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Green's Functions and Energy Decay on Homogeneous Spaces (**)

ABSTRACT. — We consider a homogeneous space $X = (X, d, m)$ of dimension $\nu \geq 1$ and a local regular Dirichlet form in $L^2(X, m)$. We prove that if a Poincaré inequality holds on every pseudo-ball $B(x, R)$ of X , with local characteristic constant $c_0(x)$ and $c_1(r)$, then a Green's function estimate from above and below is obtained. A Saint-Venant-like principle is recovered in terms of the Energy's decay.

Funzioni di Green e decadimento dell'energia in spazi omogenei

SUNTO. — Si considera uno spazio omogeneo $X = (X, d, m)$ di dimensione $\nu \geq 1$ e una forma di Dirichlet locale regolare in $L^2(X, m)$. Si dimostra che se una diseguaglianza di Poincaré vale su ogni pseudo-sfera $B(x, R)$ di X , con costanti caratteristiche locali $c_0(x)$ e $c_1(r)$, allora si ricava una stima della funzione di Green da sopra e da sotto. Il principio tipo Saint-Venant viene ottenuto in termini di decadimento dell'Energia.

1. - INTRODUCTION AND RESULTS

We consider a *connected, locally compact topological space* X . We suppose that a *distance* d is defined on X and we suppose that the balls

$$B(x, r) = \{y \in X : d(x, y) < r\}, \quad r > 0,$$

form a basis of open neighborhoods of $x \in X$. Moreover, we suppose that a (positive) Radon measure m is given on X , with $\text{supp } m = X$. The triple (X, d, m) is assumed to

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satisfy the following property: there exist some constants $0 < R_0 \leq +\infty$, $\nu > 0$ and a positive function $c_0(x)$ together with $c_0^{-1}(x)$ which belong to $L_{\text{loc}}^\infty(X_0)$, where X_0 is a relatively compact open subset of X , such that for every $x \in X$ and every $0 < r \leq R < R_0$

$$(1.1) \quad 0 < c_0(x) \left(\frac{r}{R} \right)^\nu m(B(x, R)) \leq m(B(x, r)).$$

Such a triple (X, d, m) will be called a homogeneous space of dimension ν . We point out, however, that a given exponent ν occurring in (1.1) should be considered, more precisely, as an upper bound of the «homogeneous dimension», hence we should better call (X, d, m) a homogeneous space of dimension less or equal than ν . We further suppose that we are also given a *strongly local, regular, Dirichlet form* a in the Hilbert space $L^2(X, m)$ - in the sense of M. Fukushima [4], - whose domain in $L^2(X, m)$ we shall denote by $\mathcal{O}[a]$. Furthermore, we shall restrict our study to Dirichlet forms of diffusion type, that is to forms a that have the following *strong local property*: $a(u, v) = 0$ for every $u, v \in \mathcal{O}[a]$ with v constant on a neighborhood of $\text{supp } u$. We recall that the following integral representation of the form a holds

$$a(u, v) = \int_X a(u, v)(dx), \quad u, v \in \mathcal{O}[a],$$

where $a(u, v)$ is a uniquely defined signed Radon measure on X , such that $a(d, d) \leq m$, with $d \in \mathcal{O}_{\text{loc}}[a]$: this last condition is fundamental for the existence of cut off functions associated to the distance. Moreover, the restriction of the measure $a(u, v)$ to any open subset Ω of X , with $\overline{\Omega} \subset X_0$, depends only on the restrictions of the functions u, v to Ω . Therefore, the definition of the measure $a(u, v)$ can be unambiguously extended to all m -measurable functions u, v on X that coincide m -a.e. on every compact subset of Ω with some functions of $\mathcal{O}[a]$. The space of all such functions will be denoted by $\mathcal{O}_{\text{loc}}[a, \Omega]$. Moreover we denote by $\mathcal{O}[a, \Omega]$ the closure of $\mathcal{O}[a] \cap C_0[\Omega]$ in $\mathcal{O}[a]$. The homogeneous metric d and the energy form a associated to the *energy measure* α , both given on X_0 , are then assumed to be mutually related by the following basic assumption:

There exists a constant $k \geq 1$ such that $\forall x \in X_0, \forall R$ with $0 < R < R_0$ the following Poincaré inequality holds [1]:

$$(1.2) \quad \int_{B(x, R)} |u - \bar{u}_{B(x, R)}|^2 dm \leq c_1 R^2 \int_{B(x, kR)} a(u, u)(dx)$$

for all $u \in D[a, B(x, kR)]$, where

$$\bar{u}_{B(x, R)} = \frac{1}{m(B(x, R))} \int_{B(x, R)} u dm.$$

By assumption (1.2), it can be shown the validity of the following Sobolev type inequality of exponent s for every $x \in X_0$ and every $0 < R < R_0$:

$$(1.3) \quad \left(\frac{1}{m(B(x, R))} \int_{B(x, R)} |u|^s dm \right)^{1/s} \leq c_1 R \left(\int_{B(x, R)} \alpha(u, u)(dx) \right)^{1/2},$$

where $u \in D[\alpha, B(x, kR)]$ and $\text{supp } u \subset B(x, R)$. Let us consider the following simple generalization of the Poincaré inequality

$$(1.4) \quad \int_{B(x, R)} |u - \bar{u}_{B(x, R)}|^2 dm \leq c_1^2(R) R^2 \int_{B(x, kR)} \alpha(u, u)(dx)$$

where $c_1(r)$ is a decreasing function of r , then the following Sobolev inequality of exponent s

$$(1.5) \quad \left(\frac{1}{m(B(x, R))} \int_{B(x, R)} |u|^s dm \right)^{1/s} \leq \tau^3(x) c_1(R) R \left(\int_{B(x, R)} \alpha(u, u)(dx) \right)^{1/2}$$

can be proved, where we have defined $\tau(x) = \left(\sup_{B(x, 2R)} c_0^{-1}(x) \right)^{1/2}$. In this paper we will fix our attention on Green function estimates from above and below by using an Harnack's inequality obtained in Ref. [5]. Moreover, we will give the behaviour of the *Energy decay* related to the Green function under consideration. We begin here by recalling the results given in [5].

THEOREM 1 [Harnack]: *Let (1.1), (1.4), (1.5) hold, and let u be a non-negative solution of $a(u, v) = 0$. Let \mathcal{O} be an open subset of X_0 and $u \in \mathcal{O}_{\text{loc}}[\mathcal{O}]$, $\forall v \in \mathcal{O}_0[\mathcal{O}]$ with $B(x, r) \subset \mathcal{O}$, then*

$$\text{ess sup}_{B_{1/2}} \tilde{u} \leq \exp \gamma \mu \text{ess inf}_{B_{1/2}} \tilde{u},$$

where \tilde{u} is the function in $L^2(m, B)$ associated with u , $\gamma \equiv \gamma(\nu, k)$, with k a positive constant and $\mu(x, r) = \tau^4(x) c_1 \left(\frac{r}{2} \right)$. A standard consequence of the previous Theorem is the following

COROLLARY 1: *Suppose that*

$$(1.6) \quad \int_r^R e^{-\gamma \mu(x, \varrho)} \frac{d\varrho}{\varrho} \rightarrow \infty \quad \text{for} \quad r \rightarrow 0$$

then the solution is continuous in the point under consideration. In particular, if

$\mu(x, \varrho) \approx o\left(\log \log \frac{1}{\varrho}\right)$, then there exists $c > 0$ such that

$$(1.7) \quad \underset{B(x, r)}{\text{osc}} u \leq c \frac{\left(\log \frac{1}{R}\right)}{\left(\log \frac{1}{r}\right)} \underset{B(x, R)}{\text{osc}} u.$$

Before presenting the main results we assume that $\partial B(x, r)$ be connected and we prove the following

LEMMA 1: Let $\bar{X} \equiv B\left(x, \frac{9}{4}r\right) - B\left(x, \frac{3}{4}r\right)$ be a connected set and let l be equal to $\sup_{x \in B(x_0, 2R)} c_0^{-1}(x) 16^{-v}$. Then there exists a finite number l of overlapping balls of radius $r/8$ joining two arbitrary points x_1 and x_l of \bar{X} which is at a distance greater than $r/2$ from the origin.

PROOF: \bar{X} can be covered with a finite set \tilde{X} of balls of radius $r/16$. We can assume that the ball $B_1 \equiv B(x_1, r/16)$ is in the considered covering and we now assume that every ball in the covering \tilde{X} intersects \bar{X} , i.e. $B_i \cap \bar{X} \neq \emptyset$ for every $i = 1 \dots l$. Since \bar{X} is a connected set, there exists a second ball $B_2 \equiv B(x_2, r/16) \subset \tilde{X}$ with $B_2 \cap \bar{X} \neq \emptyset$ s.t. the closure of B_1 and the closure of B_2 do intersect, namely $\overline{B_1} \cap \overline{B_2} \neq \emptyset$. Consider the set $X_{12} = \overline{B_1 \cup B_2}$. By hypothesis on \bar{X} and \tilde{X} , there exists a ball $B_3 \equiv B(x_3, r/16) \subset \tilde{X}$, $B_3 \cap \bar{X} \neq \emptyset$ s.t. $X_{12} \cap \overline{B_3} \neq \emptyset$. Now we can consider the new set X_{123} defined by $X_{123} = \overline{B_1 \cup B_2 \cup B_3}$. By repeating the previous steps, we can say that there exists a ball $B_4 \equiv B(x_4, r/16) \subset \tilde{X}$, $B_4 \cap \bar{X} \neq \emptyset$ s.t. $X_{123} \cap \overline{B_4} \neq \emptyset$. By iterating this procedure we can construct two sets $X_{1\dots n}$ and $Y_{n+1\dots l}$ s.t. $X_{1\dots n} = \overline{\bigcup_{i=1\dots n} B(x_i, r/16)}$, $Y_{n+1\dots l} = \overline{\bigcup_{i=n+1\dots l} B(x_i, r/16)}$ with

$$X_{1\dots n} \cap Y_{n+1\dots l} \neq \emptyset.$$

Indeed if $X_{1\dots n}$ and $Y_{n+1\dots l}$ were s.t. $X_{1\dots n} \cap Y_{n+1\dots l} = \emptyset$ this would mean that the space \bar{X} is not connected against the hypothesis. The chain of balls is obtained by joining the centers of the balls $B(x_i, r/16)$ forming the set

$$X_{1\dots l} = \overline{\bigcup_{i=1\dots l} B(x_i, r/16)}$$

previously built which starts from x_1 and stops to x_l . To obtain the overlapping of the balls is sufficient to consider $B(x_i, r/8)$ and the new sets

$$\widehat{X}_{1\dots n} = \overline{\bigcup_{i=1\dots n} B(x_i, r/8)}; \quad \widehat{Y}_{n+1\dots l} = \overline{\bigcup_{i=n+1\dots l} B(x_i, r/8)};$$

with

$$\bigcap_{1 \dots n} \widehat{X} \cap \bigcap_{n+1 \dots l} \widehat{Y} \neq \emptyset$$

for every n and the Lemma is proved. Now we can state our main results

THEOREM 2 [Size of the Green function]: $\forall B(x_0, R) \subset B(x_0, 20R) \subset X_0$ and $\forall r \in \left(0, \frac{R}{16}\right]$, the following estimate holds for all $x \in \partial B(x_0, r)$

$$\int_r^R \frac{c_0(x) \exp(-l\gamma\mu(x, s))}{m(B(x, s))} s^2 \frac{ds}{s} \leq G_{B(x, R)}^x(y) \leq \int_r^R \frac{\tau^6(x) c_1^2(r) \exp(l\gamma\mu(x, s))}{m(B(x, s))} s^2 \frac{ds}{s},$$

where $l = \sup_{x \in B(x_0, 2R)} c_0^{-1}(x) 16^{-v}$ is a finite number of balls of radius $r/8$ covering $\partial B(x_0, r)$.

THEOREM 3 (Saint-Venant-like principle): Let u be a local solution in X_0 and $B(x_0, 4R_0) \subset X_0$ with $R_0 = k^2 R$. Let

$$(1.8) \quad \psi(r) = \int_{B(x_0, r)} G_{B(x_0, 2r/q)}^{x_0} a(u, u)(dx).$$

Then

$$(1.9) \quad \begin{aligned} \psi(r) &\leq c c_1^4(qr) \psi(R_0) \exp\left(-\int_r^R \exp\left(-2\gamma\mu(x_0, \varrho) \frac{d\varrho}{\varrho}\right)\right) + \\ &+ \exp\left(-\int_r^R \exp(-5l\gamma\mu(x_0, \varrho)) \frac{d\varrho}{\varrho}\right) \psi(R_0). \end{aligned}$$

In particular, if $\gamma\mu(x, r) \leq o\left(\log \log \frac{1}{r}\right)$ then

$$(1.10) \quad \psi(r) \leq \left(\frac{\log \frac{1}{R}}{\log \frac{1}{r}} \right) \psi(R_0),$$

with $k \geq 1$.

From the point of view of partial differential equations these results can be applied to two important classes of operators on \mathbb{R}^n :

a) *Doubly Weighted uniformly elliptic operators* in divergence form with measurable coefficients, whose coefficient matrix $A = (a_{ij})$ satisfies

$$w(x) |\xi|^2 \leq \langle A\xi, \xi \rangle \leq v(x) |\xi|^2.$$

Here $\langle \cdot, \cdot \rangle$ denotes the usual dot product; w and v are weight functions, respectively belonging to A_2 and D_∞ such that the following *Poincaré* inequality

$$\left(\frac{1}{|v(B)|} \int_B |f(x) - f_B|^2 v dx \right)^{1/2} \leq cr \left(\frac{1}{|w(B)|} \int_B |\nabla f|^2 w dx \right)^{1/2}$$

holds.

b) *Doubly Weighted Hörmander type operators* [3], whose form is $L = X_k^*(\alpha^{bk}(x) X_b)$ where $X_b, b = 1, \dots, m$ are smooth vector fields in \mathbb{R}^n that satisfy the Hörmander condition and $\alpha = (\alpha^{bk})$ is any symmetric $m \times m$ matrix of measurable functions on \mathbb{R}^n , such that

$$w(x) \sum_i \langle X_i, \xi \rangle^2 \leq \sum_{i,j} \alpha_{ij}(x) \xi_i \xi_j \leq v(x) \sum_i \langle X_i, \xi \rangle^2,$$

where $X_i \xi(x) = \langle X_i, \nabla \xi \rangle$, $i = 1, \dots, m$, $\nabla \xi$ is the usual gradient of ξ and $\langle \cdot, \cdot \rangle$ denotes the usual inner product on \mathbb{R}^n . Then the following *Poincaré* inequality for vector fields

$$\left(\frac{1}{|v(B)|} \int_B |f(x) - f_B|^2 v dx \right)^{1/2} \leq cr \left(\frac{1}{|w(B)|} \int_B \left(\sum_j |\langle X_j, \nabla f(x) \rangle|^2 \right)^{1/2} w dx \right)^{1/2},$$

holds, with $w \in A_2$ and $v \in D_\infty$.

2. - ESTIMATES OF THE GREEN'S FUNCTIONS AND CAPACITIES OF BALLS

We define the *Green* function G_Ω^x for the problem

$$(2.1) \quad \begin{cases} a(u, v) = \int_\Omega fv m(dx), \\ u \in \mathcal{D}_0[\Omega], \quad \forall v \in \mathcal{D}_0[\Omega], \end{cases}$$

Ω is a given ball $B(x_0, R_0) \subset X_0$ and $x \in \Omega$. The regularized Green function $G_{\varrho, \Omega}^x$ associated with (2.1) is

$$(2.2) \quad \begin{cases} a(G_{\varrho, \Omega}^x, v) = - \oint_{B(z, \varrho)} vm(dx), \\ G_{\varrho, \Omega}^x \in \mathcal{D}_0[\Omega], \quad \forall v \in \mathcal{D}_0[\Omega], \end{cases}$$

where we have defined $\fint_{B(x, \varrho)} = \left(\frac{1}{m(B(x, \varrho))} \right) \int_{B(x, \varrho)}$, with $\varrho > 0$ and $B(x, \varrho) \subset \Omega$. We define the *capacity* of the ball $B(x, r)$ with respect to the ball $B(x, dr)$, $d > 1$, relative to the form a , by setting

$$\text{cap}(B(x, r), B(x, dr)) = \min \{a(v, v) : v \in \mathcal{O}_0[B(x, dr)], v \geq 1 \text{ m-a.e. on } B(x, r)\}.$$

By Sobolev-Poincaré's inequality (jj), the minimum is achieved and the unique minimizer $u \equiv u_{B(x, r)}$ is called the *equilibrium potential* of $B(x, r)$ with respect to $B(x, dr)$, relative to the form a .

THEOREM 4: Let $G_{B(x, dr)}^x$ be the Green function of problem (2.1), $\Omega = B(x, dr)$, with singularity at x , $d \geq 2$, $B(x, 4r) \subset X_0$. Suppose $\partial B(x, r)$ be connected, then the following estimates hold: $\forall y \in \partial B(x, r)$

$$(2.3) \quad \frac{e^{-l\gamma\mu(x, r)}}{\text{cap}(B(x, r), B(x, dr))} \leq G_{B(x, dr)}^x(y) \leq \frac{e^{l\gamma\mu(x, r)}}{\text{cap}(B(x, r), B(x, dr))}$$

and

$$(2.4) \quad c \frac{(d-1)^2 c_0(x)}{m(B(x, r))} r^2 \leq (\text{cap}(B(x, r), B(x, dr)))^{-1} \leq \frac{d^2 r^2 c_1^2(r) \tau^6(x)}{m(B(x, r))},$$

where $c_1(r)$ is the decreasing function of assumption (1.4) $l = \sup_{x \in B(x_0, 2R)} c_0^{-1}(x) 16^{-v}$.

PROOF: Let us consider the cut-off function φ of $B(x, r)$ in $B\left(x, \left(1 + \frac{d-1}{2}\right)r\right)$ as a test function, then

$$(2.5) \quad \text{cap}(B(x, r), B(x, dr)) \leq \int_{B(x, dr)} a(\varphi, \varphi)(dy) \leq \frac{40 d^v}{c_0(x)(d-1)^2} \frac{m(B(x, r))}{r^2}.$$

There exists a positive Radon measure $\nu \equiv \nu_{B(x, r)}$ called the *equilibrium measure* of $B(x, r)$ in $B(x, dr)$ relative to the form a , such that

$$(2.6) \quad a(u_{B(x, r)}, v) = \int_{B(x, dr)} \tilde{v}(y) \nu_{B(x, r)}(dy)$$

$\forall v \in \mathcal{O}_0[B(x, dr)]$. \tilde{v} is the q.c. version of v , $\text{supp } \nu_{B(x, r)} \subset \partial B(x, r)$ and

$$(2.7) \quad \text{cap}(B(x, r), B(x, dr)) = a(u_{B(x, r)}, u_{B(x, r)}) = \nu_{B(x, r)}(\partial B(x, r)).$$

Since $u_{B(x, r)} \equiv 1$, m-a.e. on $B(x, r)$, $\varrho < \frac{r}{2}$, we have

$$G_{\varrho, B(x, dr)}^x \in C \left(B(x, dr) - B \left(x, \frac{r}{2} \right) \right) \cap Q_0[B(x, dr)].$$

Then

$$(2.8) \quad a(u_{B(x, r)}, G_{\varrho, B(x, dr)}^x) = \int a(u_{B(x, r)}, G_{\varrho, B(x, dr)}^x)(dy) = \mathop{\int}\limits_{B(z, \varrho)} = 1.$$

But

$$(2.9) \quad \begin{aligned} a(u_{B(x, r)}, G_{\varrho, B(x, dr)}^x) &= \int_{B(x, dr)} G_{\varrho, B(x, dr)}^x(y) \nu_{B(x, r)}(dy) = \\ &= \int_{\partial B(x, r)} G_{\varrho, B(x, dr)}^x(y) \nu_{B(x, r)}(dy) = 1 \end{aligned}$$

and

$$(2.10) \quad \begin{aligned} 1 &= \int_{\partial B(x, r)} G_{\varrho, B(x, dr)}^x(y) \nu_{B(x, r)}(dy) \geq \inf_{\partial B(x, r)} G_{\varrho, B(x, dr)}^x \nu_{B(x, r)}(\partial B(x, r)) \Rightarrow \\ &\Rightarrow 1 \geq \inf_{\partial B(x, r)} G_{\varrho, B(x, dr)}^x \text{cap}(B(x, r), B(x, dr)) \end{aligned}$$

Therefore

$$(2.11) \quad \frac{1}{\text{cap}(B(x, r), B(x, dr))} \geq \inf_{\partial B(x, r)} G_{\varrho, B(x, dr)}^x \nu_{B(x, r)}.$$

But

$$1 = \int_{\partial B(x, r)} G_{\varrho, B(x, dr)}^x(y) \nu_{B(x, r)}(dy) \leq \sup_{\partial B(x, r)} G_{\varrho, B(x, dr)}^x \nu_{B(x, r)}(\partial B(x, r)),$$

this implies that

$$1 \leq \sup_{\partial B(x, r)} G_{\varrho, B(x, dr)}^x \text{cap}(B(x, r), B(x, dr))$$

that is

$$(2.12) \quad \frac{1}{\text{cap}(B(x, r), B(x, dr))} \leq \sup_{\partial B(x, r)} G_{\varrho, B(x, dr)}^x.$$

Collecting together (2.11) and (2.12), we obtain

$$(2.13) \quad \inf_{\partial B(x, r)} G_{\varrho, B(x, dr)}^x \leq \frac{1}{\text{cap}(B(x, r), B(x, dr))} \leq \sup_{\partial B(x, r)} G_{\varrho, B(x, dr)}^x.$$

By Sobolev-Poincaré inequality, for $d \geq 2$, we have

$$(2.14) \quad \text{cap}(B(x, r), B(x, dr)) = a(u_{B(x, r)}, u_{B(x, r)}) \geq$$

$$\geq \frac{1}{d^2 r^2 c_1^2(r) \tau^6(x)} \int_{B(x, r)} u_{B(x, r)}^2 m(dx) = \frac{1}{d^2 r^2 c_1^2(r) \tau^6(x)} m(B(x, r)),$$

which together with (2.5) shows that

$$(2.15) \quad \frac{(d-1)^2 c_0(x)}{d^v 40} \frac{r^2}{m(B(x, r))} \leq \frac{1}{\text{cap}(B(x, r), B(x, dr))} \leq \frac{d^2 r^2 c_1^2(r) \tau^6(x)}{m(B(x, r))}.$$

By Lemma 1, $\partial B(x, r)$ can be covered by a finite number l of overlapping balls of radius $r/8$ with distance greater than $r/4$ from x . This set has been denoted as

$$\widehat{X} = \bigcup_{i=1 \dots l} B(x_i, r/8).$$

Thus on each ball of \widehat{X} , a Harnack's inequality holds

$$(2.16) \quad \sup_{B_i} G_{\varrho, B(x_i, dr)}^{x_i} \leq e^{\gamma \mu(x, r)} \inf_{B_i} G_{\varrho, B(x_i, dr)}^{x_i}.$$

Let $u(x) = G_{\varrho, B(x, dr)}^x$. We begin with $B(x_1, r/8)$ and $B(x_2, r/8)$ both included in \widehat{X} , then we obtain

$$\begin{aligned} u(x_1) &\leq \sup_{B_1} u(x) \leq e^{\gamma \mu(x, r)} \inf_{B_1} u(x) \leq e^{\gamma \mu(x, r)} u(\tilde{x}_1) \leq \\ &\leq e^{\gamma \mu(x, r)} \sup_{B_2} u(x) \leq (e^{\gamma \mu(x, r)})^2 \inf_{B_2} u(x) \leq (e^{\gamma \mu(x, r)})^2 u(x_2), \end{aligned}$$

where $\tilde{x}_1 \in B(x_1, r/8) \cap B(x_2, r/8)$. Let us consider the ball $B(x_3, r/8) \subset \widehat{X}$ as in Lemma 1, then if $B(x_3, r/8) \cap B(x_1, r/8) \neq \emptyset$ then

$$u(x_1) \leq (e^{\gamma \mu(x, r)})^2 u(x_1),$$

otherwise if $B(x_3, r/8) \cap B(x_1, r/8) = \emptyset$ one gets

$$u(x_1) \leq (e^{\gamma \mu(x, r)})^3 u(x_1).$$

By iterating the process to the l balls of the chain, we get

$$u(x_1) \leq \dots \leq (e^{l\gamma\mu(x, r)})^l \inf_{B_l} u(x) \leq (e^{l\gamma\mu(x, r)})^l u(x_l).$$

Then, collecting together the inequality chain and taking into account that $u(x) = G_{Q, B(x, dr)}^x$, we obtain

$$(2.17) \quad \sup_{\partial B(x, r)} G_{Q, B(x, dr)}^x \leq e^{l\gamma\mu(x, r)} \inf_{\partial B(x, r)} G_{Q, B(x, dr)}^x.$$

Therefore, by previous results we have

$$\frac{1}{\text{cap}(B(x, r), B(x, dr))} \leq \sup_{\partial B(x, r)} G_{Q, B(x, dr)}^x \leq e^{l\gamma\mu(x, r)} \inf_{\partial B(x, r)} G_{Q, B(x, dr)}^x,$$

then

$$(2.18) \quad \frac{e^{-l\gamma\mu(x, r)}}{\text{cap}(B(x, r), B(x, dr))} \leq \inf_{\partial B(x, r)} G_{Q, B(x, dr)}^x$$

On the other hand

$$\begin{aligned} \inf_{\partial B(x, r)} G_{Q, B(x, dr)}^x &\leq \frac{1}{\text{cap}(B(x, r), B(x, dr))} \leq \sup_{\partial B(x, r)} G_{Q, B(x, dr)}^x \leq \\ &\leq e^{l\gamma\mu(x, r)} \inf_{\partial B(x, r)} G_{Q, B(x, dr)}^x \leq \frac{e^{l\gamma\mu(x, r)}}{\text{cap}(B(x, r), B(x, dr))}, \end{aligned}$$

then

$$(2.19) \quad \sup_{\partial B(x, r)} G_{Q, B(x, dr)}^x \leq \frac{e^{l\gamma\mu(x, r)}}{\text{cap}(B(x, r), B(x, dr))}.$$

Putting together (2.18) and (2.19), we get

$$(2.20) \quad \frac{e^{-l\gamma\mu(x, r)}}{\text{cap}(B(x, r), B(x, dr))} \leq G_{B(x, dr)}^x(y) \leq \frac{e^{l\gamma\mu(x, r)}}{\text{cap}(B(x, r), B(x, dr))}.$$

PROOF OF THEOREM 2: Let $n \in \mathbb{N}$ be such that $2^n r < R < 2^{n+1} r$ $\forall j = 0, 1, \dots, n$ and let G_j^x be the Green function in $B(x, 2^j r)$ with singularity at x . Then by estimating from above and by Theorem 4 we have

$$G_j^x(y) \leq \tau^6(x) \frac{c_1^2(2^{j-1}r) \exp(l\gamma\mu(x, 2^{j-1}r))}{m(B(x, 2^{j-1}r))} d^2(2^{j-1}r)^2,$$

with $y \in \partial B(x, 2^{j-1}r)$. We introduce the function

$$u_j := G_j^x - G_{j-1}^x \quad \text{in } B(x, 2^{j-1}r),$$

which is a solution of $a(u_j, v) = 0$. Indeed,

$$a(u_j, v) = a(G_j^x, v) - a(G_{j-1}^x, v) = \int\limits_{B(x, \varrho)} v m(dx) - \int\limits_{B(x, \varrho)} v m(dx) = 0,$$

with $u_j \in \mathcal{O}_{\text{loc}}[B(x, 2^{j-1}r)]$, $\forall v \in \mathcal{O}_0[B(x, 2^{j-1}r)]$.

$$\tilde{u}_j(y) - G_j^x(y) = -G_{j-1}^x(y) = 0 \quad \text{q.e., } y \in \partial B(x, 2^{j-1}r) \Rightarrow$$

$$\Rightarrow \tilde{u}_j(y) \leq \frac{c_j(r, x)}{m(B(x, 2^{j-1}r))} (2^{j-1}r)^2 \quad \text{q.e. on } \partial B(x, 2^{j-1}r), \quad \forall j = 1, \dots, n$$

and $c_j(r, x) = \tau^6(x) c_1^2(2^{j-1}r) \exp(l\gamma\mu(x, 2^{j-1}r))$. By the maximum principle

$$u_j(y) \leq \frac{c_j(r, x)}{m(B(x, 2^{j-1}r))} (2^{j-1}r)^2 \quad m\text{-a.e. in } B(x, 2^{j-1}r), \quad j = 1, \dots, n$$

if

$$u := G_{B(x, R)}^x - G_n^x \quad \text{in } B(x, 2^n r),$$

we find

$$u(y) \leq \frac{c_n(r, x)}{m(B(x, 2^n r))} (2^n r)^2 \quad m\text{-a.e. in } B(x, 2^n r).$$

This yields to

$$\begin{aligned} G_{B(x, R)}^x(y) &\leq u(y) + \sum_{j=1}^n u_j(y) \leq \\ &\leq \frac{c_n(r, x)}{m(B(x, 2^n r))} (2^n r)^2 + \sum_{j=1}^n \frac{c_j(r, x)}{m(B(x, 2^{j-1}r))} (2^{j-1}r)^2 \leq \sum_{j=0}^n \frac{c_j(r, x)}{m(B(x, 2^{j-1}r))} (2^{j-1}r)^2 \Rightarrow \\ \Rightarrow G_{B(x, R)}^x &\leq \sum_{j=0}^n \tau^6(x) \frac{c_1^2(2^j r) \exp(l\gamma\mu(x, 2^j r))}{m(B(x, 2^j r))} d^2(2^j r)^2 \Rightarrow \\ \Rightarrow G_{B(x, R)}^x(y) &\leq \int_r^R \tau^6(x) \frac{c_1^2(r) \exp(l\gamma\mu(x, s))}{m(B(x, s))} s^2 \frac{ds}{s}. \end{aligned}$$

On the other hand, if we proceed to estimate the Green's function from below, we

have to consider the following initial inequality

$$G_j^x(y) \geq \frac{c_0(x) \exp(-l\gamma\mu(x, 2^{j-1}r))}{m(B(x, 2^{j-1}r))} d^2(2^{j-1}r)^2.$$

By repeating the same steps of the estimate from above we arrive at

$$(2.21) \quad G_{B(x, R)}^x(y) \geq \int_r^R \frac{c_0(x) \exp(-l\gamma\mu(x, s))}{m(B(x, s))} s^2 \frac{ds}{s}$$

and the desired estimate from above and below of the Green function becomes

$$\int_r^R \frac{c_0(x) \exp(-l\gamma\mu(x, s))}{m(B(x, s))} s^2 \frac{ds}{s} \leq G_{B(x, R)}^x(y) \leq \int_r^R \tau^6(x) \frac{c_1^2(r) \exp(l\gamma\mu(x, s))}{m(B(x, s))} s^2 \frac{ds}{s},$$

where we have taken the smallest radius on $c_1(r)$ because of its decreasing property.

3. - ENERGY'S DECAY

We first prove a *weighted Caccioppoli's inequality*.

PROPOSITION 1: *Let v be a local solution in $B(x_0, 4r)$,*

$$\begin{cases} a(v, w) = 0 \\ v \in \mathcal{O}_{loc}[B(x_0, 4r)], \quad \forall w \in \mathcal{O}_0[B(x_0, 4r)] \end{cases}$$

then

$$(3.1) \quad \int_{B(x_0, qr)} G^{x_0} a(v, v)(dx) + \sup_{B(x_0, qr)} v^2 \leq c_q \frac{s^{3l}(x_0, r) c_1^4(qr) s_0^{18}(x_0)}{m(B(x_0, r))} \int_{B(x_0, r) - B(x_0, qr)} v^2 m(dx),$$

where c_q is a constant depending only by q and where we have defined

$$(3.2) \quad s(x_0, r) = \sup_{z \in B(x_0, r) - B(x_0, qr)} e^{\gamma\mu(z, r)}$$

and

$$(3.3) \quad s_0(x_0) = \sup_{z \in B(x_0, qr)} \tau(z),$$

PROOF: Let $z \in B(x_0, r)$ and $B(z, sr) \subset B(z, tr) \subset B(x_0, r)$, $s < t < 1$ and let φ be the cut-off function of $B(z, sr)$ w.r.t. $B(z, tr)$. We choose as test function $\varphi^2 v G_\varrho^z$, where G_ϱ^z denotes regularized Green function relative to z and to the ball $B(z, 2r)$. Since $\varphi, v, G_\varrho^z \in \mathcal{O}_{\text{loc}}[B(x_0, 2r)] \cap L^\infty(B(x_0, 2r), m)$ we have

$$\begin{aligned}
 (3.4) \quad 0 &= a(v, \varphi^2 v G_\varrho^z) = \int_{B(z, tr)} a(v, \varphi^2 v G_\varrho^z)(dx) = \\
 &= \int_{B(z, tr)} \varphi^2 G_\varrho^z \alpha(v, v)(dx) + \int_{B(z, tr)} \varphi^2 v \alpha(v, G_\varrho^z)(dx) + \int_{B(z, tr)} v G_\varrho^z \alpha(v, \varphi^2)(dx) = \\
 &= \int_{B(z, tr)} \varphi^2 G_\varrho^z \alpha(v, v)(dx) + \int_{B(z, tr)} \varphi^2 v \alpha(v, G_\varrho^z)(dx) + \int_{B(z, tr)} v G_\varrho^z 2 \varphi \alpha(v, \varphi)(dx) = \\
 &= \int_{B(z, tr)} \varphi^2 G_\varrho^z \alpha(v, v)(dx) + \frac{1}{2} \int_{B(z, tr)} a(\varphi^2 v^2, G_\varrho^z)(dx) - \int_{B(z, tr)} v^2 \varphi \alpha(\varphi, G_\varrho^z)(dx) + \\
 &\quad + 2 \int_{B(z, tr)} v G_\varrho^z \varphi \alpha(v, \varphi)(dx).
 \end{aligned}$$

This implies that

$$\begin{aligned}
 (3.5) \quad &\int_{B(z, tr)} \varphi^2 G_\varrho^z \alpha(v, v)(dx) + \frac{1}{2} \int_{B(z, tr)} a(\varphi^2 v^2, G_\varrho^z)(dx) = \\
 &= \frac{1}{2} \int_{B(z, tr)} 2 v^2 \varphi \alpha(\varphi, G_\varrho^z)(dx) - 2 \int_{B(z, tr)} v G_\varrho^z \varphi \alpha(v, \varphi)(dx) \leqslant \\
 &\leqslant \int_{B(z, tr)} |v^2 \varphi| |\alpha(\varphi, G_\varrho^z)| (dx) + 2 \int_{B(z, tr)} |v G_\varrho^z \varphi| |\alpha(v, \varphi)| (dx) \leqslant \\
 &\leqslant \frac{1}{4} \int_{B(z, tr)} \varphi^2 G_\varrho^z \alpha(v, v)(dx) + 4 \int_{B(z, tr)} v^2 G_\varrho^z \alpha(\varphi, \varphi)(dx) + \\
 &\quad + \frac{1}{4\epsilon} \int_{B(z, tr) - B(z, sr)} v^2 G_\varrho^z \alpha(\varphi, \varphi)(dx) + \epsilon \int_{B(z, tr) - B(z, sr)} v^2 \varphi^2 (G_\varrho^z)^{-1} \alpha(G_\varrho^z, G_\varrho^z)(dx).
 \end{aligned}$$

We will now estimate the last term at the r.h.s. of (3.5). Let σ be the cut-off function of the annulus $B(z, tr) - B(z, sr)$ w.r.t. the balls $B\left(z, \frac{s^2}{t} r\right)$ and $B(z, (2t-s)r)$,

then by applying Lemma (7.2) of Ref. [1] with $f = \sigma\varphi v$, we find

$$(3.6) \quad \int_{B(z, tr)} \sigma^2 v^2 \varphi^2 (G_\varrho^z)^{-1} \alpha(G_\varrho^z, G_\varrho^z)(dx) \leq 4 \int_{B(z, tr)} v^2 \varphi^2 (G_\varrho^z)^2 \alpha(\sigma\varphi v, \sigma\varphi v)(dx).$$

By means of Schwarz rule applied to $\alpha(\sigma\varphi v, \sigma\varphi v)$, we have

$$(3.7) \quad \begin{aligned} \int_{B(z, tr)} (G_\varrho^z)^2 \alpha(\sigma\varphi v, \sigma\varphi v)(dx) &\leq 3 \int_{B(z, tr) - B(z, s^* r)} (G_\varrho^z)^2 \varphi^2 \alpha(v, v)(dx) + \\ &+ \frac{120}{\frac{s^2}{t^2} (t-s)^2 r^2} \int_{B(z, tr) - B(z, s^* r)} v^2 (G_\varrho^z)^2 m(dx). \end{aligned}$$

Substituting (3.7) in (3.6), it follows

$$(3.8) \quad \begin{aligned} \int_{B(z, tr) - B(z, s^* r)} v^2 \varphi^2 \alpha(G_\varrho^z, G_\varrho^z)(dx) &\leq \\ &\leq \frac{480}{\frac{s^2}{t^2} (t-s)^2 r^2} \int_{B(z, tr) - B(z, s^* r)} v^2 (G_\varrho^z)^2 m(dx) + 12 \int_{B(z, tr) - B(z, s^* r)} \varphi^2 (G_\varrho^z)^2 \alpha(v, v)(dx). \end{aligned}$$

Let us consider

$$\frac{\sup G_\varrho^z}{\inf G_\varrho^z},$$

where the sup and the inf are taken on $B(z, tr) - B(z, s^* r)$ with $\varrho < s^* r$, we have that

$$(3.9) \quad 0 < \frac{\sup G_\varrho^z}{\inf G_\varrho^z} \leq \tilde{c} \tau^{10}(z) \exp(2l\gamma\mu(z, r)) (t/s^*)^{\nu-2} c_1^2(r).$$

By multiplying inequality (3.8) by $\sup_{z \in B(z, tr) - B(z, s^* r)} (G_\varrho^z)^{-1}$ and taking account of estimates (2.3), we obtain

$$(3.10) \quad \begin{aligned} \int_{B(z, tr) - B(z, s^* r)} v^2 \varphi^2 (G_\varrho^z)^{-1} \alpha(G_\varrho^z, G_\varrho^z)(dx) &\leq \\ &\leq \frac{\tau^{10}(z) 480 \tilde{c} e^{2l\gamma\mu(z, r)} c_1^2(r)}{(s/t)^{2\nu-2} (t-s)^2 r^2} \int_{B(z, tr) - B(z, s^* r)} v^2 G_\varrho^z m(dx) + \\ &+ 12 \tilde{c} \frac{\tau^{10}(z) e^{2l\gamma\mu(z, r)} c_1^2(r)}{(s/t)^{2\nu-4}} \int_{B(z, tr)} \varphi^2 G_\varrho^z \alpha(v, v)(dx). \end{aligned}$$

Putting (3.8) in (3.5), taking account of the properties of φ and choosing $\varepsilon\tilde{c} = \exp(-2l\gamma\mu(z, r))(s/t)^{2\nu-2}\tau^{-10}(z)/(24c_1^2(r))$, we obtain

$$\begin{aligned}
 (3.11) \quad & \int_{B(z, tr)} \varphi^2 G_\varphi^z \alpha(v, v)(dx) + \frac{1}{2} \int_{B(z, tr)} \alpha(v^2 \varphi^2, G_\varphi^z)(dx) \leq \\
 & \leq \frac{1}{4} \int_{B(z, tr)} \varphi^2 G_\varphi^z \alpha(v, v)(dx) + 4 \int_{B(z, tr)} v^2 G_\varphi^z \alpha(\varphi, \varphi)(dx) + \\
 & + 6 \tilde{c} \tau^{10}(z) e^{2l\gamma\mu(z, r)} c_1^2(r) (s/t)^{2\nu-2} \int_{B(z, tr) - B(z, s^*r)} v^2 G_\varphi^z \alpha(\varphi, \varphi)(dx) + \\
 & + \frac{20}{(t-s)^2 r^2} \int_{B(z, tr) - B(z, s^*r)} v^2 G_\varphi^z m(dx) + \frac{1}{2} \frac{s^2}{t^2} \int_{B(z, tr)} \varphi^2 G_\varphi^z \alpha(v, v)(dx) \\
 & \leq \frac{c\tau^{10}(z) e^{2l\gamma\mu(z, r)} c_1^2(r) (s/t)^{2\nu-2}}{(t-s)^2 r^2} \int_{B(z, tr) - B(z, s^*r)} v^2 G_\varphi^z m(dx).
 \end{aligned}$$

Since $\varphi \equiv 1$ on $B(z, sr)$ and by the definition of G_φ^z and recalling that $\int_{B(z, \varrho)} = \left(\frac{1}{m(B(z, \varrho))}\right) \int_{B(z, \varrho)}$, we have

$$\begin{aligned}
 (3.12) \quad & \int_{B(z, tr)} \varphi^2 (G_\varphi^z)^2 \alpha(v, v)(dx) + \frac{1}{2} \int_{B(z, \varrho)} v^2 m(dx) \leq \\
 & \leq \frac{c\tau^{10}(z) e^{2l\gamma\mu(z, r)} c_1^2(r) (s/t)^{2\nu-2}}{(t-s)^2 r^2} \int_{B(z, tr) - B(z, s^*r)} v^2 G_\varphi^z m(dx).
 \end{aligned}$$

Take the limit $\varrho \rightarrow 0^+$, then $G_\varrho^z \rightarrow G^z$ uniformly in $B(z, tr) - B(z, s^*r)$, $\forall s^*$ fixed. By the Lebesgue theorem in Ref. [2], we obtain for m -a.e.

$$\begin{aligned}
 (3.13) \quad & \int_{B(z, sr)} G^z \alpha(v, v)(dx) + \frac{1}{2} \tilde{v}(z)^2 \leq \\
 & \leq \frac{c\tau^{10}(z) e^{2l\gamma\mu(z, r)} c_1^2(r) (s/t)^{2\nu-2}}{(t-s)^2 r^2} \int_{B(z, tr) - B(z, s^*r)} v^2 G^z m(dx).
 \end{aligned}$$

We take the sup for $z \in B(x_0, qr)$ by choosing $q \in \left(0, \frac{1}{3}\right)$, $s = [2q(1-q)]^{1/2}$, $t = 1 - q$.

Then $B(z, tr) - B(z, s^*r) \subset B(x_0, r) - B(x_0, qr)$ and $\forall z \in B(x_0, qr)$ we get

$$\begin{aligned}
 (3.14) \quad & \int_{B(x_0, qr)} G^{x_0} \alpha(v, v)(dx) + \frac{1}{2} \sup_{B(x_0, qr)} v^2 \leq \\
 & \leq c_q \frac{s_0^{10}(x_0) s^{2l}(x_0, r) c_1^2(r)}{r^2} \sup_{z \in B(x_0, qr)} \int_{B(z, r) - B(z, qr)} v^2 G^z m(dx) \\
 & \leq c_q \frac{s^{3l}(x_0, r) c_1^4(qr) s_0^{16}(x_0)}{m(B(z, qr))} \int_{B(x_0, r) - B(x_0, qr)} v^2 m(dx) \leq \\
 & \leq c_q \frac{s^{3l}(x_0, r) c_1^4(qr) s_0^{18}(x_0)}{m(B(x_0, r))} \int_{B(x_0, r) - B(x_0, qr)} v^2 m(dx)
 \end{aligned}$$

with $c_q = cq^{-v} (s/t)^{2v-2} / (t-s)^2$.

PROOF OF THEOREM 3: Let us consider the test function $w = (u-k) G_\varrho^z \varphi$, where G_ϱ^z is the regularized Green function relative to z and to the ball $B(z, tr)$, φ is the capacitory potential of $B(z, sr)$ w.r.t. $B(z, tr)$. $z \in B(x_0, qr)$, $s < t < 1$, q to be fixed. k is a constant. We have

$$\begin{aligned}
 (3.15) \quad 0 &= \int_{B(z, tr)} \alpha(u, (u-k) \varphi G_\varrho^z)(dx) \cdot \\
 &\cdot \int_{B(z, tr)} \varphi G_\varrho^z \alpha(u, u)(dx) + \int_{B(z, tr)} (u-k) \varphi \alpha(u, G_\varrho^z)(dx) + \int_{B(z, tr)} (u-k) G_\varrho^z \alpha(u, \varphi)(dx) + \\
 &= \int_{B(z, tr)} \varphi G_\varrho^z \alpha(u, u)(dx) + \frac{1}{2} \int_{B(z, tr)} \alpha((u-k)^2 \varphi, G_\varrho^z)(dx) + \\
 &\quad + \int_{B(z, tr)} (u-k) G_\varrho^z \alpha(u, \varphi)(dx) - \frac{1}{2} \int_{B(z, tr)} (u-k)^2 \alpha(\varphi, G_\varrho^z)(dx).
 \end{aligned}$$

Then for $\varrho < r$, we get

$$\begin{aligned}
 (3.16) \quad & \int_{B(z, tr)} \varphi G_\varrho^z \alpha(u, u)(dx) + \frac{1}{2} \int_{B(z, \varrho)} (u-k)^2 m(dx) = \\
 &= \frac{1}{2} \int_{B(z, tr)} (u-k)^2 \alpha(\varphi, G_\varrho^z)(dx) - \int_{B(z, tr)} (u-k) G_\varrho^z \alpha(u, \varphi)(dx) = \\
 &= \frac{1}{2} \int_{B(z, tr)} \alpha(\varphi, G_\varrho^z (u-k)^2)(dx) - 2 \int_{B(z, tr)} (u-k) G_\varrho^z \alpha(u, \varphi)(dx) =
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \int_{B(z, tr)} G_\varrho^z (\tilde{u} - k)^2 d\nu_{B(z, sr)} - 2 \int_{B(z, tr)} (u - k) G_\alpha^z(u, \varphi)(dx) \leqslant \\
&\leqslant \frac{1}{2} \sup_{B(z, tr)} (u - k)^2 + 2 \int_{B(z, tr)} (u - k) G_\varrho^z \alpha(u, \varphi)(dx) \leqslant \\
&\leqslant \frac{1}{2} \sup_{B(z, tr)} (u - k)^2 + \frac{1}{\eta} \int_{B(z, tr)} G_\varrho^z \alpha(u, u)(dx) + \eta \int_{B(z, tr)} (u - k)^2 G_\varrho^z \alpha(\varphi, \varphi)(dx) \leqslant \\
&\leqslant \frac{1}{2} \sup_{B(z, tr)} (u - k)^2 + \frac{1}{\eta} \int_{B(z, tr)} G_\varrho^z \alpha(u, u)(dx) + \\
&\quad + \eta \sup_{B(z, tr)} (u - k)^2 \sup_{B(z, tr) - B(z, sr)} G_\varrho^z \text{cap}(B(z, sr), B(z, tr)).
\end{aligned}$$

Then, by the max principle and by Theorem (4), we have for arbitrary $\eta > 0$

$$\begin{aligned}
(3.17) \quad &\int_{B(z, tr)} \varphi G_\varrho^z \alpha(u, u)(dx) + \frac{1}{2} \int_{B(z, \varrho)} (u - k)^2 m(dx) \leqslant \\
&\leqslant \frac{1}{2} \sup_{B(z, tr)} (u - k)^2 + \frac{1}{\eta} \int_{B(z, tr) - B(z, sr)} G_\varrho^z \alpha(u, u)(dx) \\
&\quad + \eta \sup_{B(z, tr)} (u - k)^2 \sup_{B(z, tr) - B(z, sr)} G_\varrho^z \text{cap}(B(z, sr), B(z, tr)).
\end{aligned}$$

Moreover, the application of Harnack's inequality on Green's function gives

$$\begin{aligned}
(3.18) \quad &\sup_{B(z, tr) - B(z, sr)} G_\varrho^z \text{cap}(B(z, sr), B(z, tr)) \leqslant \\
&\leqslant e^{l\gamma\mu(z, r)} \inf_{B(z, tr) - B(z, sr)} G_\varrho^z \text{cap}(B(z, sr), B(z, tr)) \leqslant c\tau^8(z) c_1^2(r) e^{2l\gamma\mu(z, r)},
\end{aligned}$$

where $l = 16^{-v} \sup_{x \in B(x_0, 2R)} c_0^{-1}(x)$. This implies that

$$\begin{aligned}
(3.19) \quad &\int_{B(z, tr)} \varphi G_\varrho^z \alpha(u, u)(dx) + \frac{1}{2} \int_{B(z, \varrho)} (u - k)^2 m(dx) \leqslant \\
&\leqslant \frac{1}{2} \sup_{B(z, tr)} (u - k)^2 + \eta c\tau^8(z) c_1^2(r) \sup_{z \in B(z, tr) - B(z, sr)} e^{2l\gamma\mu(z, r)} \sup_{B(z, tr)} (u - k)^2 + \\
&\quad + \frac{1}{\eta} \int_{B(z, tr) - B(z, sr)} G_\varrho^z \alpha(u, u)(dx)
\end{aligned}$$

Then, passing to the limit as $\varrho \rightarrow 0$ we obtain, by Lebesgue theorem in [2], for m -a.e. z

$$(3.20) \quad \begin{aligned} & \int_{B(z, tr)} \varphi G_{B(x, tr)}^z \alpha(u, u)(dx) + \frac{1}{2} (\tilde{u}(z) - k)^2 \leq \\ & \leq \left(\frac{1}{2} + \eta cct^8(z) c_1^2(r) \sup_{z \in B(z, tr) - B(z, sr)} e^{2l\gamma\mu(z, r)} \right) \sup_{B(z, tr)} (u - k)^2 + \\ & + \frac{1}{\eta} \int_{B(z, tr) - B(z, sr)} G^z \alpha(u, u)(dx). \end{aligned}$$

We now choose $t = 1 - q$, $s = 2q$, with $q \in \left(0, \frac{1}{6}\right]$. From

$$(3.21) \quad \begin{aligned} & \sup_{B(z, tr)} (u - k)^2 \leq \left(1 + 2\eta cct^8(z) c_1^2(r) \sup_{z \in B(z, tr) - B(z, sr)} e^{2l\gamma\mu(z, r)} \right) \sup_{B(z, tr)} (u - k)^2 + \\ & + \frac{1}{\eta} \int_{B(z, tr) - B(z, sr)} G_\varrho^z \alpha(u, u)(dx), \end{aligned}$$

we take the supremum for $z \in B(x_0, qr)$. Then from Theorem (4) we have

$$(3.22) \quad \begin{aligned} & \sup_{B(x_0, qr)} (u - k)^2 \leq (1 + 2\eta cts_0^8(x_0) c_1^2(r) s^{2l}(x_0, r)) \sup_{B(x_0, r)} (u - k)^2 + \\ & + \frac{1}{\eta} \left(\frac{s^{2l}(x_0, r)}{\text{cap}(B(x_0, r), B(x_0, qr))} \right) \int_{B(x_0, r) - B(x_0, qr)} \alpha(u, u)(dx) \leq \\ & \leq (1 + 2\eta cts_0^8(x_0) c_1^2(r) s^{2l}(x_0, r)) \sup_{B(x_0, r)} (u - k)^2 + \\ & + \frac{s^{2l}(x_0, r)}{\eta} \int_{B(x_0, r) - B(x_0, qr)} e^{l\gamma\mu(x_0, r)} G_{B(x_0, 2r)}^z \alpha(u, u)(dx), \end{aligned}$$

with $s(x_0, r)$ defined as in Eq. (3.2). From Proposition (1), it follows that

$$(3.23) \quad \begin{aligned} & \int_{B(x_0, qr)} G^{x_0} \alpha(v, v)(dx) + \frac{1}{2} \sup_{B(x_0, qr)} v^2 \leq \\ & \leq c_q \frac{s^{3l}(x_0, r) c_1^4(qr) s_0^{18}(x_0)}{m(B(x_0, r))} \int_{B(x_0, r) - B(x_0, qr)} v^2 m(dx) \leq \\ & \leq c_q c_1^4(qr) s^{3l}(x_0, r) s_0^{18}(x_0) \sup_{B(x_0, r)} v^2, \end{aligned}$$

where $s_0(x_0)$ has been defined in Eq. (3.3). Thus

$$(3.24) \quad \int_{B(x_0, qr)} G_{B(x_0, 2r)}^{x_0} a(u, u)(dx) \leq c_q c_1^4(qr) s^{3l}(x_0, r) s_0^{18}(x_0) \sup_{B(x_0, r)} (u - k)^2.$$

Then, by (3.22) we obtain

$$(3.25) \quad \begin{aligned} & \int_{B(x_0, qr)} G_{B(x_0, 2r)}^{x_0} a(u, u)(dx) + \sup_{B(x_0, qr)} (u - k)^2 \leq \\ & \leq [c_q c_1^4(qr) s^{3l}(x_0, r) s_0^{18}(x_0) + 1 + 2\eta c s_0^8(x_0) c_1^2(r) s^{2l}(x_0, r)] \sup_{B(x_0, r)} (u - k)^2 + \\ & + \frac{s^{3l}(x_0, r)}{\eta} \int_{B(x_0, r) - B(x_0, qr)} G_{B(x_0, 2r)}^{x_0} a(u, u)(dx). \end{aligned}$$

By «hole filling» after having multiplied by η , we obtain

$$(3.26) \quad \begin{aligned} & (s^{3l}(x_0, r) + \eta) \int_{B(x_0, qr)} G_{B(x_0, 2r)}^{x_0} a(u, u)(dx) + \eta \sup_{B(x_0, qr)} |u - k|^2 \leq \\ & \leq \eta \sup_{B(x_0, r)} (u - k)^2 [c_q c_1^4(qr) s^{3l}(x_0, r) s_0^{18}(x_0) + 1 + 2\eta c s_0^8(x_0) c_1^2(r) s^{2l}(x_0, r)] + \\ & + s^{3l}(x_0, r) \int_{B(x_0, r)} G_{B(x_0, 2r)}^{x_0} a(u, u)(dx). \end{aligned}$$

We now study the last term at the right hand side of (3.26)

$$(3.27) \quad \begin{aligned} & \int_{B(x_0, r)} G_{B(x_0, 2r)}^{x_0} a(u, u)(dx) = \int_{B(x_0, 2q^{-1}r)} G_{B(x_0, 2q^{-1}r)}^{x_0} a(u, u)(dx) - \\ & - \int_{B(x_0, r)} (G_{B(x_0, 2q^{-1}r)}^{x_0} - G_{B(x_0, 2r)}^{x_0}) a(u, u)(dx). \end{aligned}$$

Here we have taken into account that

$$F = G_{B(x_0, 2q^{-1}r)}^{x_0} - G_{B(x_0, 2r)}^{x_0}$$

is a solution of the problem

$$a(F, v) = 0, \quad \forall v \in \mathcal{O}[B(x_0, 2r)],$$

therefore by the maximum principle and Theorem (4)

$$(3.28) \quad \inf_{B(x_0, r)} F \geq \inf_{\partial B(x_0, 2r)} \tilde{F} = \inf_{\partial B(x_0, 2r)} G_{B(x_0, 2q^{-1}r)}^{x_0} \geq \\ \geq c \exp(-l\gamma\mu(x_0, r)) \left(\frac{c_0(x_0) r^2}{m(B(x_0, r))} \right).$$

Therefore, by Poincaré inequality, we also have for arbitrary $\bar{q} \in (0, 1)$

$$(3.29) \quad \int_{B(x_0, r)} G_{B(x_0, 2r)}^{x_0} \alpha(u, u)(dx) \leq \int_{B(x_0, r)} G_{B(x_0, 2q^{-1}r)}^{x_0} \alpha(u, u)(dx) - \\ - c \exp(-l\gamma\mu(x_0, r)) \left(\frac{c_0(x_0) r^2}{m(B(x_0, r))} \right) \int_{B(x_0, r)} \alpha(u, u)(dx) \leq \\ \leq \int_{B(x_0, r)} G_{B(x_0, 2q^{-1}r)}^{x_0} \alpha(u, u)(dx) - \\ - c \left(\frac{c_0(x_0) \exp(-\gamma\mu(x_0, r)) r^2}{m(B(x_0, r)) c_1(\kappa^{-1}\bar{q}r)(\kappa^{-1}\bar{q}r)^2} \right) \int_{B(x_0, \kappa^{-1}\bar{q}r)} |u - \bar{u}|^2 m(dx),$$

where \bar{u} denotes the average of u on $B(x_0, \kappa^{-1}\bar{q}r)$. By choosing \bar{q} such that $\kappa^{-1}\bar{q} = q$, we find

$$(3.30) \quad \int_{B(x_0, r)} G_{B(x_0, 2r)}^{x_0} \alpha(u, u)(dx) \leq \int_{B(x_0, r)} G_{B(x_0, 2q^{-1}r)}^{x_0} \alpha(u, u)(dx) - \\ - c \left(\frac{c_0(x_0) \exp(-\gamma\mu(x_0, r))}{m(B(x_0, r))} \right) \frac{1}{c_1(qr)} \sup_{B(x_0, qr)} |u - \bar{u}|^2 m(B(x_0, qr)),$$

while taking the doubling property of m into account,

$$\frac{m(B(x_0, qr))}{m(B(x_0, r))} \geq c_0(x_0) q^\nu,$$

we obtain that

$$(3.31) \quad \int_{B(x_0, r)} G_{B(x_0, 2r)}^{x_0} \alpha(u, u)(dx) \leq \int_{B(x_0, r)} G_{B(x_0, 2q^{-1}r)}^{x_0} \alpha(u, u)(dx) - \\ - cq^{\nu-2} (\exp(-l\gamma\mu(x_0, r))) \frac{c_0^2(x_0)}{c_1(qr)} \sup_{B(x_0, qr)} |u - \bar{u}|^2.$$

Taking into account (3.31) and choosing $\bar{u} = k$ in (3.26), we obtain

$$(3.32) \quad (s^{3l}(x_0, r) + \eta) \int_{B(x_0, qr)} G_{B(x_0, 2r)}^{x_0} a(u, u)(dx) + \\ + \left(\eta + cq^{\nu-2} (\exp(-l\gamma\mu(x_0, r))) \frac{c_0^2(x_0)}{c_1(qr)} \right) \sup_{B(x_0, qr)} |u - \bar{u}|^2 \leq$$

$$(3.33) \quad \leq \eta \sup_{B(x_0, r)} |u - \bar{u}|^2 [c_q c_1^4(qr) s^{3l}(x_0, r) s_0^{18}(x_0) + 1 + 2\eta c s_0^8(x_0) c_1^2(r) s^{2l}(x_0, r)] + \\ + s^{3l}(x_0, r) \int_{B(x_0, r)} G_{B(x_0, 2r)}^{x_0} a(u, u)(dx).$$

Since

$$(3.34) \quad s(x_0, r) = \sup_{z \in B(x_0, r) - B(x_0, qr)} e^{\gamma\mu(x_0, r)} = \exp \bar{s}(x_0, r)$$

with $\bar{s}(x_0, r) = \sup_{z \in B(x_0, r) - B(x_0, qr)} \gamma\mu(x_0, r)$, we now choose $\eta = \exp(-2l\bar{s}(x_0, r))$ and dividing by

$$(3.35) \quad \exp(3l\bar{s}(x_0, r)) + \eta,$$

we have

$$(3.36) \quad \int_{B(x_0, qr)} G_{B(x_0, 2r)}^{x_0} a(u, u)(dx) + \\ + \frac{\left(e^{-2l\bar{s}(x_0, r)} + cq^{\nu-2} e^{-l\gamma\mu(x_0, r)} \frac{c_0^2(x_0)}{c_1(qr)} \right)}{e^{3l\bar{s}(x_0, r)} + e^{-2l\bar{s}(x_0, r)}} \sup_{B(x_0, qr)} |u - \bar{u}|^2 \leq \\ \leq \frac{[c_q c_1^4(qr) e^{3l\bar{s}(x_0, r)} s_0^{18}(x_0) + 1 + 2\eta c s_0^8(x_0) c_1^2(r) e^{2l\bar{s}(x_0, r)}]}{e^{5l\bar{s}(x_0, r)} + 1} \sup_{B(x_0, r)} |u - \bar{u}|^2 + \\ + \frac{1}{1 + e^{-5l\bar{s}(x_0, r)}} \int_{B(x_0, r)} G_{B(x_0, 2r)}^{x_0} a(u, u)(dx) \leq \frac{3c_q c_1^2(qr) s_0^{18}(x_0) e^{3l\bar{s}(x_0, r)}}{e^{5l\bar{s}(x_0, r)} + 1} \sup_{B(x_0, r)} |u - \bar{u}|^2 + \\ + \frac{1}{1 + e^{-5l\bar{s}(x_0, r)}} \int_{B(x_0, r)} G_{B(x_0, 2r)}^{x_0} a(u, u)(dx).$$

On the other hand the first member of previous inequality becomes

$$\int_{B(x_0, qr)} G_{B(x_0, 2r)}^{x_0} \alpha(u, u)(dx) + \frac{\left(1 + cq^{\nu-2} e^{2l\bar{s}(x_0, r)} e^{-l\gamma\mu(x_0, r)} \frac{c_0^2(x_0)}{c_1(qr)}\right)}{e^{5l\bar{s}(x_0, r)} + 1} \sup_{B(x_0, qr)} |u - \bar{u}|^2.$$

This means that

$$(3.37) \quad \begin{aligned} \int_{B(x_0, qr)} G_{B(x_0, 2r)}^{x_0} \alpha(u, u)(dx) &\leq \frac{3c_q c_1^2(qr) s_0^{18}(x_0) e^{3l\bar{s}(x_0, r)}}{e^{5l\bar{s}(x_0, r)} + 1} \sup_{B(x_0, r)} |u - \bar{u}|^2 + \\ &+ \frac{1}{1 + e^{-5l\bar{s}(x_0, r)}} \int_{B(x_0, r)} G_{B(x_0, 2r)}^{x_0} \alpha(u, u)(dx) \leq 2cc_1^2(qr) e^{-2l\bar{s}(x_0, r)} \sup_{B(x_0, r)} |u - \bar{u}|^2 + \\ &+ \frac{1}{1 + e^{-5l\bar{s}(x_0, r)}} \int_{B(x_0, r)} G_{B(x_0, 2r)}^{x_0} \alpha(u, u)(dx). \end{aligned}$$

In the last inequality $\sup_{B(x_0, r)} |u - \bar{u}|^2$ can be related to the energy by [6]

$$(3.38) \quad \begin{aligned} \sup_{B(x_0, r)} |u - \bar{u}|^2 &\leq c \left(\operatorname{osc}_{B(x_0, R)} u \right)^2 \exp \left(- \int_r^R \exp \left(-2\gamma\mu(x_0, \varrho) \frac{d\varrho}{\varrho} \right) \right) \leq \\ &\leq \frac{1}{m(B(x_0, kR))} \exp \left(- \int_r^R \exp \left(-2\gamma\mu(x_0, \varrho) \frac{d\varrho}{\varrho} \right) \right) \int_{B(x_0, kR)} |u - \bar{u}|^2 m(dx) \leq \\ &\leq ce^{2l\gamma\mu(x, r)} c_1^2(R) \int_{B(x_0, k^2 R)} G_{B(x_0, 2q^{-1}r)}^{x_0} \alpha(u, u)(dx) \cdot \\ &\quad \cdot \exp \left(- \int_r^R \exp \left(-2\gamma\mu(x_0, \varrho) \frac{d\varrho}{\varrho} \right) \right). \end{aligned}$$

Then

$$(3.39) \quad \begin{aligned} \int_{B(x_0, qr)} G_{B(x_0, 2r)}^{x_0} \alpha(u, u)(dx) &\leq cc_1^2(qr) c_1^2(R) \exp \left(- \int_r^R \exp \left(-2\gamma\mu(x_0, \varrho) \frac{d\varrho}{\varrho} \right) \right) \cdot \\ &\quad \cdot \int_{B(x_0, k^2 R)} G_{B(x_0, 2q^{-1}r)}^{x_0} \alpha(u, u)(dx) + \frac{1}{1 + e^{-5l\bar{s}(x_0, r)}} \int_{B(x_0, r)} G_{B(x_0, 2q^{-1}r)}^{x_0} \alpha(u, u)(dx). \end{aligned}$$

If we consider $u \in \mathcal{O}[a, B(x_0, R_0)]$, $B(x_0, R_0) \subset X$ and $R < kR$, one gets

$$(3.40) \quad \int_{B(x_0, r)} G_{B(x_0, 2q^{-1}r)}^{x_0} a(u, u)(dx) \leq \int_{B(x_0, kR)} G_{B(x_0, 2q^{-1}r)}^{x_0} a(u, u)(dx)$$

and

$$(3.41) \quad \begin{aligned} & \int_{B(x_0, qr)} G_{B(x_0, 2r)}^{x_0} a(u, u)(dx) \leq \\ & \leq cc_1^4(qr) \exp\left(-\int_r^R \exp\left(-2\gamma\mu(x_0, \varrho) \frac{d\varrho}{\varrho}\right)\right) \int_{B(x_0, k^2R)} G_{B(x_0, 2q^{-1}r)}^{x_0} a(u, u)(dx) + \\ & + \frac{1}{e^{5l\tilde{\gamma}(x_0, r)} + 1} \int_{B(x_0, kR)} G_{B(x_0, 2q^{-1}r)}^{x_0} a(u, u)(dx) \leq \\ & \leq cc_1^4(qr) \exp\left(-\int_r^R \exp\left(-2\gamma\mu(x_0, \varrho) \frac{d\varrho}{\varrho}\right)\right) \int_{B(x_0, k^2R)} G_{B(x_0, 2q^{-1}r)}^{x_0} a(u, u)(dx) + \\ & + \exp\left(-\int_r^R \exp(-5l\gamma\mu(x_0, \varrho)) \frac{d\varrho}{\varrho}\right) \int_{B(x_0, k^2R)} G_{B(x_0, 2q^{-1}r)}^{x_0} a(u, u)(dx). \end{aligned}$$

If we make the choice $\gamma\mu(x, r) \leq o\left(\log\log\frac{1}{r}\right)$, we obtain by the previous inequality

$$(3.42) \quad \psi(r) \leq 2 \exp\left(-\int_r^R \exp\left(-\log\log\frac{1}{\varrho}\right) \frac{d\varrho}{\varrho}\right) \psi(k^2R) \leq \left(\frac{\log\frac{1}{R}}{\log\frac{1}{r}}\right) \psi(k^2R),$$

where

$$(3.43) \quad \psi(r) = \int_{B(x_0, r)} G_{B(x_0, 2q^{-1}r)}^{x_0} a(u, u)(dx).$$

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