (2.13) *in* $B = B(x_0, r)$, $0 < r < r_0$, *with* $4B \subset \Omega$, *then*

$$\sup_B u \leqslant C \inf_B u$$

where C does not depend on x_0, r [8, 10].

To end this section we give the following result which is an easy corollary of Proposition 2.4.

Let w_0 be the solution of the problem

$$w_0 \in H_0^{1,p}(\Omega, X); \quad \langle A w_0, v\rangle = \int_\Omega v\, dx, \quad \forall v \in H_0^{1,p}(\Omega, X) \tag{2.14}$$

COROLLARY 2.6: *We have* $w_0 > 0$ *q.e. in* Ω

PROOF: We have that w_0 is a non-negative supersolution in Ω relative to A.

Assume that for a ball $B = B(x_0, r)$, $0 < r < r_0$, $4B \subset \Omega$, we have $\inf_B w_0 = 0$; then $\inf_B (w_0 + \varepsilon) = \varepsilon$.

Since $w_0 + \varepsilon$ is a nonnegative supersolution in Ω relative to A from Proposition 2.4 we have

$$\left(\frac{1}{m(B)} \int_B w_0^s\, dx\right)^{1/s} \leqslant C\varepsilon$$

where C does not depend on ε.

Letting $\varepsilon \to 0$ we obtain

$$\int_B w_0^s\, dx = 0.$$

Since w_0 is non-negative in B we have $w_0 = 0$ a.e. and then q.e. in B.

We have

$$A w_0 = 0 \quad \text{in} \quad \mathcal{D}'(B) \tag{2.15}$$

and (2.15) contradicts (2.14). Then $w_0 > 0$ q.e. in Ω. (From [8, 10] we also have that $w_0 \in C(\Omega)$ then $w_0 > 0$ in Ω). ■

Let $u_k \in H^{1,p}(\Omega, X)$ be the solutions of the problem

$$\langle Au, v\rangle = \langle f_k, v\rangle + \int_\Omega v\, d\mu_k$$

where $f_k \in H^{-1,q}(\Omega, X)$, $\frac{1}{p} + \frac{1}{q} = 1$, and μ_k are Radon measures on Ω.

Assume that

(2.16) $$u_k \to u \text{ weakly in } H^{1,p}(\Omega, X)$$

(2.17) $$f_k \to f \text{ in } H^{-1,q}(\Omega, X)$$

$$\mu_k \to \mu \text{ weakly* in the Radon measures on } \Omega$$

Corollary 2.6: *Assume that* (2.16)-(2.18) *are satisfied. Then Xu_k converges to Xu strongly in $L^r(\Omega)$, $1 \leqslant r < p$.*

We have also that $u_k \to u$ in $H^{1,r}(\Omega, X)$ and that (at least after extraction of subsequences) $a(x, Xu_k)$ converges strongly in $L^s(\Omega)$ to $a(x, Xu)$, $1 \leqslant s < q$, moreover $Xu_k \to Xu$ and $a(x, Xu_k) \to a(x, Xu)$ a.e. in Ω.

Proof: The proof of the first part of the Theorem is given in the Appendix at the end of the paper.

For the second part of the theorem we recall that from embedding theorems we have $u_k \to u$ in $L^p(\Omega)$ then $u_k \to u$ in $L^r(\Omega)$, since Ω is bounded; so $u_k \to u$ in $H^{1,r}(\Omega, X)$.

At least after extraction of subsequences we have $Xu_k \to Xu$ a.e. in Ω and $\int\limits_\Omega |a(x, Xu_k)|^q \, dx \leqslant C$, so we have $a(x, Xu_k) \to a(x, Xu)$ a.e. in Ω and in $L^s(\Omega)$. ■

3. - Relaxed dirichlet problems

Estimates for the solutions.

Let $\mu \in \mathcal{M}_0^p(\Omega, X)$, $f \in H^{-1,q}(\Omega, X)$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we consider the following relaxed Dirichlet problem

(3.1) $$\begin{gathered} u \in H_0^{1,p}(\Omega, X) \cap L_\mu^p(\Omega) \\ \langle Au, v \rangle + \int\limits_\Omega |u|^{p-2} uv \, d\mu = \langle f, v \rangle \\ \forall v \in H_0^{1,p}(\Omega, X) \cap L_\mu^p(\Omega) \end{gathered}$$

More generally for $\psi \in H^{1,p}(\Omega, X) \cap L_\mu^p(\Omega)$ we consider a problem of type (3.1) with non-homogeneous boundary conditions

(3.2) $$\begin{gathered} u \in H^{1,p}(\Omega, X) \cap L_\mu^p(\Omega), \qquad (u - \psi) \in H_0^{1,p}(\Omega, X) \\ \langle Au, v \rangle + \int\limits_\Omega |u|^{p-2} uv \, d\mu = \langle f, v \rangle \\ \forall v \in H_0^{1,p}(\Omega, X) \cap L_\mu^p(\Omega) \end{gathered}$$

THEOREM 3.1: *Let* $\mu \in \mathfrak{M}_0^p(\Omega, X)$, $\psi \in H^{1,p}(\Omega, X) \cap L_\mu^p(\Omega)$. *The problem* (3.2) *has a unique solution. Moreover the solution of* (3.2) *satisfies the estimate*

$$\int_\Omega |Xu|^p\, dx + \int_\Omega |u|^p\, d\mu \leq C\left(\|f\|^q_{H^{-1,q}(\Omega,X)} + \int_\Omega |X\psi|^p\, dx + \int_\Omega |\psi|^p\, d\mu\right) \tag{3.3}$$

where C *is a structural constant.*

PROOF: Let $B : H_0^{1,p}(\Omega, X) \cap L_\mu^p(\Omega) \to (H_0^{1,p}(\Omega, X) \cap L_\mu^p(\Omega))'$ be the operator defined as

$$\langle Bz, v\rangle = \langle A(z+\psi), v\rangle + \int_\Omega |z+\psi|^{p-2}(z+\psi)\, d\mu$$

where $v \in H_0^{1,p}(\Omega, X) \cap L_\mu^p(\Omega)$. The operator B is monotone, continuous and coercive; then there exists a solution $z \in H^{1,p}(\Omega, X) \cap L_\mu^p(\Omega)$ of the problem $Bz = f$ and $u = z + \psi$ is a solution of (3.2).

We take $v = (u - \psi)$ as test function in (3.2); we obtain

$$\langle Au, u-\psi\rangle + \int_\Omega |u|^{p-2} u(u-\psi)\, d\mu \leq \langle f, u-\psi\rangle$$

then

$$C_0 \int_\Omega |Xu|^p\, dx + \int_\Omega |u|^p\, d\mu \leq$$

$$\leq \|f\|^q_{H^{-1,q}(\Omega,X)} \|u-\psi\|^q_{H_0^{1,p}(\Omega,X)} + \left(\int_\Omega |Xu|^p\, dx\right)^{1/q} \left(\int_\Omega |X\psi|^p\, dx\right)^{1/p} +$$

$$+ C_1 \left(\int_\Omega |u|^p\, d\mu\right)^{1/q} \left(\int_\Omega |\psi|^p\, d\mu\right)^{1/p}$$

which implies (3.3) by Young's inequality. ■

The following lemma will be used to prove the continuous dependence on f of the solution of (3.2).

THEOREM 3.2: *Let* $\mu \in \mathfrak{M}_0^p(\Omega, X)$; *let* $u_1, u_2 \in H^{1,p}(\Omega, X) \cap L_\mu^p(\Omega)$, *let* $\phi \in H^{1,p}(\Omega, X) \cap L^\infty(\Omega)$, $\phi \geq 0$ *q.e. in* Ω.

If $2 \leq p < +\infty$

$$C_0 \int_\Omega |Xu_1 - Xu_2|^p \phi\, dx + 2^{2-p} \int_\Omega |u_1 - u_2|^p \phi\, d\mu \leq \tag{3.4}$$

$$\leq \int_\Omega (|Xu_1|^{p-2} Xu_1 - |Xu_2|^{p-2} Xu_2)\, \phi\, dx + \int_\Omega (|u_1|^{p-2} u_1 - |u_2|^{p-2} u_2)\, \phi\, d\mu.$$

If $1 < p < 2$

$$(3.5)\qquad C_0\left(\int_\Omega |Xu_1 - Xu_2|^p \phi\, dx\right)^{2/p} + \leqslant$$

$$\leqslant K_1(u_1, u_2, \phi)\int_\Omega (|Xu_1|^{p-2}Xu_1 - |Xu_2|^{p-2}Xu_2)(Xu_1 - Xu_2)\,\phi\, dx$$

$$(3.5')\qquad \int_\Omega (|u_1 - u_2|^p \phi\, d\mu)^{2/p} \leqslant$$

$$\leqslant K_2(u_1, u_2, \phi)\int_\Omega (|u_1|^{p-2}u_1 - |u_2|^{p-2})(u_1 - u_2)\,\phi\, d\mu\,.$$

where

$$(3.6)\qquad K_1(u_1, u_2, \phi) = 2\left(\int_\Omega |Xu_1|^p \phi\, dx + \int_\Omega |Xu_2|^p \phi\, dx\right)^{\frac{2-p}{p}}$$

$$K_2(u_1, u_2, \phi) = 2\left(\int_\Omega |u_1|^p \phi\, d\mu + \int_\Omega |u_2|^p \phi\, d\mu\right)^{(2-p)/p}$$

PROOF: The proof is the same as in [17] and is founded on inequalities (2.4)-(2.7) and on the Hölder inequality. ■

The following result shows that the continuous dependence on f of the solutions of (3.2) is uniform with respect to μ.

THEOREM 3.3: *Let* $\mu \in \mathcal{M}_0^p(\Omega, X)$; *let* $f_1, f_2 \in H^{-1,q}(\Omega, X)$, $\frac{1}{p} + \frac{1}{q} = 1$, *let* u_1, u_2 *be the solutions of* (3.2) *corresponding to* $f = f_1$ *and* $f = f_2$.

If $p \geqslant 2$, *then*

$$(3.7)\qquad \|u_1 - u_2\|^p_{H_0^{1,p}(\Omega, X)} + \|u_1 - u_2\|^p_{L^p_\mu(\Omega)} \leqslant C\|f_1 - f_2\|^q_{H^{-1,q}(\Omega, X)}$$

If $1 < p < 2$, *then*

$$(3.8)\qquad \|u_1 - u_2\|^p_{H_0^{1,p}(\Omega, X)} + \|u_1 - u_2\|^p_{L^p_\mu(\Omega)} \leqslant C\,\Gamma(f_1, f_2, \psi)\|f_1 - f_2\|^2_{H^{-1,q}(\Omega, X)}$$

where C *is a structural constant and*

$$\Gamma(f_1, f_2, \psi) = \left(\|f_1\|^q_{H^{-1,q}(\Omega, X)} + \|f_2\|^q_{H^{-1,q}(\Omega, X)} + \int_\Omega |X\psi|^p\, dx + \int_\Omega |\psi|^p\, d\mu\right)^{\frac{2(2-p)}{p}}$$

PROOF: Let $p \geqslant 2$; we use $v = u_1 - u_2$ as test function in (3.2) and we obtain

$$(3.9) \qquad \langle Au_1 - Au_2, u_1 - u_2 \rangle + \int_\Omega (|u_1|^{p-2} u_1 - |u_2|^{p-2} u_2)(u_1 - u_2)\, d\mu =$$

$$= \langle f_1 - f_2, u_1 - u_2 \rangle \leqslant \|f_1 - f_2\|_{H^{-1,q}(\Omega, X)} \|u_1 - u_2\|_{H_0^{1,p}(\Omega, X)}$$

If $p \geqslant 2$ the result follows from (2.4) and (2.5) by a Young inequality.

Let us consider the case $1 < p < 2$. From (3.5) and (3.9) we obtain by (2.6), (2.7) and Theorem 3.2.

$$(3.10) \qquad C_0 \|u_1 - u_2\|^2_{H_0^{1,p}(\Omega, X)} + \|u_1 - u_2\|^p_{L^2_\mu(\Omega)} \leqslant$$

$$\leqslant K(u_1, u_2, 1) \|f_1 - f_2\|_{H^{-1,q}(\Omega, X)} \|u_1 - u_2\|_{H_0^{1,p}(\Omega, X)}$$

where the constant $K(u_1, u_2, 1)$ is given by the Theorem 3.2.

By (3.3) we have $K(u_1, u_2, 1)^2 \leqslant C\Gamma(f_1, f_2, \psi)$, so (3.8) follows from (3.10). ■

A connection between classical Dirichlet problems on open subsets of Ω and relaxed Dirichlet problems of type (3.1) is given by the following remark.

REMARK 3.4: If U is an open subset of Ω and v is a function in $H_0^{1,p}(\Omega, X)$ such that $v=0$ q.e. in $\Omega \backslash U$ then the restriction of v to U belongs to $H_0^{1,p}(U, X)$, [23].

Conversely if we extend a function $v \in H_0^{1,p}(U, X)$ by setting $v = 0$ in $\Omega \backslash U$, then v is p-quasi-continuous and belongs to $H_0^{1,p}(\Omega, X)$. Therefore if μ is the measure defined by

$$\mu(B) = 0 \text{ if } \operatorname{cap}_p(B \backslash U, \Omega, X) = 0; \qquad \mu(B) = +\infty \text{ otherwise}$$

where B is a Borel set, then $u \in H_0^{1,p}(\Omega, X)$ is a solution of the problem (3.1) if and only if the restriction of u to U is the solution of the classical boundary value problem

$$u \in H_0^{1,p}(U, X), \qquad Au = f \text{ in } \mathcal{D}'(U)$$

and in addition $u = 0$ q.e. on $\Omega \backslash U$.

Estimates for the solutions.

The solutions of relaxed Dirichlet problems satisfy the comparison principles given in the following propositions.

Proposition 3.5: *Let* $\mu \in \mathcal{M}_0^p(\Omega, X)$; *let* $f \in H^{-1,q}(\Omega, X)$, $\frac{1}{p} + \frac{1}{q} = 1$, *and let* u *be the solution of* (3.1). *If* $f \geq 0$ *in* Ω *then* $u \geq 0$ *q.e. in* Ω

Proof: Let $v = u \wedge 0$ then $v \in H_0^{1,p}(\Omega, X) \cap L_\mu^p(\Omega)$.
Using v as test function in (3.1) we obtain $\|v\|_{H_0^{1,p}(\Omega, X)} = 0$, then $v = 0$ q.e. in Ω. ■

Proposition 3.6: *Let* $\mu_1, \mu_2 \in \mathcal{M}_0^p(\Omega, X)$; *let* $f_1, f_2 \in H^{-1,q}(\Omega, X)$, $\frac{1}{p} + \frac{1}{q} = 1$, *and let* u_1, u_2 *be the solution of* (3.1) *corresponding to* f_1, μ_1 *and* f_2, μ_2. *Assume* $0 \leq \leq f_1 \leq f_2$ *and* $\mu_2 \leq \mu_1$ *in* Ω. *Then* $u_1 \leq u_2$ *q.e. in* Ω

Proof: By Proposition 3.5. $u_2 \geq 0$ q.e. in Ω.
Let $v = (u_1 - u_2)^+$. Since $0 \leq v \leq u_1^+$ and $\mu_2 \leq \mu_1$ we have $v \in L_{\mu_1}^p(\Omega) \subset L_{\mu_2}^p(\Omega)$. Then we can use v as test function in both the relaxed Dirichlet problems and we obtain

$$\langle Au_1 - Au_2, v\rangle + \int_\Omega (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)\, v\, d\mu = \langle f_1 - f_2, v\rangle \leq 0$$

which implies

$$\int_{u_1 > u_2} (|Xu_1|^{p-2}Xu_1 - |Xu_2|^{p-2}Xu_2)(Xu_1 - Xu_2)\, dx \leq 0$$

Then $\|(u_1 - u_2)^+\|_{H_0^{1,p}(\Omega, X)} = 0$ q.e. in Ω, so $u_1 \leq u_2$ q.e. in Ω. ■

Proposition 3.7: *Let* $\mu_1, \mu_2 \in \mathcal{M}_0^p(\Omega, X)$; *let* $f_1, f_2 \in H^{-1,q}(\Omega, X)$, $\frac{1}{p} + \frac{1}{q} = 1$, *be Radon measures and let* u_1, u_2 *be the solution of* (3.1) *corresponding to* f_1, μ_1 *and* f_2, μ_2. *If* $|f_1| \leq f_2$ *and* $\mu_2 \leq \mu_1$, *then* $|u_1| \leq u_2$.

Proof: By Proposition 3.6 we have $u_1 \leq u_2$ q.e. in Ω. We observe that the function $-u_1$ is the solution of (3.1) corresponding to $-f_1$ and μ_1; so by Proposition 3.6 we obtain also $-u_1 \leq u_2$ q.e. in Ω.

Estimates involving auxiliary Radon measures.

We consider now some further estimates for the gradient of the solution u of (3.2). We begin by proving that if $f \in L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$, then the solutions of (3.2) are actually solutions in distribution sense of a new equation involving a Radon measure λ, which depends on u, μ, f, and whose variation on compact sets can be estimated in terms of $\|f\|_{L^q(\Omega)}$ and $\|Xu\|_{L^p(\Omega)}$.

Proposition 3.8: *Let* $\mu \in \mathcal{M}_0^p(\Omega, X)$; *let* $f \in L^q(\Omega)$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, *and let* u *be the solution of* (3.2) *for some* $\psi \in H^{1,p}(\Omega, X) \cap L^p_\mu(\Omega)$. *Let* $\lambda, \lambda_1, \lambda_2$ *be elements of* $H^{-1,q}(\Omega, X)$ *defined by* $Au + \lambda = f$, $Au^+ + \lambda_1 = f^+$, $A(-u^-) - \lambda_2 = -f^-$. *Then* $\lambda, \lambda_1, \lambda_2$ *are Radon measures,* $\lambda_1, \lambda_2 \geqslant 0$, $\lambda = \lambda_1 - \lambda_2$, $|\lambda| \leqslant \lambda_1 + \lambda_2$. *Moreover for every compact set* K *we have*

$$|\lambda(K)| \leqslant \mathrm{cap}_p(K, \Omega; X)^{1/p} (2c_1 \|Xu\|^{p-1}_{L^p(\Omega)} + c_{p,\Omega} \|f\|_{L^q(\Omega)}) \tag{3.11}$$

where c_1 *and* $c_{p,\Omega}$ *are structural constants.*

Proof: Let $v \in H_0^{1,p}(\Omega, X)$, $v \geqslant 0$ q.e. in Ω and let $v_n = \left(\dfrac{1}{n} v\right) \wedge u^+$. Then $v_n \geqslant 0$ q.e., $v_n \in H_0^{1,p}(\Omega, X) \cap L^p_\mu(\Omega)$. As $|u|^{p-2} u v_n \geqslant 0$ q.e. in Ω and $f v_n \leqslant f^+ v_n$ a.e. in Ω, by taking v_n as test function in (3.2) we obtain

$$\langle Au, v_n \rangle \leqslant \int_\Omega f^+ v_n \, dx \leqslant \frac{1}{n} \int_\Omega f^+ v \, dx .$$

Since $Xv_n = \dfrac{1}{n} Xv$ a.e. in $\{v < nu^+\}$ and $Xv_n = Xu^+$ a.e. in $\{v \geqslant nu^+\}$ we obtain

$$\frac{1}{n} \int_{\{v < nu^+\}} |Xu|^{p-2} Xu Xv \, dx + \int_{\{v \geqslant nu^+\}} |Xu^+|^p \, dx \leqslant \frac{1}{n} \int_\Omega f^+ v \, dx$$

so

$$\int_{\{v < nu^+\}} |Xu|^{p-2} Xu Xv \, dx \leqslant \int_\Omega f^+ v \, dx .$$

Taking the limit as $n \to +\infty$ we obtain

$$\int_{\{u^+ > 0\}} |Xu|^{p-2} Xu Xv \, dx \leqslant \int_\Omega f^+ v \, dx .$$

Then

$$\int_\Omega |Xu|^{p-2} Xu Xv \, dx \leqslant \int_\Omega f^+ v \, dx$$

for every $v \in H_0^{1,p}(\Omega, X)$, $v \geqslant 0$ q.e. in Ω. This implies $\lambda_1 \geqslant 0$ so λ_1 is a Radon measure.

In a similar way we deduce that also λ_2 is a non-negative Radon measure, hence $\lambda = \lambda_1 - \lambda_2$ is also a Radon measure and $|\lambda| \leqslant \lambda_1 + \lambda_2$.

We have

$$\|\lambda_1\|_{H_0^{-1,q}(\Omega,X)} \leqslant c_1 \|Xu^+\|_{L^p(\Omega)} + c_{p,\Omega}\|f^+\|_{L^q(\Omega)}.$$

The same estimate holds also for λ_2. To prove (3.11) for every $\varepsilon > 0$ we fix a function z in $H_0^{1,p}(\Omega, X)$ such that $z \geqslant 0$ q.e. in Ω, $z \geqslant 1$ q.e. in a neighborhood of K and $\|z\|^p_{H_0^{1,p}(\Omega,X)} \leqslant \mathrm{cap}_p(K, \Omega; X) + \varepsilon$. Then

$$|\lambda|(K) \leqslant \int_\Omega z\, d\lambda_1 + \int_\Omega z\, d\lambda_2 \leqslant \|z\|^p_{H_0^{1,p}(\Omega,X)}(\|\lambda_1\|_{H_0^{-1,q}(\Omega,X)} + \|\lambda_2\|_{H_0^{-1,q}(\Omega,X)}) \leqslant$$

$$\leqslant 2(\mathrm{cap}_p(K, \Omega; X) + \varepsilon)^{1/p}(c_1\|Xu\|^{p-1}_{L^p(\Omega)} + c_{p,\Omega}\|f\|_{L^q(\Omega)})$$

Taking the limit as $\varepsilon \to 0$ we obtain (3.11). ●

REMARK 3.9: Under the assumptions of Proposition 3.8, if $f \geqslant 0$ then $u = u^+$ and $\lambda = \lambda_1$. Therefore in this case $\lambda \geqslant 0$, hence $Au \leqslant f$ in Ω in the sense of $H^{-1,q}(\Omega, X)$.

The following theorem together with Proposition 3.8 will be used in the proof of the main result (Theorem 3.11) of this section.

THEOREM 3.10: *Let g_n be a sequence in $H^{-1,q}(\Omega, X)$, $\frac{1}{p} + \frac{1}{q} = 1$, let λ_n be a sequence of Radon measures and for every n let $u_n \in H_0^{1,p}(\Omega, X)$ be a solution of the equation*

$$Au_n = g_n + \lambda_n$$

Assume that u_n converges weakly in $H_0^{1,p}(\Omega, X)$ to some function u, g_n converges strongly in $H^{-1,q}(\Omega, X)$ and λ_n is bounded in the space of Radon measures (i.e. for every compact set $K \subset \Omega$ there exists a constant C_K such that $|\lambda_n(K)| \leqslant C_K$).

Then u_n converges strongly in $H^{-1,r}(\Omega, X)$, $1 < r < p$, $|Xu_n|^{p-2}Xu_n$ converges to $|Xu|^{p-2}Xu$ weakly in $L^q(\Omega)$ and strongly in $L^s(\Omega)$, $1 < s < q$.

PROOF: The sequence u_n converges to u strongly in $L^r(\Omega)$. Moreover in the Appendix we prove that Xu_n converges to Xu weakly in $L^p(\Omega)$ and strongly in $L^r(\Omega)$, then u_n converges to u in $H^{1,r}(\Omega, X)$. Let us fix a subsequence still denoted by u_n such that u_n converges to u and Xu_n converges to Xu pointwise a.e., then $|Xu_n|^{p-2}Xu_n$ converges to $|Xu|^{p-2}Xu$ pointwise a.e.. The sequence $|Xu_n|^{p-2}Xu_n$ is bounded in $L^q(\Omega)$. By Vitali's convergence theorem $|Xu_n|^{p-2}Xu_n$ converges to $|Xu|^{p-2}Xu$ strongly in $L^s(\Omega)$, then weakly in $L^q(\Omega)$. ■

As a consequence of Proposition 3.8 and Theorem 3.10 we have the following result:

THEOREM 3.11: *Let g_n be a sequence in $H^{-1,q}(\Omega, X)$, $\frac{1}{p} + \frac{1}{q} = 1$, which converges* *to some g in* $H^{-1,q}(\Omega, X)$, *let* μ_n *be a sequence in* $\mathcal{M}_0^p(\Omega, X)$ *and let* ψ_n *be a sequence bounded in* $H^{1,p}(\Omega, X) \cap L^p_{\mu_n}(\Omega)$ *such that* $\int |\psi_n| d\mu_n \leq M$. *Assume that the solution* u_n *of* (3.2) *corresponding to* $\mu = \mu_n$, $f = g_n$, $\psi = \psi_n$ *converges weakly in* $H^{1,p}(\Omega, X)$ *to some function u. Then* u_n *converges to u in* $H^{1,r}(\Omega, X)$, $1 < r < p$, *and* $|Xu_n|^{p-2} Xu_n$ *converges to* $|Xu|^{p-2}Xu$ *weakly in* $L^q(\Omega)$ *and strongly in* $L^s(\Omega)$, $1 < s < q$.

Proof: For the proof of the result we follow the proof of the analogous result in elliptic framework given in [17].

Given $\varepsilon \in (0, 1)$ we fix a function $h \in L^q(\Omega)$ such that $\|h - g\|H^{-1,q}(\Omega, X) \leq \varepsilon$ and we consider the solution z_n of (3.2) corresponding to $\mu = \mu_n$, $f = h$, $\psi = \psi_n$. By Theorem 3.3 we have

$$\|z_n - u_n\|_{H_0^{1,p}(\Omega, X)} \leq C\|h - g_n\|^{\alpha}_{H^{-1,q}(\Omega, X)} \tag{3.12}$$

where $\alpha = \dfrac{1}{p-1}$ if $p \geq 2$ and $\alpha = 1$ if $1 < p < 2$, while C is a constant depending on structural constants M, and $\sup\limits_n \|g_n\|_{H^{-1,q}(\Omega, X)}$. This implies that z_n is bounded in $H^{1,p}(\Omega, X)$. Therefore, at least after extraction of subsequences, we may assume that z_n converges weakly in $H^{1,p}(\Omega, X)$ to some function z and (3.12) gives

$$\|z - u\|_{H_0^{1,p}(\Omega, X)} \leq C\|h - g\|^{\alpha}_{H^{-1,q}(\Omega, X)} \leq C\varepsilon^{\alpha}. \tag{3.13}$$

By Proposition 3.8 there exists a sequence λ_n of Radon measures in $H^{-1,q}(\Omega, X)$ such that $Az_n + \lambda_n = h$ in Ω.

By (3.11) for every compact set $K \subset \Omega$ the sequence $(|\lambda_n|(K))$ is bounded. Therefore Theorem 3.10 implies that z_n converges strongly to z in $H^{1,r}(\Omega, X)$. Using Poincaré's and Hölder's inequality we obtain

$$\|u_n - u\|_{H^{1,r}(\Omega, X)} \leq C_{p,\Omega}(\|u_n - z_n\|_{H_0^{1,p}(\Omega, X)} + \|u - z\|_{H_0^{1,p}(\Omega, X)}) + \|z_n - z\|_{H^{1,r}(\Omega, X)}$$

where $C_{p,\Omega}$ ia constant depending on structural constants and Ω. The above inequality together with (3.12)and (3.13) gives

$$\limsup_{n \to +\infty} \|u_n - u\|_{H^{1,r}(\Omega, X)} \leq 2C_{p,\omega}\varepsilon^{\alpha}.$$

As ε is arbitrary we have that u_n converges to u in $H^{1,r}(\Omega, X)$; then we can easily prove that $|Xu_n|^{p-2}Xu_n$ converges to $|Xu|^{p-2}Xu$ weakly in $L^q(\Omega)$ and strongly in $L^s(\Omega)$. ■

4. - Corrector result

Definition of the corrector.

Let μ_n be a sequence in $\mathcal{M}_0^p(\Omega, X)$ and let $f \in L^{\infty}(\Omega)$ Let us consider u_n as the solution of the problem (3.1) with $\mu = \mu_n$. By the estimate (3.3) u_n is bounded in

$H_0^{1,p}(\Omega, X)$, thus we may assume that u_n converges weakly in $H_0^{1,p}(\Omega, X)$ to some function u. By Theorem 3.11 Xu_n converges to Xu in $H^{1,r}(\Omega, X)$, $1 < r < p$. We may inprove the convergence of Xu_n and obtain the convergence in $L^p(\Omega)$ by means of a corrector; we define a Borel function $P_n \colon \Omega \to R^m$ depending on the sequence μ_n but independent of f, u, u_n. such that if R_n is defined by

$$Xu_n = Xu + uP_n + R_n \tag{4.1}$$

then the sequence R_n converges to 0 strongly in $L^p(\Omega)$.

In order to construct P_n, let us consider the solution w_n of the problem (3.1) with $\mu = \mu_n$ and $f = 1$. By the estimate (3.3) w_n is bounded in $H_0^{1,p}(\Omega, X)$, then we may assume that w_n converges weakly in $H_0^{1,p}(\Omega, X)$ to some function w. Then we define the Borel function $P_n \colon \Omega \to R^m$

$$P_n(x) = \frac{Xw_n - Xw}{w} \ \text{ if } w(x) > 0, \qquad P_n(x) = 0 \ \text{ if } w(x) = 0 \tag{4.2}$$

We are now in position to state the main theorem of this section:

THEOREM 4.1: *Let μ_n be a sequence of measures in $\mathcal{M}_0^p(\Omega, X)$. Let u_n and w_n the solution of problem* (3.1) *with $\mu = \mu_n$ and $\mu = \mu_n$, $f = 1$. Assume that u_n and w_n converge weakly in $H_0^{1,p}(\Omega, X)$ to some function u and w and define P_n and R_n by* (4.1) *and* (4.2). *Then R_n converges to* 0 *in $L^p(\Omega)$.*

REMARK 4.2: Let w_0 be the unique function in $H_0^{1,p}(\Omega, X)$ such that $Aw_0 = 1$ in Ω. By The comparison principle in Proposition 3.7 and the homogeneity of A we have $|u_n| \leqslant Cw_n \leqslant Cw_0$ q.e. in Ω with $C = \|f\|_{L^\infty(\Omega)}^{1/(p-1)}$, hence $|u| \leqslant Cw \leqslant Cw_0$ q.e. in Ω. As $w_0 \in L^\infty(\Omega)$ the sequences u_n and w_n are bounded in $L^\infty(\Omega)$.

REMARK 4.3: Before proving Theorem 4.1 let us observe that if $f \in L^\infty(\Omega)$ the sequence R_n defined by (4.1) (4.2) converges to 0 weakly in $L^p(\Omega)$ and strongly in $L^r(\Omega)$, $1 < r < p$. Indeed $\frac{u}{w} \in L^\infty(\{w > 0\})$ by Remark 4.2 and

$$R_n = Xu_n - Xu \ \text{ in } \{w = 0\} \tag{4.3}$$

$$R_n = (Xu_n - Xu) + \frac{u}{w}(Xw_n - Xw) \ \text{ in } \{w > 0\} \tag{4.4}$$

while $(Xu_n - Xu)$ and $(Xw_n - Xw)$ converge to 0 weakly in $L^p(\Omega)$ and strongly in $L^r(\Omega)$, $1 < r < p$.

The corrector result of Theorem 4.1 is formally equivalent to the strong convergence of $\left(u_n - \frac{uw_n}{w}\right)$ to 0 in $H^{1,p}(\Omega, X)$. This assertion, which is only formal since w may be 0 on a set of positive Lebesgue measure, becomes correct in $H^{1,p}(U, X)$ if U is an open subset of Ω where $w \geqslant \varepsilon > 0$.

Preliminary results.

To prove Theorem 4.1 we use the following lemmas:

LEMMA 4.4: *Assume that the conditions in Theorem* 4.1 *hold. For every* $\varepsilon > 0$ *define* $U_\varepsilon = \{w > \varepsilon\} \cup \{|u| > \varepsilon w\}$. *Then for every* $\varepsilon > 0$ *the functions* $\frac{uw_n}{w \vee \varepsilon}$ *belong to* $H_0^{1,p}(\Omega, X) \cap L^p_{\mu_n}(\Omega)$ *and one has*

$$\lim_{n \to +\infty} \left(\int_{U_\varepsilon} \left| Xu_n - X\left(\frac{uw_n}{w \vee \varepsilon} \right) \right|^p dx + \int_{U_\varepsilon} \left| u_n - \frac{uw_n}{w \vee \varepsilon} \right|^p d\mu_n \right) = 0 \tag{4.5}$$

PROOF: Define for every $\varepsilon > 0$ the functions

$$u_n^\varepsilon = \frac{uw_n}{w \vee \varepsilon}, \qquad r_n^\varepsilon = u_n - u_n^\varepsilon$$

First step. We will prove that u_n^ε, $r_n^\varepsilon \in H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega) \cap L^p_{\mu_n}(\Omega)$ and investigate their convergence as $n \to +\infty$ for $\varepsilon > 0$ fixed.

We observe that the functions u and $\frac{1}{w \vee \varepsilon}$ are in $H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$ and that the sequences u_n and w_n are bounded in $L^\infty(\Omega)$ (see Remark 4.2) and converge to u and w weakly in $H_0^{1,p}(\Omega, X)$. We recall that $H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega) \subset L^\infty_\mu(\Omega)$ for every $\mu \in \mathcal{M}_0^p(\Omega, X)$ so $\frac{u}{w \vee \varepsilon} \in L^\infty_\mu(\Omega)$. The functions u_n^ε and r_n^ε are bounded in $L^\infty(\Omega)$ and converge to $\frac{u}{w \vee \varepsilon}$ and $u - \frac{u}{w \vee \varepsilon}$ weakly in $H_0^{1,p}(\Omega, X)$. By Theorem 3.11 u_n and w_n converge to u and w strongly in $H^{1,r}(\Omega, X)$, $1 < r < p$; so u_n^ε converges to $\frac{uw}{w \vee \varepsilon}$ strongly in $H^{1,r}(\Omega, X)$. At least after extraction of subsequences we have that u_n, w_n, Xu_n, Xw_n, Xu_n^ε converge a.e. to u, w, Xu, Xw, $X\left(\frac{uw}{w \vee \varepsilon}\right)$; then $|Xu_n|^{p-2} Xu_n$, $|Xw_n|^{p-2} Xw_n$, $|Xu_n^\varepsilon|^{p-2} Xu_n^\varepsilon$ converge to $|Xu|^{p-2} Xu$, $|Xw|^{p-2} Xw$, $\left| X\left(\frac{uw}{w \vee \varepsilon}\right) \right|^{p-2} X\left(\frac{uw}{w \vee \varepsilon}\right)$ weakly in $L^q(\Omega)$ and a.e. in Ω. As $u - \frac{uw}{w \vee \varepsilon} = 0$ a.e. in U_ε, we obtain that r_n^ε converges to 0 strongly in $L^p(U_\varepsilon)$ and $|Xu_n^\varepsilon|^{p-2} Xu_n^\varepsilon$ converges to $|Xu|^{p-2} Xu$ weakly in $L^q(U_\varepsilon)$ Consider now a Lipschitz function Φ_ε defined by $\Phi_\varepsilon(t) = 0$ for $t \leqslant \varepsilon$, $\Phi_\varepsilon(t) = \frac{t}{\varepsilon} - 1$ for $\varepsilon \leqslant t \leqslant 2\varepsilon$, $\Phi_\varepsilon(t) = 1$ for $t \geqslant 2\varepsilon$; we define $\phi = \Phi_\varepsilon(w)\, \Phi_\varepsilon\left(\frac{|u|}{w \vee \varepsilon}\right)$. We have $\phi \in H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$, $0 \leqslant \phi \leqslant 1$ in Ω q.e., $\phi = 1$ in $U_{2\varepsilon}$, $\phi = 0$ in $\Omega \setminus U_\varepsilon$. By the previous remarks the sequence $r_n^\varepsilon \phi$ converges to 0 weakly in $H_0^{1,p}(\Omega, X)$ and strongly in $L^p(\Omega)$.

Second step. We define

$$\mathcal{E}_n^\varepsilon = \int_\Omega (|Xu_n|^{p-2} Xu_n - |Xu_n^\varepsilon|^{p-2} Xu_n^\varepsilon)\, Xr_n^\varepsilon \phi\, dx +$$

$$+ \int_\Omega (|u_n|^{p-2} u_n - |u_n^\varepsilon|^{p-2} u_n^\varepsilon)\, r_n^\varepsilon \phi\, d\mu_n .$$

In this step we prove that for ε fixed we have

$$\lim_{n\to+\infty} \mathcal{E}_n^\varepsilon = 0 .$$

We write $\mathcal{E}_n^\varepsilon$ as

$$(4.6)\qquad \mathcal{E}_n^\varepsilon = \int_\Omega (|Xu_n|^{p-2} Xu_n - |Xu_n^\varepsilon|^{p-2} Xu_n^\varepsilon)\, X(r_n^\varepsilon \phi)\, dx +$$

$$+ \int_\Omega (|u_n|^{p-2} u_n - |u_n^\varepsilon|^{p-2} u_n^\varepsilon)\, r_n^\varepsilon \phi\, d\mu_n - \int_{U_\varepsilon} (|Xu_n|^{p-2} Xu_n - |Xu_n^\varepsilon|^{p-2} Xu_n^\varepsilon)\, X\phi\, r_n^\varepsilon\, dx =$$

$$= \int_\Omega |Xu_n|^{p-2} Xu_n X(r_n^\varepsilon \phi)\, dx + \int_\Omega |u_n|^{p-2} u_n r_n^\varepsilon \phi\, d\mu_n -$$

$$- \int_\Omega \left| \frac{u}{w \vee \varepsilon} Xw_n \right|^{p-2} \frac{u}{w \vee \varepsilon} Xw_n X \left(r_n^\varepsilon \phi\, dx - \int_\Omega |u_n^\varepsilon|^{p-2} u_n^\varepsilon \right) r_n^\varepsilon \phi\, d\mu_n +$$

$$+ \int_{U_\varepsilon} \left(\left| \frac{u}{w \vee \varepsilon} Xw_n \right|^{p-2} \frac{u}{w \vee \varepsilon} Xw_n - |Xu_n^\varepsilon|^{p-2} Xu_n^\varepsilon \right) X(\phi r_n^\varepsilon)\, dx -$$

$$- \int_{U_\varepsilon} (|Xu_n|^{p-2} Xu_n - |Xu_n^\varepsilon|^{p-2} Xu_n^\varepsilon)\, X\phi\, r_n^\varepsilon\, dx .$$

We have $\phi = 0$ q.e. in $\Omega \backslash U_\varepsilon$, then the function $z = \left| \frac{u}{w \vee \varepsilon} \right|^{p-2} \frac{u}{w \vee \varepsilon} \phi$ belongs to $H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$ (in the case $1 < p < 2$ we have to take into account that $\phi = 0$ if $\frac{u}{w \vee \varepsilon} \leqslant \varepsilon$).

We have

$$- \int_\Omega \left| \frac{u}{w \vee \varepsilon} Xw_n \right|^{p-2} \frac{u}{w \vee \varepsilon} Xw_n X(r_n^\varepsilon \phi)\, dx =$$

$$- \int_\Omega |Xw_n|^{p-2} Xw_n \left| \frac{u}{w \vee \varepsilon} \right|^{p-2} \frac{u}{w \vee \varepsilon} X(\phi r_n^\varepsilon)\, dx =$$

$$- \int_\Omega |Xw_n|^{p-2} Xw_n X \left(\left| \frac{u}{w \vee \varepsilon} \right|^{p-2} \frac{u}{w \vee \varepsilon} \phi r_n^\varepsilon \right) dx +$$

$$+ (p-1) \int_\Omega |Xw_n|^{p-2} Xw_n X \left(\frac{u}{w \vee \varepsilon} \right) \left| \frac{u}{w \vee \varepsilon} \right|^{p-2} \phi r_n^\varepsilon\, dx .$$

We have $w_n \dfrac{u}{w \vee \varepsilon} = u_n^\varepsilon$, taking as test function $v = \left| \dfrac{u}{w \vee \varepsilon} \right|^{p-2} \dfrac{u}{w \vee \varepsilon} \phi r_n^\varepsilon$ in the equation defining w_n we obtain

$$-\int_\Omega |Xw_n|^{p-2} Xw_n X\left(\left| \frac{u}{w \vee \varepsilon} \right|^{p-2} \frac{u}{w \vee \varepsilon} \phi r_n^\varepsilon\right) dx -$$

$$- \int_\Omega |u_n^\varepsilon|^{p-2} u_n^\varepsilon \phi r_n^\varepsilon \, d\mu_n = \int_{U_\varepsilon} \left| \frac{u}{w \vee \varepsilon} \right|^{p-2} \frac{u}{w \vee \varepsilon} \phi r_n^\varepsilon \, dx .$$

Taking $v = \phi r_n^\varepsilon$ as test function in the equation defining u_n from (4.6) we obtain

$$\mathcal{E}_n^\varepsilon = \int_{U_\varepsilon} f \phi r_n^\varepsilon \, dx - \int_{U_\varepsilon} \left| \frac{u}{w \vee \varepsilon} \right|^{p-2} \frac{u}{w \vee \varepsilon} \phi r_n^\varepsilon \, dx +$$

$$+ (p-1) \int_{U_\varepsilon} |Xw_n|^{p-2} Xw_n X\left(\frac{u}{w \vee \varepsilon}\right) \left| \frac{u}{w \vee \varepsilon} \right|^{p-2} \phi r_n^\varepsilon \, dx +$$

$$+ \int_{U_\varepsilon} \left(\left| \frac{u}{w \vee \varepsilon} Xw_n \right|^{p-2} \left(\frac{u}{w \vee \varepsilon} Xw_n \right) - |Xu_n^\varepsilon|^{p-2} Xu_n^\varepsilon \right) X(\phi r_n^\varepsilon) \, dx -$$

$$- \int_{U_\varepsilon} (|Xu_n|^{p-2} Xu_n - |Xu_n^\varepsilon|^{p-2} Xu_n^\varepsilon) \, X\phi r_n^\varepsilon \, dx =$$

$$= \mathfrak{I}_n^1 - \mathfrak{I}_n^2 + \mathfrak{I}_n^3 + \mathfrak{I}_n^4 - \mathfrak{I}_n^5 .$$

Since $\dfrac{u}{w \vee \varepsilon} \in L^\infty(U_\varepsilon)$, r_n^ε is bounded in $L^\infty(\Omega)$ and converges strongly to 0 in $L^p(U_\varepsilon)$, while the sequences $|Xw_n|^{p-2} Xw_n$, $|Xu_n|^{p-2} Xu_n$, $|Xu_n^\varepsilon|^{p-2} Xu_n^\varepsilon$ converge weakly in $L^q(U_\varepsilon)$ it follows that $\mathfrak{I}_n^1, \mathfrak{I}_n^2, \mathfrak{I}_n^3, \mathfrak{I}_n^5$ converge to 0. To conclude the proof of our result it is enough to show that

$$\lim_{n \to +\infty} \mathfrak{I}_n^4 = 0 \tag{4.7}$$

Since Xw_n and Xu_n^ε converge to Xw and Xu a.e. in U_ε it follows that

$$\lim_{n \to +\infty} \left(\left| \frac{u}{w \vee \varepsilon} Xw_n \right|^{p-2} \left(\frac{u}{w \vee \varepsilon} Xw_n \right) - |Xu_n^\varepsilon|^{p-2} Xu_n^\varepsilon \right) = \tag{4.8}$$

$$= \left(\left| \frac{u}{w \vee \varepsilon} Xw \right|^{p-2} \left(\frac{u}{w \vee \varepsilon} Xw \right) - |Xu|^{p-2} Xu \right)$$

a.e. in U_ε. Let us prove that $\left|\left(\left|\frac{u}{w\vee\varepsilon}Xw_n\right|^{p-2}\left(\frac{u}{w\vee\varepsilon}Xw_n\right)-|Xu_n^\varepsilon|^{p-2}Xu_n^\varepsilon\right)\right|^q$ is equi-integrable.

Consider the case $p\geq 2$. Since $\frac{u}{w\vee\varepsilon}\in H_0^{1,p}(\Omega, X)\cap L^\infty(\Omega)$ and w_n is bounded in $L^\infty(\Omega)$ there exists a constant C such that

$$\left|\left(\left|\frac{u}{w\vee\varepsilon}Xw_n\right|^{p-2}\left(\frac{u}{w\vee\varepsilon}Xw_n\right)-|Xu_n^\varepsilon|^{p-2}Xu_n^\varepsilon\right)\right|^q\leqslant \tag{4.9}$$

$$\leqslant C_1^q\left(2\left|\frac{u}{w\vee\varepsilon}Xw_n\right|+\left|w_nX\left(\frac{u}{w\vee\varepsilon}\right)\right|\right)^{q(p-2)}\left|w_nX\left(\frac{u}{w\vee\varepsilon}\right)\right|^q\leqslant$$

$$\leqslant C\left(|Xw_n|^{q(p-2)}\left|X\left(\frac{u}{w\vee\varepsilon}\right)\right|^q+\left|X\left(\frac{u}{w\vee\varepsilon}\right)\right|^p\right)$$

where we use Lagrange's formula.

By Hölder's inequality for every measurable set $E\subset\Omega$

$$\int_E|Xw_n|^{q(p-2)}\left|X\left(\frac{u}{w\vee\varepsilon}\right)\right|^q dx\leqslant\left(\int_E|Xw_n|^p dx\right)^{\frac{p-2}{(p-1)}}\left(\int_E\left|X\left(\frac{u}{w\vee\varepsilon}\right)\right|^p\right)^{\frac{q}{p}}.$$

By (4.9) the equi-integrability is proved.

In the case $1<p<2$ we have

$$\left|\left(\left|\frac{u}{w\vee\varepsilon}Xw_n\right|^{p-2}\left(\frac{u}{w\vee\varepsilon}Xw_n\right)-|Xu_n^\varepsilon|^{p-2}Xu_n^\varepsilon\right)\right|^q\leqslant C_1^q\left|w_nX\left(\frac{u}{w\vee\varepsilon}\right)\right|^p$$

then the sequence in the left hand side is equi-integrable.

By the Dominated Convergence Theorem (4.8) implies that

$$\left(\frac{u}{w\vee\varepsilon}Xw_n\right|^{p-2}\left(\frac{u}{w\vee\varepsilon}Xw_n\right)-|Xu_n^\varepsilon|^{p-2}Xu_n^\varepsilon\Bigg)\to$$

$$\to\left(\frac{u}{w\vee\varepsilon}|Xw|^{p-2}\left(\frac{u}{w\vee\varepsilon}Xw\right)-|Xu|^{p-2}Xu\right)$$

in $L^q(\Omega)$. As $X(r_n^\varepsilon \phi)$ converges to 0 weakly in $L^P(\Omega)$ we obtain (4.7) which implies (4.5).

Third step. If $2 \leq p$ then Lemma 3.2 gives

$$\int_\Omega |Xr_n^\varepsilon|^p \phi \, dx + 2^{2-p} \int_\Omega |r_n^\varepsilon|^p \phi \, d\mu_n \leq \mathcal{E}_n^\varepsilon. \tag{4.10}$$

If $1 < p < 2$, we observe that the sequences $\|u_n\|_{L^p_{\mu_n}(\Omega)}$ and $\|w_n\|_{L^p_{\mu_n}(\Omega)}$ are bounded by the estimate (3.3). Since u and $\dfrac{1}{w \vee \varepsilon}$ belong to $H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$ we conclude that $\|u_n^\varepsilon\|_{L^p_{\mu_n}(\Omega)}$ is bounded too.

Since u_n and u_n^ε are bounded in $H_0^{1,p}(\Omega, X)$ by Lemma 3.2 there exists a constant K such that

$$\int_\Omega |Xr_n^\varepsilon|^p \phi \, dx + 2^{2-p} \int_\Omega |r_n^\varepsilon|^p \phi \, d\mu_n \leq K \mathcal{E}_n^\varepsilon. \tag{4.11}$$

Taking (4.10) and (4.11) into account we obtain from (4.5) that

$$\lim_{n \to +\infty} \left(\int_\Omega |Xr_n^\varepsilon|^p \phi \, dx + 2^{2-p} \int_\Omega |r_n^\varepsilon|^p \phi \, d\mu_n \right) = 0$$

hence

$$\lim_{n \to +\infty} \left(\int_\Omega |Xr_n^\varepsilon|^p \, dx + 2^{2-p} \int_\Omega |r_n^\varepsilon|^p \, d\mu_n \right) = 0. \tag{4.12}$$

As $w \vee 2\varepsilon = w \vee \varepsilon$ q.e. in $U_{2\varepsilon}$ we have $r_n^\varepsilon = u_n - \dfrac{uw_n}{w \vee 2\varepsilon}$ q.e. in $U_{2\varepsilon}$ and $Xr_n^\varepsilon = Xu_n - X\left(\dfrac{uw_n}{w \vee 2\varepsilon}\right)$ a.e. in $U_{2\varepsilon}$. Therefore (4.12) implies (4.5) with ε replaced by 2ε. ■

LEMMA 4.5: *Let $f \in L^\infty(\Omega)$, let u_n be solution of* (3.1) *with $\mu = \mu_n$. For every $\varepsilon > 0$ define $V_\varepsilon = \{w \leq \varepsilon\}$. Then*

$$\lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \left(\int_{V_\varepsilon} |Xu_n|^p \, dx + \int_{V_\varepsilon} |u_n|^p \, d\mu_n \right) = 0. \tag{4.13}$$

PROOF: For every $\varepsilon > 0$ let Φ^ε be the Lipschitz function defined at the end of the first step of the proof of Lemma 4.4 and let $z^\varepsilon \in H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$ be the function defined by $z^\varepsilon = 1 - \Phi^\varepsilon(w)$.

As $z^\varepsilon \geqslant 0$ q.e. in Ω and $z^\varepsilon = 1$ q.e. in V_ε by (3.1) we have

$$\int_{V_\varepsilon} |Xu_n|^p\,dx + \int_{V_\varepsilon} |u_n|^p\,d\mu_n \leqslant \int_\Omega |Xu_n|^p z^\varepsilon\,dx + \int_\Omega |u_n|^p z^\varepsilon\,d\mu_n =$$

$$= \int_\Omega |Xu_n|^{p-2} Xu_n X(u_n z^\varepsilon)\,dx + \int_\Omega |u_n|^p z^\varepsilon\,d\mu_n - \int_\Omega (|Xu_n|^{p-2} Xu_n Xz^\varepsilon)\,u_n\,dx =$$

$$= \int_\Omega f u_n z^\varepsilon\,dx - \int_\Omega (|Xu_n|^{p-2} Xu_n Xz^\varepsilon)\,u_n\,dx.$$

Since u_n converges strongly to u in $L^p(\Omega)$ and is bounded in $L^\infty(\Omega)$ (Remark 4.2) while $|Xu_n|^{p-2}Xu_n$ converges to $|Xu|^{p-2}Xu$ weakly in $L^q(\Omega)$ (Theorem 3.11), we can take the limit of the last two terms as $n \to +\infty$ obtaining

$$\limsup_{n\to+\infty}\left(\int_{V_\varepsilon} |Xu_n|^p\,dx + \int_{V_\varepsilon} |u_n|^p\,d\mu_n\right) \leqslant \tag{4.14}$$

$$\leqslant \int_\Omega f u z^\varepsilon\,dx - \int_\Omega (|Xu|^{p-2} Xu Xz^\varepsilon)\,u\,dx.$$

As z^ε is bounded in $L^\infty(\Omega)$ and converges to the characteristic function of the set $\{w=0\}$, while $u=0$ a.e. in $\{w=0\}$ (Remark 4.2), we have that uz^ε converges to 0 strongly in $L^p(\Omega)$.

In the other hand by Remark 4.2 we have $|u| \leqslant Cw$ q.e. in Ω, then

$$\int_\Omega |u|^p\,|Xz^\varepsilon|^p\,dx \leqslant \frac{C^p}{\varepsilon^p}\int_{\{\varepsilon<w<2\varepsilon\}} w^p\,|Xw|^p\,dx \leqslant (2C)^p \int_{\{\varepsilon<w<2\varepsilon\}} |Xw|^p\,dx$$

so that uXz^ε converges to 0 in $L^p(\Omega)$. Taking the limit in (4.14) as $\varepsilon \to 0$ we obtain (4.13). ■

LEMMA 4.6: *Let $f \in L^\infty(\Omega)$ and u_n be the solution of* (3.1) *for $\mu = \mu_n$. For every $\varepsilon > 0$ define $W_\varepsilon = \{w > \varepsilon\} \cap \{|u| \leqslant \varepsilon w\}$. Then*

$$\lim_{\varepsilon\to 0}\limsup_{n\to+\infty}\left(\int_{W_\varepsilon} |Xu_n|^p\,dx + \int_{W_\varepsilon} |u_n|^p\,d\mu_n\right) = 0 \tag{4.15}$$

PROOF: For every $\varepsilon > 0$ let Φ^ε be the Lipschitz function defined at the end of the first step of the proof of Lemma 4.4. As $\dfrac{u}{w \vee \varepsilon} \in H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$ (Remark 4.2) the function $z^\varepsilon = 1 - \Phi^\varepsilon\left(\dfrac{|u|}{w \vee \varepsilon}\right)$ belongs to $H^{1,p}(\Omega, X) \cap L^\infty(\Omega)$. As $z^\varepsilon \geqslant 0$ q.e. in

Ω and $z^{\varepsilon} = 1$ on W_{ε} by the same computations as in Lemma 4.5 we obtain

$$\left(\int_{W_\varepsilon} |Xu_n|^p \, dx + \int_{W_\varepsilon} |u_n|^p \, d\mu_n \right) \leq \int_{\Omega} fuz^{\varepsilon} \, dx - \int_{\Omega} (|Xu_n|^{p-2} Xu_n Xz^{\varepsilon}) \, u_n \, dx .$$

Since u_n converges strongly to u in $L^p(\Omega)$ and is bounded in $L^{\infty}(\Omega)$ (Remark 4.2) while $|Xu_n|^{p-2} Xu_n$ converges to $|Xu|^{p-2} Xu$ weakly in $L^q(\Omega)$ (Theorem 3.11) we can take the limit of the last two terms as $n \to +\infty$ obtaining

$$\limsup_{n \to +\infty} \left(\int_{W_\varepsilon} |Xu_n|^p \, dx + \int_{W_\varepsilon} |u_n|^p \, d\mu_n \right) \leq \tag{4.16}$$

$$\leq \int_{\Omega} fuz^{\varepsilon} \, dx - \int_{\Omega} (|Xu|^{p-2} XuXz^{\varepsilon}) \, u \, dx .$$

As z_ε is bounded in $L^{\infty}(\Omega)$ and converges to the characteristic function of $\{u = 0\}$, we have that uz^{ε} converges strongly to 0 in $L^p(\Omega)$. Moreover

$$\int_{\Omega} |u|^p \, |Xz^{\varepsilon}|^p \, dx \leq \frac{1}{\varepsilon^p} \int_{\{0 < |u| < 2\varepsilon(w \vee \varepsilon)\}} |u|^p \left| X\left(\frac{|u|}{w \vee \varepsilon} \right) \right|^p dx \leq$$

$$\leq \frac{2^{p-1}}{\varepsilon^p} \int_{\{0 < |u| < 2\varepsilon(w \vee \varepsilon)\}} \left(\left(\frac{|u|}{w \vee \varepsilon} \right)^p |Xu|^p + \left(\frac{|u|}{w \vee \varepsilon} \right)^{2p} |Xw|^p \right) dx \leq$$

$$\leq 2^{p-1} \int_{\{0 < |u| < 2\varepsilon(w \vee \varepsilon)\}} (|Xu|^p + ((2\varepsilon)^{2p} \, |Xw|^p)) \, dx$$

and so uXz^{ε} converges strongly to 0 in $L^p(\Omega)$ as $\varepsilon \to 0$. Therefore (4.15) follows from (4.16) taking the limit as $\varepsilon \to 0$. ■

PROOF OF THEOREM 4.1: Recall that $U_\varepsilon \cup V_\varepsilon \cup W_\varepsilon = \Omega$, then

$$\int_{\Omega} |R_n|^p \, dx = \int_{U_\varepsilon} |R_n|^p \, dx + \int_{V_\varepsilon} |R_n|^p \, dx + \int_{W_\varepsilon} |R_n|^p \, dx$$

Since $r_n = Xu_n - Xu - \frac{u}{w}(Xw_n - Xw)$ in $\{w > 0\}$ we deduce from (4.4) that for $\varepsilon > 0$ fixed

$$\lim_{n \to +\infty} \int_{U_\varepsilon} |R_n|^p \, dx = 0$$

On the other hand we shall prove

$$\limsup_{n\to+\infty} \int_{V_\varepsilon} |R_n|^p\,dx = 0 \tag{4.17}$$

$$\limsup_{n\to+\infty} \int_{W_\varepsilon} |R_n|^p\,dx = 0\,. \tag{4.18}$$

Since $|u| \leqslant Cw$ q.e. in Ω (Remark 4.2) we have $Xu = Xw = 0$ in $\{w = 0\}$.

This fact with Lemma 4.5 (applied to the sequences u_n and w_n) allows us to obtain (4.17) from the previous inequality.

As $|u| \leqslant Cw$ q.e. in Ω (Remark 4.2), we have $|R_n| \leqslant |Xu_n - Xu| + \varepsilon|Xw_n - Xw|$ q.e. in W_ε (Remark 4.3). Therefore

$$4^{1-p}\limsup_{n\to+\infty} \int_{W_\varepsilon} |R_n|^p\,dx \leqslant$$

$$\leqslant \limsup_{n\to+\infty}\left[\int_{W_\varepsilon} |Xu_n|^p\,dx + \int_{W_\varepsilon} |Xu|^p\,dx + \varepsilon^p\left(\int_{W_\varepsilon} |Xw_n|^p\,dx + \int_{W_\varepsilon} |Xw|^p\,dx\right)\right].$$

As the characteristic function of W_ε converges to the characteristic function of $\{w > 0\} \cap \{u = 0\}$ and $Xu = 0$ a.e. on the set $\{u = 0\}$, the previous inequality and the Lemma 4.6 give (4.17), so the proof is concluded. ■

Theorem 4.1 gives the correction of Xu_n to obtain strong convergence. We observe that in general the function $Xu + uP_n$ is not a X-gradient. The following result gives a corrector in $H_0^{1,p}(\Omega, X)$ for the functions u_n.

THEOREM 4.7: *Let μ_n be a sequence of measures in $\mathcal{M}_0^p(\Omega, x)$ and let $f \in L^\infty(\Omega)$.*

Assume that u_n and w_n are the solutions to are the solution of problem (3.1) *corresponding to $\mu = \mu_n$ and to $\mu = \mu_n$, $f = 1$ and that u_n and w_n converge weakly in $H_0^{1,p}(\Omega, X)$ to some function u and w. Define r_n^ε by*

$$u_n = \frac{|u|\,w_n}{w \vee \varepsilon} + r_n^\varepsilon\,.$$

Then

$$\lim_{\varepsilon\to 0}\ \limsup_{n\to+\infty} \|r_n^\varepsilon\|_{H_0^{1,p}(\Omega, X)} = 0$$

PROOF: Since $U_\varepsilon \cup V_\varepsilon \cup W_\varepsilon = \Omega$ we have

$$\int_\Omega |r_n^\varepsilon|^p\,dx = \int_{U_\varepsilon} |r_n^\varepsilon|^p\,dx + \int_{V_\varepsilon} |r_n^\varepsilon|^p\,dx + \int_{W_\varepsilon} |r_n^\varepsilon|^p\,dx \tag{4.19}$$

By the Lemma 4.4 Xr_n^ε converges to 0 strongly in $L^p(U_\varepsilon)$ as $n \to 0$, so we have only to estimate the last two terms in (4.19) As

$$Xr_n^\varepsilon = Xu_n - \frac{u}{w \vee \varepsilon} Xw_n - \frac{w_n}{w \vee \varepsilon} Xu + \frac{uw_n}{(w \vee \varepsilon)^2} X(w \vee \varepsilon)$$

and $|u| \leq Cw$ q.e. in Ω (Remark 4.2) we have

$$4^{1-p}\,|Xr_n^\varepsilon|^p \leq |Xu_n|^p + C^p\,|Xw_n|^p + \left(\frac{w_n}{w \vee \varepsilon}\right)^p |Xu|^p + C^p\left(\frac{w_n}{w \vee \varepsilon}\right)^p |Xw|^p.$$

We observe that w_n is bounded in $L^\infty(\Omega)$ (Remark 4.2) and converges to w weakly in $H_0^{1,p}(\Omega, X)$, then

$$4^{1-p} \limsup_{n \to +\infty} \int_{V_\varepsilon} |Xr_n^\varepsilon|^p\,dx \leq$$

$$\leq \limsup_{n \to +\infty} \left[\left(\int_{V_\varepsilon} |Xu_n|^p\,dx + \int_{V_\varepsilon} |Xu|^p\,dx\right) + C^p\left(\int_{V_\varepsilon} |Xw_n|^p\,dx + \int_{V_\varepsilon} |Xw|^p\,dx\right)\right].$$

Since $|u| \leq Cw$ q.e. in Ω (Remark 4.2) we have $Xu = Xw = 0$ in $\{w = 0\}$.

This fact with Lemma 4.5 (applied to the sequences u_n and w_n) gives

$$\lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \int_{V_\varepsilon} |Xr_n^\varepsilon|^p\,dx = 0$$

Since $w = w \vee \varepsilon$ and $|u| \leq \varepsilon w$ q.e. on W_ε we have

$$4^{1-p}\,|Xr_n^\varepsilon|^p \leq |Xu_n|^p + \varepsilon^p\,|Xw_n|^p + \left(\frac{w_n}{w}\right)^p |Xu|^p + C^p\left(\frac{w_n}{w}\right)^p |Xw|^p$$

q.e. in W_ε and thus

$$4^{1-p} \limsup_{n \to +\infty} \int_{V_\varepsilon} |Xr_n^\varepsilon|^p\,dx \leq$$

$$\leq \limsup_{n \to +\infty} \left[\left(\int_{W_\varepsilon} |Xu_n|^p\,dx + \int_{W_\varepsilon} |Xu|^p\,dx\right) + \varepsilon^p\left(\int_\Omega |Xw_n|^p\,dx + \int_\Omega |Xw|^p\,dx\right)\right].$$

As the characteristic function of W_ε converges to the characteristic function of $\{w>0\} \cap \{u=0\}$ and $Xu=0$ a.e. on the set $\{u=0\}$, the term $\int_{W_\varepsilon} |Xu_n|^p\,dx$ converges to 0 as $\varepsilon \to 0$; then by Lemma 4.6 we have

$$\lim_{\varepsilon\to 0} \limsup_{n\to+\infty} \int_{W_\varepsilon} |Xr_n^\varepsilon|^p\,dx = 0\,.$$

The result then follows. ■

5. - Corrector result

Let μ_n be a sequence of measures in $\mathcal{M}_0^p(\Omega, X)$ and $f \in L^\infty(\Omega)$ Assume that u_n and w_n are the solutions to are the solution of problem (3.1) corresponding to $\mu = \mu_n$ and to $\mu = \mu_n$, $f=1$ and that u_n and w_n converge weakly in $H_0^{1,p}(\Omega, X)$ to some function u and w. In this section we will study the behavior of the following sequences

$$\langle Au_n,\, w_n^\beta \phi\rangle - \left\langle Aw_n,\, \left|\frac{u}{w\vee\varepsilon}\right|^{p-2} \frac{u}{w\vee\varepsilon}\, w_n^\beta \phi\right\rangle \tag{5.1}$$

$$\int_\Omega |u_n|^{p-2} u_n w_n^\beta \phi\, d\mu_n - \int_\Omega \left|\frac{u}{w\vee\varepsilon}\right|^{p-2} \frac{u}{w\vee\varepsilon}\, w_n^{p-1+\beta} \phi\, d\mu_n \tag{5.2}$$

where $\beta \geq (p-1)\vee 1$ and $\phi \in H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$. The estimates will be useful in the proof of the main result of section 6. For $1<p<2$ the function $\left|\frac{u}{w\vee\varepsilon}\right|^{p-2} \frac{u}{w\vee\varepsilon}$ does not belong to $H_0^{1,p}(\Omega, X)$, then the formula (5.1) (5.2) are not correct. We introduce the locally Lipschitz function $\Psi_\varepsilon(t)$ defined by

$$\Psi_\varepsilon(t) = |t|^{p-2} t \ \text{ if } \ |t| > \varepsilon, \qquad \Psi_\varepsilon(t) = |\varepsilon|^{p-2} t \ \text{ if } \ |t| \leq \varepsilon \tag{5.3}$$

and we replace in (5.1) (5.2) $\left|\frac{u}{w\vee\varepsilon}\right|^{p-2} \frac{u}{w\vee\varepsilon}$ by $\Psi_\varepsilon\left(\frac{u}{w\vee\varepsilon}\right)$. We begin with an estimate in the set $U_\varepsilon = \{w>\varepsilon\} \cap \{|u| > \varepsilon w\}$.

Lemma 5.1: *Let μ_n be a sequence of measures in $\mathcal{M}_0^p(\Omega, x)$ and let $f \in L^\infty(\Omega)$.*

Assume that u_n and w_n are the solutions to are the solution of problem (3.1) *corresponding to $\mu = \mu_n$ and to $\mu = \mu_n$, $f=1$ and that u_n and w_n converge weakly in $H_0^{1,p}(\Omega, X)$ to some function u and w. Let $\varepsilon > 0$ and $\beta \geq 1$, define $v_\varepsilon = \Psi_\varepsilon\left(\frac{u}{w\vee\varepsilon}\right) \in H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$. Then the sequence*

$$|Xu_n|^{p-2} Xu_n X(w_n^\beta) - |Xw_n|^{p-2} Xw_n X(v_\varepsilon w_n^\beta)$$

converges weakly in $L^1(U_\varepsilon)$ as $n \to \infty$ to the function

$$|Xu|^{p-2} XuX(w_n^\beta) - |Xw|^{p-2} XwX(v_\varepsilon w^\beta)$$

PROOF: By Theorem 4.1 we have

$$Xu_n = Xu + \frac{u}{w} Xw_n - \frac{u}{w} Xw + R_n \quad \text{a.e. in } U_\varepsilon \tag{5.4}$$

where R_n converges strongly to 0 in $L^p(\Omega)$. Since $v_\varepsilon = \left| \frac{u}{w} \right|^{p-2} \frac{u}{w}$ a.e. in U_ε, we have

$$\begin{aligned} &|Xu_n|^{p-2} Xu_n X(w_n^\beta) - |Xw_n|^{p-2} Xw_n X(v_\varepsilon w_n^\beta) = \\ &\qquad = \beta w_n^{\beta-1} |Xu_n|^{p-2} Xu_n Xw_n - \beta - w_n^{\beta-1} \left| \frac{u}{w} \right|^{p-2} \frac{u}{w} |Xw_n|^p - \\ &\qquad\qquad - w_n^\beta |Xw_n|^{p-2} Xw_n Xv_\varepsilon \end{aligned} \tag{5.5}$$

a.e. in U_ε.

In a similar way we obtain

$$\begin{aligned} &|Xu|^{p-2} XuX(w^\beta) - |Xw|^{p-2} XwX(v_\varepsilon w^\beta) = \\ &\quad = \beta w^{\beta-1} |Xu|^{p-2} XuXw - \beta w^{\beta-1} \left| \frac{u}{w} \right|^{p-2} \frac{u}{w} |Xw|^p - w^\beta |Xw|^{p-2} XwXv_\varepsilon \end{aligned} \tag{5.6}$$

a.e. in U_ε.

By Theorem 3.11 the sequences u_n and w_n converge to u and w in $H_0^{1,r}(\Omega, X)$, $1 < r < p$ and so, at least after extraction of subsequences we may assume that u_n, w_n, Xu_n, Xw_n converge to u, w, Xu, Xw a.e. in Ω. This implies that $|Xu_n|^{p-2} Xu_n$, $|Xw_n|^{p-2} Xw_n$ converge to $|Xu|^{p-2} Xu$, $|Xw|^{p-2} Xw$ a.e. in U_ε. So we have that $|Xu_n|^{p-2} Xu_n$, $\left| \frac{u}{w} Xw_n \right|^{p-2} \frac{u}{w} Xw_n$ converge to $|Xu|^{p-2} Xu$, $\left| \frac{u}{w} Xw \right|^{p-2} \frac{u}{w} Xw$ a.e. in U_ε.

We prove now that $|Xu_n|^{p-2} Xu_n - \frac{u}{w} Xw_n|^{p-2} \frac{u}{w} Xw_n$ converges to $|Xu|^{p-2} Xu - \frac{u}{w} Xw|^{p-2} \frac{u}{w} Xw$ strongly in $L^q(U_\varepsilon)$. It is enough to prove that the sequence $|Xu_n|^{p-2} Xu_n - \frac{u}{w} Xw_n|^{p-2} \frac{u}{w} Xw_n$ is equi-integrable.

Consider the case $p \geqslant 2$. We recall that $\frac{u}{w} \in L^\infty(U_\varepsilon)$ (Remark 4.2); by (5.4) there exists a constant C such that

$$\begin{aligned} &\left| |Xu_n|^{p-2} Xu_n - \left| \frac{u}{w} Xw_n \right|^{p-2} \frac{u}{w} Xw_n \right|^q \leqslant \\ &\leqslant C_1^q \left(2 \left| \frac{u}{w} Xw_n \right| + \left| Xu - \frac{u}{w} Xw + R_n \right| \right)^{q(p-2)} \left| Xu - \frac{u}{w} Xw + R_n \right|^q \leqslant \\ &\qquad \leqslant C \left(|Xw_n|^{q(p-2)} \left| Xu - \frac{u}{w} Xw + R_n \right|^q + \left| Xu - \frac{u}{w} Xw + R_n \right|^p \right) \end{aligned} \tag{5.7}$$

a.e. in U_ε (where we use Lagrange's formula).

We integrate on an arbitrary measurable set $E \subset \Omega$; by Hölder's inequality we obtain

$$\int_E |Xw_n|^{q(p-2)} \left| Xu - \frac{u}{w} Xw + R_n \right|^q dx \leqslant$$

$$\leqslant \left(\int_\Omega |Xw_n|^p \right)^{\frac{p-2}{p-1}} \left(\int_E \left| Xu - \frac{u}{w} Xw + R_n \right|^p dx \right)^{\frac{q}{p}}.$$

We recall that Xw_n is bounded in $L^p(\Omega)$ and that R_n converges to 0 strongly in $L^p(\Omega)$, so the previous inequality and (5.7) gives the result.

Consider now the case $1 < p < 2$. We have

$$\left| |Xu_n|^{p-2} Xu_n - \left| \frac{u}{w} Xw_n \right|^{p-2} \frac{u}{w} Xw_n \right|^q \leqslant C_1^q \left| Xu - \frac{u}{w} Xw + R_n \right|^p.$$

The result follows from the strong convergence of R_n to 0 in $L^p(\Omega)$. Then $|Xu_n|^{p-2} Xu_n - \left| \frac{u}{w} Xw_n \right|^{p-2} \frac{u}{w} Xw_n$ converges (in both the cases) to $|Xu|^{p-2} Xu - \left| \frac{u}{w} Xw \right|^{p-2} \frac{u}{w} Xw$ strongly in $L^q(U_\varepsilon)$.

We recall that Xw_n converges to Xw in $L^p(U_\varepsilon)$ and that w_n is bounded in $L^\infty(\Omega)$ (Remark 4.2) and converges to w a.e. in Ω, then

$$\beta w^{\beta-1} \left(|Xu_n|^{p-2} Xu_n - \left| \frac{u}{w} Xw_n \right|^{p-2} \frac{u}{w} Xw_n \right)$$

converges to

$$\beta w^{\beta-1} \left(|Xu|^{p-2} Xu - \left| \frac{u}{w} Xw \right|^{p-2} \frac{u}{w} Xw \right)$$

weakly in $L^1(U_\varepsilon)$.

We have that $|Xw_n|^{p-2} Xw_n$ converges weakly in $L^q(\Omega)$ to $|Xw|^{p-2} Xw$, then $w_n^\beta |Xw_n|^{p-2} Xw_n Xv_\varepsilon$ converges to $w^\beta |Xw|^{p-2} Xw Xv_\varepsilon$ weakly in $L^1(\Omega)$. The result follows now from (5.5) (5.6). ■

Lemma 5.2: *Let μ_n be a sequence of measures in $\mathcal{M}_0^p(\Omega, X)$.*

Assume that u_n and w_n are the solutions to are the solution of problem (3.1) *corresponding to $\mu = \mu_n$ and to $\mu = \mu_n$, $f = 1$ and that u_n and w_n converge weakly in $H_0^{1,p}(\Omega, X)$ to some function u and w. Let $\varepsilon > 0$ and $\beta \geqslant 1$, define $v_\varepsilon =$*

$= \Psi_\varepsilon\left(\frac{u}{w \vee \varepsilon}\right) \in H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$. *Then*

$$\langle Au_n, w_n^\beta \phi\rangle - \langle Aw_n, v_\varepsilon w_n^\beta \phi\rangle = \langle Au, w^\beta \phi\rangle - \langle Aw, v_\varepsilon w^\beta \phi\rangle + \mathcal{R}_n^\varepsilon$$

where $\lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \mathcal{R}_n^\varepsilon = 0$.

Proof: For every $\varepsilon > 0$ we have

$$\langle Au_n, w_n^\beta \phi\rangle - \langle Aw_n, v_\varepsilon w_n^\beta \phi\rangle = \mathcal{A}_n^\varepsilon + \mathcal{B}_n^\varepsilon + \mathcal{C}_n^\varepsilon$$

where

$$\mathcal{A}_n^\varepsilon = \int_{U_\varepsilon} \phi |Xu_n|^{p-2} Xu_n Xw_n^\beta\, dx - \int_{U_\varepsilon} \phi |Xw_n|^{p-2} Xw_n X(v_\varepsilon w_n^\beta)\, dx$$

$$\mathcal{B}_n^\varepsilon = \int_{V_\varepsilon \cup W_\varepsilon} \phi |Xu_n|^{p-2} Xu_n Xw_n^\beta\, dx - \int_{V_\varepsilon \cup W_\varepsilon} \phi |Xw_n|^{p-2} Xw_n X(v_\varepsilon w_n^\beta)\, dx$$

$$\mathcal{C}_n^\varepsilon = \int_\Omega w^\beta |Xu_n|^{p-2} Xu_n X\phi\, dx - \int_{U_\varepsilon} v_\varepsilon w_n^\beta |Xw_n|^{p-2} Xw_n X(\phi)\, dx.$$

In a similar way we define $\mathcal{A}^\varepsilon, \mathcal{B}^\varepsilon, \mathcal{C}^\varepsilon$ by replacing u_n and w_n by u and w, so

$$\langle Au, w^\beta \phi\rangle - \langle Aw, v_\varepsilon w^\beta \phi\rangle = \mathcal{A}^\varepsilon + \mathcal{B}^\varepsilon + \mathcal{C}^\varepsilon.$$

By the Lemma 5.1 we have

$$\lim_{n \to +\infty} \mathcal{A}_n^\varepsilon = \mathcal{A}^\varepsilon \tag{5.8}$$

for every $\varepsilon > 0$.

We have that $|Xu_n|^{p-2} Xu_n$, $|Xw_n|^{p-2} Xw_n$ converges weakly in $L^q(\Omega)$ to $|Xu|^{p-2} Xu$, $|Xw|^{p-2} Xw$ (Theorem 3.11) and that w_n is bounded in $L^\infty(\Omega)$ (Remark 4.2) and converges strongly to w in $L^p(\Omega)$, while $v_\varepsilon \in L^\infty(\Omega)$ (Remark 4.2) we conclude that

$$\lim_{n \to +\infty} \mathcal{C}_n^\varepsilon = \mathcal{C}^\varepsilon. \tag{5.9}$$

We now consider the term $\mathcal{B}_n^\varepsilon - \mathcal{B}^\varepsilon$. For every measurable set $B \subset \Omega$ we define

$$\mathfrak{I}_n^1(B) = \beta \int_B \phi w_n^{\beta-1} |Xu_n|^{p-2} Xu_n Xw_n\, dx$$

$$\mathfrak{I}_n^{\varepsilon,2}(B) = \beta \int_B \phi v_\varepsilon w_n^{\beta-1} |Xw_n|^{p-2} Xw_n X(w_n)\, dx$$

$$\mathfrak{I}_n^{\varepsilon,3}(B) = \int_B \phi w_n^\beta |Xw_n|^{p-2} Xw_n X(v_\varepsilon)\, dx$$

In a similar way we define $\mathfrak{J}^1$, $\mathfrak{J}^{\varepsilon,2}$, $\mathfrak{J}^{\varepsilon,3}$ by replacing u_n and w_n by u and w. We have

$$(5.10)\qquad |\mathcal{B}_n^\varepsilon - \mathcal{B}^\varepsilon| \leq |\mathfrak{J}_n^1(V_\varepsilon \cup W_\varepsilon)| + |\mathfrak{J}^1(V_\varepsilon \cup W_\varepsilon)| + \\ + |\mathfrak{J}_n^{\varepsilon,2}(V_\varepsilon)| + |\mathfrak{J}^{\varepsilon,2}(V_\varepsilon)| + |\mathfrak{J}_n^{\varepsilon,2}(W_\varepsilon)| + \\ + |\mathfrak{J}^{\varepsilon,2}(W_\varepsilon)| + |\mathfrak{J}_n^{\varepsilon,3}(V_\varepsilon \cup W_\varepsilon) - \mathfrak{J}^{\varepsilon,3}(V_\varepsilon \cup W_\varepsilon)|\,.$$

Since $\beta \geq 1$ the sequence $w_n^{\beta-1}$ is bounded in $L^\infty(\Omega)$ (Remark 4.2). Moreover $|u| \leq \leq Cw$ (Remark 4.2), by (5.3) we have $v_\varepsilon \leq (C \vee \varepsilon)^{p-1}$ q.e. in Ω. Moreover there exists a constant K such that

$$|\mathfrak{J}_n^1(V_\varepsilon \cup W_\varepsilon)| + |\mathfrak{J}_n^{\varepsilon,2}(V_\varepsilon)| \leq K\left(\int_{V_\varepsilon \cup W_\varepsilon} |Xu_n|^{p-1}|Xw_n|\,dx + \int_{V_\varepsilon} |Xw_n|^p\,dx\right) \leq$$

$$\leq K\left[\left(\int_{V_\varepsilon \cup W_\varepsilon} |Xu_n|^p\,dx\right)^{1/q}\left(\int_\Omega |Xw_n|^p\,dx\right)^{1/p} + \int_{V_\varepsilon} |Xw_n|^p\,dx\right]$$

Then by Lemma 4.5 and 4.6 we have

$$(5.11)\qquad \lim_{\varepsilon\to 0}\limsup_{n\to+\infty}\left(|\mathfrak{J}_n^1(V_\varepsilon \cup W_\varepsilon)| + |\mathfrak{J}_n^{\varepsilon,2}(V_\varepsilon)|\right) = 0$$

In a similar way we prove

$$(5.12)\qquad \lim_{\varepsilon\to 0}\left(|\mathfrak{J}^1(V_\varepsilon \cup W_\varepsilon)| + |\mathfrak{J}^{\varepsilon,2}(V_\varepsilon)|\right) = 0$$

We have $|u| \leq \varepsilon w$ q.e. in W_ε, so we have also $|v_\varepsilon| \leq \varepsilon^{p-1}$ q.e. in W_ε. The boundness of $w_n^{\beta-1}$ in $L^\infty(\Omega)$ (Remark 4.2) implies

$$|\mathfrak{J}_n^{\varepsilon,2}(W_\varepsilon)| \leq K\varepsilon^{p-1}\int_\Omega |Xw_n|^p\,dx$$

for a suitable constant K. We recall that w_n is bounded in $H_0^{1,p}(\Omega, X)$, hence we conclude that

$$(5.13)\qquad \lim_{\varepsilon\to 0}\limsup_{n\to+\infty}|\mathfrak{J}_n^{\varepsilon,2}(W_\varepsilon)| = 0$$

In a similar way we prove

$$(5.14)\qquad \lim_{\varepsilon\to 0}\limsup_{n\to+\infty}|\mathfrak{J}_n^{\varepsilon,2}(W_\varepsilon)| = 0$$

Since $|Xw_n|^{p-2}Xw_n$ converges to $|Xw|^{p-2}Xw$ weakly in $L^q(\Omega)$ and w_n is bounded in $L^\infty(\Omega)$ (Remark 4.2) and converges strongly in $L^p(\Omega)$, we conclude that

$$(5.15)\qquad \lim_{n\to+\infty}\mathfrak{J}_n^{\varepsilon,3}(V_\varepsilon \cup W_\varepsilon) = \mathfrak{J}^{\varepsilon,3}(V_\varepsilon \cup W_\varepsilon).$$

From (5.10)-(5.15) we have

$$\lim_{\varepsilon \to 0} \limsup_{n \to +\infty} | \mathcal{B}_n^\varepsilon - \mathcal{B}^\varepsilon | = 0 . \tag{5.16}$$

We recall that $\mathcal{R}_n^\varepsilon = \mathcal{A}_n^\varepsilon - \mathcal{A}^\varepsilon + \mathcal{B}^\varepsilon - \mathcal{B}^\varepsilon + \mathcal{C}_n^\varepsilon - \mathcal{C}^\varepsilon$ the result follows from (5.8), (5.9) and (5.16).

LEMMA 5.3: *Let μ_n be a sequence of measures in $\mathcal{M}_0^p(\Omega, X)$.*

Assume that u_n and w_n are the solutions to are the solution of problem (3.1) *corresponding to $\mu = \mu_n$ and to $\mu = \mu_n$, $f = 1$ and that u_n and w_n converge weakly in $H_0^{1,p}(\Omega, X)$ to some function u and w. Let $\varepsilon > 0$ and $\beta \geq (p-1) \vee 1$, define $u_n^\varepsilon = \dfrac{u w_n}{w \vee \varepsilon}$ as in Lemma* 4.4. *Then*

$$\int_{U_\varepsilon} |u_n|^{p-2} u_n w_n^\beta \phi \, d\mu_n - \int_{U_\varepsilon} |u_n^\varepsilon|^{p-2} u_n^\varepsilon w_n^\beta \phi \, d\mu_n$$

tends to 0 *as $n \to +\infty$ for every $\phi \in H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$.*

PROOF: Let $\phi \in H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$ and $r_n^\varepsilon = u_n - u_n^\varepsilon$. We recall that the sequences u_n and u_n^ε are bounded in $L^\infty(\Omega)$ (Remark 4.2), then there exists a constant C such that

$$\big| |u_n|^{p-2} u_n \phi - |u_n^\varepsilon|^{p-2} u_n^\varepsilon \phi \big| \leq C r_n^\varepsilon$$

We recall that w_n is bounded in $L^\infty(\Omega)$ (Remark 4.2), then there exists a constant K such that $w_n^\beta \leq K w_n^{(p-1) \vee 1}$, then

$$\left| \int_{U_\varepsilon} |u_n|^{p-2} u_n w_n^\beta \phi \, d\mu_n - \int_{U_\varepsilon} |u_n^\varepsilon|^{p-2} u_n^\varepsilon w_n^\beta \phi \, d\mu_n \right| \leq$$

$$\leq CK \int_{U_\varepsilon} |r_n^\varepsilon|^{(p-1) \vee 1} w_n^{(p-1) \vee 1} d\mu_n \leq CK \left(\int_{U_\varepsilon} |r_n^\varepsilon|^p \, d\mu_n \right)^{\frac{1}{p \vee q}} \left(\int_{U_\varepsilon} w_n^p \, d\mu_n \right)^{\frac{1}{p \wedge q}}$$

The result follows now from the estimate (3.3) and from the Lemma 4.3. ■

LEMMA 5.4: *Let μ_n be a sequence of measures in $\mathcal{M}_0^p(\Omega, X)$.*

Assume that u_n and w_n are the solutions to are the solution of problem (3.1) *corresponding to $\mu = \mu_n$ and to $\mu = \mu_n$, $f = 1$ and that u_n and w_n converge weakly in $H_0^{1,p}(\Omega, X)$ to some function u and w. Let $\varepsilon > 0$ and $\beta \geq (p-1) \vee 1$, define $v_\varepsilon = \Psi_\varepsilon \left(\dfrac{u}{w \vee \varepsilon} \right) \in H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$ and let*

$$\mathcal{E}_n^\varepsilon = \int_\Omega |u_n|^{p-2} u_n w_n^\beta \phi \, d\mu_n - \int_\Omega v_\varepsilon w_n^{\beta + p - 1} \phi \, d\mu_n$$

Then $\lim_{\varepsilon \to 0} \limsup_{n \to +\infty} | \mathcal{E}_n^\varepsilon | = 0.$

PROOF: We observe that $w_n^{p-1} v_\varepsilon = |u_n^\varepsilon|^{p-2} u_n^\varepsilon$ q.e. in u_ε. By Lemma 5.3 for every $\varepsilon > 0$ the sequence

$$\int_{U_\varepsilon} |u_n|^{p-2} u_n w_n^\beta \phi \, d\mu_n - \int_{U_\varepsilon} v_\varepsilon w_n^{\beta+p-1} \phi \, d\mu_n$$

tends to 0 as $n \to +\infty$.

As ϕ is bounded for the proof of the result is enough to prove

$$\lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \int_{V_\varepsilon \cup W_\varepsilon} |u_n|^{p-1} w_n^\beta \, d\mu_n = 0 \tag{5.17}$$

$$\lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \int_{V_\varepsilon} v_\varepsilon w_n^{\beta+p-1} \, d\mu_n = 0 \tag{5.18}$$

$$\lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \int_{W_\varepsilon} v_\varepsilon w_n^{\beta+p-1} \, d\mu_n = 0 . \tag{5.19}$$

Taking into account that $\beta \geqslant 1$ we have $\|w_n^{\beta-1}\|_{L^\infty(\Omega)} \leqslant K$ for a suitable constant K (Remark 4.2), then

$$\int_{V_\varepsilon \cup W_\varepsilon} |u_n|^{p-1} w_n^\beta \, d\mu_n \leqslant K \int_{V_\varepsilon \cup W_\varepsilon} |u_n|^{p-1} w_n \, d\mu_n \leqslant$$

$$\leqslant K \left(\int_{V_\varepsilon \cup W_\varepsilon} |u_n|^p \, d\mu_n \right)^{1/q} \left(\int_\Omega |w_n|^p \, d\mu_n \right)^{1/p}$$

thus (5.17) follows from (3.3) and from the Lemmas 4.5 and 4.6.

We recall that $|u| \leqslant Cw$ (Remark 4.2), so by (5.3) we have $|v_\varepsilon| \leqslant (C \vee \varepsilon)^{p-1}$ q.e. in Ω; then

$$\int_{V_\varepsilon} |v_\varepsilon| w_n^{\beta+p-1} \, d\mu_n \leqslant (C \vee \varepsilon)^{p-1} K \int_{V_\varepsilon} w_n^p \, d\mu_n$$

The relation (5.18) follows from Lemma 4.5.

We recall that $|u| \leqslant \varepsilon w$ q.e. in W_ε, then

$$\int_{W_\varepsilon} |v_\varepsilon| w_n^{\beta+p-1} \, d\mu_n \leqslant \varepsilon^{p-1} \int_{W_\varepsilon} w_n^p \, d\mu_n$$

and (5.19) follows from (3.3). ■

6. - The case $f = 1$

In this section we will study the properties of the set $\mathcal{K}(\Omega)$ of the function w such that

$$w \in H_0^{1,p}(\Omega, X), \quad w \geq 0 \text{ q.e. in } \Omega \text{ and } Aw \leq 1 \ \ w \geq 0 \text{ in } \mathcal{D}'(\Omega).$$

The results of the present section will be used in the proofs of theorems 7.3, 7.5, and are independent of the results in sections 4 and 5.

For every $w \in \mathcal{K}(\Omega)$ we have

$$\int_\Omega |Xw|^p \, dx \leq \langle Aw, w \rangle \leq \int_\Omega w \, dx$$

the function w in $\mathcal{K}(\Omega)$ are uniformly bounded, so $\mathcal{K}(\Omega)$ is bounded and weakly relatively compact in $H_0^{1,p}(\Omega, X)$.

We also observe that if w_0 is the solution of the Dirichlet problem

$$w_0 \in H_0^{1,p}(\Omega, X), \quad Aw_0 = 1 \text{ in } \mathcal{D}'(\Omega)$$

by Proposition 3.6 we have $0 \leq w \leq w_0$, $\forall w \in \mathcal{K}(\Omega)$ (and it is easily proved that $w_0 \in L^\infty(\Omega)$).

Given $w \in \mathcal{K}(\Omega)$ we define

$$\sigma = 1 - Aw$$

By the definition of $\mathcal{K}(\Omega)$ we have $\sigma \geq 0$ in $\mathcal{D}'(\Omega)$, so σ is a non-negative Radon measure. As $Aw \in H^{-1,q}(\Omega, X)$, $\frac{1}{p} + \frac{1}{q} = 1$, we have $\sigma \in H^{-1,q}(\Omega, X)$.

Our aim in this section is to prove the following characterization of $\mathcal{K}(\Omega)$ as the set of the solutions of all relaxed Dirichlet problems corresponding to $f = 1$

Theorem 6.1: *The set $\mathcal{K}(\Omega)$ is compact in the weak topology of $H_0^{1,p}(\Omega, X)$. A function $w \in H_0^{1,p}(\Omega, X)$ belongs to $\mathcal{K}(\Omega)$ if and only if there exists a measure $\mu \in \mathcal{M}_0^p(\Omega, X)$ such that w is the solution of the problem*

$$w \in H_0^{1,p}(\Omega, X) \cap L^p_\mu(\Omega)$$

$$\langle Aw, v \rangle + \int_\Omega |w|^{p-2} wv \, d\mu = \int_\Omega v \, dx \tag{6.1}$$

$$\forall v \in H_0^{1,p}(\Omega, X) \cap L^p_\mu(\Omega)$$

The measure μ is uniquely determined by $w \in \mathcal{K}(\Omega)$. More precisely for every

$w \in \mathcal{K}(\Omega)$ and for every Borel set $B \subset \Omega$ we have

(6.2)
$$\mu(B) = \int_B \frac{d\sigma}{w^{p-1}} \quad \text{if } \operatorname{cap}_p(B \cap \{w = 0\}, \Omega; X) = 0,$$
$$\mu(B) = +\infty \quad \text{if } \operatorname{cap}_p(B \cap \{w = 0\}, \Omega; X) > 0$$

where σ is the non-negative Radon measure in $H^{-1,q}(\Omega, X)$ defined by $\sigma = 1 - Aw$.

Observe that from (5.2) we have

$$\sigma(B \cap \{w > 0\}) = \int_B w^{p-1}\, d\mu \tag{6.3}$$

for every Borel set $B \subset \Omega$.

To prove the Theorem we need some preliminary results:

Lemma 6.2: *Let $\mu \in \mathcal{M}_0^p(\Omega, X)$ and let $u \in H_0^{1,p}(\Omega, X) \cap L_\mu^p(\Omega)$. Let u_n be the solution of the problem*

$$u_n \in H_0^{1,p}(\Omega, X) \cap L_\mu^p(\Omega)$$
$$\langle Au_n, v\rangle + \int_\Omega |u_n|^{p-2} u_n v\, d\mu + n \int_\Omega |u_n|^{p-2} u_n v\, dx = n \int_\Omega |u|^{p-2} u\, dx \tag{6.4}$$
$$\forall v \in H_0^{1,p}(\Omega, X) \cap L_\mu^p(\Omega)$$

Then u_n converges to u strongly in $H_0^{1,p}(\Omega, X)$ and in $L_\mu^p(\Omega)$

Proof: The proof is in some way standard using the algebraic inequality on our operator and we give the details for sake of completeness. We use $u_n - u$ as test function in (5.4) and we obtain

$$\langle Au_n, u_n - u\rangle + \int_\Omega |u_n|^{p-2} u_n (u_n - u)\, d\mu +$$
$$+ n \int_\Omega (|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u)\, dx = 0.$$

Hence

$$\langle Au_n - Au, u_n - u\rangle + \int_\Omega (|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u)\, d\mu + \tag{6.5}$$
$$+ n \int_\Omega (|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u)\, dx = -\langle Au, u_n - u\rangle - \int_\Omega |u|^{p-2} u(u_n - u)\, d\mu.$$

If $2 \leqslant p$ we have

$$(6.6)\qquad C_0 \|u_n - u\|^p_{H_0^{1,p}(\Omega, X)} + 2^{2-p}(\|u_n - u\|^p_{L^p_\mu(\Omega)} + n\|u_n - u\|^p_{L^p(\Omega)}) \leqslant$$

$$\leqslant -\langle Au, u_n - u\rangle - \int_\Omega |u|^{p-2} u(u_n - u)\, d\mu$$

hence

$$C_0 \|u_n - u\|^p_{H_0^{1,p}(\Omega, X)} + 2^{2-p}(\|u_n - u\|^p_{L^p_\mu(\Omega)} + n\|u_n - u\|^p_{L^p(\Omega)}) \leqslant$$

$$\leqslant \|Au\|_{H^{-1,q}(\Omega, X)} \|u_n - u\|_{H_0^{1,p}(\Omega, X)} + \|u\|^{p/q}_{L^p_\mu(\Omega)} \|u_n - u\|_{L^p_\mu(\Omega)}.$$

By Young's inequality we obtain

$$\frac{C_0}{q} \|u_n - u\|^p_{H_0^{1,p}(\Omega, X)} + \frac{2^{2-p}}{q} \|u_n - u\|^p_{L^p_\mu(\Omega)} + 2^{2-p} n\|u_n - u\|^p_{L^p(\Omega)}) \leqslant$$

$$\leqslant \frac{C_0^{q-1}}{q} \|Au\|^q_{H^{-1,q}(\Omega, X)} + \frac{2}{q} \|u\|^p_{L^p_\mu(\Omega)}.$$

Then u_n converges to u weakly in $H_0^{1,p}(\Omega, X)$ and in $L^p_\mu(\Omega)$; so (6.6) gives the strong convergence in both the spaces.

Consider now the case $1 < p < 2$. From Theorem 3.2 (where we take μ as the Lebesgue measure) we have

$$\|u_n - u\|^2_{L^p(\Omega)} \leqslant 2(\|u_n\|^{2-p}_{L^p(\Omega)}) + \|u\|^{2-p}_{L^p(\Omega)}) \int_\Omega (|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u)\, dx$$

Then from Theorem 3.2 and (6.5) we have

$$C_0 \|u_n - u\|^p_{H_0^{1,p}(\Omega, X)} + \|u_n - u\|^p_{L^p_\mu(\Omega)} + n\|u_n - u\|^p_{L^p(\Omega)} \leqslant$$

$$H(u_n, u)\left(|\langle Au, u_n - u\rangle| + \left| \int_\Omega |u|^{p-2} u(u_n - u)\, d\mu \right| \right) \leqslant$$

$$H(u_n, u)(\|Au\|_{H^{-1,q}(\Omega, X)} \|u_n - u\|_{H_0^{1,p}(\Omega, X)} + \|u\|^{p-1}_{L^p_\mu(\Omega)} \|u_n - u\|_{L^p_\mu(\Omega)})$$

where

$$H(u_n, u) = 2(\|u_n\|^{2-p}_{H_0^{1,p}(\Omega, X)} + \|u_n\|^{2-p}_{L^p_\mu(\Omega)} + \|u_n\|^{2-p}_{L^p(\Omega)}) +$$

$$+ 2(\|u\|^{2-p}_{H_0^{1,p}(\Omega, X)} + \|u\|^{2-p}_{L^p_\mu(\Omega)} + \|u\|^{2-p}_{L^p(\Omega)}) \leqslant$$

$$2(\|u_n - u\|^{2-p}_{H_0^{1,p}(\Omega, X)} + \|u_n - u\|^{2-p}_{L^p_\mu(\Omega)} + \|u_n - u\|^{2-p}_{L^p(\Omega)}) +$$

$$+ 4(\|u\|^{2-p}_{H_0^{1,p}(\Omega, X)} + \|u\|^{2-p}_{L^p_\mu(\Omega)} + \|u\|^{2-p}_{L^p(\Omega)}).$$

By Young's inequality we have

$$\|u_n - u\|^p_{H_0^{1,p}(\Omega, X)} + \|u_n - u\|^p_{L^p_\mu(\Omega)} + n\|u_n - u\|^p_{L^p(\Omega)} \leqslant$$

$$\leqslant K_1(u)(\|u_n - u\|^{3-p}_{H_0^{1,p}(\Omega, X)} + \|u_n - u\|^{3-p}_{L^p_\mu(\Omega)} + n\|u_n - u\|^{3-p}_{L^p(\Omega)}) \leqslant$$

$$\leqslant \frac{1}{2}(\|u_n - u\|^p_{H_0^{1,p}(\Omega, X)} + \|u_n - u\|^p_{L^p_\mu(\Omega)} + n\|u_n - u\|^p_{L^p(\Omega)}) + K_2(u)$$

where $K_1(u)$, $K_2(u)$ are constant depending on u.

We have

$$\|u_n - u\|^p_{H_0^{1,p}(\Omega, X)} + \|u_n - u\|^p_{L^p_\mu(\Omega)} + (2n-1)\|u_n - u\|^p_{L^p(\Omega)} \leqslant 2K_2(u)$$

then u_n converges to u weakly in $H_0^{1,p}(\Omega, X)$ and in $L^p_\mu(\Omega)$. The constant $H(u_n, u)$ is bounded (with respect to n, then (6.7) give the strong convergence in $H_0^{1,p}(\Omega, X)$ and in $L^p_\mu(\Omega)$. ■

The weak regularity condition in the definition of $\mathcal{M}_0^p(\Omega, X)$ has a fundamental role in the following Lemma:

LEMMA 6.3: *Let $\mu \in \mathcal{M}_0^p(\Omega, X)$ and let w be the solution of* (6.1). *Then $\mu(B) = +\infty$ for every Borel set $B \subset \Omega$ with* $\mathrm{cap}_p(B \cap \{w = 0\}, \Omega; X) > 0$.

PROOF: Let $u \in H_0^{1,p}(\Omega, X) \cap L^p_\mu(\Omega)$. We have $0 \leqslant u \leqslant 1$ q.e. in Ω. Let u_n be the solution of (6.4). By the comparison principles we have $0 \leqslant u_n \leqslant n^{\frac{1}{p-1}} w$ q.e. in Ω (recall that $u \leqslant 1$ q.e. in Ω), so we have $u_n = 0$ q.e. in $\{w = 0\}$. By Lemma 6.2 u_n converges strongly to u in $H_0^{1,p}(\Omega, X)$, then $u = 0$ q.e. in $\{w = 0\}$.

Let U be a p-quasi-open set in Ω such that $\mu(U) < +\infty$.

We recall that from Lemma 1.7 we have an increasing sequence z_n of non-negative functions in $H_0^{1,p}(\Omega, X)$ that converge to $\mathbf{1}_U$ q.e. in Ω (we observe that $0 \leqslant z_n \leqslant 1$ q.e. in Ω).

Since $\mu(U) < +\infty$ we have $z_n \in L^p_\mu(\Omega)$, then by the previous step $z_n = 0$ q.e. in $\{w = 0\}$, this implies $\mathrm{cap}_p(B \cap \{w = 0\}, \Omega; X) = 0$.

Consider now a Borel set $B \subset \Omega$ such that $\mathrm{cap}_p(B \cap \{w = 0\}, \Omega; X) > 0$. For every p-quasi-open set containing B we have $\mathrm{cap}_p(U \cap \{w = 0\}, \Omega; X) > 0$, so $\mu(U) = +\infty$ by the previous step. The weak regularity property of the measure μ gives $\mu(B) = +\infty$. ■

LEMMA 6.4: *Let $\lambda, \mu \in \mathcal{M}_0^p(\Omega, X)$. Assume that there is a function w in $H_0^{1,p}(\Omega, X) \cap L^p_\lambda(\Omega) \cap L^p_\mu(\Omega)$ such that*

$$\langle Aw, v\rangle + \int_\Omega |w|^{p-2} wv \, d\lambda = \int_\Omega v \, dx \tag{6.8}$$

$$\langle Aw, v\rangle + \int_\Omega |w|^{p-2} wv \, d\mu = \int_\Omega v \, dx. \tag{6.9}$$

Then $\lambda = \mu$.

Proof: From the comparison principles we have $w \geqslant 0$ q.e. in Ω. Consider the measures λ_0 and μ_0 defined as

$$\lambda_0(B) = \int_B w^{p-1}\, d\lambda, \quad \mu_0(B) = \int_B w^{p-1}\, d\mu$$

where B is a Borel set in Ω.

The first step of the proof is prove that $\lambda_0 = \mu_0$.

For every $\varepsilon > 0$ we define the measures λ_ε, μ_ε by

$$\lambda_\varepsilon(B) = \int_{B \cap \{w > \varepsilon\}} w^{p-1}\, d\lambda, \; \mu_\varepsilon(B) = \int_{B \cap \{w > \varepsilon\}} w^{p-1}\, d\mu$$

where B is a Borel set in Ω.

To prove that $\lambda_0 = \mu_0$ it is enough to prove that $\lambda_\varepsilon = \mu_\varepsilon$ $\forall \varepsilon > 0$. We have that $w \in L^p_\lambda(\Omega) \cap L^p_\mu(\Omega)$, then λ_ε, μ_ε are bounded measures, then to prove the result we have to prove that $\lambda_\varepsilon(U) = \mu_\varepsilon(U)$ for every open set U in Ω.

Let us fix U and define $U_\varepsilon = U \cap \{w > \varepsilon\}$. The set $U_\varepsilon =$ is p-quasi-open. We recall that from Lemma 1.7 we have an increasing sequence z_n of non-negative functions in $H^{1,p}_0(\Omega, X)$ that converge to $\mathbf{1}_{U_\varepsilon}$ q.e. in Ω (we observe that $0 \leqslant z_n \leqslant 1$ q.e. in Ω). Since $w \in L^p_\lambda(\Omega) \cap L^p_\mu(\Omega)$ and $w > \varepsilon$ q.e. on u_ε, we obtain $\lambda(U_\varepsilon)$, $\mu(U_\varepsilon) < +\infty$, then $z_n \in L^p_\lambda(\Omega) \cap L^p_\mu(\Omega)$. From (6.8) (6.9) we have

$$\int_\Omega w^{p-1} z_n\, d\lambda = \int_\Omega w^{p-1} z_n\, d\mu .$$

Taking the limit as $n \to +\infty$ we have

$$\lambda_\varepsilon(U) = \int_{U_\varepsilon} w^{p-1} z_n\, d\lambda = \int_{U_\varepsilon} w^{p-1} z_n\, d\mu = \mu_\varepsilon(U) .$$

The above relation prove $\lambda_\varepsilon = \mu_\varepsilon$ $\forall \varepsilon > 0$, then $\lambda_0 = \mu_0$. For every Borel set B contained in $\{w > 0\}$ we have

$$\lambda(B) = \int_B \frac{1}{w^{p-1}}\, d\lambda_0, \; \mu(B) = \int_B \frac{1}{w^{p-1}}\, d\mu_0 .$$

Consider now a Borel set contained in $\{w = 0\}$ and $\mathrm{cap}_p(B, \Omega; X) > 0$, by Lemma 6.3 we have $\lambda(B) = \mu(B) = +\infty$. For an arbitrary Borel set B in Ω we have

$$\lambda(B) = \lambda(B \cap \{w > 0\}) + \lambda(B \cap \{w = 0\}) =$$

$$= \mu(B \cap \{w > 0\}) + \mu(B \cap \{w = 0\}) = \mu(B)$$

and the result is proved. ■

PROOF OF THEOREM 6.1: Let us prove that $\mathcal{K}(\Omega)$ is compact in the weak topology of $H_0^{1,p}(\Omega, X)$. Let w_n be a sequence in $\mathcal{K}(\Omega)$. Since $\mathcal{K}(\Omega)$ is bounded in $H_0^{1,p}(\Omega, X)$, we may assume that w_n converges weakly in $H_0^{1,p}(\Omega, X)$ to a function w.

We have to prove that $w \in \mathcal{K}(\Omega)$.

Consider the measures $\sigma_n = 1 - Aw_n$, σ_n is a sequence of Radon measures in $H^{-1,q}(\Omega, X)$, which is bounded in $H^{-1,q}(\Omega, X)$. Since $\sigma_n \geqslant 0$, we have that $\sigma_n(K)$ is bounded for every compact set $K \subset \Omega$. By Theorem 3.10 we have that Aw_n converges to Aw weakly in $H^{-1,q}(\Omega, X)$, then $Aw \leqslant 1$ and $w \in \mathcal{K}(\Omega)$.

Assume $\mu \in \mathcal{M}_0^p(\Omega, X)$ and let w be the solution of (6.1), then from comparison principles we have $w \geqslant 0$ q.e. in Ω.

From Proposition 3.8 and Remark 3.9 we have that $Aw \leqslant 1$ in $\mathcal{D}'(\Omega)$, then $w \in \mathcal{K}(\Omega)$.

Assume now $w \in \mathcal{K}(\Omega)$. Define $\sigma = 1 - Aw$ and let μ the measure defined by (6.2).

We first prove that $\mu \in \mathcal{M}_0^p(\Omega, X)$.

The measure σ is in $H^{-1,q}(\Omega, X)$ and non-negative, then $\sigma(B) = \mu(B) = 0$ for every Borel set B with $\text{cap}_p(B, \Omega; X) = 0$. We have to prove

$$\mu(B) = \inf\{\mu(U),\ U\ p\text{-quasi-open}, B \subset U\} \tag{6.10}$$

for every Borel set in Ω with $\mu(B) < +\infty$.

We define the measure μ_n by $\mu_n(B) = \mu\left(B \cap \left\{w > \frac{1}{n}\right\}\right)$.

Observe that

$$\mu_n(\Omega) = \mu\left(\left\{w > \frac{1}{n}\right\}\right) \leqslant n^{p-1}\sigma\left(\left\{w > \frac{1}{n}\right\}\right) \leqslant n^p \int_\Omega w\, d\sigma = \langle 1 - Aw, w\rangle.$$

We fix now a Borel set B with $\mu(B) < +\infty$. From the definition of μ we have $\text{cap}_p(B \cap \{w = 0\}, \Omega; X) = 0$. For $n \geqslant 2$ define $B_n = B \cap \left\{\frac{1}{n} < w \leqslant \frac{1}{n-1}\right\}$ and $B_1 = B \cap \{w > 1\}$, then $\mu(B) = \sum_n \mu(B_n)$.

We have $\mu_n(\Omega) < +\infty$ for every $\varepsilon > 0$, moreover for every n there exists an open set V_n with $B_n \subset V_n \subset \Omega$ and $\mu(V_n) < \mu_n(B_n) + 2^{-n}\varepsilon = \mu(B_n) + 2^{-n}\varepsilon$. Define $U_n = V_n \cap \left\{\frac{1}{n} < w\right\}$. Since w is p-quasi-continuous we have that the set U_n is p-quasi-open. We have also $B_n \subset U_n$ and $\mu(U_n) \leqslant \mu(V_n) \leqslant \mu(B_n) + 2 - n\varepsilon$. Define $U_0 = U \cap \{w = 0\}$, then $U = \cup_{n \geqslant 0} U_n$. Since U_n are p-quasi-open, U is p-quasi-open and we have $\mu(U) \leqslant \mu(B) + \varepsilon$. We recall that $\varepsilon > 0$ is arbitrary, then (6.10) follows.

We now prove that w is the solution of (6.1).

From (6.2) we have

$$\int_\Omega w^p\,d\mu = \int_{\{w>0\}} w^p\,d\mu = \int_{\{w>0\}} w\,d\sigma = \langle 1 - Aw,\, w\rangle < +\infty$$

so $w \in L^p_\mu(\Omega)$. Let $v \in H^{1,p}_0(\Omega, X) \cap L^p_\mu(\Omega)$, from (6.2) we have $v = 0$ q.e. on $\{w = 0\}$. We have

$$\langle Aw,\, v\rangle + \int_\Omega |w|^{p-2} wv\,d\mu = \langle Aw,\, v\rangle + \int_{\{w>0\}} w^{p-1} v\,d\mu =$$

$$= \langle Aw,\, v\rangle + \int_{\{w>0\}} v\,d\sigma = \langle Aw,\, v\rangle + \int_\Omega v\,d\sigma = \int_\Omega v\,dx$$

which proves (6.1). The uniqueness of μ follows from Lemma 6.4. ∎

LEMMA 6.4: *Let $\mu \in \mathcal{M}^p_0(\Omega, X)$, let w be the solution of* (6.1) *and let $\beta \geq 1$. Then the set $\{w^\beta \phi,\, \phi \in \mathcal{D}(\Omega)\}$ is dense in $H^{1,p}_0(\Omega, X) \cap L^p_\mu(\Omega)$.*

PROOF: We have $w \in H^{1,p}_0(\Omega, X) \cap L^\infty(\Omega) \cap L^p_\mu(\Omega)$ (Remark 4.2) and $\beta \geq 1$, then the function $w^\beta \phi$ is in $H^{1,p}_0(\Omega, X) \cap L^\infty(\Omega) \cap L^p_\mu(\Omega)$ for every $\phi \in \mathcal{D}(\Omega)$.

To prove the result for every $u \in H^{1,p}_0(\Omega, X) \cap L^p_\mu(\Omega)$ we have to find a sequence $\phi_n \in \mathcal{D}(\Omega)$ such that $w^\beta \phi_n$ converges to u both in $H^{1,p}_0(\Omega, X)$ and in $L^p_\mu(\Omega)$.

By an approximation by truncation we may assume $u \in L^\infty(\Omega)$ and $u \geq 0$ q.e. in Ω.

Let u_n be the solution of (6.4). By comparison principles we have $0 \leq u_n \leq Cw$ q.e. in Ω where $C^{p-1} = n\|u\|^{p-1}_{L^\infty(\Omega)}$. From the Lemma 6.2 u_n converges to u both in $H^{1,p}_0(\Omega, X)$ and in $L^p_\mu(\Omega)$. As consequence we may assume without loss of generality that there exists a constant C such that $0 \leq u \leq Cw$ q.e. in Ω. We observe that $\{(u - C\varepsilon)^+ > 0\} \subset \{w > \varepsilon\}$ and that $(u - C\varepsilon)^+$ converges as $\varepsilon \to 0$ to u both in $H^{1,p}_0(\Omega, X)$ and in $L^p_\mu(\Omega)$, then we may assume also that $\{u > 0\} \subset \{w > \varepsilon\}$ for some $\varepsilon > 0$ so $\frac{u}{w^\beta} = \frac{u}{(w \vee \varepsilon)^\beta}$. We recall that $u \in H^{1,p}_0(\Omega, X) \cap L^\infty(\Omega)$, then $\frac{u}{w^\beta} \in H^{1,p}_0(\Omega, X) \cap L^\infty(\Omega)$.

There exists a sequence $\phi_n \in \mathcal{D}(\Omega)$ bounded in $L^\infty(\Omega)$ which converges to $z = \frac{u}{w^\beta}$ in $H^{1,p}_0(\Omega, X)$ and q.e. in Ω, then also μ-a.e. in Ω.

We recall that $w \in H^{1,p}_0(\Omega, X) \cap L^\infty(\Omega)$ and $\beta \geq 1$, then $w^\beta \phi_n$ converges to $w^\beta z = u$ in $H^{1,p}_0(\Omega, X)$. We have $w \in L^\infty_\mu(\Omega) \cap L^p_\mu(\Omega)$ (Remark 4.2) and $\beta \geq 1$, then w^β in $L^\infty_\mu(\Omega) \cap L^p_\mu(\Omega)$.

We recall that ϕ_n are bounded and converge to $z = \frac{u}{w^\beta}$ μ-a.e., so we have that the sequence $w^\beta \phi_n$ converges to $w^\beta z = u$ strongly in $L^p_\mu(\Omega)$ (use the Dominated Convergence Theorem). ∎

7. - THE γ^A-convergence

Definition of the γ_A-convergence.

In this section we introduce the notion of γ^A convergence in $\mathcal{M}_p^0(\Omega, X)$, which is defined as the convergence of the solutions f the corresponding relaxed Dirichlet problems.

DEFINITION 7.1: *Let μ_n be a sequence in $\mathcal{M}_p^0(\Omega, X)$ and $\mu \in \mathcal{M}_p^0(\Omega, X)$. We say that μ_n γ^A-converges to μ if for every $f \in H^{-1,q}(\Omega, X)$ the solutions of the problem*

$$(7.1)\qquad \begin{gathered} u_n \in H_0^{1,p}(\Omega, X) \cap L^p_{\mu_n}(\Omega) \\ \langle Au_n, v\rangle + \int_\Omega |u_n|^{p-2} u_n v \, d\mu_n = \langle f, v\rangle \\ \forall v \in H_0^{1,p}(\Omega, X) \cap L^p_{\mu_n}(\Omega) \end{gathered}$$

converge weakly in $H_0^{1,p}(\Omega, X)$ as $n \to +\infty$ to the solution u of the problem

$$(7.2)\qquad \begin{gathered} u \in H_0^{1,p}(\Omega, X) \cap L^p_{\mu}(\Omega) \\ \langle Au, v\rangle + \int_\Omega |u|^{p-2} uv \, d\mu = \langle f, v\rangle \\ \forall v \in H_0^{1,p}(\Omega, X) \cap L^p_{\mu}(\Omega) \end{gathered}$$

Let us emphasize the fact that the notion of γ^A-convergence depends on the operator A.

Although the definition depends also on Ω and on the boundary conditions we shall see in Theorems 7.11, 7.12 that the boundary condition on $\partial\Omega$ does not play an important role in this problem.

DEFINITION 7.2: *The solutions of the problem* (6.1) *depends continuously on f uniformly with respect to μ* (*Theorem* 3.3). *Then a sequence μ_n γ^A-converges to μ if the solution of* (6.1) *weakly converges in $H_0^{1,p}(\Omega, X)$ to the solution of* (6.2) *for every f in a dense subset of $H^{-1,q}(\Omega, X)$.*

Let μ_n be a sequence of measures in $\mathcal{M}_p^0(\Omega, X)$, let w_n be the solution of the problem

$$(7.3)\qquad \begin{gathered} w_n \in H_0^{1,p}(\Omega, X) \cap L^p_{\mu_n}(\Omega) \\ \langle Aw_n, v\rangle + \int_\Omega |w_n|^{p-2} w_n v \, d\mu_n = \int_\Omega v \, dx \\ \forall v \in H_0^{1,p}(\Omega, X) \cap L^p_{\mu_n}(\Omega) \end{gathered}$$

and let w the solution of the problem

$$w \in H_0^{1,p}(\Omega, X) \cap L^p_\mu(\Omega)$$

(7.4) $$\langle Aw, v\rangle + \int_\Omega |w|^{p-2} wv\, d\mu = \int_\Omega v\, dx$$

$$\forall v \in H_0^{1,p}(\Omega, X) \cap L^p_\mu(\Omega).$$

The following result characterize the γ^A-convergence of the μ_n to μ in terms of the weak convergence in $H_0^{1,p}(\Omega, X)$ of w_n to w.

THEOREM 7.3: *Let μ_n be a sequence of measures in $\mathcal{M}_p^0(\Omega, X)$, let w_n (w) be the solution of the problem* (7.3) ((7.4)). *The following conditions are equivalent:*
(a) *w_n weakly converges to w in $H_0^{1,p}(\Omega, X)$*
(b) *μ_n γ^A-converges to μ*

PROOF: The fact $(b) \Rightarrow (a)$ derives from the definition of γ^A-convergence taking $f = 1$.

Assume that (a) holds. Given $f \in L^\infty(\Omega)$ let u_n be the solutions of problem (7.1). From (3.3) we have that u_n is bounded in $H_0^{1,p}(\Omega, X)$, then we may assume that u_n weakly converge in $H_0^{1,p}(\Omega, X)$ to some function u.

We have to prove that u is a solution of (7.2).

By the comparison principles we have $|u_n| \leq Cw_n$ where $C = \|f\|^{\frac{1}{p-1}}_{L^\infty(\Omega)}$. As $n \to + + \infty$ we have $|u| \leq Cw$ q.e. in Ω.

For $\varepsilon > 0$ Ψ_ε will be the locally Lipschitz function defined by (5.3) and define $v_\varepsilon = = \Psi_\varepsilon\left(\frac{u}{w \vee \varepsilon}\right)$. We have $v_\varepsilon \in H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$. Fix $\beta \geq (p-1) \vee 1$ and $\phi \in \mathcal{D}(\Omega)$.

We recall that $w_n \in H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega)$ (Remark 4.2); we take $v = w_n^\beta \phi$ as test function in (7.1) and $v = v_\varepsilon w_n^\beta \phi$ as test function in (7.3). We obtain

$$\langle Au_n, w_n^\beta \phi\rangle + \int_\Omega |u_n|^{p-2} u_n w_n^\beta \phi\, d\mu_n = \int_\Omega f, w_n^\beta \phi\, dx$$

$$\langle Aw_n, v_\varepsilon w_n^\beta \phi\rangle + \int_\Omega |w_n|^{p-2} w_n v_\varepsilon w_n^\beta \phi\, d\mu_n = \int_\Omega v_\varepsilon w_n^\beta \phi\, dx$$

Then

(7.5) $$\langle Au_n, w_n^\beta \phi\rangle - \langle Aw_n, v_\varepsilon w_n^\beta \phi\rangle + \int_\Omega |u_n|^{p-2} u_n w_n^\beta \phi\, d\mu_n - \int_\Omega w_n^{\beta+p-1} v_\varepsilon \phi\, d\mu_n =$$

$$= \int_\Omega f, w_n^\beta \phi\, dx - \int_\Omega v_\varepsilon w_n^\beta \phi\, dx$$

From Lemmas 5.2 and 5.4 we obtain

$$(7.6)\qquad \langle Au_n, w_n^\beta \phi\rangle - \langle Aw_n, v_\varepsilon w_n^\beta \phi\rangle + \int_\Omega |u_n|^{p-2} u_n w_n^\beta \phi \, d\mu_n - \int_\Omega w_n^{\beta+p-1} v_\varepsilon \phi \, d\mu_n =$$

$$= \langle Au_n, w_n^\beta \phi\rangle - \langle Aw_n, v_\varepsilon w_n^\beta \phi\rangle + \mathcal{R}_n^\varepsilon$$

with

$$\lim_{\varepsilon\to 0} \limsup_{n\to+\infty} |\mathcal{R}_n^\varepsilon| = 0$$

We recall that w_n is bounded in $L^\infty(\Omega)$ then converges strongly in $L^p(\Omega)$ to w. For every $\varepsilon > 0$ we have

$$\lim_{n\to+\infty} \left(\int_\Omega f, w_n^\beta \phi \, dx - \int_\Omega v_\varepsilon w_n^\beta \phi \, dx \right) = \int_\Omega f, w^\beta \phi \, dx - \int_\Omega v_\varepsilon w^\beta \phi \, dx$$

The above relation with (6.5) (6.6) gives

$$\langle Au, w^\beta \phi\rangle - \langle Aw, v_\varepsilon w^\beta \phi\rangle = \int_\Omega f, w^\beta \phi \, dx - \int_\Omega v_\varepsilon w^\beta \phi \, dx + \mathcal{R}^\varepsilon$$

where

$$\lim_{\varepsilon\to 0} |\mathcal{R}^\varepsilon| = 0.$$

Define $\sigma = 1 - Aw$; from Theorem 6.1 we have that σ is a non-negative Radon measure in $H^{-1,q}(\Omega, X)$. We have

$$(7.7)\qquad \langle Au, w^\beta \phi\rangle + \int_\Omega v_\varepsilon w^\beta \phi \, d\sigma = \operatorname*{int}_\Omega f, w^\beta \phi \, dx + \mathcal{R}^\varepsilon$$

We recall that $|u| \leqslant Cw$ q.e. in Ω (Remark 4.2), then from (5.3) we have $v_\varepsilon \leqslant$ $\leqslant (C \vee \varepsilon)^{p-1}$ q.e. in Ω. Recalling the definition of Ψ_ε we obtain the convergence q.e. in Ω of $v_\varepsilon w^\beta$ to $|u|^{p-2} u w^{(\beta-p+1)}$. We recall that σ is a non-negative Radon measure in $H^{-1,q}(\Omega, X)$ and w^β is bounded, so we have $w^\beta \phi \in L^1_\sigma(\Omega)$ and

$$\lim_{\varepsilon\to 0} \int_\Omega v_\varepsilon w^\beta \phi \, d\sigma = \int_\Omega |u|^{p-2} u w^{(\beta-p+1)} \, d\sigma.$$

Then from (7.7) we have

$$(7.8)\qquad \langle Au, w^\beta \phi\rangle + \int_\Omega |u|^{p-2} u w^{(\beta-p+1)} \phi \, d\sigma = \int_\Omega f, w^\beta \phi \, dx$$

We recall that $|u| \leq Cw$ q.e. in Ω (Remark 4.2) and $w \in L^p_\mu(\Omega)$, so $u \in L^p_\mu(\Omega)$ and

$$\int_\Omega |u|^{p-2} uw^{(\beta-p+1)} \phi \, d\sigma = \int_{\{w>0\}} |u|^{p-2} uw^{(\beta-p+1)} \phi \, d\sigma = \int_\Omega |u|^{p-2} uw^\beta \phi \, d\mu$$

then from (7.8)

$$\langle Au, w^\beta \phi \rangle + \int_\Omega |u|^{p-2} uw^\beta \phi \, d\mu = \int_\Omega f, w^\beta \phi \, dx$$

We recall that the set $\{w^\beta \phi, \phi \in \mathcal{D}(\Omega)\}$ is dense in $H^{1,p}_0(\Omega, X) \cap L^p_\mu(\Omega)$, then u is the solution of (7.2) and μ_n γ^A-converges to μ (Remark 7.2). ■

REMARK 7.4: The uniqueness of the γ^A-limit is an easy consequence of Theorem 7.3 and Lemma 6.4.

Compactness and density results.

The following result proves the compactness of $\mathcal{M}^0_p(\Omega, X)$ with respect to the γ^A-convergence

THEOREM 7.5: *Every sequence in $\mathcal{M}^0_p(\Omega, X)$ contains a γ^A-convergent subsequence.*

PROOF: Let μ_n be a sequence in $\mathcal{M}^0_p(\Omega, X)$ and let w_n be the solutions of (7.3). By Theorem 6.1 $w_n \in \mathcal{K}(\Omega)$ (where $\mathcal{K}(\Omega)$ is defined at the beginning of section 6). We recall that $\mathcal{K}(\Omega)$ is compact in the weak topology of $H^{1,p}_0(\Omega, X)$, then a subsequence of w_n converges weakly in $H^{1,p}_0(\Omega, X)$ to some function $w \in \mathcal{K}(\Omega)$. By Theorem 6.1 there is a measure $\mu \in \mathcal{M}^0_p(\Omega, X)$, such that w is solution of (7.4). The result follows from Theorem 6.3. ■

The case of Dirichlet problems in perforated domains is a particular case and it is considered in the following theorem:

THEOREM 7.6: *Let Ω_n be an arbitrary sequence of open subsets of Ω. Then there exists a subsequence, still denoted by Ω_n, and a measure $\mu \in \mathcal{M}^0_p(\Omega, X)$ such that for every $f \in H^{-1,q}(\Omega, X)$ the solution u_n of the problem*

$$u_n \in H^{1,p}_0(\Omega_n, X), \quad Au_n = f \text{ in } \mathcal{D}'(\Omega_n)$$

extended by 0 *to Ω, converges weakly in $H^{1,p}_0(\Omega_n, X)$ to the solution u of problem* (7.2).

PROOF: The conclusion follows easily from Theorem 7.5 and Remark 3.4. ■

Using Theorem 7.3 we prove now the following density result in $\mathcal{M}^0_p(\Omega, X)$:

Theorem 7.7: *Every measure* $\mu \in \mathcal{M}_p^0(\Omega, X)$ *is the* γ^A*-limit of a sequence* μ_n *of Radon measures in* $\mathcal{M}_p^0(\Omega, X)$ *such that the solution* w_n *of* (7.3) *converges strongly in* $H_0^{1,p}(\Omega, X)$ *to the solution of* (7.4).

Proof: By (6.2) a measure μ in $\mathcal{M}_p^0(\Omega, X)$ is a Radon measure if the solution w of (7.4) is such that

$$\inf_K w > 0 \quad \text{for every compact set } K \subset \Omega \tag{7.9}$$

We denote by $w_0 \in H_0^{1,p}(\Omega, X)$ the solution of the equation $Aw_0 = 1$, then w_0 satisfies (7.9) [8].

Fix $\mu \in \mathcal{M}_p^0(\Omega, X)$ and denote by $w \in \mathcal{K}(\Omega)$ the solution of (7.4). We define $w_n = w \vee \frac{1}{n} w_0$. It is easy to see that w_n is a non-negative subsolution of the equation $Au = 1$, so $w_n \in \mathcal{K}(\Omega)$. Moreover the function w_n satisfies (7.9) and converges strongly to w in $H_0^{1,p}(\Omega, X)$. Then the measures μ_n associated with w_n, which are Radon measures according to (7.9), γ^A converge to μ by Theorem 7.3. ■

Strong convergence and correctors.

The following result deals with the convergence of solutions, momenta and energies, when also f varies.

Theorem 7.8: *Let* μ_n *be a sequence of measures in* $\mathcal{M}_p^0(\Omega, X)$, *which* γ^A*-converges to the measure* $\mu \in \mathcal{M}_p^0(\Omega, X)$ *and let* f_n *be a sequence in* $H^{-1,q}(\Omega, X)$, *which converges to* f *in* $H^{-1,q}(\Omega, X)$. *Define* u_n *as the solution of the problem*

$$\begin{gathered} u_n \in H_0^{1,p}(\Omega, X) \cap L^p_{\mu_n}(\Omega) \\ \langle Au_n, v\rangle + \int_\Omega |u_n|^{p-2} u_n v \, d\mu_n = \langle f_n, v\rangle \\ \forall v \in H_0^{1,p}(\Omega, X) \cap L^p_{\mu_n}(\Omega) \end{gathered} \tag{7.10}$$

and u *as the solution of problem* (7.2). *Then the sequence* u_n *converges to* u *weakly in* $H_0^{1,p}(\Omega, X)$ *and strongly in* $H_0^{1,r}(\Omega, X)$, $1 < r < p$. *Moreover* $|Xu_n|^{p-2} Xu_n$ *converges to* $|Xu|^{p-2}Xu$ *weakly in* $L^q(\Omega)$ *and strongly in* $L^s(\Omega)$, $1 < s < q$. *Finally the energies* $|Xu_n|^p\, dx + |u_n|^p\, d\mu_n$ *converge to* $|Xu|^p\, dx + |u|^p\, d\mu$ *weakly* in the sense of Radon measures on* Ω, *i.e.*

$$\lim_{n \to +\infty} \left(\int_\Omega |Xu_n|^p \phi \, dx + \int_\Omega |u_n|^p \phi \, d\mu_n \right) = \left(\int_\Omega |Xu|^p \phi \, dx + \int_\Omega |u|^p \phi \, d\mu \right) \tag{7.11}$$

for every $\phi \in C_0(\Omega)$.

PROOF: Define v_n as the solution of problem (7.1). From Theorem 3.3 the sequence $(u_n - v_n)$ converges to 0 strongly in $H_0^{1,p}(\Omega, X)$. Using Theorem 3.11 we easily obtain that u_n converges to u weakly in $H_0^{1,p}(\Omega, X)$ and strongly in $H_0^{1,r}(\Omega, X)$, $1 < r < p$. Moreover $|Xu_n|^{p-2}Xu_n$ converges to $|Xu|^{p-2}Xu$ weakly in $L^q(\Omega)$ and strongly in $L^s(\Omega)$, $1 < s < q$.

By (7.10) for every $\phi \in \mathcal{D}(\Omega)$ we have

$$\int_\Omega |Xu_n|^p \phi\, dx + \int_\Omega |u_n|^p \phi\, d\mu_n =$$

$$= \int_\Omega |Xu_n|^{p-2} Xu_n X(u_n\phi)\, dx + \int_\Omega |u_n|^p \phi\, d\mu_n - \int_\Omega u_n |Xu_n|^{p-2} Xu_n X\phi\, dx =$$

$$= \langle f_n, u_n\phi \rangle - \int_\Omega u_n |Xu_n|^{p-2} Xu_n X\phi\, dx.$$

We recall that f_n converges to f in $H^{-1,q}(\Omega, X)$, u_n converges to u weakly in $H_0^{1,p}(\Omega, X)$ and $|Xu_n|^{p-2}Xu_n$ converges to $|Xu|^{p-2}Xu$ weakly in $L^q(\Omega)$ then

$$\lim_{n\to+\infty}\left(\int_\Omega |Xu_n|^p \phi\, dx + \int_\Omega |u_n|^p \phi\, d\mu_n\right) =$$

$$= \langle f, u\phi \rangle - \int_\Omega u|Xu|^{p-2} Xu X\phi\, dx = \left(\int_\Omega |Xu|^p \phi\, dx + \int_\Omega |u|^p \phi\, d\mu\right)$$

for every $\phi \in \mathcal{D}(\Omega)$ (in the second equality we use the fact that u is the solution of (7.2)).

An easy approximation give now (7.11) for all $\phi \in C_0(\Omega)$. ■

We consider now a corrector result for the strong convergence in $H_0^{1,p}(\Omega, X)$.

THEOREM 7.9: *Under the assumptions of Theorem* 7.8 *let* P_n *be the correctors defined by* (4.4), *where* u_n *and* w *are the solutions of* (7.3), (7.4). *Then for every* $\varepsilon > 0$ *there exists a function* u^ε *in* $H_0^{1,p}(\Omega, X) \cap L^\infty(\Omega) \cap L_\mu^p(\Omega)$ *such that* $\|u^\varepsilon - u\|_{H_0^{1,p}(\Omega, X)}$ *and* $|u^\varepsilon| \leq C^\varepsilon w$ *for same constant* C^ε, *such that the sequence* R_n^ε *defined by*

$$Xu_n = Xu + u^\varepsilon P_n + R_n^\varepsilon \tag{7.12}$$

satisfies

$$\limsup_{n\to+\infty} \|R_n^\varepsilon\|_{L^p(\Omega)} \leq \varepsilon. \tag{7.13}$$

If $f \in L^\infty(\Omega)$ *we can take* $\varepsilon \geq 0$ *and* $u^\varepsilon = u$.

PROOF: If $f \in L^\infty(\Omega)$ the result follows from Theorem 4.1. When $f \in H^{-1,q}(\Omega, X)$ for every $\varepsilon > 0$, $\alpha > 0$, $K > 0$ we can choose $f^\varepsilon \in L^\infty(\Omega)$ such that $\|f - f^\varepsilon\|_{H^{-1,q}(\Omega, X)} \leq$ $\leq (K\varepsilon)^\alpha$. If $p \geq 2$ we choose $\alpha = p - 1$ and $K = \frac{1}{2} C^{1/p}$, where C is the constant appearing in (7.9). If $1 < p < 2$ we choose $\alpha = 1$, $K = M \wedge 1$ where

$$M = \frac{1}{2} C^{-1} 2^{\frac{p-2}{p}} (\|f\|_{H^{-1,q}(\Omega, X)} + \varepsilon)^{\frac{p-2}{p-1}}$$

and C is the constant appearing in (3.10).

We define v_n as the solution of problem (7.1), v_n^ε as the solution of the analogous problem relative to f^ε and u^ε the solution of problem (7.2) relative to f^ε. For $\varepsilon > 0$ fixed the sequence v_n^ε converges to u^ε weakly in $H_0^{1,p}(\Omega, X)$.

From Theorem 3.3 we deduce that

$$\|u - u^\varepsilon\|_{H_0^{1,p}(\Omega, X)} \leq \frac{\varepsilon}{2} \tag{7.14}$$

$$\|v_n - v_n^\varepsilon\|_{H_0^{1,p}(\Omega, X)} \leq \frac{\varepsilon}{2} \tag{7.14'}$$

and from Remark 4.2 we have that $u^\varepsilon \in L^\infty(\Omega)$ and $u^\varepsilon \leq C^\varepsilon w$ q.e. in Ω with $C^\varepsilon = \|f^\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}$. The result of Theorem 4.1 gives that the sequence Q_n^ε defined by

$$X v_n^\varepsilon = X u^\varepsilon + u^\varepsilon P_n + Q_n^\varepsilon \tag{7.15}$$

converges to 0 strongly in $L^p(\Omega)$ for $\varepsilon > 0$ fixed and $n \to +\infty$.

From (7.12) (7.15) we obtain

$$R_n^\varepsilon = Q_n^\varepsilon + (Xu^\varepsilon - Xu) + (Xv_n^\varepsilon - Xv_n) + (Xu_n - Xv_n).$$

Then (7.13) follows from (7.14), (7.14'), (7.15) and from the fact that $(u_n - v_n)$ converges to 0 strongly in $H_0^{1,p}(\Omega, X)$ by Theorem 3.3. ■

THEOREM 7.10: *Under the assumptions of Theorem* 7.8 *if the solution* w_n *of* (7.3) *converges strongly in* $H_0^{1,p}(\Omega, X)$ *to the solution* w *of* (7.4), *then* u_n *converges strongly to* u *in* $H_0^{1,p}(\Omega, X)$.

PROOF: For every ε let u^ε be the function introduced in Theorem 7.9. The function $\frac{u^\varepsilon}{w}$ is bounded on $\{w > 0\}$, if w_n converges to w strongly in $H_0^{1,p}(\Omega, X)$; then for $\varepsilon > 0$ fixed $u^\varepsilon P_n$ converges strongly to 0 in $L^p(\Omega)$.

The result follows from (7.12) (7.13). ■

Localization properties.

We end the section by proving the local character of the γ^A-convergence. The following result deals with local solutions in an open subset U of Ω and we do not pay any care to the boundary conditions on ∂U. For every open set $U \subset \Omega$ the duality pairing between $H^{-1,q}(U, X)$ and $H_0^{1,p}(U, X)$ is denoted by $\langle .,. \rangle_U$. The operator

$\sum_{i=1}^{m} X_i^\star(|Xv|^{p-2}Xv)$ as operator from $H_0^{1,p}(U, X)$ to $H^{-1,q}(U, X)$, will still be denoted by A.

THEOREM 7.11: *Let μ_n be a sequence of measures in $\mathcal{M}_p^0(\Omega, X)$, which γ^A-converges to the measure $\mu \in \mathcal{M}_p^0(\Omega, X)$. Let U be an open subset of Ω, let f_n be a sequence in $H^{-1,q}(U, X)$, which converges to f in $H^{-1,q}(U, X)$ and let u_n be a sequence in $H^{1,p}(U, X)$, which converges weakly to some u in $H^{1,p}(U, X)$.*

Suppose that

$$u_n \in L^p_{\mu_n}(U'), \quad \forall U' \subset\subset U$$

$$\langle Au_n, v\rangle_U + \int_U |u_n|^{p-2} u_n v \, d\mu_n = \langle f_n, v\rangle_U \tag{7.16}$$

$$\forall v \in H_0^{1,p}(U, X) \cap L^p_{\mu_n}(U) \text{ with } supp(v) \subset\subset U$$

$$u \in L^p_{\mu}(U'), \quad \forall U' \subset\subset U$$

$$\langle Au, v\rangle_U + \int_U |u|^{p-2} uv \, d\mu = \langle f, v\rangle_U \tag{7.17}$$

$$\forall v \in H_0^{1,p}(U, X) \cap L^p_{\mu}(U) \text{ with } supp(v) \subset\subset U$$

We have that u_n converges to u strongly in $H^{1,r}(U, X)$, $1 < r < p$ and $|Xu_n|^{p-2}Xu_n$ converges to $|Xu|^{p-2}Xu$ weakly in $L^q(U)$ and strongly in $L^s(U)$, $1 < s < q$. Finally the energy $|Xu_n|^p dx + |u_n|^p d\mu_n$ converges to $|Xu|^p dx + |u|^p d\mu$ weakly* in the sense of Radon measures.

PROOF: Fix an open set $U' \subset\subset U$ and a function $\zeta \in Lip(U; X)$ such that $\zeta \geq 0$ on U, $\zeta = 1$ on U', $supp(\zeta) \subset U$.

We use $v = \zeta u_n$ as test function in (7.16) and we obtain

$$\int_{U'} |u_n|^p d\mu_n \leq \langle f_n, \zeta u_n\rangle_U - \langle Au, \zeta u_n\rangle_U \leq M$$

for a suitable constant M. By Theorem 3.11 the sequence u_n converges to u weakly in $H^{1,p}(U', X)$ and strongly in $H^{1,r}(U', X)$, $1 < r < p$; moreover $|Xu_n|^{p-2}Xu_n$ converges to $|Xu|^{p-2}Xu$ weakly in $L^q(U')$ and strongly in $L^s(U')$, $1 < s < q$. We recall that u_n is bounded in $H^{1,p}(U, X)$ and $|Xu_n|^{p-2}Xu_n$ is bounded in $L^q(U)$; then, since $U' \subset\subset U$ is arbitrary, we have that u_n converges to u weakly in $H^{1,p}(U, X)$ and strongly in $H_0^{1,r}(U, X)$, $1 < r < p$; moreover $|Xu_n|^{p-2}Xu_n$ converges to $|Xu|^{p-2}Xu$ weakly in $L^q(U)$ and strongly in $L^s(U)$, $1 < s < q$.

Define $\phi(x) = exp\left(1 - \frac{1}{\zeta(x)}\right)$ if $\zeta(x) > 0$ and $\phi(x) = 0$ if $\zeta(x) = 0$. Then $\phi \in Lip(U; X) \cap H_0^{1,p}(U, X)$, $\phi \geq 0$ in U, $\phi = 1$ in U' and $\phi^{p-1} \in Lip(U; X) \cap H_0^{1,p}(U, X)$, so $\phi^{p-1} v \in H_0^{1,p}(U, X)$ for every v in $H_0^{1,p}(U, X)$.

Let us define $z_n = \phi u_n$, $z = \phi u$ and

$$\psi_n = |\phi X u_n + u_n X\phi|^{p-2}(\phi X u_n + u_n X\phi) - |\phi X u_n|^{p-2}\phi X u_n$$
$$\psi = |\phi X u + u X\phi|^{p-2}(\phi X u + u X\phi) - |\phi X u|^{p-2}\phi X u.$$

For every $v \in H_0^{1,p}(\Omega, X)$ we have

$$|Xz_n|^{p-2} Xz_n Xv = \psi_n Xv + \phi^{p-1}|Xu_n|^{p-2}Xu_n =$$
$$= \psi_n Xv + |Xu_n|^{p-2}Xu_n X(\phi^{p-1}v) - v|Xu_n|^{p-2}Xu_n X(\phi^{p-1})$$

The function z_n is the solution of the problem

$$\begin{gathered} z_n \in H_0^{1,p}(\Omega, X) \cap L^p_{\mu_n}(\Omega) \\ \langle Az_n, v\rangle_\Omega + \int_\Omega |z_n|^{p-2} z_n v \, d\mu_n = \langle g_n, v\rangle_\Omega \\ \forall v \in H_0^{1,p}(\Omega, X) \cap L^p_{\mu_n}(\Omega) \end{gathered} \tag{7.18}$$

where $g_n \in H^{-1,q}(\Omega, X)$ is defined as

$$\langle g_n, v\rangle_\Omega = \int_U \psi_n Xv \, dx + \langle f_n, \phi^{p-1}v\rangle_U - \int_U v|Xu_n|^{p-2}Xu_n X(\phi^{p-1})\, dx$$

Define $g \in H^{-1,q}(\Omega, X)$ as

$$\langle g, v\rangle_\Omega = \int_U \psi Xv \, dx + \langle f, \phi^{p-1}v\rangle_U - \int_U v|Xu|^{p-2}XuX(\phi^{p-1})\, dx$$

We prove now that g_n converges to g in $H^{-1,q}(\Omega, X)$.

We have that $|Xu_n|^{p-2}Xu_n$ converges to $|Xu|^{p-2}Xu$ weakly in $L^q(U)$, then the last two terms in g_n converges strongly in $H^{-1,q}(\Omega, X)$ to the corresponding terms in g. To conclude we have to prove that ψ_n converges to ψ strongly in $L^q(U)$. We recall that u_n converges to u strongly in $H^{1,r}(\Omega, X)$, $1 < r < p$, so we may assume that u_n, Xu_n converge to u, Xu a.e. in U. Then ψ_n converges to ψ a.e. in U.

It remain to prove that the sequence $|\psi_n|^q$ is equi-integrable.

If $p \geqslant 2$ there exists a constant C such that

$$|\psi_n|^q \leqslant C(|Xu_n|^{q(p-2)} + |u_n|^{q(p-2)})|u_n|^q = C(|Xu_n|^{q(p-2)}|u_n|^q + |u_n|^p)$$

a.e. in U. We have $supp(\psi_n) \subset supp(\phi)$; then for every measurable $E \subset U$ the Hölder inequality gives

$$\int_E |\psi_n|^q dx \leqslant C\left[\left(\int_U |Xu_n|^p dx\right)^{\frac{p-2}{p-1}}\left(\int_{E\cap K} |u_n|^p dx\right)^{\frac{q}{p}} + \left(\int_{E\cap K} |u_n|^p dx\right)\right]$$

where $K = supp(\phi)$. The sequence u_n converges strongly in $L^p(K)$; then the sequence $|\psi_n|^q$ is equi-integrable.

If $1 < p < 2$ we have

$$|\psi_n|^q \leqslant C_1^q |X\phi|^p |u_n|^p$$

a.e. in U; then the sequence $|\psi_n|^q$ is equi-integrable.

The Dominated Convergence Theorem gives that ψ_n converges to ψ in $L^q(U)$.

We recall that z_n converges to z weakly in $H_0^{1,p}(\Omega, X)$, then by (7.18) and Theorem 6.8 z is the solution of the problem

$$
\begin{gathered}
z \in H_0^{1,p}(\Omega, X) \cap L_\mu^p(\Omega) \\
\langle Az, v\rangle_\Omega + \int_\Omega |z|^{p-2} zv \, d\mu = \langle g, v\rangle_\Omega \qquad (7.19) \\
\forall v \in H_0^{1,p}(\Omega, X) \cap L_\mu^p(\Omega)
\end{gathered}
$$

Since $\phi = 1$ in U' we have $u = z$ in U' then $u \in L_\mu^p(U')$. Moreover if $v \in H_0^{1,p}(\Omega, X) \cap L_\mu^p(\Omega)$ with $supp(v) \subset\subset U'$, then $\langle g, v\rangle_\Omega = \langle g, v\rangle_U$, then (7.17) follows from (7.19).

The convergence of the energies follows as in Theorem 7.8. ∎

THEOREM 7.12: *Let μ_n be a sequence of measures in $\mathcal{M}_p^0(\Omega, X)$, which γ^A-converges to the measure $\mu \in \mathcal{M}_p^0(\Omega, X)$ and let U an open subset of Ω then μ_n γ^A-converges to the measure μ*

PROOF: Fix $f \in H^{-1,q}(\Omega, X)$ and denote by u_n the solution of the problem (7.1) with Ω replaced by U. There is a subsequence, still denoted by u_n, that converges weakly in $H_0^{1,p}(U, X)$ to a function $u \in H_0^{1,p}(U, X)$. From Theorem 7.11 $u \in L_\mu^p(U')$ for every open set $U' \subset\subset U$ and u is a solution of (7.17).

To conclude the proof we have to prove that $u \in L_\mu^p(U)$. We consider a sequence v_n such that v_n converges strongly to u in $H_0^{1,p}(U, X)$, $supp(v_n) \subset\subset U$, $|v_n| \leqslant |u|$ q.e. in U and $uv_n \geqslant 0$ q.e. in U. We recall that $u \in L_\mu^p(U')$ for every open set $U' \subset\subset U$; then $v_n \in L_\mu^p(U)$. We may also assume that v_n converges to u q.e. in U, then

$$\int_U |u|^p \, d\mu = \liminf_{n \to +\infty} \int_U |u|^{p-2} uv_n \, d\mu .$$

Use v_n as test function in (7.17) we obtain

$$\int_U |u|^{p-2} uv_n \, d\mu = \langle f, v_n\rangle_U - \langle Au, v_n\rangle_U$$

Then

$$\int_U |u|^p \, d\mu = \langle f, u\rangle_U - \langle Au, u\rangle_U < +\infty$$

so $u \in L_\mu^p(U)$ and u is the solution of problem (7.2) with Ω replaced by U. From the uniqueness of the solution of problem (7.2) all the sequences u_n converges to u and the proof is complete. ■

Corollary 7.13: *Let $\mu, \mu_n \in \mathcal{M}_p^0(\Omega, X)$ and Ω_i a family of open subsets of Ω, which covers Ω. Then μ_n γ^A-converges to the measure μ in Ω if and only if μ_n γ^A-converges to μ in Ω_i for every i.*

Proof: The conclusion follows by Theorems 7.5, 7.12 and from the uniqueness of the γ^A-limit. ■

Appendix

Here we generalize to the subelliptic framework some results given for the euclidean framework in [6] (see also [18]).

We consider a sequence of subelliptic Leray-Lions operators on $H^{1,p}(\Omega, X)$ of the form

$$\sum_{i=1}^{m} X_i^\star (a_k(x, u, Xu))$$

where a_k: $\Omega \times R \times R^m \to R$ satisfies a Carathéodory conditions, i.e. $a_k(., y, \xi)$ is measurable for every $y \in R$ and $\xi \in R^m$ and $a_k(x,.,.)$ is continuous on $R \times R^M$ for a.e. $x \in \Omega$. We also assume that

$$\text{(A.1)} \qquad |a_k(x, y, \xi)| \leq \alpha(x) + A(|y|^{p-1} + |\xi|^{|} p - 1)$$

$$\text{(A.2)} \qquad (a_k(x, y, \zeta) - a_k(x, y, \xi))(\zeta - \xi) > 0$$

where $\alpha \in L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$. We assume that there exists $a : \Omega \times R \times R^m \to R$ such that for a.e. $x \in \Omega$

$$a_k(x,.,.) \to a(x,.,.)$$

uniformly on compact sets of $R \times R^m$, then also a satisfies the condition (A.2). Let $u_k \in H^{1,p}(\Omega, X)$, $f_k \in H^{-1,q}(\Omega, X)$ and μ_k Radon measures be sequences such that

$$u_k \to u \text{ weakly in } H^{1,p}(\Omega, X)$$

$$f_k \to f \text{ weakly in } H^{-1,q}(\Omega, X)$$

$$\mu_k \to \mu \text{ weakly}^\star \text{ in the space of Radon measures.}$$

Finally we assume

$$\sum_{i=1}^{m} X_i^\star a_k(x, u_k, Xu_k) = f_k + \mu_k \text{ in } \mathcal{D}'(\Omega)$$

i.e.

$$\int_{\Omega} a_k(x, u_k, Xu_k)\, Xv\, dx = \langle f_k, u_k \rangle + \int_{\Omega} v\, d\mu_k$$

The following result holds:

THEOREM A.1: *Let the above assumptions hold. Then Xu_k converges to Xu strongly in $L^r(\Omega)$, $1 < r < p$.*

PROOF: We observe that since the sequence u_k weakly converges to u in $H_{1,p}(\Omega, X)$ then the sequence Xu_k is bounded in $L^p(\Omega)$. Then to prove the result it is enough to prove that every subsequence of Xu_k contains a subsequence which converges to Xu a.e. in Ω. We denote

$$g_k = (a_k(x, u_k, Xu_k) - a_k(x, u, Xu))(Xu_k - Xu)$$

To prove the result it is enough to prove that g_k converges to 0 a.e. in Ω.

We have for $K \subset \Omega$ compact and $\delta > 0$, $0 < \theta < 1$

$$\int_K g_k^\theta\, dx = \int_{K \cap \{|u_k - u| < \delta\}} g_k^\theta\, dx + \int_{K \cap \{|u_k - u| \geq \delta\}} g_k^\theta\, dx \leq$$

$$\leq m(\Omega)^{1-\theta} \left(\int_{K \cap \{|u_k - u| < \delta\}} g_k\, dx \right)^\theta + m(K \cap \{|u_k - u| \geq \delta\})^{1-\theta} \left(\int_K g_k\, dx \right)^\theta$$

The above relation implies

$$\lim_{k \to +\infty} \int_K g_k^\theta\, dx \leq m(\Omega)^{1-\theta} \lim_{k \to +\infty} \left(\int_{K \cap \{|u_k - u| < \delta\}} g_k\, dx \right)^\theta.$$

Let ψ of class $C^1(R)$ such that

$$\psi(y) = y \text{ for } |y| \leq 1;\ \psi(y) = 0 \text{ for } |y| \geq 2;\ \psi(y) \leq 2$$
$$|\psi'| \leq M$$

We denote

$$\psi_\delta = \delta \psi \left(\frac{y}{\delta} \right)$$

Then

$$\psi_\delta(y) = y \text{ for } |y| \leq \delta;\ \psi_\delta(y) = 0 \text{ for } |y| \geq 2\delta;\ \psi_\delta(y) \leq 2\delta$$
$$|X\psi_\delta| \leq M.$$

In the set $\{|u_k - u| < \delta\}$ we have

$$g_k = (a_k(x, u_k, Xu_k) - a_k(x, u, Xu))\, X(\psi_\delta(u_k - u)).$$

Let $\phi \in \mathcal{D}(\Omega)$, $\phi = 1$ on K, $0 \leqslant \phi \leqslant 1$ on Ω, we denote $U = supp(\phi)$. We have

$$\text{(A.1)} \quad \int\limits_{K \cap \{|u_k - u| < \delta\}} g_k\, dx = \int\limits_{K \cap \{|u_k - u| < \delta\}} g_k \phi\, dx =$$

$$= \int\limits_{K \cap \{|u_k - u| < \delta\}} (a_k(x, u_k, Xu_k) - a_k(x, u, Xu))\, X(\psi_\delta(u_k - u))\, \phi\, dx =$$

$$= \int\limits_\Omega (a_k(x, u_k, Xu_k) - a_k(x, u, Xu))\, X(\psi_\delta(u_k - u))\, \phi\, dx +$$

$$+ \int\limits_{\{\delta \leqslant |u_k - u| < 2\delta\}} (a_k(x, u_k, Xu_k) - a_k(x, u, Xu))\, X(\psi_\delta(u_k - u))\, \phi\, dx \leqslant$$

$$\leqslant \int\limits_\Omega (a_k(x, u_k, Xu_k) - a_k(x, u, Xu))\, X(\psi_\delta(u_k - u))\, \phi\, dx +$$

$$+ M \int\limits_{U \cap \{\delta \leqslant |u_k - u| < 2\delta\}} (|a_k(x, u_k, Xu_k)| + |a_k(x, u, Xu)|)(|Xu_k| + |Xu|)\, dx$$

We estimate now the first term in the left hand side.

$$\int\limits_\Omega (a_k(x, u_k, Xu_k) - a_k(x, u, Xu)) X(\psi_\delta(u_k - u)) \phi\, dx =$$

$$= \int\limits_\Omega a_k(x, u_k, Xu_k)\, X(\psi_\delta(u_k - u)\, \phi)\, dx -$$

$$- \int\limits_\Omega \psi_\delta(u_k - u)\, a_k(x, u_k, Xu_k)\, X(\phi)\, dx - \int\limits_\Omega a_k(x, u, Xu)\, X(\psi_\delta(u_k - u))\, \phi\, dx =$$

$$= \langle f_k + \mu_k, \psi_\delta(u_k - u)\, \phi \rangle -$$

$$- \int\limits_\Omega \psi_\delta(u_k - u)\, a_k(x, u_k, Xu_k)\, X(\phi)\, dx - \int\limits_\Omega a_k(x, u, Xu)\, X(\psi_\delta(u_k - u))\, \phi\, dx.$$

Let $\mu_k(U) \leqslant M_U$, then

$$|\langle \mu_k, \psi_\delta(u_k - u)\, \phi \rangle| \leqslant 2\delta M_U$$

So

$$\int\limits_\Omega (a_k(x, u_k, Xu_k) - a_k(x, u, Xu))\, X(\psi_\delta(u_k - u))\, \phi\, dx \leqslant$$

$$2\delta M_U + |\langle f_k, \psi_\delta(u_k - u) \phi \rangle| +$$

$$\left| \int\limits_\Omega \psi_\delta(u_k - u)\, a_k(x, u_k, Xu_k)\, X(\phi)\, dx \right| + \left| \int\limits_\Omega a_k(x, u, Xu)\, X(\psi_\delta(u_k - u))\, \phi\, dx \right|.$$

We obtain

$$\limsup_{k \to +\infty} \int_{\Omega} (a_k(x, u_k, Xu_k) - a_k(x, u, Xu))\, X(\psi_\delta(u_k - u))\, \phi\, dx = 2\delta M_U$$

where we use the Dominated Convergence Theorem.

We now consider the second term in the right hand side of (A.1). Let

$$h_k = M(|a_k(x, u_k, Xu_k)| + |a_k(x, u, Xu)|)(|Xu_k| + |Xu|)$$

We have

$$\|h_k\|_{L^1(U)} \leqslant C$$

Moreover $\forall \varepsilon > 0$, there are $S > 0$ integer such that $\frac{C}{S} < \varepsilon$ and $\eta > 0$ with $2^S \eta < \varepsilon$. We have

$$\sum_{i=1}^{S} \int_{U \cap \{2^{i-1}\eta \leqslant |u_k - u| < 2^i \eta\}} h_k\, dx \leqslant \int_U h_k\, dx \leqslant C$$

There at least one term in the left hand side less than $\frac{C}{S}$, i.e. there exists i_k such that

$$\int_{U \cap \{2^{i_k - 1}\eta \leqslant |u_k - u| < 2^{i_k}\eta\}} h_k\, dx \leqslant \frac{C}{S} < \varepsilon .$$

Denote $\delta_k = 2^{i_k - 1}\eta$, then $2\delta_k \leqslant 2^S \eta < \varepsilon$ and

$$\int_{U \cap \{\delta_k \leqslant |u_k - u| < 2\delta_k\}} h_k\, dx < \varepsilon .$$

Choose now in (A.1) $\delta = \eta \leqslant \delta_k$ we obtain

$$\limsup_{k \to +\infty} \int_{K \cap \{|u_k - u| < \eta\}} g_k\, dx \leqslant 2\eta M_U + \varepsilon \leqslant (M_U + 1)\varepsilon .$$

Then

$$\limsup_{k \to +\infty} \int_K g_k^\theta\, dx \leqslant$$

$$m(\Omega)^{1-\theta} \left(\limsup_{k \to +\infty} \int_{K \cap \{|u_k - u| < \delta\}} g_k \right)^{\theta} \leqslant$$

$$\leqslant m(\Omega)^{1-\theta} (M_U + 1)^\theta \varepsilon^\theta .$$

As $\varepsilon > 0$ is arbitrary we obtain

$$\lim_{k \to +\infty} \int_K g_k^\theta\, dx = 0$$

so g_k converges to 0 in $L^1(K)$. Since K is an arbitrary compact set in Ω we have that g_k

converges to 0 in $L^1_{\mathrm{loc}}(\Omega)$, so every subsequence of g_k contains a further subsequence converging to 0 a.e. in Ω. Then every subsequence of Xu_k contains a further subsequence converging to Xu a.e. in Ω and this implies that every subsequence of u_k contains a further subsequence converging to u in $H^{1,r}(\Omega, X)$, $1 < r < p$ and this concludes the proof. ∎

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