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On the spectrum of the Laplace-Beltrami operator for p-forms on asymptotically hyperbolic manifolds (**)

SUMMARY. — Under suitable conditions on the asymptotic decay of the metric, we compute the essential spectrum of the Laplace-Beltrami operator acting on p-forms on asymptotically hyperbolic manifolds.

Sullo spettro dell'operatore di Laplace-Beltrami per le p-forme su varietá asintoticamente iperboliche

RIASSUNTO. — Sotto opportune ipotesi sull'andamento asintotico della metrica, si calcola lo spettro essenziale dell'operatore di Laplace-Beltrami per le p-forme su varietá asintoticamente iperboliche.

1. - INTRODUCTION

The spectrum of the Laplace-Beltrami operator on complete noncompact Riemannian manifolds in its relationships with the geometric properties of the manifold has been investigated by many authors. In the case of a general Riemannian manifold the problem turns out to be very difficult, because of the lack of powerful analytic tools such as the Fourier transform. Hence the attention has mainly focused on particular classes of Riemannian manifolds, in which these difficulties can be bypassed thanks to the presence of symmetries or to the imposition of a «controlled» asymptotic behaviour of the Riemannian metric.

This is the case for manifolds endowed with rotationally symmetric Riemannian metrics, where a decomposition technique introduced by Dodziuk in [2] and then

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employed by Eichhorn ([4]) and Donnelly ([3]) considerably simplifies the problem. By this technique, Dodziuk obtained in [2] results on the existence and multiplicity of L^2 harmonic forms for a Riemannian metric which can be expressed, in geodesic coordinates, as

$$dt^2 + g(t)\, d\theta^2, \tag{1.1}$$

where $g(t)$ is a positive function and $d\theta^2$ is the standard metric on the sphere $\mathbb{S}^{N-1}$. These techniques were then employed by Eichhorn in [4] for his results on the discreteness of the spectrum of the Laplace-Beltrami operator for Riemannian metrics of type (1.1), and by Donnelly in [3] in his computation of the spectrum of the Laplace-Beltrami operator on the hyperbolic space $\mathbb{H}^n$.

A completely different approach to this kind of problems can be found in [5], [6] and [7], where the essential spectrum is determined on conformally compact Riemannian manifolds through the sophisticated machinery of the pseudodifferential calculus on manifolds developed by Melrose (see [8] and the references therein).

In the present paper we consider a noncompact Riemannian N-dimensional manifold endowed with a Riemannian metric of type

$$ds^2 = f(t)\, dt^2 + g(t)\, d\theta^2, \tag{1.2}$$

where $t \in [0, +\infty)$, $d\theta^2$ is the standard metric on $\mathbb{S}^{N-1}$, $f(t) > 0$ and $g(t) > 0$. We suppose that ds^2 is asymptotically hyperbolic, that is $f(t) \to 1$ and $g(t) \to \sinh^2 t$ as $t \to + \infty$. As for the behaviour at $t = 0$, we suppose that $f(t) = 1$ and $g(t) = t^2$ in a neighbourhood of 0. Via decomposition and perturbation techniques, we compute the essential spectrum of the Laplace-Beltrami operator on p-forms, under suitable hypothesis on the rate of convergence of the metric (1.2) to the hyperbolic metric

$$dt^2 + \sinh^2 t\, d\theta^2.$$

The main result is the following (Theorem 5.11). Let us define

$$\tilde{f}(t) := f(t) - 1,$$

$$\tilde{g}(t) := g(t) - \sinh^2 t;$$

if for $t \gg 0$

$$|\tilde{g}(t)| \leq \frac{C}{t}, \quad \left| \frac{\partial \tilde{g}}{\partial t} \right| \leq \frac{C}{t}, \quad \left| \frac{\partial^2 \tilde{g}}{\partial t^2} \right| \leq \frac{C}{t}, \tag{1.3}$$

$$|\tilde{f}(t)| \leq \frac{C}{t}, \quad \left| \frac{\partial \tilde{f}}{\partial t} \right| \leq \frac{C}{t}, \quad \left| \frac{\partial^2 \tilde{f}}{\partial t^2} \right| \leq \frac{C}{t}, \tag{1.4}$$

then the essential spectrum of the Laplace-Beltrami operator is the interval

$$\left[\min\left\{\left(\frac{N-2p-1}{2}\right)^2, \left(\frac{N-2p+1}{2}\right)^2\right\}, +\infty\right)$$

if $N \neq 2p$, whilst for $N = 2p$ it is equal to

$$\{0\} \cup \left[\frac{1}{4}, +\infty\right).$$

The assumptions (1.4), (1.3), though rather general, can be probably weakened. It would be interesting to get to a more precise knowledge of the spectrum of Δ_M, in particular as concerns the absolutely continuous spectrum; however, this seems difficult, because of the lack of a completely developed Fourier theory for p-forms on the hyperbolic space $\mathbb{H}^N$, which would permit to understand whether a perturbation of the Laplace-Beltrami operator is trace-class or not.

The paper is organized as follows. In section 2, we construct an explicit model of asymptotically hyperbolic manifold, endowing the interior of the unit ball B^N in $\mathbb{R}^N$ with a Riemannian metric of type (1.2), where $t = \operatorname{settanh}(\|\bar{x}\|)$. Moreover, we introduce notations and some preliminaries which will be useful in the subsequent sections. In section 3 we prove a generalization of the result by Dodziuk in [2] to the case of a metric of type (1.2); slightly modifying Dodziuk's proof we give necessary and sufficient conditions for the existence of L^2 harmonic p-forms on M, and we determine their multiplicity. We then apply the result to the present situation, proving that for an asymptotically hyperbolic Riemannian manifold $0 \in \sigma_p(\Delta_M)$ if and only if $p = \frac{N}{2}$. Moreover, we show that in this case 0 belongs also to the essential spectrum since it is an eigenvalue of infinite multiplicity. In section 4, we first introduce an orthogonal decomposition of $L^2_p(M)$ analogous to those employed by Eichhorn and by Donnelly (see [4] and [3]). The decomposition is obtained in two steps; first, thanks to the Hodge decomposition on $\mathbb{S}^{N-1}$, we write any p-form ω as

$$\omega = \omega_{1\delta} \oplus \omega_{2d} \wedge dt \oplus (\omega_{1d} \oplus \omega_{2\delta} \wedge dt),$$

where $\omega_{1\delta}$ (resp. ω_{1d}) is a coclosed (resp. closed) p-form on $\mathbb{S}^{N-1}$ parametrized by t, and $\omega_{2\delta}$ (resp. ω_{2d}) is a coclosed (resp. closed) $(p-1)$-form on $\mathbb{S}^{N-1}$ parametrized by t. The decomposition is orthogonal in L^2 and Δ_M splits accordingly as

$$\Delta_M = \Delta_{M1} \oplus \Delta_{M2} \oplus \Delta_{M3}.$$

This allows to reduce ourselves to the study of the spectral properties of Δ_{Mi}, $i = 1, 2, 3$.

The second step consists in decomposing $\omega_{1\delta}$ (resp. ω_{2d}, $\omega_{2\delta}$) according to an orthonormal basis of coclosed p-eigenforms (resp. closed $(p-1)$-eigenforms, coclosed $(p-1)$-eigenforms) of $\Delta_{S^{N-1}}$. In this way, up to a unitary equivalence, the spectral analysis of Δ_{Mi}, $i = 1, 2, 3$, can be reduced to the investigation of the spectra of a countable number of Sturm-Liouville operators $D_{i\lambda}$ on the half line, parametrized by the eigenvalues λ of $\Delta_{S^{N-1}}$.

In [4] J. Eichhorn proved that for a complete Riemannian metric over a noncompact manifold the essential spectrum of Δ_M coincides with the essential spectrum of the Friedrichs extension Δ_M^F of the restriction of Δ_M to any exterior domain in M. This allows to consider the Sturm-Liouville operators $D_{i\lambda}$ on $[c, +\infty)$, for $c > 0$, and to overcome the difficulties due to the presence of singular potentials at $t = 0$.

In section 5, under the assumptions (1.3), (1.4), we compute the essential spectrum of Δ_M. First, through classical perturbation theory, we compute the spectrum of $D_{1\lambda}^F$ for every λ, and we show that

$$\left[\left(\frac{N-2p-1}{2}\right)^2, +\infty\right) \subseteq \sigma_{\mathrm{ess}}(\Delta_{M1}).$$

Then we show that $\sigma_{\mathrm{ess}}(\Delta_{M1})$ is exactly the interval $\left[\left(\frac{N-2p-1}{2}\right)^2, +\infty\right)$. By duality, we find that $\sigma_{\mathrm{ess}}(\Delta_{M2}) = \left[\left(\frac{N-2p+1}{2}\right)^2, +\infty\right)$. As for the essential spectrum of Δ_{M3}, first we compute the essential spectrum of $D_{3\lambda}^F$ for every λ, proving that

$$\left[\min\left\{\left(\frac{N-2p-1}{2}\right)^2, \left(\frac{N-2p+1}{2}\right)^2\right\}, +\infty\right) \subseteq \sigma_{\mathrm{ess}}(\Delta_{M3}).$$

Then we show that any positive number μ such that

$$\mu < \min\left\{\left(\frac{N-2p-1}{2}\right)^2, \left(\frac{N-2p+1}{2}\right)^2\right\},$$

can not belong to the essential spectrum of Δ_{M3}. Hence,

$$\sigma_{\mathrm{ess}}(\Delta_M) \setminus \{0\} = \left[\min\left\{\left(\frac{N-2p-1}{2}\right)^2, \left(\frac{N-2p+1}{2}\right)^2\right\}, +\infty\right).$$

Finally, recalling the results of Section 3, we fully determine the essential spectrum of Δ_M.

2. - Preliminary facts

For $N \geqslant 2$, let $\overline{B^N}$ denote the closed unit ball

$$\overline{B^N} = \left\{ \overline{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N \,|\, x_1^2 + \ldots + x_N^2 \leqslant 1 \right\},$$

and let $\mathbb{S}^{N-1}$ denote the sphere

$$\mathbb{S}^{N-1} = \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N \,|\, x_1^2 + \ldots + x_N^2 = 1 \right\},$$

endowed with a coordinate system (U_i, Θ_i), $i = 2, \ldots, k+1$, $\Theta_i \colon U_i \to \mathbb{R}^{N-1}$.

Let us consider the interior of $\overline{B^N}$,

$$B^N = \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N \,|\, x_1^2 + \ldots + x_N^2 < 1 \right\},$$

with the coordinate system (V_i, Φ_i), for $i = 1, \ldots, k+1$, defined in the following way: in a neighbourhood of 0, for some $\delta > 0$,

$$V_1 = \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N \,|\, x_1^2 + \ldots + x_N^2 < \delta \right\}$$

and

$$\Phi_1(x_1, \ldots, x_N) = (x_1, \ldots, x_N),$$

whilst for $i > 1$, $\overline{x} \neq 0$,

$$V_i = \left\{ \overline{x} \in \mathbb{R}^N \,|\, \frac{\overline{x}}{\|\overline{x}\|} \in U_i \right\},$$

$$\Phi_i \colon V_i \to (0, +\infty) \times \Theta_i(U_i),$$

$$\Phi_i(x_1, \ldots, x_N) = \left(2 \ \mathrm{settanh}\,(\|\overline{x}\|), \ \Theta_i\left(\frac{\overline{x}}{\|\overline{x}\|} \right) \right) =: (t, \theta_i).$$

We denote by M the manifold B^N, endowed with a Riemannian metric ds^2 such that on $\Phi_i(V_i)$, for $i > 1$,

$$ds^2 := f(t)\, dt^2 + g(t)\, d\theta^2, \tag{2.1}$$

where $f(t) > 0$, $g(t) > 0$ for every $t \in (0, +\infty)$ and $d\theta^2$ is the standard metric on $\mathbb{S}^{N-1}$. ds^2 is well-defined on $B^N \setminus \{0\}$.

We suppose that the metric is asymptotically hyperbolic, that is, as $t \to +\infty$,

$$f(t) \to 1, \qquad g(t) \to \sinh^2 t. \tag{2.2}$$

As for the behaviour as $t \to 0$, we suppose that for $t \in (0, \varepsilon)$ ($\varepsilon = 2$ settanh(δ))

$$f(t) \equiv 1, \qquad g(t) = t^2. \tag{2.3}$$

This assures that ds^2 can be extended to a smooth Riemannian metric on all M; indeed, for $t \in (0, \varepsilon)$, ds^2 is the expression, in polar coordinates, of the Euclidean metric on $\mathbb{R}^N$. As already remarked in the Introduction, the essential spectrum of the Laplace-Beltrami operator acting on p-forms on a complete noncompact Riemannian manifold does not change under perturbations of the Riemannian metric on compact sets ([4]). As a consequence, condition (2.3) does not modify essentially the spectral properties of the Laplace-Beltrami operator on M.

The manifold M, endowed with the Riemannian metric ds^2, is complete. Indeed, in view of (2.7) and (2.6), there exist $C_1, C_2 > 0$, $D_1, D_2 > 0$ such that for every $t > 0$

$$C_1 \leq f(t) \leq C_2,$$

$$D_1 \sinh^2 t \leq g(t) \leq D_2 \sinh^2 t;$$

hence the distance d_M induced by ds^2, given by

$$d_M(p_1, p_2) = \inf_{\gamma \in \Gamma(p_1, p_2)} \int_0^1 \left(f(t(s)) \left(\frac{d\gamma^1}{ds} \right)^2 + g(t(s)) \left\| \frac{d\gamma^i}{ds} \right\|^2_{\mathbb{S}^{N-1}} \right)^{1/2} ds,$$

is equivalent to the distance induced by the hyperbolic metric, which is complete.

For $p = 0, \ldots, N$, we will denote by $C^\infty(\Lambda^p(M))$ the space of all smooth p-forms on M, and by $C_c^\infty(\Lambda^p(M))$ the set of all smooth, compactly supported p-forms on M. For any $\omega \in C^\infty(\Lambda^p(M))$, we will denote by $|\omega(t, \theta)|$ the norm induced by the Riemannian metric on the fiber over (t, θ), given in local coordinates by

$$|\omega(t, \theta)|^2 = g^{i_1 j_1}(t, \theta) \ldots g^{i_p j_p}(t, \theta)\, \omega_{i_1 \ldots i_p}(t, \theta)\, \omega_{j_1 \ldots j_p}(t, \theta),$$

where g^{ij} is the expression of the Riemannian metric in local coordinates. We will denote by d_M, $*_M$, δ_M, respectively, the differential, the Hodge $*$ operator and the codifferential on M, defined as in [1]. Δ_M will stand for the Laplace-Beltrami operator acting on p-forms

$$\Delta_M = d_M \delta_M + \delta_M d_M,$$

which is expressed in local coordinates by the Weitzenböck formula

$$((\Delta_M)\, \omega)_{i_1 \ldots i_p} = - g^{ij} \nabla_i \nabla_j \omega_{i_1 \ldots i_p} + \sum_j R_j^{\alpha}\, \omega_{i_1 \ldots \alpha \ldots i_p} + \sum_{j,\, l \neq j} R^{\alpha\beta}_{i_j\ \ i_l} \omega_{\alpha i_1 \ldots \beta \ldots i_p},$$

where $\nabla_i \omega$ is the covariant derivative of ω with respect to the Riemannian metric, and R_j^i, $R^i{}_k{}^j{}_l$ denote respectively the local components of the Ricci tensor and the Riemann tensor induced by the Riemannian metric. As usual, $L_p^2(M)$ will denote the completion of $C_c^\infty(\Lambda^p(M))$ with respect to the norm $\|\omega\|_{L_p^2(M)}$ induced by the scalar product

$$\langle \omega, \widetilde{\omega} \rangle_{L_p^2(M)} := \int_M \omega \wedge *_M \widetilde{\omega} \, ;$$

$\|\omega\|_{L_p^2(M)}$ reads also

$$\|\omega\|_{L_p^2(M)}^2 = \int_M |\omega(t, \theta)|^2 \, dV_M ,$$

where dV_M is the volume element of (M, ds^2).

It is well-known that, since the Riemannian metric on M is complete, the Laplace-Beltrami operator is essentially selfadjoint on $C_c^\infty(\Lambda^p(M))$, for $p = 0, \ldots, N$. We will denote by Δ_M also its closure.

Now, given $\omega \in C^\infty(\Lambda^p(M))$, let us write

$$\omega = \omega_1 + \omega_2 \wedge dt , \tag{2.4}$$

where ω_1 and ω_2 are respectively a p-form and a $(p-1)$-form on $\mathbb{S}^{N-1}$ depending on t. An easy computation shows that $*_M \omega$ can be expressed in terms of (2.4) as

$$\begin{aligned} *_M \omega = (-1)^{N-p} g^{\frac{N-2p+1}{2}}(t)\, f^{-\frac{1}{2}}(t) \, *_{\mathbb{S}^{N-1}} \omega_2 \, + \\ + g^{\frac{N-2p-1}{2}}(t)\, f^{\frac{1}{2}}(t) \, *_{\mathbb{S}^{N-1}} \omega_1 \wedge dt , \end{aligned} \tag{2.5}$$

where $*_{\mathbb{S}^{N-1}}$ denotes the Hodge $*$ operator on $\mathbb{S}^{N-1}$. Moreover, d_M and δ_M split respectively as

$$d_M \omega = d_{\mathbb{S}^{N-1}} \omega_1 + \left\{ (-1)^p \frac{\partial \omega_1}{\partial t} + d_{\mathbb{S}^{N-1}} \omega_2 \right\} \wedge dt , \tag{2.6}$$

$$\begin{aligned} \delta_M \omega = g^{-1}(t)\, \delta_{\mathbb{S}^{N-1}} \omega_1 + (-1)^p f^{-\frac{1}{2}} g^{\frac{-N-1+2p}{2}} \frac{\partial}{\partial t} \left(f^{-\frac{1}{2}} g^{\frac{N+1-2p}{2}} \omega_2 \right) + \\ + g^{-1} \delta_{\mathbb{S}^{N-1}} \omega_2 \wedge dt , \end{aligned} \tag{2.7}$$

where p is the degree of ω, $d_{\mathbb{S}^{N-1}}$ is the differential on $\mathbb{S}^{N-1}$ and $\delta_{\mathbb{S}^{N-1}}$ is the codifferential on $\mathbb{S}^{N-1}$.

Moreover, the L^2-norm of $\omega \in C^\infty(\Lambda^p(M)) \cap L^2_p(M)$ can be written as

$$\|\omega\|^2_{L^2_p(M)} = \int_0^{+\infty} g^{\frac{N-2p-1}{2}}(s)\, f^{\frac{1}{2}}(s) \|\omega_1(s)\|^2_{L^2_p(\mathbb{S}^{N-1})}\, ds + \tag{2.8}$$

$$+ \int_0^{+\infty} g^{\frac{N+1-2p}{2}}(s)\, f^{-\frac{1}{2}}(s) \|\omega_2(s)\|^2_{L^2_{p-1}(\mathbb{S}^{N-1})}\, ds,$$

where $\|.\|_{L^2_p(\mathbb{S}^{N-1})}$ is the L^2-norm for p-forms on $\mathbb{S}^{N-1}$.

3. - Zero in the spectrum

In the present section we will investigate whether 0 belongs or not to the point (and essential) spectrum of Δ_M, for differential forms of degree $p = 0, \dots, N$. The main tool employed is the following generalization of a result of Dodziuk ([2]):

Theorem 3.1: *Let us consider, for* $N \geq 2$, *the manifold* M *endowed with a complete Riemannian metric of type* (2.1), *satisfying condition* (2.3) *for* $t \in (0, \varepsilon)$; *then, if we denote by* $\mathcal{H}^p(M)$, *for* $p = 0, \dots, N$, *the space of* L^2 *harmonic p-forms on* M, *we have*

1) *for* $p \notin \{0, N, N/2\}$, $\mathcal{H}^p(M) = \{0\}$;

2) *if* $\int_0^\infty f^{\frac{1}{2}}(s)\, g^{\frac{N-1}{2}}(s)\, ds = +\infty$, $\mathcal{H}^N(M) \simeq \mathcal{H}^0(M) = \{0\}$; *if on the contrary* $\int_0^\infty f^{\frac{1}{2}}(s)\, g^{\frac{N-1}{2}}(s)\, ds < +\infty$, $\mathcal{H}^N(M) \simeq \mathcal{H}^0(M) = \mathbb{R}$;

3) *if* $p = \frac{N}{2}$, $\mathcal{H}^p(M) = \{0\}$ *if* $\int_1^{+\infty} f^{\frac{1}{2}}(s)\, g^{-\frac{1}{2}}(s)\, ds = +\infty$; *if on the other hand* $\int_1^{+\infty} f^{\frac{1}{2}}(s)\, g^{-\frac{1}{2}}(s)\, ds < +\infty$, $\mathcal{H}^{\frac{N}{2}}(M)$ *is a Hilbert space of infinite dimension.*

Proof: The proof follows very closely the argument in [2]; it will be exposed here for the sake of completeness.

An L^2-form on M is harmonic if and only it is closed and coclosed. Hence, $\omega \in \mathcal{H}^p(M)$ if and only if

$$\|\omega\|_{L^2_p(M)} < \infty, \qquad d\omega = 0, \qquad d *_M \omega = 0. \tag{3.1}$$

Moreover, $*_M$ gives an isomorphism between $\mathcal{H}^p(M)$ and $\mathcal{H}^{N-p}(M)$.

The proof of 2) is immediate; if ω is a harmonic function, not identically vanishing, ω is constant on M, hence $\omega \in L^2(M)$ if and only if the total volume of M, given by $\int_0^\infty f^{\frac{1}{2}}(s)\, g^{\frac{N-1}{2}}(s)\, ds$, is finite.

We now come to the proof of 1). Let $\omega \in \mathcal{H}^p(M)$, for $p \neq 0, N$, and let us consider its decomposition (2.4). Then, in view of (2.6), $d_M \omega = 0$ implies

$$d_{S^{N-1}} \omega_1 = 0, \qquad d_{S^{N-1}} \omega_2 + (-1)^p \frac{\partial \omega_1}{\partial t} = 0,$$

whilst $d_M *_M \omega = 0$ yields

$$d_{S^{N-1}} *_{S^{N-1}} \omega_2 = 0,$$

$$g^{\frac{N-2p-1}{2}} f^{\frac{1}{2}} d_{S^{N-1}} *_{S^{N-1}} \omega_1 + \frac{\partial}{\partial t}\left(g^{\frac{N-2p+1}{2}} f^{\frac{1}{2}} *_{S^{N-1}} \omega_2 \right) = 0. \tag{3.2}$$

In view of (2.8), the boundedness of the L^2-norm of ω reads

$$\int_0^{+\infty} \int_{S^{N-1}} \Big(g^{\frac{N-2p-1}{2}} f^{\frac{1}{2}} |w_1(t,\theta)|^2 + g^{\frac{N-2p+1}{2}} f^{-\frac{1}{2}} |w_2(t,\theta)|^2 \Big)\, dV_{S^{N-1}}\, dt < +\infty; \tag{3.3}$$

moreover, since $|\omega(t,\theta)|$ is bounded in a neighbourhood of 0, we have that

$$|\omega(t,\theta)|^2 = g(t)^{-p} |\omega_1(t,\theta)|^2 + f(t)^{-1} g(t)^{1-p} |\omega_2(t,\theta)|^2 \leqslant C$$

for some $C > 0$ for $t \in (0, \varepsilon]$.

Applying $*_{S^{N-1}}$ to both sides of (3.2), we find the following set of conditions:

$$d_{S^{N-1}} \omega_1 = 0; \tag{3.4}$$

$$d_{S^{N-1}} *_{S^{N-1}} \omega_2 = 0; \tag{3.5}$$

$$d_{S^{N-1}} \omega_2 + (-1)^p \frac{\partial \omega_1}{\partial t} = 0; \tag{3.6}$$

$$\frac{\partial}{\partial t}\left(g^{\frac{N-2p+1}{2}}(t)\, f^{-\frac{1}{2}}(t)\, \omega_2 \right) + (-1)^p f^{\frac{1}{2}}(t)\, g^{\frac{N-2p-1}{2}}(t)\, \delta_{S^{N-1}} \omega_1 = 0; \tag{3.7}$$

$$g^{-p}(t) |\omega_1(t,\theta)|^2 + f^{-1}(t)\, g^{1-p}(t) |\omega_2(t,\theta)|^2 \leqslant C \;\; \forall t \in (0, \varepsilon]; \tag{3.8}$$

$$\int_0^{+\infty} \int_{S^{N-1}} \Big(g^{\frac{N-2p-1}{2}} f^{\frac{1}{2}} |w_1(t,\theta)|^2 + g^{\frac{N-2p+1}{2}} f^{-\frac{1}{2}} |w_2(t,\theta)|^2 \Big)\, dV_{S^{N-1}}\, dt < +\infty. \tag{3.9}$$

Now, it can be shown that if $\omega \in \mathcal{H}^p(M)$ and $\omega_1 = 0$, then $\omega_2 = 0$; indeed, if

$\omega_2 \wedge dt \in \mathcal{H}^p(M)$, in view of (3.5) and (3.6) ω_2 is a harmonic form on $\mathbb{S}^{N-1}$ for every $t > 0$. Since $0 \leqslant p - 1 \leqslant N - 2$, $\omega_2(t, \theta)$ can be nonzero only if $p - 1 = \deg \omega_2 = 0$, that is, only if ω_2 is a function not depending on θ. On the other hand, (3.7) implies

$$\frac{\partial}{\partial t}\left(g^{\frac{N-1}{2}} f^{-\frac{1}{2}} \omega_2\right) = 0,$$

that is, $\omega_2 = Cg(t)^{-\frac{N-1}{2}} f(t)^{\frac{1}{2}}$, which diverges as $t \to 0$, in contradiction with (3.8), unless $C = 0$.

Hence, if $\omega \neq 0$ and $\omega \in \mathcal{H}^p(M)$, then $\omega_1 \neq 0$. Now, applying $d_{\mathbb{S}^{N-1}}$ to both sides of (3.7), since $d_{\mathbb{S}^{N-1}}$ commutes with $\frac{\partial}{\partial t}$, we get

$$\frac{\partial}{\partial t}\left(g(t)^{\frac{N-2p+1}{2}} f^{-\frac{1}{2}} d_{\mathbb{S}^{N-1}} \omega_2\right) + (-1)^p f(t)^{\frac{1}{2}} g(t)^{\frac{N-2p-1}{2}} d_{\mathbb{S}^{N-1}} \delta_{\mathbb{S}^{N-1}} \omega_1 = 0,$$

whence, in view of (3.6),

$$\frac{\partial}{\partial t}\left(g(t)^{\frac{N-2p+1}{2}} f(t)^{-\frac{1}{2}} \frac{\partial \omega_1}{\partial t}\right) = f(t)^{\frac{1}{2}} g(t)^{\frac{N-2p-1}{2}} d_{\mathbb{S}^{N-1}} \delta_{\mathbb{S}^{N-1}} \omega_1 .$$

Taking, for fixed $t > 0$, the scalar product of both sides of the last equation with ω_1, we get

$$\left\langle \frac{\partial}{\partial t}\left(g(t)^{\frac{N-2p+1}{2}} f(t)^{-\frac{1}{2}} \frac{\partial \omega_1}{\partial t}\right), \omega_1 \right\rangle_{L^2_p(\mathbb{S}^{N-1})} \langle \delta_{\mathbb{S}^{N-1}} \omega_1, \delta_{\mathbb{S}^{N-1}} \omega_1 \rangle_{L^2_p(\mathbb{S}^{N-1})} \geqslant 0,$$

whence

$$\frac{\partial}{\partial t}\left\langle g(t)^{\frac{N-2p+1}{2}} f(t)^{\frac{1}{2}} \frac{\partial \omega_1}{\partial t}, \omega_1 \right\rangle_{L^2_p(\mathbb{S}^{N-1})} =$$

$$= \left\langle \frac{\partial}{\partial t}\left(g(t)^{\frac{N-2p+1}{2}} f(t)^{-\frac{1}{2}} \frac{\partial \omega_1}{\partial t}\right), \omega_1 \right\rangle_{L^2_p(\mathbb{S}^{N-1})} + g(t)^{\frac{N-2p+1}{2}} f(t)^{-\frac{1}{2}} \left\langle \frac{\partial \omega_1}{\partial t}, \frac{\partial \omega_1}{\partial t} \right\rangle_{L^2_p(\mathbb{S}^{N-1})} \geqslant 0.$$

Due to the boundedness of $|\omega|$ near 0 and to (2.3), $|\omega_1(t, \theta)|_{\mathbb{S}^{N-1}} = O(t^{2p})$ for small t. As a consequence,

$$\left\langle f(t)^{-\frac{1}{2}} g(t)^{\frac{N-2p+1}{2}} \frac{\partial \omega_1}{\partial t}, \omega_1 \right\rangle_{L^2_p(\mathbb{S}^{N-1})} = O(t^N),$$

hence

$$\frac{\partial}{\partial t}\langle \omega_1, \omega_1 \rangle_{L^2_p(\mathbb{S}^{N-1})} = 2 \left\langle \frac{\partial \omega_1}{\partial t}, \omega_1 \right\rangle_{L^2_p(\mathbb{S}^{N-1})} \geq 0$$

for every $t > 0$, that is, $\|\omega_1(t)\|_{L^2_p(\mathbb{S}^{N-1})}$ is a nondecreasing function of t.

Now, let $\omega_1 \neq 0$; since $\|\omega_1(t)\|_{L^2_p(\mathbb{S}^{N-1})}$ is nondecreasing and $\|\omega\|_{L^2_p(M)} < +\infty$,

$$\int_1^{+\infty} g(s)^{\frac{N-2p-1}{2}} f(s)^{\frac{1}{2}}\, ds \leq C \int_1^{+\infty} g(s)^{\frac{N-2p-1}{2}} f(s)^{\frac{1}{2}} \|\omega_1(s)\|^2_{L^2_p(\mathbb{S}^{N-1})}\, ds \leq \|\omega\|^2_{L^2_p(M)} < +\infty .$$

Hence for $p \neq 0, N$, $\mathcal{H}^p(M) \neq \{0\}$ implies

$$\int_1^{+\infty} g(s)^{\frac{N-2p-1}{2}} f(s)^{\frac{1}{2}}\, ds < +\infty ,$$

and, by duality,

$$\int_1^{+\infty} g(s)^{\frac{-N+2p-1}{2}} f(s)^{\frac{1}{2}}\, ds < +\infty .$$

If $N = 2p$, the two integrands coincide. If, on the contrary, $N - 2p \neq 0$, then, since $(N-2p-1)(-N+2p-1) = 1-(N-2p)^2$, either one of the exponents is zero, or the two exponents have opposite signs; in both cases one of the integrals diverges. Hence, for $p \notin \{0, N, N/2\}$, $\mathcal{H}^p(M) = \{0\}$.

Finally we come to 3). For $p = N/2$, if $\int_1^{+\infty} g(s)^{-1/2} f(s)^{1/2}\, ds = +\infty$, $\mathcal{H}^p(M) = \{0\}$. This proves the first half of 3). We still have to prove that if $\int_1^{+\infty} g(s)^{-1/2} f(s)^{1/2}\, ds < + +\infty$, $\mathcal{H}^{N/2}(M)$ has infinite dimension. To this purpose, let us recall that if $N = 2p$ the Hodge $*$ operator acting on forms of degree p depends only on the conformal structure of the manifold. Hence the conditions $\|\omega\|_{L^2_p} < +\infty$, $d\omega = 0$, $d^*\omega = 0$ are conformally invariant.

Now, let us suppose that $\int_1^{+\infty} g(s)^{-1/2} f(s)^{1/2}\, ds < +\infty$, and let us denote by $B(0, r)$ the open ball in $\mathbb{R}^N$ with radius

$$r = \exp\left(\int_1^{+\infty} g(s)^{-1/2} f(s)^{1/2}\, ds \right)$$

centered in 0, endowed with polar coordinates. Then consider the mapping:

$$F : M \backslash \{0\} \to \mathbb{R}^N \backslash \{0\}$$

given by

$$F(t,\,\theta) := \left(\exp\left(\int_1^t g(s)^{-1/2} f(s)^{1/2}\, ds\right),\, \theta\right).$$

In view of condition (2.3), F can be extended to a C^1-diffeomorphism of M into $B(0,\,r)$, which is actually C^∞ on $M\backslash\{0\}$. Moreover, an easy computation shows that F is conformal from M, endowed with the metric (2.1), to $B(0,\,r)$, endowed with the Euclidean metric.

Let us denote by $\mathcal{H}$ the (infinite-dimensional) space of all smooth p-forms on $B(0, r)$ harmonic with respect to the Euclidean metric; since F is conformal and $N=2p$, $F^*\mathcal{H}$ consists of forms of degree p, square-summable on M, smooth on M (up to modifications at 0) and harmonic. As a consequence, $\mathcal{H}^{N/2}(M)$ has infinite dimension. ■

In our case, since $f(t) \to 1$ and $g(t) \to \sinh^2 t$ as $t \to +\infty$, then

$$\int_0^{+\infty} f(s)^{\frac{1}{2}}\, g(s)^{\frac{N-1}{2}}\, ds = +\infty\,,$$

whilst

$$\int_1^{+\infty} f(s)^{\frac{1}{2}}\, g(s)^{-\frac{1}{2}}\, ds < +\infty\,.$$

As a consequence we can easily deduce the following

THEOREM 3.2: *For $N \geq 2$, let us consider the manifold M, endowed with a Riemannian metric of type* (2.1), *satisfying conditions* (2.2) *and* (2.3). *Then*

1. *if $p \neq N/2$, then $0 \notin \sigma_p(\Delta_M)$;*
2. *if $p = N/2$, $\mathcal{H}^p(M)$ is a Hilbert space of infinite dimension, hence $0 \in \sigma_{\mathrm{ess}}(\Delta_M) \cap \sigma_p(\Delta_M)$.*

4. - HODGE DECOMPOSITION AND UNITARY EQUIVALENCE

From (2.6) and (2.7), a lengthy but straightforward computation gives

$$\Delta_M \omega = (\Delta_M \omega)_1 + (\Delta_M \omega)_2 \wedge dt,$$

where

$$(4.1) \qquad (\Delta_M \omega)_1 = g^{-1}(t)\, \Delta_{\mathbb{S}^{N-1}} \omega_1 + (-1)^p f^{-1}(t)\, g^{-1}(t)\, \frac{\partial g}{\partial t}\, d_{\mathbb{S}^{N-1}} \omega_2 + \\ - f^{-\frac{1}{2}}(t)\, g^{\frac{-N+1+2p}{2}}(t)\, \frac{\partial}{\partial t}\left(f^{-\frac{1}{2}}(t)\, g^{\frac{N-1-2p}{2}}(t)\, \frac{\partial \omega_1}{\partial t}\right)$$

and

$$(4.2)\qquad (\Delta_M \omega)_2 = g^{-1}(t)\,\Delta_{\mathbb{S}^{N-1}}\omega_2 + (-1)^p g^{-2}(t)\,\frac{\partial g}{\partial t}\,\delta_{\mathbb{S}^{N-1}}\omega_1 +$$

$$- \frac{\partial}{\partial t}\left\{ f^{-\frac{1}{2}}(t)\, g^{\frac{-N-1+2p}{2}}(t)\, \frac{\partial}{\partial t}\left(f^{-\frac{1}{2}}(t)\, g^{\frac{N+1-2p}{2}}(t)\,\omega_2 \right)\right\}.$$

Here we denote by $\Delta_{\mathbb{S}^{N-1}}$ the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$.

Since for every $\omega \in C^\infty(\Lambda^p(M)) \cap L^2_p(M)$ we have that $\omega_1 \in L^2_p(M)$, $\omega_2 \wedge dt \in L^2_p(M)$ and

$$\langle \omega_1, \omega_2 \wedge dt \rangle_{L^2_p(M)} = 0,$$

(2.4) gives rise to an orthogonal decomposition of $L^2_p(M)$ into two closed subspaces. However, (4.1) and (4.2) show that Δ_M is not invariant under this decomposition. As a consequence, further decompositions are required.

It is well-known that, for $0 \leqslant p \leqslant N-1$,

$$C^\infty(\Lambda^p(\mathbb{S}^{N-1})) = dC^\infty(\Lambda^{p-1}(\mathbb{S}^{N-1})) \oplus \delta C^\infty(\Lambda^{p+1}(\mathbb{S}^{N-1})) \oplus \mathcal{H}^p(\mathbb{S}^{N-1}),$$

where $\mathcal{H}^p(\mathbb{S}^{N-1})$ is the space of harmonic p-forms on $\mathbb{S}^{N-1}$ (empty if $p \neq 0, N-1$), and the decomposition is orthogonal in $L^2_p(\mathbb{S}^{N-1})$. Hence, for $0 \leqslant p \leqslant N-1$,

$$L^2_p(\mathbb{S}^{N-1}) = \overline{dC^\infty(\Lambda^{p-1}(\mathbb{S}^{N-1}))} \oplus \overline{\delta C^\infty(\Lambda^{p+1}(\mathbb{S}^{N-1}))} \oplus \mathcal{H}^p(\mathbb{S}^{N-1}).$$

Thus, for $1 \leqslant p \leqslant N-1$, every $\omega \in L^2_p(M)$ can be written as

$$(4.3)\qquad \omega = \omega_{1\delta} \oplus \omega_{2d} \wedge dt \oplus (\omega_{1d} \oplus \omega_{2\delta} \wedge dt),$$

where $\omega_{1\delta}$ (resp. ω_{1d}) is a coclosed (resp. closed) p-form on $\mathbb{S}^{N-1}$ parametrized by t, and $\omega_{2\delta}$ (resp. ω_{2d}) is a coclosed (resp. closed) $(p-1)$-form on $\mathbb{S}^{N-1}$ parametrized by t. In this way we get the orthogonal decomposition

$$L^2_p(M) = \mathcal{L}^1(M) \oplus \mathcal{L}^2(M) \oplus \mathcal{L}^3(M),$$

where for every $\omega \in L^2_p(M)$, $\omega_{1\delta} \in \mathcal{L}^1(M)$, $\omega_{2d} \wedge dt \in \mathcal{L}^2(M)$ and $\omega_{1d} \oplus (\omega_{2\delta} \wedge dt) \in \mathcal{L}^3(M)$. Since

$$d_{\mathbb{S}^{N-1}}\Delta_{\mathbb{S}^{N-1}} = \Delta_{\mathbb{S}^{N-1}} d_{\mathbb{S}^{N-1}}, \qquad \delta_{\mathbb{S}^{N-1}}\Delta_{\mathbb{S}^{N-1}} = \Delta_{\mathbb{S}^{N-1}}\delta_{\mathbb{S}^{N-1}},$$

$$\frac{\partial}{\partial t} d_{\mathbb{S}^{N-1}} = d_{\mathbb{S}^{N-1}} \frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial t}\delta_{\mathbb{S}^{N-1}} = \delta_{\mathbb{S}^{N-1}} \frac{\partial}{\partial t},$$

the Laplace-Beltrami operator is invariant under this decomposition, and can be written as the orthogonal sum

$$\Delta_M = \Delta_{M1} \oplus \Delta_{M2} \oplus \Delta_{M3}.$$

It is easy to see that, for $i = 1, 2, 3$, Δ_{Mi} is essentially selfadjoint on $C_c^\infty(\Lambda^p(M)) \cap \mathcal{L}^i(M)$. We denote again by Δ_{Mi} its closure.

Since the orthogonal sum is finite, for $1 \leq p \leq N-1$,

$$\sigma_{\text{ess}}(\Delta_M) = \bigcup_{i=1}^{3} \sigma_{\text{ess}}(\Delta_{Mi}),$$

$$\sigma_p(\Delta_M) = \bigcup_{i=1}^{3} \sigma_p(\Delta_{Mi}).$$

For $p = 0$ (resp. $p = N$), any $\omega \in L^2(M)$ can be written as $\omega = \omega_{1\delta}$ (resp. $\omega = \omega_{2d} \wedge dt$), where $\omega_{1\delta}$ (resp. ω_{2d}) is a coclosed (resp. closed) 0-form (resp. $(N-1)$-form) parametrized by t on $\mathbb{S}^{N-1}$. Hence $L_0^2(M) = \mathcal{L}^1(M)$ (resp. $L_{N-1}^2(M) = \mathcal{L}^2(M)$) and $\Delta_M = \Delta_{M1}$ (resp. $\Delta_M = \Delta_{M2}$).

As a consequence, in order to determine the spectrum of Δ_M it suffices to study the spectral properties of Δ_{Mi}, $i = 1, 2, 3$.

Then, let us introduce a further decomposition. First of all, we decompose $\omega_{1\delta}$ according to an orthonormal basis $\{\tau_{1k}\}_{k \in \mathbb{N}}$ of coclosed p-eigenforms of $\Delta_{\mathbb{S}^{N-1}}$; this yields

$$\omega_{1\delta} = \oplus_k h_k(t)\, \tau_{1k}, \tag{4.4}$$

where $h_k(t)\, \tau_{1k} \in L_p^2(M)$ for every $k \in \mathbb{N}$, and the sum is orthogonal in $L_p^2(M)$, thanks to (2.1). We will call p-form of type I any p-form $\omega \in L_p^2(M)$ such that

$$\omega = h(t)\, \tau_1,$$

where τ_1 is a coclosed normalized p-eigenform of $\Delta_{\mathbb{S}^{N-1}}$, corresponding to some eigenvalue λ. For every $k \in \mathbb{N}$, let us denote by λ_k^p the eigenvalue of $\Delta_{\mathbb{S}^{N-1}}$ associated to τ_{1k}. Since for every $k \in \mathbb{N}$

$$\Delta_{M1}(h(t)\, \tau_{1k}) = \frac{\lambda_k^p}{g(t)} h(t)\, \tau_{1k} - \tag{4.5}$$
$$- f(t)^{-\frac{1}{2}} g(t)^{\frac{-N+1+2p}{2}} \frac{\partial}{\partial t}\left(f(t)^{-\frac{1}{2}} g(t)^{\frac{N-1-2p}{2}} \frac{\partial h}{\partial t} \right) \tau_{1k},$$

Δ_{M1} is invariant under the decomposition (4.4), and, since if $\omega = h(t)\, \tau_{1k}$

$$\|\omega\|^2_{L_p^2(M)} = \int_0^\infty g(s)^{\frac{N-2p-1}{2}} f(s)^{\frac{1}{2}} h(s)^2\, ds,$$

Δ_{M1} is unitarily equivalent to the direct sum with respect to $k \in \mathbb{N}$ of the operators

$$\Delta_{1\lambda_k^p}\colon \mathcal{D}(\Delta_{1\lambda_k^p}) \subset L^2(\mathbb{R}^+, g^{\frac{N-2p-1}{2}} f^{\frac{1}{2}}) \to L^2(\mathbb{R}^+, g^{\frac{N-2p-1}{2}} f^{\frac{1}{2}})$$

(4.6)
$$\Delta_{1\lambda_k^p} h = \left\{ \frac{\lambda_k^p}{g(t)} h(t) - f(t)^{-\frac{1}{2}} g(t)^{\frac{-N+1+2p}{2}} \frac{\partial}{\partial t} \left(f(t)^{-\frac{1}{2}} g(t)^{\frac{N-1-2p}{2}} \right) \right\}.$$

If we introduce the transformation

$$w(t) = h(t)\ f(t)^{\frac{1}{4}} g(t)^{\frac{N-2p-1}{4}}, \tag{4.7}$$

a direct (but lengthy) computation shows that Δ_{M1} is unitarily equivalent to the direct sum, over $k \in \mathbb{N}$, of the operators

$$D_{1\lambda_k^p}\colon \mathcal{D}(D_{1\lambda_k^p}) \subset L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$$

given by

$$\begin{aligned} D_{1\lambda_k^p} w = &- \frac{\partial}{\partial t}\left(\frac{1}{f} \frac{\partial w}{\partial t} \right) + \left\{ - \frac{7}{16} \frac{1}{f^3} \left(\frac{\partial f}{\partial t} \right)^2 + \frac{1}{4} \frac{1}{f^2} \frac{\partial^2 f}{\partial t^2} - \right. \\ &- \frac{1}{2} \frac{1}{f^2} \frac{\partial f}{\partial t} \frac{(N-1-2p)}{4} \frac{1}{g} \frac{\partial g}{\partial t} + \frac{1}{f} \frac{(N-2p-1)}{4} \frac{(N-2p-5)}{4} \frac{1}{g^2} \left(\frac{\partial g}{\partial t} \right)^2 + \\ &\left. + \frac{1}{f} \frac{(N-2p-1)}{4} \frac{1}{g} \frac{\partial^2 g}{\partial t^2} + \frac{\lambda_k^p}{g} \right\} w. \end{aligned} \tag{4.8}$$

Analogously, we decompose ω_{2d} according to an orthonormal basis of closed $(p-1)$-eigenforms $\{\tau_{2k}\}_{k \in \mathbb{N}}$ of $\Delta_{S^{N-1}}$:

$$\omega_{2d} \wedge dt = \oplus_k h_k(t)\ \tau_{2k} \wedge dt\,. \tag{4.9}$$

We will call p-form of type II a p-form $\omega \in L^2_p(M)$ such that

$$\omega = h(t)\ \tau_2 \wedge dt,$$

where τ_2 is a coclosed normalized $(p-1)$-eigenform, corresponding to some eigenvalue λ of $\Delta_{S^{N-1}}$. For every $k \in \mathbb{N}$

$$\Delta_{M2}(h(t)\ \tau_{2k} \wedge dt) = (\Delta_{2\lambda_k^{p-1}} h)\ \tau_{2k} \wedge dt,$$

where

$$\begin{aligned} \Delta_{2\lambda_k^{p-1}} h = &\frac{\lambda_k^{p-1}}{g(t)} h(t) - \\ &- \frac{\partial}{\partial t} \left\{ f(t)^{-\frac{1}{2}} g(t)^{\frac{-N-1+2p}{2}} \frac{\partial}{\partial t} \left(f(t)^{-\frac{1}{2}} g(t)^{\frac{N+1-2p}{2}} h(t) \right) \right\}. \end{aligned} \tag{4.10}$$

Here, again, for every $k \in \mathbb{N}$ we denote by λ_k^{p-1} the eigenvalue of $\Delta_{S^{N-1}}$ corresponding to the eigenform τ_{2k}. Since if $\omega = h(t)\, \tau_{2k} \wedge dt$

$$\|\omega\|^2_{L_p^2(M)} = \int_0^\infty g(s)^{\frac{N-2p+1}{2}} f(s)^{-\frac{1}{2}} h(s)^2\, ds,$$

introducing the transformation

$$w(t) = h(t) f(t)^{-\frac{1}{4}} g(t)^{\frac{N+1-2p}{4}}, \tag{4.11}$$

we find that Δ_{M2} is unitarily equivalent to the direct sum, with respect to $k \in \mathbb{N}$, of the operators

$$D_{2\lambda_k^{p-1}} \colon \mathcal{D}(D_{2\lambda_k^{p-1}}) \subset L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$$

$$\begin{aligned} D_{2\lambda_k^{p-1}} w = -\frac{\partial}{\partial t}\left(\frac{1}{f}\frac{\partial w}{\partial t}\right) + \Bigg\{ &-\frac{7}{16}\frac{1}{f^3}\left(\frac{\partial f}{\partial t}\right)^2 + \frac{1}{4}\frac{1}{f^2}\frac{\partial^2 f}{\partial t^2} - \\ -\frac{1}{2}\frac{1}{f^2}\frac{\partial f}{\partial t}\frac{(N-1+2p)}{4}\frac{1}{g}\frac{\partial g}{\partial t} &+ \frac{1}{f}\frac{(N-2p+1)}{4}\frac{(N-2p+5)}{4}\frac{1}{g^2}\left(\frac{\partial g}{\partial t}\right)^2 + \\ &+ \frac{1}{f}\frac{(-N+2p-1)}{4}\frac{1}{g}\frac{\partial^2 g}{\partial t^2} + \frac{\lambda_k^{p-1}}{g}\Bigg\} w\,. \end{aligned} \tag{4.12}$$

Finally, we decompose $\omega_{2\delta}$ with respect to an orthonormal basis of coclosed $(p-1)$-eigenforms $\{\tau_{3k}\}_{k\in\mathbb{N}}$ of $\Delta_{S^{N-1}}$. For every $k \in \mathbb{N}$ we denote by λ_k^{p-1} the eigenvalue corresponding to the eigenform τ_{3k}; then $\left\{\frac{1}{\sqrt{\lambda_k^{p-1}}} d_{S^{N-1}} \tau_{3k}\right\}_{k\in\mathbb{N}}$ is an orthonormal basis of closed eigenforms of $\Delta_{S^{N-1}}$ for closed p-forms. Hence, we get the following decomposition for $\omega_{1d} \oplus \omega_{2\delta} \wedge dt$:

$$\omega_{1d} \oplus \omega_{2\delta} \wedge dt = \oplus_k \left(\frac{1}{\sqrt{\lambda_k^{p-1}}} h_{1k}\, d_{S^{N-1}} \tau_{3k} \oplus (-1)^p h_{2k} \tau_{3k} \wedge dt \right).$$

We call p-form of type III any p-form ω such that

$$\omega = \frac{1}{\sqrt{\lambda}} h_1(t)\, d_{S^{N-1}} \tau_3 \oplus_M (-1)^p h_2(t)\, \tau_3 \wedge dt,$$

where τ_3 is a normalized coclosed $(p-1)$-eigenform of $\Delta_{S^{N-1}}$, corresponding to the eigenvalue λ. A direct computation shows that

$$\Delta_{M3}\left(\frac{1}{\sqrt{\lambda_k^{p-1}}}h_1(t)\,d_{\mathbb{S}^{N-1}}\tau_{3k}\oplus_M(-1)^p h_2(t)\,\tau_{3k}\wedge dt\right)= \tag{4.14}$$

$$=\left(\Delta_{1\lambda_k^{p-1}}h_1+\frac{1}{f(t)}\,\frac{1}{g(t)}\,\frac{\partial g}{\partial t}\sqrt{\lambda_k^{p-1}}h_2\right)\left(\frac{1}{\sqrt{\lambda_k^{p-1}}}d_{\mathbb{S}^{N-1}}\tau_{3k}\right)\oplus$$

$$\oplus\left(\Delta_{2\lambda_k^{p-1}}h_2+\frac{1}{g^2(t)}\,\frac{\partial g}{\partial t}\sqrt{\lambda_k^{p-1}}h_1\right)((-1)^p\tau_{3k}\wedge dt);$$

moreover, if $\omega=\frac{1}{\sqrt{\lambda_k^{p-1}}}h_1(t)\,d_{\mathbb{S}^{N-1}}\tau_{3k}\oplus_M(-1)^p h_2(t)\,\tau_{3k}\wedge dt$, then

$$\|\omega\|^2_{L^2_p(M)}=\int_0^{+\infty}g(s)^{\frac{N-2p-1}{2}}f(s)^{\frac{1}{2}}h_1(s)^2\,ds+\int_0^{+\infty}g(s)^{\frac{N+1-2p}{2}}f(s)^{-\frac{1}{2}}h_2(s)^2\,ds.$$

Hence, introducing the transformation

$$\begin{aligned} w_1(t)&=g^{\frac{N-2p-1}{4}}(t)\,f^{\frac{1}{4}}(t)\,h_1(t)\\ w_2(t)&=g^{\frac{N-2p+1}{4}}(t)\,f^{-\frac{1}{4}}(t)\,h_2(t), \end{aligned} \tag{4.15}$$

we find that Δ_{M3} is unitarily equivalent to the direct sum, with respect to $k\in\mathbb{N}$, of the operators

$$D_{3\lambda_k^{p-1}}\colon \mathcal{D}(D_{3\lambda_k^{p-1}})\subset L^2(\mathbb{R}^+)\oplus L^2(\mathbb{R}^+)\to L^2(\mathbb{R}^+)\oplus L^2(\mathbb{R}^+)$$

$$D_{3\lambda_k^{p-1}}(w_1\oplus w_2)=\left(D_{1\lambda_k^{p-1}}w_1+g(t)^{-\frac{3}{2}}f(t)^{-\frac{1}{2}}\,\frac{\partial g}{\partial t}\sqrt{\lambda_k^{p-1}}w_2\right)\oplus \tag{4.16}$$

$$\oplus\left(D_{2\lambda_k^{p-1}}w_2+g(t)^{-\frac{3}{2}}f(t)^{-\frac{1}{2}}\,\frac{\partial g}{\partial t}\sqrt{\lambda_k^{p-1}}w_1\right).$$

As remarked in the Introduction, J. Eichhorn proved in [4] that for a complete Riemannian metric over a noncompact manifold the essential spectrum of Δ_M coincides with the essential spectrum of the Friedrichs extension Δ_M^F of the restriction of Δ_M to any exterior domain in M. Thus, if we consider, for $0<\eta<1$, the Friedrichs extension $\Delta^F_{M,\eta}$ of the operator

$$\Delta'_{M,\eta}\colon C_c^\infty(\Lambda^p(M\backslash B(0,\eta)))\to L^2(M\backslash B(0,\eta))$$

$$\Delta'_{M,\eta}\omega=\Delta_M\omega\,,$$

we have that

$$\sigma_{\mathrm{ess}}(\Delta_M) = \sigma_{\mathrm{ess}}(\Delta^F_{M,\eta})$$

for every η, $0 < \eta < 1$. Hence, in order to compute the essential spectrum of Δ_M it suffices to determine the essential spectrum of $\Delta^F_{M,\eta}$ for some η, $0 < \eta < 1$. For the sake of simplicity we will write Δ^F_M instead of $\Delta^F_{M,\eta}$.

The same orthogonal decompositions obtained for Δ_M hold also for Δ^F_M: namely, we have a decomposition

$$L^2_p(M \backslash B(0,\eta)) = \mathcal{L}^1(M \backslash B(0,\eta)) \oplus \mathcal{L}^2(M \backslash B(0,\eta)) \oplus \mathcal{L}^3(M \backslash B(0,\eta))$$

analogous to (4.3), and Δ^F_M splits accordingly as

$$\Delta^F_M = \Delta^F_{M1} \oplus \Delta^F_{M2} \oplus \Delta^F_{M3},$$

where, for $i = 1, 2, 3$, Δ^F_{Mi} is the Friedrichs extension of the restriction of Δ_M to $C^\infty_c(\Lambda^p(M \backslash B(0,\eta))) \cap \mathcal{L}^i(M \backslash B(0,\eta))$. Moreover (see [4]), for $i = 1, 2, 3$, $\sigma_{\mathrm{ess}}(\Delta_{Mi}) = \sigma_{\mathrm{ess}}(\Delta^F_{Mi})$.

Let $c = \operatorname{settanh}(\eta)$; again, it is possible to show that, for $i = 1, 2$, Δ^F_{Mi} is unitarily equivalent to the direct sum, over $k \in \mathbb{N}$, of the Friedrichs extensions $D^F_{i\lambda^p_k}$ of the operators

$$D'_{i\lambda^p_k}\colon C^\infty_c(c, +\infty) \to L^2(c, +\infty)$$

given by (4.8) if $i = 1$ and by (4.12) if $i = 2$.

Analogously, Δ^F_{M3} is unitarily equivalent to the direct sum, over $k \in \mathbb{N}$, of the Friedrichs extensions $D^F_{3\lambda^{p-1}_k}$ of the operators

$$D'_{3\lambda^{p-1}_k}\colon C^\infty_c(c, +\infty) \oplus C^\infty_c(c, +\infty) \to L^2(c, +\infty) \oplus L^2(c, +\infty)$$

given by (4.16). Moreover, for every $i = 1, 2, 3$, for every $k \in \mathbb{N}$ and for every $c > 0$, we have that $\sigma_{\mathrm{ess}}(D_{i\lambda_k}) = \sigma_{\mathrm{ess}}(D^F_{i\lambda_k})$.

Thus, much information about the essential spectrum of Δ_M can be recovered by the investigation of the essential spectra of the selfadjoint operators $D^F_{1\lambda^p_k}$, $D^F_{2\lambda^{p-1}_k}$ and $D^F_{3\lambda^{p-1}_k}$ for arbitrarily large c. Since the Hodge $*$ operator isometrically maps p-forms of type I onto $(N-p)$-forms of type II, it suffices to consider the cases $i = 1$ and $i = 3$. We remark that, since the direct sums in (4.4) and (4.13) have an infinite number of summands, for $i = 1, 3$

$$\sigma_{\mathrm{ess}}(\Delta_{Mi}) \supset \bigcup_k \sigma_{\mathrm{ess}}(D^F_{i\lambda_k})$$

but we cannot argue that

$$\sigma_{\mathrm{ess}}(\Delta_{Mi}) = \bigcup_k \sigma_{\mathrm{ess}}(D^F_{i\lambda_k}).$$

5. - The essential spectrum

In the present section, we will compute the essential spectrum of Δ_M, under suitable assumptions on the asymptotic behaviour of f and g. Namely, if

$$\tilde{f}(t) := f(t) - 1\ ,$$

$$\tilde{g}(t) := g(t) - \sinh^2 t\ ;$$

we will suppose that for $t \gg 0$

$$(5.1)\qquad |\tilde{g}(t)| \leqslant \frac{C}{t}\,, \qquad \left|\frac{\partial \tilde{g}}{\partial t}\right| \leqslant \frac{C}{t}\,, \qquad \left|\frac{\partial^2 \tilde{g}}{\partial t^2}\right| \leqslant \frac{C}{t}\,,$$

$$(5.2)\qquad |\tilde{f}(t)| \leqslant \frac{C}{t}\,, \qquad \left|\frac{\partial \tilde{f}}{\partial t}\right| \leqslant \frac{C}{t}\,, \qquad \left|\frac{\partial^2 \tilde{f}}{\partial t^2}\right| \leqslant \frac{C}{t}\,.$$

For $i = 1,2,3$ and for every $k \in \mathbb{N}$, let Δ_{Mi} and $D_{i\lambda_k}$ be defined as in Section 4.

First of all, we will determine the essential spectrum of Δ_{M1}. To this purpose, let us recall some basic facts.

Definition 5.1: ([9]) Let A be a selfadjoint operator on a Hilbert space $\mathcal{H}$. An operator C such that $\mathcal{D}(A) \subset \mathcal{D}(C)$ is called relatively compact with respect to A if and only if $C(A+iI)^{-1}$ is compact.

In terms of the Hilbert space $\mathcal{D}(A)$ endowed with the norm $\|\phi\|_A$ given by

$$\|\phi\|_A^2 = \|\phi\|_{\mathcal{H}}^2 + \|A\phi\|_{\mathcal{H}}^2,$$

C is relatively compact if and only if C is compact from $\mathcal{D}(A)$ with the norm $\|.\|_A$ to $\mathcal{H}$ with the norm $\|.\|_{\mathcal{H}}$. Moreover, we recall the following Lemma (for a proof see [9]):

Lemma 5.2: *Let A be a selfadjoint operator on a Hilbert space $\mathcal{H}$, and let C be a symmetric operator such that C is a relatively compact perturbation for A^n for some positive integer n. Suppose further that $B = A + C$ is selfadjoint on $\mathcal{D}(A)$, Then*

$$\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B).$$

Finally, we recall that, given a selfadjoint operator A on a Hilbert space $\mathcal{H}$, $\mu \in \sigma_{\mathrm{ess}}(A)$ if and only if there exists a Weyl sequence $\{w_n\} \subset \mathcal{D}(A)$ for μ, that is, a sequence $\{w_n\} \subset \mathcal{D}(A)$ with no convergent subsequences in $\mathcal{H}$, bounded in $\mathcal{H}$ and such that

$$\lim_{n \to +\infty} (A - \mu)\, w_n = 0 \quad \text{in } \mathcal{H}.$$

We are now in position to prove our first result.

LEMMA 5.3: *Let M be endowed with a Riemannian metric of type* (2.1), *satisfying conditions* (5.1), (5.2); *then for* $0 \leqslant p \leqslant N-1$,

$$\sigma_{\text{ess}}(D^F_{1\lambda^p_k}) = \left[\left(\frac{N-2p-1}{2}\right)^2, +\infty\right)$$

for every $k \in \mathbb{N}$.

PROOF: Let us consider the Friedrichs extension D^F_{10} of the operator with constant coefficients

$$D_{10}\colon C^\infty_c(c, +\infty) \to L^2(c, +\infty)$$

$$D_{10}\, w = -\frac{\partial^2}{\partial t^2} w + \frac{(N-1-2p)^2}{4} w\,. \tag{5.3}$$

It is well-known that $\sigma_{\text{ess}}(D^F_{10}) = \left[\left(\frac{N-2p-1}{2}\right)^2, +\infty\right)$. We will show that $D^F_{1\lambda^p_k} - D^F_{10}$ is a relatively compact perturbation of $(D^F_{10})^2$ for every $k \in \mathbb{N}$. This, thanks to Lemma 5.2, will give the conclusion.

First of all, it is not difficult to see that for every $k \in \mathbb{N}$,

$$\mathcal{D}((D^F_{10})^2) \subset \mathcal{D}(D^F_{1\lambda^p_k} - D^F_{10});$$

indeed, comparing the domains of $D^F_{1\lambda^p_k}$ and of D^F_{10}, we find that

$$\mathcal{D}(D^F_{10}) = \mathcal{D}(D^F_{1\lambda^p_k}).$$

We still have to check that for every sequence $\{w_n\} \subset \mathcal{D}((D^F_{10})^2)$ such that

$$\|w_n\|^2_{L^2} + \|(D^F_{10})^2 w_n\|^2_{L^2} \leqslant C \tag{5.4}$$

there exists a subsequence $\{w_{n_l}\}$ such that $\{(D^F_{1\lambda^p_k} - D^F_{10})\, w_{n_l}\}$ converges in $L^2(c, +\infty)$.

To this purpose, let us observe that conditions (5.1) and (5.2) yield:

$$\left(1 - \frac{1}{f}\right) \in L^2(c, +\infty) \cap L^\infty(c, +\infty); \tag{5.5}$$

$$\frac{1}{f^2}\,\frac{\partial f}{\partial t} \in L^2(c, +\infty) \cap L^\infty(c, +\infty); \tag{5.6}$$

$$W_1(t) \in L^2(c, +\infty) \cap L^\infty(c, +\infty), \tag{5.7}$$

where

$$(5.8)\qquad W_1(t) := \left\{ -\frac{7}{16}\frac{1}{f^3}\left(\frac{\partial f}{\partial t}\right)^2 + \frac{1}{4}\frac{1}{f^2}\frac{\partial^2 f}{\partial t^2} + \right.$$

$$-\frac{1}{2}\frac{1}{f^2}\frac{\partial f}{\partial t}\frac{(N-1-2p)}{4}\frac{1}{g}\frac{\partial g}{\partial t} + \frac{1}{f}\frac{(N-2p-1)}{4}\frac{(N-2p-5)}{4}\frac{1}{g^2}\left(\frac{\partial g}{\partial t}\right)^2 +$$

$$\left. + \frac{1}{f}\frac{(N-2p-1)}{4}\frac{1}{g}\frac{\partial^2 g}{\partial t^2} + \frac{\lambda}{g}\right\} - \frac{(N-2p-1)^2}{4}\,.$$

Moreover, by (5.4), the sequence $\{w_n\}$ is bounded in $W^{3,2}(c, +\infty)$, and the Sobolev embedding theorem implies that $\{w_n\}$, $\left\{\frac{\partial w_n}{\partial t}\right\}$, $\left\{\frac{\partial^2 w_n}{\partial t^2}\right\}$ are bounded sequences in $L^\infty(c, +\infty)$.

Now, for every $n, m \in \mathbb{N}$

$$(5.9)\qquad \|(D^F_{1\lambda^p_k} - D^F_{10})(w_n - w_m)\|_{L^2(c, +\infty)} \leq$$

$$\leq \left\| \left(1 - \frac{1}{f}\right)\frac{\partial^2}{\partial t^2}(w_n - w_m)\right\|_{L^2(c, +\infty)} + \left\|\frac{\partial f}{\partial t}(w_n - w_m)\right\|_{L^2(c, +\infty)} +$$

$$+ \|W_1(w_n - w_m)\|_{L^2(c, +\infty)}.$$

Let us begin with the third summand. For any compact subset $K \subset (c, +\infty)$ and for every $n, m \in \mathbb{N}$

$$\|W_1(w_n - w_m)\|^2_{L^2(c, +\infty)} \leq C\int_K (w_n - w_m)^2\, ds + C\int_{(c, +\infty)\setminus K} W_1^2(s)\, ds,$$

where C is a positive constant independent of K. Indeed, $W_1^2 \in L^\infty(c, +\infty)$ and $(w_n - w_m)^2$ is bounded in $L^\infty(c, +\infty)$.

Let us consider a sequence $\{c_h\} \subset (c, +\infty)$ such that $c_h \to +\infty$ as $h \to +\infty$ and for every $h \in \mathbb{N}$

$$C\int_{c_h}^{+\infty} W_1^2(s)\, ds < \frac{1}{h}\,.$$

For $h = 1$, thanks to the Rellich-Kondrachov theorem, there exists a subsequence $\{w_{n(1)}\}$ such that $\{(w_{n(1)})_{|(c, c_1)}\}$ converges in $L^2(c, c_1)$. Hence, for every $\delta > 0$ there exists $\overline{n}(1)$ such that for every $n, m > \overline{n}(1)$

$$\int_c^{+\infty} W_1^2 (w_{n(1)} - w_{m(1)})^2 \, ds < \frac{\delta}{3} + 1 .$$

Analogously, for $h = 2$ there exists a subsequence $\{w_{n(2)}\} \subseteq \{w_{n(1)}\}$ such that for every $\delta > 0$ there exists $\overline{n}(2)$ such that for every $n, m > \overline{n}(2)$

$$\int_c^{+\infty} W_1^2 (w_{n(2)} - w_{m(2)})^2 \, ds < \frac{\delta}{3} + \frac{1}{2} .$$

Going on in this way, for every $h \in \mathbb{N}$ we can find a subsequence $\{w_{n(h)}\} \subseteq \{w_{n(h-1)}\}$ such that for every $\delta > 0$ there exists $\overline{n}(h)$ such that for every $n, m > \overline{n}(h)$

$$\int_c^{+\infty} W_1^2 (w_{n(h)} - w_{m(h)})^2 \, ds < \frac{\delta}{3} + \frac{1}{h} .$$

Through a Cantor diagonal process, then, we can find a subsequence $\{w_{n_l}\} \subseteq \{w_n\}$ such that $\{W_1 w_{n_l}\}$ is a Cauchy sequence in $L^2(c, +\infty)$.

As for the estimates of the other two summands, recalling (5.5) and (5.6), since $\left\{ \frac{\partial w_{n_l}}{\partial t} \right\}$ and $\left\{ \frac{\partial^2 w_{n_l}}{\partial t^2} \right\}$ are bounded in $L^{\infty}(c, +\infty)$ and in $W^{1,2}(K)$ for any compact set $K \subset (c, +\infty)$, we can apply the same procedure. As a consequence, we can extract a subsequence, again denoted by $\{w_{n_l}\}$, such that $\{(D^F_{1\lambda^p_k} - D^F_{10})\, w_{n_l}\}$ converges in $L^2(c, +\infty)$. This yields the conclusion. ∎

As a consequence,

$$\left[\left(\frac{N - 2p - 1}{2} \right)^2, +\infty \right) \subset \sigma_{\text{ess}}(\Delta_{M1}) .$$

On the other hand, the following Lemma holds:

LEMMA 5.4: *Let M be endowed with a Riemannian metric of type* (2.1), *such that* $f(t) \to 1$ *and* $g(t) \to \sinh^2 t$ *as* $t \to +\infty$; *for* $0 \leq p \leq N-1$, *if* $\mu < \left(\frac{N-2p-1}{2} \right)^2$, *then* $\mu \notin \sigma_{\text{ess}}(\Delta_{M1})$.

PROOF: First of all, for every $k \in \mathbb{N}$ let us write $D'_{1\lambda^p_k}$ as

$$D'_{1\lambda^p_k} w = - \frac{\partial}{\partial t} \left(\frac{1}{f} \frac{\partial w}{\partial t} \right) + \left(V_1(t) + \frac{\lambda^p_k}{g} \right) w ,$$

where

$$(5.10)\qquad V_1(t) := \left\{ -\frac{7}{16}\frac{1}{f^3}\left(\frac{\partial f}{\partial t}\right)^2 + \frac{1}{4}\frac{1}{f^2}\frac{\partial^2 f}{\partial t^2} - \right.$$

$$-\frac{1}{2}\frac{1}{f^2}\frac{\partial f}{\partial t}\frac{(N-1-2p)}{4}\frac{1}{g}\frac{\partial g}{\partial t} + \frac{1}{f}\frac{(N-2p-1)}{4}\frac{(N-2p-5)}{4}\frac{1}{g^2}\left(\frac{\partial g}{\partial t}\right)^2 +$$

$$\left. + \frac{1}{f}\frac{(N-2p-1)}{4}\frac{1}{g}\frac{\partial^2}{g\partial t^2}\right\}.$$

Now, let $\mu < \left(\frac{N-1-2p}{2}\right)^2$. Since for every $k \in \mathbb{N}$ the essential spectrum of $D^F_{1\lambda^p_k}$ does not depend on c and since $V_1(t)$ converges to $\left(\frac{N-2p-1}{2}\right)^2 > \mu$ as $t \to +\infty$, we can choose $c > 0$ such that for every $t > c$

$$V_1(t) - \mu > C$$

for some positive constant $C > 0$.

If $\mu \in \sigma_{\mathrm{ess}}(\Delta_{M1}) = \sigma_{\mathrm{ess}}(\Delta^F_{M1})$, there exists a Weyl sequence for μ, that is a sequence $\{\omega_k\} \subset \mathcal{D}(\Delta^F_{M1})$ such that

$$\langle \omega_k, \omega_k \rangle_{L^2_p(M)} \leq C,$$

$$\lim_{k \to +\infty} (\Delta^F_{M1}\omega_k - \mu\omega_k) = 0,$$

from which it is not possible to extract any subsequence converging in $L^2_p(M)$. Moreover, we can suppose that

$$\omega_k = h_k(t)\,\tau_{1k},$$

where τ_{1k} is a coclosed normalized p-eigenform of $\Delta_{S^{N-1}}$ corresponding to λ^p_k and $\lambda^p_k \to +\infty$ as $k \to +\infty$. Hence, via unitary equivalence, there exists a sequence $\{w_k\} \subset \mathcal{D}(D^F_{1\lambda^p_k})$ such that

$$\|w_k\|_{L^2(c,+\infty)} \leq C$$

$$(5.11)\qquad \lim_{k \to +\infty} \|D^F_{1\lambda^p_k} w_k - \mu w_k\|_{L^2(c,+\infty)} = 0,$$

from which we cannot extract any L^2-converging subsequence. Then

$$\langle D^F_{1\lambda^p_k} w_k - \mu w_k, w_k \rangle_{L^2(c,+\infty)} \to 0$$

as $k \to +\infty$, and, since for every $k \in \mathbb{N}$

$$\mathcal{D}(D^F_{1\lambda^p_k}) \subset W^{1,2}_0(c, +\infty),$$

we get

$$(5.12)\qquad \int_c^{+\infty} \frac{1}{f(s)} \left(\frac{\partial w_k}{\partial s} \right)^2 (s)\, ds + \int_c^{+\infty} [V_1(s) - \mu]\, w_k^2(s)\, ds + \int_c^{+\infty} \frac{\lambda_k^p}{g(s)}\, w_k^2(s)\, ds \to 0$$

as $k \to +\infty$. Since all the terms are positive, we have

$$\int_c^{+\infty} [V_1(s) - \mu]\, w_k^2(s)\, ds \to 0$$

as $k \to +\infty$, whence

$$\int_c^{+\infty} w_k^2(s)\, ds \to 0$$

as $k \to +\infty$, because

$$\int_c^{+\infty} w_k^2(s)\, ds \leq \frac{1}{C} \int_c^{+\infty} [V_1(s) - \mu]\, w_k^2(s)\, ds\, .$$

This yields a contradiction. Hence, if $\mu \leq \left(\frac{N - 2p - 1}{2} \right)^2$, $\mu \notin \sigma_{\text{ess}}(\Delta_{M1})$. ■

As a consequence

Proposition 5.5: *Let M be endowed with a Riemannian metric of type* (2.1), *satisfying conditions* (5.1), (5.2); *then, for* $0 \leq p \leq N - 1$,

$$\sigma_{\text{ess}}(\Delta_{M1}) = \left[\left(\frac{N - 2p - 1}{2} \right)^2, +\infty \right).$$

By duality,

Proposition 5.6: *Let M be endowed with a Riemannian metric of type* (2.1), *satisfying conditions* (5.1), (5.2); *then, for* $1 \leq p \leq N$,

$$\sigma_{\text{ess}}(\Delta_{M2}) = \left[\left(\frac{N - 2p + 1}{2} \right)^2, +\infty \right).$$

We still have to determine the essential spectrum of Δ_{M3} for $1 \leq p \leq N - 1$. First of all, we compute the essential spectrum of $D^F_{3\lambda_k^{p-1}}$ for every $k \in \mathbb{N}$:

LEMMA 5.7: *Let M be endowed with a Riemannian metric of type* (2.1), *satisfying conditions* (5.1) *and* (5.2); *then, for* $1 \leqslant p \leqslant N-1$,

$$\sigma_{\mathrm{ess}}(D^F_{3\lambda_k^{p-1}}) = \left[\min\left\{\left(\frac{N-2p-1}{2}\right)^2, \left(\frac{N-2p+1}{2}\right)^2\right\}, +\infty\right)$$

for every $k \in \mathbb{N}$.

PROOF: Let us consider the Friedrichs extension D^F_{30} of the operator

$$D_{30}\colon C_c^\infty(c, +\infty) \oplus C_c^\infty(c, +\infty) \to L^2(c, +\infty) \oplus L^2(c, +\infty)$$

$$D_{30}(w_1 \oplus w_2) := \left(-\frac{\partial^2 w_1}{\partial t^2} + \left(\frac{N-2p-1}{2}\right)^2 w_1\right) \oplus \oplus \left(-\frac{\partial^2 w_2}{\partial t^2} + \left(\frac{N-2p+1}{2}\right)^2 w_2\right). \tag{5.13}$$

It is not difficult to see that

$$\sigma_{\mathrm{ess}}(D^F_{30}) = \left[\min\left\{\left(\frac{N-2p-1}{2}\right)^2, \left(\frac{N-2p+1}{2}\right)^2\right\}, +\infty\right).$$

As in Lemma 5.3, we will show that $D^F_{3\lambda_k^{p-1}} - D^F_{30}$ is a relatively compact perturbation of $(D^F_{30})^2$. First of all, $\mathcal{D}((D^F_{30})^2) \subset \mathcal{D}(D^F_{30}) = \mathcal{D}(D^F_{3\lambda_k^{p-1}} - D^F_{30})$; indeed, an explicit comparison of the domains shows that $\mathcal{D}(D^F_{30}) = \mathcal{D}(D^F_{3\lambda_k^{p-1}})$ for every $k \in \mathbb{N}$.

We still have to check that for every sequence

$$\{w_{1n} \oplus w_{2n}\} \subset \mathcal{D}((D^F_{30})^2)$$

such that

$$\|w_{1n} \oplus w_{2n}\|_{L^2 \oplus L^2} + \|(D^F_{30})^2(w_{1n} \oplus w_{2n})\|_{L^2 \oplus L^2} \leqslant C, \tag{5.14}$$

there exists a subsequence $\{w_{1n_l} \oplus w_{2n_l}\}$ such that

$$(D^F_{3\lambda_k^{p-1}} - D^F_{30})(w_{1n_l} \oplus w_{2n_l})$$

converges in $L^2(c, +\infty) \oplus L^2(c, +\infty)$.

Now, (5.14) implies that $\{w_{in}\}$ is bounded in $W^{3,2}(c, +\infty)$ for $i = 1, 2$; hence $\{w_{in}\}$, $\left\{\frac{\partial w_{in}}{\partial t}\right\}$ and $\left\{\frac{\partial^2 w_{in}}{\partial t^2}\right\}$ are bounded in $L^\infty(c, +\infty)$ and in $W^{1,2}(K)$ for every compact subset $K \subset (c, +\infty)$. For every $n, m \in \mathbb{N}$

$$\|(D^F_{3\lambda_k^{p-1}} - D^F_{30})((w_{1n} - w_{1m}) \oplus (w_{2n} - w_{2m}))\|_{L^2(c,+\infty)\oplus L^2(c,+\infty)} \leqslant$$

$$\leqslant \|(D^F_{1\lambda_k^{p-1}} - D^F_{10})(w_{1n} - w_{1m})\|_{L^2(c,+\infty)} + \|(D^F_{2\lambda_k^{p-1}} - D^F_{20})(w_{2n} - w_{2m})\|_{L^2(c,+\infty)} +$$

$$+ \|V_{3\lambda_k^{p-1}}(w_{1n} - w_{1m})\|_{L^2(c,+\infty)} + \|V_{3\lambda_k^{p-1}}(w_{2n} - w_{2m})\|_{L^2(c,+\infty)},$$

where

$$V_{3\lambda_k^{p-1}}(t) := g(t)^{-\frac{3}{2}} f(t)^{-\frac{1}{2}} \frac{\partial g}{\partial t} \sqrt{\lambda_k^{p-1}}.$$

The first two terms can be estimated as in Lemma 5.3; as for the last two terms, since under conditions (5.1) and (5.2)

$$V_{3\lambda_k^{p-1}} \in L^2(c, +\infty) \cap L^\infty(c, +\infty),$$

following the argument of Lemma 5.3 we get the conclusion. ■

We still have to check whether the essential spectrum of Δ_{M3} can contain any other $\mu \in \mathbb{R}$. The techniques of Lemma 5.4 can not be applied in this case, because $D_{3\lambda_k^{p-1}}$ is a coupled system of differential operators. Hence different techniques are needed. We have

LEMMA 5.8: *Let M be endowed with a Riemannian metric of type* (2.1), *satisfying conditions* (5.1) *and* (5.2). *For* $1 \leqslant p \leqslant N-1$, if $0 < \mu < \min\left\{\left(\frac{N-2p-1}{2}\right)^2, \left(\frac{N-2p+1}{2}\right)^2\right\}$, *then* $\mu \notin \sigma_{\text{ess}}(\Delta_{M3})$.

PROOF: We already know from Lemma 5.7 that for every $k \in \mathbb{N}$,

$$\sigma_{\text{ess}}(D^F_{3\lambda_k^{p-1}}) = \left[\min\left\{\left(\frac{N-2p-1}{2}\right)^2, \left(\frac{N-2p+1}{2}\right)^2\right\}, +\infty\right).$$

As a consequence, given a positive $\mu < \min\left\{\left(\frac{N-2p-1}{2}\right)^2, \left(\frac{N-2p+1}{2}\right)^2\right\}$, μ belongs to the essential spectrum of Δ_{M3} if and only if there exist a sequence $\{\mu_k\}$ of eigenvalues of Δ_M and a corresponding sequence $\{\Phi_k\}$ of p-forms of type III such that

$$\mu_k \to \mu \quad \text{as } k \to +\infty$$

and for every $k \in \mathbb{N}$

$$\Delta_M \Phi_k - \mu_k \Phi_k = 0.$$

Since $\mu > 0$, we can suppose, up to the choice of a subsequence, that either for every $d_M \Phi_k \neq 0$ for every $k \in \mathbb{N}$ or $\delta_M \Phi_k \neq 0$ for every $k \in \mathbb{N}$. Let us suppose to be in the first case. In view of (2.6), $d_M \Phi_k$ is a $(p+1)$-form of type II; moreover,

$$\|d_M \Phi_k\|_{L^2(M)} < C \quad \text{for every } k \in \mathbb{N}$$

$$\Delta_M d_M \Phi_k - \mu_k d_M \Phi_k = 0 \quad \text{for every } k \in \mathbb{N},$$

and

$$\mu_k \to \mu \quad \text{as } k \to +\infty .$$

Hence, $\mu \in \sigma_{\text{ess}}(\Delta_{M2})$, and, thanks to Proposition 5.6,

$$\mu > \left(\frac{N - 2(p+1) + 1}{2} \right)^2 = \left(\frac{N - 2p - 1}{2} \right)^2,$$

in contradiction with our hypothesis.

If on the contrary we are in the second case, in view of (2.7), $\delta_M \Phi_k$ is a $(p-1)$-form of type I; moreover,

$$\|\delta_M \Phi_k\|_{L^2(M)} < C \quad \text{for every } k \in \mathbb{N},$$

$$\Delta_M \delta_M \Phi_k - \mu_k \delta_M \Phi_k = 0 \quad \text{for every } k \in \mathbb{N},$$

and

$$\mu_k \to \mu \quad \text{as } k \to +\infty .$$

Hence, $\mu \in \sigma_{\text{ess}}(\Delta_{M1})$, and by Proposition 5.5

$$\mu > \left(\frac{N - 2(p-1) - 1}{2} \right)^2 = \left(\frac{N - 2p + 1}{2} \right)^2,$$

in contradiction with our hypothesis.

This yields the conclusion. ■

Hence,

PROPOSITION 5.9: *Let M be endowed with a Riemannian metric of type* (2.1), *satisfying conditions* (5.1), (5.2); *then, for* $1 \leq p \leq N-1$,

$$\sigma_{\text{ess}}(\Delta_{M3}) \setminus \{0\} = \left[\min \left\{ \left(\frac{N-2p-1}{2} \right)^2, \left(\frac{N-2p+1}{2} \right)^2 \right\}, +\infty \right).$$

REMARK 5.10: By an argument similar to that of Lemma 5.8 it is possible to show that if there exist a sequence $\{\mu_k\}$ of positive eigenvalues of Δ_M and a corresponding sequence $\{\Phi_k\}$ of p-forms of type III such that

$$\mu_k \to 0 \qquad \text{as } k \to +\infty$$

and for every $k \in \mathbb{N}$

$$\Delta_M \Phi_k - \mu_k \Phi_k = 0,$$

then $0 \notin \sigma_{\text{ess}}(\Delta_{M3})$.

Recalling the results of section 3, finally we can completely determine the essential spectrum of Δ_M:

THEOREM 5.11: *Let M be endowed with a Riemannian metric of type* (2.1) *satisfying condition* (2.3) *and conditions* (5.1), (5.2). *Then, if* $p \neq \frac{N}{2}$,

$$\sigma_{\text{ess}}(\Delta_M) = \left[\min\left\{\left(\frac{N-2p-1}{2}\right)^2, \left(\frac{N-2p+1}{2}\right)^2\right\}, +\infty\right)$$

whilst if $p = \frac{N}{2}$

$$\sigma_{\text{ess}}(\Delta_M) = \{0\} \cup \left[\frac{1}{4}, +\infty\right).$$

PROOF: Thanks to Propositions 5.5, 5.6, 5.9 we have that

$$\sigma_{\text{ess}}(\Delta_M)\setminus\{0\} = \left[\min\left\{\left(\frac{N-2p-1}{2}\right)^2, \left(\frac{N-2p+1}{2}\right)^2\right\}, +\infty\right).$$

Moreover, in view of Remark 5.10, 0 can belong to the essential spectrum of Δ_M if and only if it is an eigenvalue of Δ_M of infinite multiplicity. In view of Theorem 3.2, this happens only if $p = \frac{N}{2}$. The conclusion follows. ■

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