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# Generalized Growth and Polynomial Approximation of Generalized Biaxisymmetric Potentials (**) 

Abstract. - In this paper some results have been obtained which could provide methods to measure the rate of growth of the maximum modulus of GBASP. These measures of the rate of growth are obtained in terms of the maximum term, the rank, the coefficients and even polynomial approximation errors. The main advantage of our approach is that it carries over to the non-entire case also.

## Crescita generalizzata e approssimazione polinomiale per potenziali biassialmente simmetrici

Sunto. - Si ottengono risultati applicabili alla misurazione della velocità di crescita del massimo modulo per potenziali biassialmente simmetrici in senso generalizzato.

## 1. - Introduction

Let $F^{\alpha, \beta}$ be a real valued regular solution to the generalized biaxisymmetric potential equation

$$
\begin{equation*}
\left(\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}+\frac{2 \alpha+1}{x} \frac{\partial}{\partial x}+\frac{2 \beta+1}{y} \frac{\partial}{\partial y}\right) F^{\alpha, \beta}=0, \quad \alpha>\beta>-\frac{1}{2}, \tag{1.1}
\end{equation*}
$$

where $(\alpha, \beta)$ are fixed in a neighbourhood of the origin and the analytic Cauchy data $F_{x}^{\alpha, \beta}(0, y)=F_{y}^{\alpha, \beta}(x, 0)=0$ is satisfied along the singular lines in the open unit hy-
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pershere $\Sigma^{\alpha, \beta}$. Such functions with even harmonic functions are referred to as generalized biaxisymmetric potentials (GBASP) having local expansions of the form

$$
\begin{equation*}
F^{\alpha, \beta}(x, y)=\sum_{n=0}^{\infty} a_{n} R_{n}^{\alpha, \beta}(x, y) \tag{1.2}
\end{equation*}
$$

such that

$$
R_{n}^{\alpha, \beta}(x, y)=\left(x^{2}+y^{2}\right)^{n} P_{n}^{\alpha, \beta}\left(\left(x^{2}-y^{2}\right) /\left(x^{2}+y^{2}\right)\right) / P_{n}^{\alpha, \beta}(1), \quad n=0,1,2, \ldots
$$

where the $P_{n}^{\alpha, \beta}$ are Jacobi polynomials [1].
The GBASP are natural extensions of harmonic / analytic functions. Hence we anticipate properties similar to those of the harmonic function found from associated analytic $f$, by taking Ref, the real part of $f$.

The purpose of this paper is to obtain some theorems which could provide methods to measure the rate of growth of the maximum modulus of GBASP defined by (1.2). These measures of the rate of growth are obtained in terms of the maximum term, the rank, the coefficients $a_{n}, s$ and even polynomial approximation errors.

The motivation for this work came from the papers of several authors, [3-17]. But primarily this work is influenced by the results of Seremeta [15]. Although our work does not include Seremeta's results, it is complementary to his work. Seremeta obtains formulas for generalized orders and types separately, but our results provides generalized orders and types simultaneously. Also, McCoy [12], obtains his results for orders and types of GBASP, s in terms of coefficients and even polynomial approximation errors. Using methods and hypothesis quite different than Seremeta [15] and McCoy [12], we have acquired much more than expected. In some ways our hypothesis are more useful than those. For example, although Seremeta [15] and McCoy [12], have obtained results for multivariate case but the main advantage of our approach is that it carries over to the multivariate case with greater ease and simplicity and it also carries over to the non-entire case. Thus the scope of our work in this paper is much broader than the scope of the earlier work in that it deals with both entire and non-entire analytic functions /GBASP's of one or more complex variables in the same framework.

Let $f(z)=\sum_{k=0}^{\infty} a_{k} z^{2 n_{k}}$ be analytic in the open unit disc $D \equiv\{z:|z|<1\}, n_{0}=0$ and $\left\{n_{k}\right\}_{k=1}^{\infty}$ being the strictly increasing sequence of positive integers such that $a_{k} \neq 0$ for $k=1,2, \ldots$.

Let $\mu\left(r, F^{\alpha, \beta}\right)$ denote the maximum term of GBASP, $F^{\alpha, \beta}, v\left(r, F^{\alpha, \beta}\right)$ the rank of $\mu\left(r, F^{\alpha, \beta}\right)$ and $M\left(r, F^{\alpha, \beta}\right)$ the maximum modulus as in complex function theory [4].

Let the operator mapping unique associated even analytic function $f$ on to GBASP, $F^{\alpha, \beta}(x, y)=\sum_{n=0}^{\infty} a_{n} R_{n}^{\alpha, \beta}(x, y)$. Following McCoy [12] for Koorn winder in-
tegral for Jacobi polynomials

$$
\begin{aligned}
& F^{\alpha, \beta}(x, y)=K_{\alpha, \beta}(f)=\int_{0}^{1} \int_{0}^{\pi} f(\zeta) \mu_{\alpha, \beta}(t, s) d s d t \\
& \mu_{\alpha, \beta}(t, s)=\gamma_{\alpha, \beta}\left(1-t^{2}\right)^{\alpha-\beta-1} t^{2 \beta+1}(\sin s)^{2 \alpha} \\
& s^{2}=x^{2}-y^{2} t^{2}-i 2 x y t \cos s \\
& \gamma_{\alpha, \beta}=2 \Gamma(\alpha+1) / \Gamma(1 / 2) \Gamma(\alpha-\beta) \Gamma(\beta+1 / 2) .
\end{aligned}
$$

The inverse operator applies orthogonality of Jacobi polynomials ([1], p. 8) and Poisson kernel ([1], p. 11) to uniquely define the transform,

$$
\begin{aligned}
& f(z)=K_{\alpha, \beta}^{-1}\left(F^{\alpha, \beta}\right)=\int_{-1}^{+1} F^{\alpha, \beta}\left(\tau \xi, r\left(1-\xi^{2}\right)^{1 / 2}\right) v_{\alpha, \beta}\left((z / r)^{2}, \xi\right) d \xi \\
& \quad v_{\alpha, \beta}(\tau, \xi)=S_{\alpha, \beta}(\tau, \xi)(1-\xi)^{\alpha}(1+\xi)^{\beta} \\
& S_{\alpha, \beta}(\xi, \xi)=\eta_{\alpha, \beta} \frac{1-\tau}{(1+\tau)^{\alpha+\beta+2}} F\left(\frac{\alpha+\beta+2}{2} ; \frac{\alpha+\beta+3}{2} ; \beta+1 ; \frac{2 \tau(1+\xi)}{(1+\tau)^{2}}\right) \\
& \eta_{\alpha, \beta}=\Gamma(\alpha+\beta+2) / 2^{\alpha+\beta+1} \Gamma(\alpha+1) \Gamma(\beta+1) .
\end{aligned}
$$

The normalizations $K_{\alpha, \beta}(1)=K_{\alpha, \beta}^{-1}(1)$ are taken. The kernel $S_{\alpha, \beta}(\tau, \xi)$ is analytic on $\|\tau\|<1$ for $-1 \leqslant \xi \leqslant 1$. The local function elements $F^{\alpha, \beta}$ and $f$ are continued harmonically/analytically by contour deformation using the envelope method [2]. It was proved [2], that GBASP $F^{\alpha, \beta}$ is regular in the open unit hypersphere $\Sigma^{\alpha, \beta}$ if and only if its associate $f$ is analytic in the unit disc. Further we have

$$
f(x+i o)=F^{\alpha, \beta}(x, o), \quad|x|<1
$$

which can be analytically continued as

$$
f(z)=F^{\alpha, \beta}(z, o), \quad|z|<1
$$

Let $\xi(x)$ and $\eta(x)$ be functions of real variables $x$ with the following properties:
(i) $\xi(x)$ is defined, positive, continuous, and strictly increasing to $+\infty$ for all $x$ such that $x \geqslant x_{o}>0$.
(ii) $\eta(x)$ is defined, positive, continuous, and strictly decreasing from $+\infty$ for all value of $x$ such that $0 \leqslant x \leqslant x_{1}$. Further $\eta(0)=+\infty ; x_{1}>0$.
(iii) $\xi(c x) \cong \xi(x)$ as $x \rightarrow \infty$ for each $c>0$, $\eta(c x)=0[\eta(x)]$ as $x \rightarrow 0^{+}$for each $c>0$, $\eta[x+o(x)]=\eta(x)$ as $x \rightarrow 0^{+}$, when $o(x)>0$.
(iv) $\xi(\log x)=o[\xi(x)]$ as $x \rightarrow \infty$.

$$
\xi(1 / x)=O[\eta(x)] \text { as } x \rightarrow 0^{+} .
$$

(v) $\frac{\xi^{-1}[c \eta(x)]}{\xi^{-1}[(c+\varepsilon) \eta(x)]}=o(x)$ as $x \rightarrow 0^{+}$for each $c>o$; and each $\varepsilon>o$.

Further we define

$$
\begin{aligned}
& U= \begin{cases}\limsup _{r \rightarrow 1^{-}} \frac{\xi\left(\log \mu\left(r, F^{\alpha, \beta}\right)\right)}{\eta(-\log r)}, & \text { if } \mu\left(r, F^{\alpha, \beta}\right) \rightarrow+\infty \text { as } r \rightarrow 1^{-} \\
0 & \text { Otherwise , }\end{cases} \\
& u= \begin{cases}\liminf _{r \rightarrow 1^{-}} \frac{\xi\left(\log \mu\left(r, F^{\alpha, \beta}\right)\right)}{\eta(-\log r)}, & \text { if } \mu\left(r, F^{\alpha, \beta}\right) \rightarrow+\infty \text { as } r \rightarrow 1^{-} \\
0 & \text { Otherwise , }\end{cases} \\
& V= \begin{cases}\limsup _{r \rightarrow 1^{-}} \frac{\xi\left(\log v\left(r, F^{\alpha, \beta}\right)\right)}{\eta(-\log r)}, & \text { if } v\left(r, F^{\alpha, \beta}\right) \rightarrow+\infty \text { as } r \rightarrow 1^{-} \\
0 & \text { Otherwise , }\end{cases} \\
& v= \begin{cases}\liminf _{r \rightarrow 1^{-}} \frac{\xi\left(\log v\left(r, F^{\alpha, \beta}\right)\right)}{\eta(-\log r)}, & \text { if } v\left(r, F^{\alpha, \beta}\right) \rightarrow+\infty \text { as } r \rightarrow 1^{-} \\
0 & \text { Otherwise },\end{cases} \\
& M= \begin{cases}\lim \sup _{r \rightarrow 1^{-}} \frac{\xi\left(\log M\left(r, F^{\alpha, \beta}\right)\right)}{\eta(-\log r)}, & \text { if } M\left(r, F^{\alpha, \beta}\right) \rightarrow+\infty \text { as } r \rightarrow 1^{-} \\
0 & \text { Otherwise , }\end{cases} \\
& m= \begin{cases}\liminf _{r \rightarrow 1^{-}} \frac{\xi\left(\log M\left(r, F^{\alpha, \beta}\right)\right)}{\eta(-\log r)}, & \text { if } M\left(r, F^{\alpha, \beta}\right) \rightarrow+\infty \text { as } r \rightarrow 1^{-} \\
0 & \text { Otherwise },\end{cases}
\end{aligned}
$$

Also,

$$
A=\limsup _{k \rightarrow \infty} \frac{\xi\left(2 n_{k}\right)}{\eta\left[\log ^{+}\left|a_{k}\right|^{1 / 2 n_{k}}\right]} ; \quad a=\liminf _{k \rightarrow \infty} \frac{\xi\left(2 n_{k}\right)}{\eta\left[\log ^{+}\left|a_{k}\right|^{1 / 2 n_{k}}\right]} ;
$$

In above definitions the letters $U, u, V, v, M, m, A$, a designate the measures of the rate of growth in terms of maximum term, the rank, the maximum modulus and the coefficients in the expansion of GBASP $F^{\alpha, \beta}$.

## 2. - Auxiliary results

In this section we prove some lemmas which have been used in the sequel.
Lemma 1: From properties (i) and (iii) of the function $\xi(x)$ we may deduce that for any number $C>1$ we shall have

$$
\xi[a+b] \leqslant C[\xi(a)+\xi(b)]
$$

Provided $a>N(C)$ or $b>N(C)$.
Proof: Let $C>1$ and $a$ and $b$ in the domain of $\xi(x)$ and suppose that $a \leqslant b$. Then

$$
\xi[a+b] \leqslant \frac{\xi(2 b)}{\xi(b)}\{\xi(a)+\xi(b)\}
$$

But $\xi(2 b) \sim \xi(b)$ as $b \rightarrow+\infty$. Hence for sufficiently large b ,

$$
\frac{\xi(2 b)}{\xi(b)} \leqslant C
$$

Hence the result follows.
Lemma 2: For real valued GBASP $F^{\alpha, \beta}$ and $0 \leqslant r<t<1$, we have

$$
M\left(r, F^{\alpha, \beta}\right) \leqslant \mu\left(r, F^{\alpha, \beta}\right)\left\{v\left(\left(t, F^{\alpha, \beta}\right)+t /(t-r)\right\} .\right.
$$

Proof: Let $0 \leqslant r<t<1$ and set $N=n_{k}=\left\{v\left(\left(t, F^{\alpha, \beta}\right)\right.\right.$. Then

$$
\begin{aligned}
& M\left(r, F^{\alpha, \beta}\right) \leqslant \sum_{j=0}^{\infty}\left|a_{j}\right| r^{2 n_{j}} \leqslant N \mu\left(r, F^{\alpha, \beta}\right)+\left\{\frac{\mu\left(r, F^{\alpha, \beta}\right)}{\left|a_{k}\right| r^{2 N}}\right\} \sum_{j=0}^{\infty}\left|a_{j}\right| r^{2 n j} \leqslant \\
& \quad \leqslant N \mu\left(r, F^{\alpha, \beta}\right)+\mu\left(r, F^{\alpha, \beta}\right) \sum_{j=k}^{\infty} \frac{\left|a_{j}\right| t^{2 n_{j}}}{\left|a_{k}\right| t^{2 N}}(r / t)^{2\left(n_{j}-N\right)} \leqslant \mu\left(r, F^{\alpha, \beta}\right)\{N+t /(t-r)\} .
\end{aligned}
$$

Hence the proof is completed.
Lemma 3: If $u \geqslant 1$ and $v \geqslant 1$, then $\exp (u v) \geqslant u \exp (v)$.
Proof: Obvious
Lemma 4: If both $u \geqslant 2$ and $v \geqslant 2$, then $\exp (u v) \geqslant \exp (u) \exp (v)$.
Proof: Let $\varrho^{*}=u+v$. Then $u v=\left(\varrho^{*}-u\right) u=2 \varrho^{*}-4$ when $u=2$. But $2 \varrho^{*}-$ $-4=\varrho^{*}+\left(\varrho^{*}-4\right) \geqslant \varrho^{*}$ when $u \geqslant 2$ and $v \geqslant 2$. Therefore by symmetry for a fixed sum $\varrho^{*} \geqslant 4$ the parabola $\left(\varrho^{*}-u\right) u$ lies above the line $y=\varrho^{*}$ when $u \geqslant 2$ and
$\varrho^{*}-u \geqslant 2$. Now the result can be easily proved.

Lemma 5: Let $v\left(t, F^{\alpha, \beta}\right)$ be an increasing integrable function and set

$$
\log \mu\left(r, F^{\alpha, \beta}\right)=\log \mu\left(r_{o}, F^{\alpha, \beta}\right)+\int_{r_{o}}^{r} v\left(t, F^{\alpha, \beta}\right) d t, \quad 0<r_{o}<r<1
$$

and define

$$
\begin{aligned}
& U=\lim \sup _{r \rightarrow 1^{-}} \frac{\log \log \mu\left(r, F^{\alpha, \beta}\right)}{-\log \log (1 / r)}=\lim _{r \rightarrow 1^{-}} \frac{\log v\left(r, F^{\alpha, \beta}\right)}{-\log (1-r)} \\
& V=\limsup _{r \rightarrow 1^{-}} \frac{\log \log v\left(r, F^{\alpha, \beta}\right)}{-\log \log (1 / r)}=\limsup _{r \rightarrow 1^{-}} \frac{\log v\left(r, F^{\alpha, \beta}\right)}{-\log (1-r)},
\end{aligned}
$$

then $V=U+1$, when $U>0$ or $V>1$ and $V \leqslant 1$ when $U=0$.

Proof: We have

$$
\frac{1}{(1-r)} \sim \frac{1}{\log 1 / r} \text { and } \log [1 /(1-r)] \sim-\log \log (1 / r) \text { as } r \rightarrow 1^{-}
$$

Set $r^{\prime}=(r+1) / 2$ and assume that $\log \mu\left(r, F^{\alpha, \beta}\right)>1$.
(2.1) $\log \mu\left(r, F^{\alpha, \beta}\right) \geqslant \int_{r}^{r^{\prime}}\left(v\left(t, F^{\alpha, \beta}\right) / t\right) d t \geqslant v\left(r, F^{\alpha, \beta}\right)\left[\left(r^{\prime}-r\right) / r^{\prime}\right]$

$$
\geqslant v\left(r, F^{\alpha, \beta}\right)\left(\frac{1-r}{2}\right)
$$

Now, since $\frac{\log \left(1 / r^{\prime}\right)}{\log (1 / r)} \rightarrow 1 / 2$ as $r \rightarrow 1^{-}$, so

$$
-\log \log \left(1 / r^{\prime}\right) \sim-\log \log (1 / r) \text { as } r \rightarrow 1^{-}
$$

Thus from relation (2.1) we find that

$$
\begin{equation*}
\log \log \mu\left(r, F^{\alpha, \beta} \geqslant \log v\left(r, F^{\alpha, \beta}\right)-\log (1 / 1-r)-\log 2\right. \tag{2.2}
\end{equation*}
$$

Since $\quad-\log \log \left(1 / r^{\prime}\right) \sim-\log \log (1 / r) \sim \log [1 /(1-r)], \quad$ dividing $\quad(2.2) \quad$ by $-\log \log \left(1 / r^{\prime}\right)$, we get

$$
U \geqslant V-1 \text { or } V \leqslant U+1
$$

Now if $V<U+1$ then for some $U^{\prime}$ such that $0<U^{\prime}<U$ and $r \geqslant r_{o} \geqslant r\left(U^{\prime}\right)$, we have

$$
\begin{equation*}
\frac{\log v\left(r, F^{\alpha, \beta}\right)}{\log (1 / 1-r)}<1+U^{\prime}<1+U \tag{2.3}
\end{equation*}
$$

So we have

$$
\begin{equation*}
\log \mu\left(r, F^{\alpha, \beta}\right) \leqslant \log \mu\left(r_{o}, F^{\alpha, \beta}\right)+\int_{r_{0}}^{r}(1 / 1-t)^{1+U^{\prime}} \frac{d t}{t} \tag{2.4}
\end{equation*}
$$

or

$$
\begin{aligned}
\log \mu\left(r, F^{\alpha, \beta}\right. & \leqslant \log \mu\left(r_{o}, F^{\alpha, \beta}\right)+\frac{1}{r_{0}} \int_{r_{0}}^{r}(1-t)^{-1-U} d t \\
& \leqslant \log \mu\left(r_{o}, F^{\alpha, \beta}\right)+\frac{1}{r_{0}} \frac{(1-r)^{-U^{\prime}}}{U^{\prime}}
\end{aligned}
$$

Then

$$
\log \log \mu\left(r, F^{\alpha, \beta}\right) \leqslant K+U^{\prime} \log (1 / 1-r)
$$

Whence $U \leqslant U^{\prime}<U$, which is impossible. Hence the proof is completed.

## 3. - Main results

Theorem 1: For real valued GBASP, $F^{\alpha, \beta}$ if $\mu\left(r, F^{\alpha, \beta}\right) \rightarrow+\infty$ as $r \rightarrow 1^{-}$, then
(i) $M=U=V=A$,
(ii) $u \leqslant v$ and $u \leqslant m$,
(iii) $u=m$ if $U$ is finite.

If $\mu\left(r, F^{\alpha, \beta}\right)$ is bounded above then $M=m=U=u=V=v=o$.
Proof: Let $\mu\left(r, F^{\alpha, \beta}\right) \rightarrow+\infty$ as $r \rightarrow 1^{-}$and let

$$
A=\limsup _{k \rightarrow \infty} \frac{\xi\left(2 n_{k}\right)}{\eta\left[\log ^{+}\left|a_{k}\right|^{1 / 2 n_{k}}\right]}<+\infty
$$

Therefore for any $\varepsilon>0$ and sufficiently large $k$, we have

$$
\frac{\xi\left(2 n_{k}\right)}{\eta\left[\log ^{+}\left|a_{k}\right|^{1 / 2 n_{k}}\right]} \leqslant(A+\varepsilon)=A \text { * }
$$

or

$$
\xi\left(2 n_{k}\right) \leqslant A * \eta\left[\log ^{+}\left|a_{k}\right|^{1 / 2 n_{k}}\right], \quad \eta^{-1}\left\{\frac{\xi\left(2 n_{k}\right)}{A^{*}}\right\} \geqslant \log ^{+}\left|a_{k}\right|^{1 / 2 n_{k}} \geqslant \log \left|a_{k}\right|^{1 / 2 n_{k}}
$$

Hence for $k \geqslant k(\varepsilon)$ and for all $r$, we get

$$
\begin{equation*}
\log \left|a_{k}\right|^{1 / 2 n_{k}} \geqslant \eta^{-1}\left\{\frac{\xi\left(2 n_{k}\right)}{A^{*}}\right\}+\log r \tag{3.1}
\end{equation*}
$$

But

$$
\begin{align*}
& \eta^{-1}\left\{\frac{\xi\left(2 n_{k}\right)}{A^{*}}\right\}+\log r<Q, \quad \text { whenever }  \tag{3.2}\\
& 2 n_{k}>\xi^{-1}\left\{A^{*} \eta(\log (1 / r))\right\} \text { and } r \rightarrow 1 \tag{3.3}
\end{align*}
$$

Further, since $\mu\left(r, F^{\alpha, \beta}\right) \rightarrow+\infty$ as $r \rightarrow 1^{-}$so $v\left(r, F^{\alpha, \beta}\right) \rightarrow+\infty$ as $r \rightarrow 1^{-}$, hence by Lemma 1, we have

$$
\begin{equation*}
\frac{\log \mu\left(r, F^{\alpha, \beta}\right)}{v\left(r, F^{\alpha, \beta}\right)} \rightarrow 0 \tag{3.4}
\end{equation*}
$$

The relations (3.1), (3.2), (3.3) and (3.4) together give for $r \rightarrow 1$,

$$
\begin{equation*}
\mu\left(r, F^{\alpha, \beta}\right) \leqslant v\left(r, F^{\alpha, \beta}\right) \xi^{-1}\left\{A^{*} \eta(\log (1 / r))\right\} . \tag{3.5}
\end{equation*}
$$

In view of the relations (3.1) and (3.2) we observe that the terms in the series are bounded above by 1 for $n_{k}>\xi^{-1}\left\{A^{*} \eta(\log (1 / r))\right\}$ and $k>k(\varepsilon)$. Since $\mu\left(r, F^{\alpha, \beta}\right) \rightarrow$ $\rightarrow+\infty$ as $r \rightarrow 1^{-}$we must conclude that for $r$ near 1 the rank $v\left(r, F^{\alpha, \beta}\right)$ precedes $\xi^{-1}\{A * \eta(\log (1 / r))\}$. Applying the function $\xi$ to (3.5) we get $U \leqslant V \leqslant A$. Clearly these inequalities hold if $A=+\infty$.

Let $U=\limsup _{r \rightarrow 1^{-}} \frac{\xi\left(\log \mu\left(r, F^{\alpha, \beta}\right)\right)}{\eta(\log (1 / r))}$ be finite.
Then for $\stackrel{r \rightarrow 1^{-}}{\varepsilon>0, ~} \varepsilon^{\prime}>0$ and $r>r(\varepsilon)$, we have

$$
\begin{equation*}
\mu\left(r, F^{\alpha, \beta}\right) \leqslant \exp \left\{\xi^{-1}\left[U^{*} \eta(\log (1 / r)]\right\},\right. \tag{3.6}
\end{equation*}
$$

where $U^{*}=(U+\varepsilon)$. Therefore,

$$
\begin{equation*}
\left|a_{k} r^{2 n_{k}}\right| \leqslant \exp \left\{\xi^{-1}\left[U^{*} \eta(\log (1 / r)]\right\} \text { for all } k \text { and } r>r(\varepsilon) .\right. \tag{3.7}
\end{equation*}
$$

So

$$
\begin{equation*}
\left|a_{k}\right| \leqslant(1 / r)^{2 n_{k}} \exp \left\{\xi^{-1}\left[U^{*} \eta(\log (1 / r)]\right\}\right. \tag{3.8}
\end{equation*}
$$

or

$$
\begin{equation*}
\log ^{+}\left|a_{k}\right|^{1 / 2 n_{k}} \leqslant \log (1 / r)+\frac{\xi^{-1}\left[U^{*} \eta(\log (1 / r))\right]}{2 n_{k}} . \tag{3.9}
\end{equation*}
$$

It is easy to see that we may use $\log ^{+}$in the left hand side of (3.9) as the right hand side is non-negative.

Setting $r=r_{k}$, for sufficiently large value of $k$,

$$
\log \left(1 / r_{k}\right)=\eta^{-1}\left\{\frac{\xi\left(2 n_{k}\right)}{U^{*}+\varepsilon^{\prime}}\right\}
$$

so that as $k \rightarrow+\infty, n_{k} \rightarrow+\infty$, and $r_{k} \rightarrow 1^{-}$. Also for these large values of $k, 2 n_{k}=$ $=\xi^{-1}\left[\left(U^{*}+\varepsilon\right) \eta\left(\log \left(1 / r_{k}\right)\right]\right.$ and $r_{k}>r(\varepsilon)$.

Therefore in view of (3.9), we get

$$
\begin{equation*}
\log ^{+}\left|a_{k}\right|^{1 / 2 n_{k}} \leqslant \log \left(1 / r_{k}\right)+\frac{\xi^{-1}\left[U^{*} \eta\left(\log \left(1 / r_{k}\right)\right)\right]}{\xi^{-1}\left[\left(U^{*}+\varepsilon\right) \eta\left(\log \left(1 / r_{k}\right)\right)\right]} . \tag{3.10}
\end{equation*}
$$

Using the property (v), (3.10) can be rewritten as

$$
\begin{equation*}
\log ^{+}\left|a_{k}\right|^{1 / 2 n_{k}} \leqslant \log \left(1 / r_{k}\right)+o\left[\log \left(1 / r_{k}\right)\right] . \tag{3.11}
\end{equation*}
$$

In view of property (iii) $\eta[x+o(x)] \sim \eta(x)$, we obtain
(3.12) $\eta \log ^{+}\left|a_{k}\right|^{1 / 2 n_{k}} \leqslant[1+0(1)] \eta\left(\log \left(1 / r_{K}\right)\right]=[1+0(1)] \frac{\xi\left(2 n_{K}\right)}{U^{*}+\varepsilon^{1}}$
or

$$
\begin{equation*}
\left(U^{*}+\varepsilon^{1}\right) \geqslant[1+0(1)] \frac{\xi\left(2 n_{k}\right)}{\eta\left(\log ^{+}\left|a_{k}\right|^{1 / 2 n_{k}}\right)} . \tag{3.13}
\end{equation*}
$$

Hence $A \leqslant U$. It also holds if $U=+\infty$. Thus we have shown that if $\mu\left(r, F^{\alpha, \beta}\right) \rightarrow$ $\rightarrow+\infty$, the $U=V=A$.

We have been proved earlier that if $\mu\left(r, F^{\alpha, \beta}\right) \rightarrow+\infty$, as $r \rightarrow 1^{-}$then

$$
\begin{equation*}
\frac{\log \mu\left(r, F^{\alpha, \beta}\right)}{v\left(r, F^{\alpha, \beta}\right)} \rightarrow 0 \tag{3.14}
\end{equation*}
$$

Therefore it is easy to see that $u \leqslant v$. Further from Cauchy inequality, we have

$$
\mu\left(r, F^{\alpha, \beta}\right) \leqslant M\left(r, F^{\alpha, \beta}\right)
$$

Which gives

$$
u \leqslant m .
$$

Now it is easily seen that we may actually show that $U=V=A$ provided $\mu\left(r, F^{\alpha, \beta}\right)$ is eventually $>1$ and $\log \mu\left(r, F^{\alpha, \beta}\right)$ is bounded above we may multiply to GBASP $F^{\alpha, \beta}$ by a positive constant $C$ to ensure that these conditions are satisfied. Then we consider the fuction $g^{\alpha, \beta}(z, o)=C F^{\alpha, \beta}(z, o)$ and apply our reasoning to $g^{\alpha, \beta}(z, o)$ to obtain

$$
\begin{equation*}
O=U_{F^{\alpha, \beta}}=U_{g^{\alpha, \beta}}=V_{g^{\alpha, \beta}}=A_{g^{\alpha, \beta}} \tag{3.15}
\end{equation*}
$$

where the subscripts refer to the functions of $F^{\alpha, \beta}$ and $g^{\alpha, \beta}$.
For the second part of the proof, we have to show that in all cases $M=U$, and $u=m$ when $U$ is finite. It is clear from Cauchy inequality that $U \leqslant M$ and $u \leqslant m$.

Further, if $U=\rightarrow \infty$ then $U=M$. So taking $U<\rightarrow \infty$ and for $t=(1+r) / 2$ it is easy to show that

$$
\begin{equation*}
\log \left(1 /(1-r) \sim \log (1 / \log (1 / r)) \text { as } r \rightarrow 1^{-}\right. \tag{3.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\log (1 / t)}{\log (1 / r)} \rightarrow \frac{1}{2} \text { as } r \rightarrow 1^{-} \tag{3.17}
\end{equation*}
$$

Using Lemma 2, we get

$$
\begin{equation*}
M\left(r, F^{\alpha, \beta}\right) \leqslant \mu\left(r, F^{\alpha, \beta}\right)\left\{v\left(t, F^{\alpha, \beta}\right)+\frac{2}{1-r}\right\} . \tag{3.18}
\end{equation*}
$$

Then for $r \rightarrow 1$ with an arbitrary constant $K$, we find that

$$
\begin{equation*}
\log M\left(r, F^{\alpha, \beta}\right) \leqslant \log ^{+} \mu\left(r, F^{\alpha, \beta}\right)+\log ^{+} v\left(t, F^{\alpha, \beta}\right)+\log \left(\frac{1}{1-r}\right)+K \tag{3.19}
\end{equation*}
$$

We see that the inequality (3.19) holds in all cases:
(a) $\mu\left(r, F^{\alpha, \beta}\right)$ and $v\left(r, F^{\alpha, \beta}\right)$ both are bounded,
(b) $\mu\left(r, F^{\alpha, \beta}\right)$ bounded and $v\left(r, F^{\alpha, \beta}\right)$ unbounded,
(c) $\mu\left(r, F^{\alpha, \beta}\right)$ and $v\left(r, F^{\alpha, \beta}\right)$ both are unbounded.

Now first, we shall take the Case (c). Choose $C>1$, and applying Lemma 2 to (3.19) as $r \rightarrow 1$, we get

$$
\begin{align*}
& \xi\left(\log M\left(r, F^{\alpha, \beta}\right)<C\left\{\xi\left(\log \mu\left(r, F^{\alpha, \beta}\right)\right)+\right.\right.  \tag{3.20}\\
&+\xi\left(\log v\left(t, F^{\alpha, \beta}\right)+\xi(\log 1 /(1-r))+\xi(K)\right.
\end{align*}
$$

or
(3.21) $\lim _{r \rightarrow 1^{-}} \frac{\xi[\log (1 / 1-r)]}{\eta(\log (1 / r))}=$

$$
=\lim _{r \rightarrow 1^{-}}\left\{\frac{\xi[\log (1 / 1-r)]}{\xi[\log \{1 / \log (1 / r)\}]} \frac{\xi[\log \{1 / \log (1 / r)\}]}{\xi[1 / \log (1 / r)]} \frac{\xi(1 / \log (1 / r))}{v(\log (1 / r))}\right\}=0
$$

since $\xi\left(\log v\left(t, F^{\alpha, \beta}\right)\right)=o\left\{\xi\left(v\left(t, F^{\alpha, \beta}\right)\right)\right\}$,

$$
\limsup _{t \rightarrow 1^{-}} \frac{\xi\left(v\left(t, F^{\alpha, \beta}\right)\right)}{\eta(\log (1 / t))}=V=U<\infty
$$

and

$$
\eta(\log (1 / t))=O\left\{\eta(\log (1 / r)) \text { as } r \rightarrow 1^{-} .\right.
$$

Dividing (3.20) by $\eta(\log (1 / r))$ and passing to limits, we get $M \leqslant C U$ and $m \leqslant C u$.

The other cases (a) and (b) can be proved in a similar manner. Hence the proof of theorem 1 is completed.

Theorem 2: For the real valued function GBASP, $F^{\alpha, \beta}, a \leqslant u$, whenever

$$
\lim _{k \rightarrow \infty}\left(\frac{n_{k}}{n_{K+1}}\right)=C>0 .
$$

PRoof: Let $u=\lim _{r \rightarrow 1^{-}} \frac{\xi\left(\log \mu\left(r, F^{\alpha, \beta}\right)\right)}{\eta(\log (1 / r))}<+\infty$.
Therefore, for any $\varepsilon>0$ and for a sequence $r_{j} \rightarrow 1^{-}$we have for $j>j(\varepsilon)$
(3.22) $\left|a_{k} r_{j}^{2 n_{k}}\right| \leqslant \mu\left(r_{j}, F^{\alpha, \beta}\right) \leqslant \exp \left[\xi^{-1}\left\{u^{*} \eta(\log (1 / r))\right\}\right], u^{*}=(u+\varepsilon)$, for all $k$.
or
(3.23) $\left|a_{k}\right| \leqslant\left(1 / r_{j}\right)^{2 n_{k}} \exp \left[\xi^{-1}\left\{u^{*} \eta(\log (1 / r))\right\}\right]$, for all $k$ and $j>j(\varepsilon)$.

Now for all large $j$ we choose $n_{k}$ and $n_{k+1}$ such that

$$
\begin{equation*}
\left.2 n_{k}<\xi^{-1}\left\{u^{*}+\epsilon\right) \eta(\log (1 / r))\right\}<2 n_{k+1} . \tag{3.24}
\end{equation*}
$$

Here $k$ and $k+1$ depend on $j$. Hence

$$
\begin{equation*}
\xi\left(2 n_{k}\right) \leqslant\left(u^{*}+\epsilon\right) \eta(\log (1 / r) \tag{3.25}
\end{equation*}
$$

and

$$
\begin{equation*}
\eta^{-1}\left\{\frac{\xi\left(2 n_{k}\right)}{\left(u^{*}+\varepsilon\right)}\right\} \geqslant \log \left(1 / r_{j}\right)>\eta^{-1}\left\{\frac{\xi\left(2 n_{k+1}\right)}{u^{*}+\varepsilon}\right\} \tag{3.26}
\end{equation*}
$$

Since $\left(\frac{n_{k}}{n_{k+1}}\right)=C>0$, so we may choose $\varepsilon>0$ so that

$$
\frac{n_{k}}{n_{k+1}}>C-\varepsilon>0, \quad \text { for } k \text { is sufficiently large }
$$

Since $n_{k}$ and $n_{k+1}$ depend upon $j$, so for sufficiently large $j$, we have

$$
\begin{equation*}
(C-\varepsilon) n_{k+1}<n_{k} \tag{3.28}
\end{equation*}
$$

Further, if $j$ is large enough all the relations (3.22), (3.23) and (3.28) will hold. Therefore (3.23) becomes,

$$
\begin{equation*}
\log ^{+}\left|\mathrm{a}_{\mathrm{k}}\right|^{1 / 2 \mathrm{n}_{\mathrm{k}}} \leqslant \log \left(1 / r_{j}\right)+\frac{\xi^{-1}\left[u^{*} \eta\left(\log \left(1 / r_{j}\right)\right)\right]}{2 n_{k}} \tag{3.29}
\end{equation*}
$$

In view of relations (3.24), (3.25), (3.26) and (3.28) we have

$$
\begin{equation*}
\log ^{+}\left|a_{k}\right|^{1 / 2 n_{k}}<b^{-1}\left\{\frac{\xi\left(2 n_{k}\right)}{u^{*}+\varepsilon}\right\}+\frac{\xi^{-1}\left[u^{*} \eta\left(\log \left(1 / r_{j}\right)\right)\right]}{\left.(C-\varepsilon) \xi^{-1}\left[u^{*}+\varepsilon\right) \eta\left(\log \left(1 / r_{j}\right)\right)\right]} \tag{3.30}
\end{equation*}
$$

Let $x=\log \left(1 / r_{j}\right)$ and $y=\eta^{-1}\left\{\frac{\xi\left(2 n_{k}\right)}{u^{*}+\varepsilon}\right\}$, then from (3.26) we have

$$
x \leqslant y .
$$

Also from (v), the second term on the right of (3.20) is $o(x)$ as $x \rightarrow 0^{+}$and hence also $o(y)$.

Further, since $\eta[y+o(y)] \sim \eta(y)$ as $y \rightarrow 0^{+}$, applying the function h to both sides of (3.20) we get

$$
\begin{equation*}
\eta\left(\log ^{+}\left|a_{k}\right|^{1 / 2 n_{k}}\right) \geqslant[1+o(1)] \frac{\xi\left(2 n_{k}\right)}{u^{*}+\varepsilon} \tag{3.31}
\end{equation*}
$$

or

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \frac{\xi\left(2 n_{k}\right)}{\eta\left(\log ^{+}\left|a_{k}\right|^{1 / 2 n_{k}}\right)} \leqslant u^{*}+\varepsilon \tag{3.32}
\end{equation*}
$$

Hence the proof is completed.

## Applications

1. Let $\xi(x)=\log ^{n}(x)=\log ^{n-1}(\log x)$, where $\log ^{(o)}(x)=x$, and $\eta(x)=\log 1 / x$ then Theorems 1 and 2 are applicable when $n \geqslant 2$.

Proof: Properties (i) through (iv) will hold. Let $\varepsilon>0$ and $C>0$, then with $k \rightarrow 0$, we have

$$
\begin{aligned}
\frac{\xi^{-1}(C \eta(x))}{\xi^{-1}((C+\varepsilon) \eta(x))}=\frac{\exp ^{n-1}\left(x^{-C}\right)}{\exp ^{n-1}\left(x^{-\varepsilon} x^{-C}\right)} & \leqslant \frac{\exp \left(\exp ^{n-2}\left(x^{-C}\right)\right)}{\exp \left(x^{-C} \exp ^{n-2}\left(x^{-C}\right)\right)}= \\
& =\left(\exp ^{n-1}\left(x^{-C}\right)\right)^{1-x^{-\varepsilon}} \leqslant\left(x^{C}\right)^{x^{-C}-1}=o(x)
\end{aligned}
$$

To obtain the first inequality we used Lemma 3 and 4, we note that for $n=1$, property (v) fails to exists.
2. If $\xi(x)=\log ^{n-1}(x)$ and $\eta(x)=x^{-K}$ for $0<K<\infty$ and $n \geqslant 2$, then Theorems 1 and 2 are applicable.

Proof: Properties (i) through (iv) will hold. Let $\varepsilon>0$ and $C=0$ with $x \rightarrow 0$, then for sufficiently large $N$

$$
\begin{aligned}
& \frac{\xi^{-1}(C \eta(x))}{\xi^{-1}((C+\varepsilon) \eta(x))}=\frac{\exp ^{n-1}\left(C\left(x^{-k}\right)\right)}{\exp ^{n-2}\left(\exp \left(\varepsilon\left(x^{-k}\right)\right) \exp \left(C\left(x^{k}\right)\right)\right)} \leqslant \\
& \quad \leqslant \frac{\exp ^{n-1}\left(C\left(x^{-k}\right)\right)}{\exp \left(\varepsilon\left(x^{-k}\right) \exp ^{n-1}\left(C\left(x^{-k}\right)\right)\right.}=\exp \left(-\varepsilon\left(x^{-k}\right)\right)<N!/\left(\varepsilon\left(x^{-k}\right)\right)^{N}=o(x) .
\end{aligned}
$$

The first inequality above was obtained by repeated use of Lemma 3 and 4.
The classical case occurs when $\xi(x)=\log x$ and $\eta(x)=\log 1 / x$. Here property (v) fails to exists for each $\varepsilon>0$. However, the argument encompassing equation (3.1) through (3.5) still supplies and we obtain $U \leqslant V \leqslant A$. Now we shall prove that $A=V=U+1$, which is the content of folloing result.
3. If $\xi(x)=\log x$ and $\eta(x)=\log 1 / x$ then $A=V=U+1$, when $U>0$.

Proof: In veiw of Lemma 5 and Theorem $1, U \leqslant V=U+1 \leqslant A$. Now, although the property (v) fails to hold we do have for $\xi(x)=\log x$ and $\eta(x)=\log 1 / x$.

$$
\frac{\xi^{-1}[C \eta(x)]}{\xi^{-1}[(C+1) \eta(x)]}=\frac{\exp (-C \log x)}{\exp [(-C-1) \log x]}=\frac{x^{-C}}{x^{-C-1}}=x
$$

Since $\eta(C x) \sim \eta(x)$, we may repeat the argument involving equations (3.6) through (3.13) $\varepsilon^{\prime}=1$ to obtain $A \leqslant U^{*}+1$ or $A \leqslant U+1$. Finally then the classical case is obtained case is obtained. In this case it may be shown as before that $U=M$. Further if $U=O=M$ then $V \leqslant A \leqslant 1$, and $V \geqslant 1$ when $\mu\left(r, F^{\alpha, \beta}\right) \rightarrow+\infty$. Hence the proof is completed.

Remark 1: Application 2, yields the generalized types corresponding to the generalized orders of Application 1.

Polynomial approximation of GBASP.
Let $C(D)$ deonote the algebra of analytic functions on the unit disc $D$. Let the Chebyshev norm be defined for $f \in C(D)$ and $F^{\alpha, \beta} \in C\left(\Sigma^{\alpha, B}\right)$ as follows:
$E_{n}\left(F^{\alpha, \beta} ; \Sigma^{\alpha, \beta}\right) \equiv E_{n}\left(F^{\alpha, \beta}\right)=\inf \left\{\left\|F^{\alpha, \beta}-G^{\alpha, \beta}\right\|, G^{\alpha, \beta} \in H_{n}^{\alpha, \beta}\right\}, \quad n=0,1,2, \ldots$
$\left\|F^{\alpha, \beta}-G^{\alpha, \beta}\right\|=\sup _{x^{2}+y^{2}=1}\left|F^{\alpha, \beta}(x, y)-G^{\alpha, \beta}(x, y)\right|$.
The set $H_{n}^{\alpha, \beta}$ contains all real biaxisymmetric harmonic polynomials of degree at most $2 n$.

Now we define

$$
\begin{aligned}
& A^{* *}=\limsup _{k \rightarrow \infty} \frac{\xi\left(2 n_{k}\right)}{\eta\left[\log ^{+}\left(E_{k}\left(F^{\alpha, \beta}\right)\right)^{1 / 2 n_{k}}\right]} \\
& a^{* *}=\liminf _{k \rightarrow \infty} \frac{\xi\left(2 n_{k}\right)}{\eta\left[\log ^{+}\left(E_{k}\left(F^{\alpha, \beta}\right)\right)^{1 / 2 n_{k}}\right]}
\end{aligned}
$$

Remark 2: Theorems 1 and 2 are also hold for $A^{* *}$ and $a^{* *}$ in place of $A$ and $a$. The verification of above remark is quite easy so we omit the details.

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