Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
119° (2001), Vol. XXV, fasc. 1, pagg. 97-109

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An application of scattering theory to the spectrum of the Laplace-Beltrami operator (**)

SUMMARY. — Applying a theorem due to Belopol'ski and Birman, we show that the Laplace-Beltrami operator on 1-forms on $\boldsymbol{R}^n$ endowed with an asymptotically Euclidean metric has absolutely continuous spectrum equal to $[0, +\infty)$.

Un'applicazione della teoria dello scattering allo spettro dell'operatore di Laplace-Beltrami

RIASSUNTO. — Applicando un teorema di Belopol'ski e Birman, si dimostra che l'operatore di Laplace-Beltrami sulle 1-forme nello spazio $\boldsymbol{R}^n$ dotato di una metrica asintoticamente Euclidea ha spettro assolutamente continuo pari a $[0, +\infty)$.

1. - INTRODUCTION

The relationships between the geometric properties of a complete noncompact Riemannian manifold and the spectrum of the Laplace-Beltrami operator have been intensively investigated by many authors.

Among them, H. Donnelly since the late seventies studied the spectra of the Laplacian and of the Laplace-Beltrami operators on particular manifolds, such as the hyperbolic space ([1]), manifolds with negative sectional curvature ([3]), asymptotically euclidean manifolds ([2]). However, to our knowledge, the case of the Laplace-Beltrami operator acting on p-forms on asymptotically euclidean manifolds has been left aside up to now.

The purpose of this paper is to contribute to the investigation of this case. We

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(**) Memoria presentata il 18 maggio 2001 da Edoardo Vesentini, uno dei XL.

study the absolutely continuous spectrum of the Laplace-Beltrami operator on $\boldsymbol{R}^n$ endowed with an asymptotically euclidean metric, that is with a Riemannian metric satisfying conditions (5), (6) and (7). The tool employed is classical scattering theory in the wave operators approach. In this case, the problem is reduced to a problem of scattering for vector-valued operators.

An inherent restriction of the proof, however, that makes it difficult an extension to more general cases, is the use of the Fourier transform, which has a crucial role in our considerations, particularly in connection with Lemma 3. The lack of this tool in the more general case of manifolds with an asymptotically controlled Riemannian metric is a major obstacle to the extension of the theorem.

This paper summarizes and improves some results of my PhD Thesis, compiled under the direction of Prof. Edoardo Vesentini, to whom I am deeply grateful for his constant and careful support.

2. - Preliminaries

Let $(\boldsymbol{R}^n, e)$ be the euclidean n-dimensional space, and $(\boldsymbol{R}^n, g)$ be the same space, endowed with a complete Riemannian metric g.

We will denote by $\Lambda_c^1(\boldsymbol{R}^n)$ the vector space of all smooth, compactly supported 1-forms on $\boldsymbol{R}^n$, and by $L_1^2(\boldsymbol{R}^n, e)$ the completion of $\Lambda_c^1(\boldsymbol{R}^n)$ with respect to the norm

$$\|\omega\|^2_{L_1^2(\boldsymbol{R}^n,\, e)} = \int_{\boldsymbol{R}^n} \langle \omega, \omega \rangle_e \, dx \, , \tag{1}$$

where dx denotes the Euclidean volume element and

$$\langle \omega(x), \omega(x) \rangle_e = \sum_i \omega_i^2(x)$$

is the fiber norm for 1-forms induced by the Euclidean metric.

$L_1^2(\boldsymbol{R}^n, e)$ is the Hilbert space direct sum of n copies of $L^2(\boldsymbol{R}^n)$. The Laplace-Beltrami operator Δ_e on 1-forms $\omega \in \Lambda_c^1(\boldsymbol{R}^n)$ acts componentwise as:

$$(\Delta_e \omega)_k = -\sum_j \frac{\partial^2 \omega_k}{\partial x_j^2} \, .$$

It is well-known that Δ_e is essentially selfadjoint on $\Lambda_c^1(\boldsymbol{R}^n)$, and its closure H_0 has purely absolutely continuous spectrum equal to

$$\sigma(H_0) = [0, +\infty),$$

with constant multiplicity.

We will denote by $h_0[\omega]$ the quadratic form associated to Δ_e on $\Lambda^1_c(\boldsymbol{R}^n)$:

(2)
$$h_0[\omega] = \int_{\boldsymbol{R}^n} \langle \Delta_e \omega, \omega \rangle_e \, dx = \int_{\boldsymbol{R}^n} \sum_{i,j=1}^{n} \left(\frac{\partial \omega_i}{\partial x^j} \right)^2 dx .$$

$L^2_1(\boldsymbol{R}^n, g)$ will stand for the completion of $\Lambda^1_c(\boldsymbol{R}^n)$ with respect to the norm:

(3)
$$\|\omega\|^2_{L^2_1(\boldsymbol{R}^n, g)} = \int_{\boldsymbol{R}^n} \langle \omega, \omega \rangle_g \sqrt{g}\, dx ,$$

where $\sqrt{g}\, dx$, as usual, denotes the volume element induced by the Riemannian metric g and

$$\langle \omega(x), \omega(x) \rangle_g = g^{ij}(x)\, \omega_i(x)\, \omega_j(x)$$

is the fiber norm for 1-forms induced by g. (Here, as everywhere throughout the paper, the repeated indices convention is adopted.)

The action of the Laplace-Beltrami operator $\Delta_g = d\delta + \delta d$ on 1-forms $\omega \in \Lambda^1_c(\boldsymbol{R}^n)$ is given in local coordinates by the Weitzenböck formula:

$$(\Delta_g \omega)_k = -(g^{ij} \nabla_i \nabla_j \omega)_k + R^i_k \omega_i ,$$

where ∇_i is the covariant derivative with respect to the connection induced by the metric g, and R^i_k is the Ricci tensor.

Since the Riemannian metric g is complete, Δ_g is essentially selfadjoint on $\Lambda^1_c(\boldsymbol{R}^n)$. We will denote its closure by H_1.

Moreover, we will denote by $h_1[\omega]$ the quadratic form associated to Δ_g on $\Lambda^1_c(\boldsymbol{R}^n)$

(4)
$$h_1[\omega] = \int_{\boldsymbol{R}^n} \langle \Delta_g \omega, \omega \rangle_g \sqrt{g}\, dx = \int_{\boldsymbol{R}^n} |\nabla \omega|^2_g \sqrt{g}\, dx + \int_{\boldsymbol{R}^n} \langle R\omega, \omega \rangle_g \sqrt{g}\, dx ,$$

where

$$|\nabla \omega|^2_g = g^{ij} g^{\alpha\beta} \nabla_i \omega_\alpha \nabla_j \omega_\beta$$

and

$$\langle R\omega, \omega \rangle_g = g^{\alpha\beta} R^i_\alpha \omega_i \omega_\beta .$$

In the next section we will show how it is possible, under suitable hypothesis on the asymptotic behaviour of g, to get information about the spectrum of H_1 from the knowledge of the spectrum of H_0, proving the following

THEOREM 2.1: *Let $\boldsymbol{R}^n$ be endowed with a Riemannian metric g such that* $\left| \frac{\partial g^{il}}{\partial x_j}(x) \right|$ *is bounded and, for $|x| \gg 0$, there exists $C > 0$ such that*

1. *for every* i, j,

$$|g^{ij}(x) - \delta^{ij}| < \frac{C}{|x|^k} \tag{5}$$

for some $k > n$;

2. *for every* i, j, k, l

$$\left| \frac{\partial g_{il}}{\partial x_j} \right| < \frac{C}{|x|^k} \tag{6}$$

$$\left| \frac{\partial^2 g_{il}}{\partial x_j \partial x_k} \right| < \frac{C}{|x|^k} \tag{7}$$

for some $k > n$.

Then the Laplace-Beltrami operator Δ_g *acting on* 1*-forms has absolutely continuous spectrum equal to* $[0, +\infty)$:

$$[0, +\infty) = \sigma_{ac}(H_1) = \sigma(H_1).$$

In particular, it has no discrete spectrum. (There might be singularly continuous spectrum or embedded eigenvalues.)

The main tool for the proof is Belopol'ski-Birman theorem (see [5], [6]), which provides a sufficient condition so that two selfadjoint operators have the same absolutely continuous spectrum.

We recall it briefly:

Theorem 2.2: *Let* H_0, H_1 *be selfadjoint operators acting respectively on Hilbert spaces* $\mathcal{H}_0$, $\mathcal{H}_1$, *and let* $E_\Omega(H_0)$, $E_\Omega(H_1)$, *for* $\boldsymbol{\Omega} \subset \boldsymbol{R}$, *be the associated spectral measures.*

If $J \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ *satisfies the conditions:*

1. J *has a bounded two-sided inverse;*
2. *for every bounded interval* $I \subset \boldsymbol{R}$,

$$E_I(H_1)(H_1 J - J H_0) E_I(H_0) \in \mathfrak{I}_1(\mathcal{H}_0, \mathcal{H}_1), \tag{8}$$

where $\mathfrak{I}_1(\mathcal{H}_0, \mathcal{H}_1) = \{A \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1) \,|\, (A^*A)^{1/2} \in \mathfrak{I}_1(\mathcal{H}_0)\}$ *and* $\mathfrak{I}_1(\mathcal{H}_0)$ *denotes, as usual, the set of trace-class operators on* $\mathcal{H}_0$;

3. *for every bounded interval* $I \subseteq \boldsymbol{R}$, $(J^* J - I)\, E_I(H_0)$ *is compact;*
4. $JQ(H_0) = Q(H_1)$, *where* $Q(H_i)$ *is the form domain of the operator* H_i, *for* $i = 0, 1$,

then the wave operators $W^{\pm}(H_1, H_0; J)$ *exist, are complete, and are partial isometries with initial space* $P_{ac}(H_0)$ *and final space* $P_{ac}(H_1)$, *where* $P_{ac}(H_i)$ *denotes, as usual, the absolutely continuous space of* H_i, *for* $i = 0, 1$.

As a consequence, the absolutely continuous spectra of H_0 *and* H_1 *do coincide.*

Remark 2.3: We recall (see [4]) that if H is a densely defined, essentially selfadjoint, positive operator on a Hilbert space $\mathcal{H}$ and $\boldsymbol{h}$ is the associated quadratic form, the form domain $Q(\overline{H})$ of the selfadjoint operator $\overline{H}$ is the domain of the closure $\tilde{\boldsymbol{h}}$ of the form $\boldsymbol{h}$, that is to say: $Q(\overline{H})$ is the set of those $u \in \mathcal{H}$ such that there exists a sequence $\{u_n\} \subset D(H)$ converging to u in $\mathcal{H}$ such that

$$\boldsymbol{h}[u_n - u_m] \to 0$$

as $n, m \to +\infty$.

3. - Proof of Theorem 2.1

We will prove that, for a suitable $J : L_1^2(\boldsymbol{R}^n, e) \to L_1^2(\boldsymbol{R}^n, g)$, $H_1 = \overline{\Delta_g}$, $H_0 = \overline{\Delta_e}$ and J satisfy the conditions of Theorem 2.2, for $\mathcal{H}_0 = L^2(\boldsymbol{R}^n, e)$ and $\mathcal{H}_1 = L^2(\boldsymbol{R}^n, g)$.

We begin with the following

Lemma 3.1: *Let g be as in Theorem 2.1. Then there exist $C, C_1 > 0$, $D, D_1 > 0$ such that*

1. *for every $x \in \boldsymbol{R}^n$*

$$C \leq \sqrt{g(x)} \leq C_1; \tag{9}$$

2. *for every $x \in \boldsymbol{R}^n$, v in the cotangent space at x, $T_x^*(\boldsymbol{R}^n)$*

$$D \sum_i v_i^2 \leq g^{ij}(x)\, v_i v_j \leq D_1 \sum_i v_i^2 . \tag{10}$$

Proof: (9) follows immediately observing that, for every x, $\sqrt{g(x)}$ is strictly positive and $\sqrt{g(x)} \to 1$ as $|x| \to +\infty$.

As for (10), since the matrix $g^{ij}(x)$, which expresses the Riemannian metric g in contravariant form, is a continuous function of x and is positive, its eigenvalues $\lambda_1(x), \dots, \lambda_n(x)$ depend continuously on x and are strictly positive. Hence, the functions f and h defined by

$$f(x) := \inf_i \lambda_i(x)$$

and

$$h(x) := \sup_i \lambda_i(x),$$

are continuous and strictly positive. Moreover, since the metric g is asymptotically euclidean, $f(x) \to 1$ and $h(x) \to 1$ as $|x| \to +\infty$. As a consequence, there exist $D, D_1 > 0$

such that, for every $x \in \boldsymbol{R}^n$,

$$D \leqslant f(x) \leqslant h(x) \leqslant D_1,$$

which yields (10). ■

Lemma 3.1 implies that there is a natural identification between $L_1^2(\boldsymbol{R}^n, g)$ and $L_1^2(\boldsymbol{R}^n, e)$, and, moreover, (1) and (3) are equivalent norms. As a consequence, the identity map on $\Lambda_c^1(\boldsymbol{R}^n)$ extends to a bounded linear operator

$$J : L_1^2(\boldsymbol{R}^n, e) \to L_1^2(\boldsymbol{R}^n, g),$$

with bounded two-sided inverse, and condition 1 of Theorem 2.2 is satisfied.

In order to prove (8), we need two Lemmas:

LEMMA 3.2: *Let $A : \xi \mapsto A_\xi$ be a $n \times n$-matrix-valued function on $\boldsymbol{R}^n$, and let $\mathcal{A}$ be the linear operator*

$$\mathcal{A} : D(\mathcal{A}) \subset L^2(\boldsymbol{R}^n) \oplus \ldots \oplus L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n) \oplus \ldots \oplus L^2(\boldsymbol{R}^n)$$

of the form

$$f(x) A(-i\nabla_x),$$

where $f(x)$ is a function on $\boldsymbol{R}^n$ and $A(-i\nabla_x)$ is the operator

$$A(-i\nabla_x) = \mathcal{F} \circ \widehat{A}_\xi \circ \mathcal{F}^{-1},$$

$\mathcal{F}$ being the Fourier transform and $\widehat{A}_\xi$ the multiplication operator

$$v \mapsto \widehat{A}_\xi v$$

$$(\widehat{A}_\xi v)(\xi) = A_\xi v(\xi).$$

Let $L_\delta^2(\boldsymbol{R}^n)$ be the space of functions h such that

$$\|h\|_\delta^2 = \|(1 + |x|^2)^{\delta/2} h(x)\|_{L^2} < \infty.$$

If, for some $\delta > \frac{n}{2}$, $f(x) \in L_\delta^2(\boldsymbol{R}^n)$ and, for every pair of indices (α, β), $(A_\xi)_\alpha^\beta \in L_\delta^2(\boldsymbol{R}^n)$, then $\mathcal{A}$ is a trace-class operator.

PROOF OF LEMMA: 3.2: It suffices to show that, for every fixed (α, β), the operator $\mathcal{A}_\beta^\alpha$

$$\mathcal{A}_\beta^\alpha : D(\mathcal{A}_\beta^\alpha) \subset L^2(\boldsymbol{R}^n) \oplus \ldots \oplus L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n) \oplus \ldots \oplus L^2(\boldsymbol{R}^n)$$

$$\omega = (\omega_1, \ldots, \omega_n) \mapsto \Big(\underbrace{0, \ldots, 0, f(x)\, A_\alpha^\beta(-i\nabla_x)\, \omega_\beta}_{\alpha}, 0, \ldots 0\Big)$$

is trace-class. But this latter operator coincides with the composition

$$I_\alpha \circ (f(x)\, A_\alpha^\beta(-i\nabla_x)) \circ P_\beta,$$

where P_β is the projection

$$P_\beta\colon L^2(\boldsymbol{R}^n) \oplus \ldots \oplus L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n)$$

$$\omega = (\omega_1, \ldots, \omega_\beta, \ldots, \omega_n) \mapsto \omega_\beta,$$

I_α is the immersion

$$I_\alpha\colon L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n) \oplus \ldots \oplus L^2(\boldsymbol{R}^n)$$

$$\omega \mapsto (\underbrace{0, \ldots, 0, \omega}_{\alpha}, \ldots, 0),$$

and $A_\beta^\alpha(-i\nabla_x)$ is the operator

$$D(A_\beta^\alpha(-i\nabla_x)) \subset L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n)$$

$$A_\beta^\alpha(-i\nabla_x) = \mathcal{F} \circ (\widehat{A}_\xi)_\beta^\alpha \circ \mathcal{F}^{-1},$$

where $\mathcal{F}$ is the Fourier transform and $(\widehat{A}_\xi)_\beta^\alpha$ is the multiplication operator associated to the scalar function $(A_\xi)_\beta^\alpha$.

The conclusion follows from the fact that P_β and I_α are bounded operators and (see [5], Theorem XI.21) any operator

$$L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n)$$

of the form $f(x)\, h(-i\nabla_x)$ is trace-class if $f(x)$ and $h(\xi)$ belong to $L_\delta^2(\boldsymbol{R}^n)$. ■

Lemma 3.3: *If $f\colon \boldsymbol{R}^n \to \boldsymbol{R}$ is continuous and such that for some $k > n$*

$$|f(x)| < \frac{C}{|x|^k} \tag{11}$$

when $|x| \gg 0$, then $f \in L_\delta^2(\boldsymbol{R}^n)$ for some $\delta > \frac{n}{2}$.

Proof: Choosing $\varepsilon > 0$ such that $k > n + \varepsilon$, then

$$|f(x)| < \frac{C}{|x|^k}$$

for $|x| \gg 0$; as a consequence, a straighforward computation in polar coordinates shows that, for $\delta = \frac{n}{2} + \varepsilon$,

$$\int_{\boldsymbol{R}^n} |f(x)|^2 (1 + |x|^2)^\delta\, dx < +\infty\,. \qquad ■$$

Now, to prove (8), it suffices to see that for every bounded interval $I \subset \boldsymbol{R}$,

$$(H_1 - H_0)\, E_I(H_0) \in \mathfrak{I}_1(L^2(\boldsymbol{R}^n) \oplus \ldots \oplus L^2(\boldsymbol{R}^n)).$$

Let Γ^{α}_{ik} be the Christoffel symbols of the Riemannian connection induced by g; then the difference $H_1 - H_0$ is given by

$$\text{(12)} \qquad ((H_1 - H_0)\,\omega)_k = (-g^{ij} + \delta^{ij})\,\frac{\delta^2 \omega_k}{\delta x^i \delta x^j} + g^{ij}\Gamma^{\alpha}_{jk}\frac{\delta\omega_\alpha}{\delta x^i} + g^{ij}\Gamma^{\alpha}_{ij}\frac{\delta\omega_k}{\delta x^\alpha} +$$

$$+ g^{ij}\Gamma^{\alpha}_{ik}\frac{\delta\omega_\alpha}{\delta x^j} + g^{ij}\frac{\delta\Gamma^{\alpha}_{jk}}{\delta x^i}\omega_\alpha - g^{ij}\Gamma^{\alpha}_{ij}\Gamma^{\beta}_{\alpha k}\omega_\beta - g^{ij}\Gamma^{\alpha}_{ik}\Gamma^{\beta}_{j\alpha}\omega_\beta + R^i_k\omega_i.$$

A direct computation shows that conditions (5), (6), (7), and hypothesis 3 in Theorem 2.1 imply that $|g^{ij}\Gamma^{\alpha}_{jk}|$, $|g^{ij}\Gamma^{\alpha}_{ij}|$, $|g^{ij}\Gamma^{\alpha}_{ik}|$, $\left| g^{ij}\frac{\partial\Gamma^{\alpha}_{jk}}{\partial x_i}\right|$, $|g^{ij}\Gamma^{\alpha}_{ij}\Gamma^{\beta}_{\alpha k}|$, $|g^{ij}\Gamma^{\alpha}_{ik}\Gamma^{\beta}_{\alpha j}|$, $|R^i_k|$ are all bounded from above by $\frac{C}{|x|^k}$ for some constant $C > 0$ and some $k > n$.

$(H_1 - H_0)(E_I(H_0))$ is a sum of operators of type $f(x)\,A(-i\nabla_x)$, with $f(x) \in L^2_\delta(\boldsymbol{R}^n)$ in view of Lemma 3.3, and A_ξ smooth and compactly supported.

Thus, thanks to Lemma 3.2, $(H_1 - H_0)(E_I(H_0))$ is trace-class and condition 2 is fulfilled.

As for condition 3, first of all we observe that the adjoint of J

$$J^*\colon L^2_1(\boldsymbol{R}^n, g) \to L^2_1(\boldsymbol{R}^n, e)$$

satisfies the equation

$$\int_{\boldsymbol{R}^n} g^{ij}\omega_i\phi_j\sqrt{g}\,dx = \int_{\boldsymbol{R}^n}\delta^{ij}\omega_i(J^*\phi)_j\,dx,$$

and therefore

$$(J^*\phi)_k = \delta_{ik}\, g^{ij}\phi_j\sqrt{g}.$$

As a consequence, in local coordinates

$$((J^*J - I)\,\phi)_k = (\sqrt{g}g^{jk} - \delta^{jk})\,\phi_j;$$

now,

$$|\sqrt{g}g^{jk} - \delta^{jk}| \leqslant |\sqrt{g}|\,|g^{jk} - \delta^{jk}| + |(\sqrt{g} - 1)|\,\delta^{ij}.$$

By (5), there exists $C > 0$ such that

$$|g^{jk} - \delta^{jk}| \leqslant \frac{C}{|x|^k}$$

for some $k > n$ for $|x| \gg 0$; moreover,

$$|(\sqrt{g} - 1)| = \frac{1}{2}\,|1 - g| + o\left(\frac{1}{|x|^k}\right) \leqslant \frac{K}{|x|^k}$$

for some $K > 0$ and some $k > n$ as $|x| \to +\infty$.

Thus, $\sqrt{g}g^{jk} - \delta^{jk}$ belongs to $L^2_\delta(\boldsymbol{R}^n)$. Hence $(J^*J - I)\,E_I(H_0)$ is an operator of type $f(x)\,A(-i\nabla_x)$, with $f(x)$ in $L^2_\delta(\boldsymbol{R}^2)$ and A_ξ smooth and compactly supported; by Lemma 3.2, it is trace-class, and therefore it is compact.

As for condition 4, thanks to Remark 2.3, $Q(H_0)$ and $Q(H_1)$ can be characterized as follows:

LEMMA 3.4: *$Q(H_0)$ is the set of those $\omega \in L^2_1(\boldsymbol{R}^n, e)$ for which there exists a sequence $\{\omega^{(n)}\} \subset \Lambda^1_c(\boldsymbol{R}^n)$ such that*

$$\omega^{(n)} \to \omega \quad \textit{in } L^2_1(\boldsymbol{R}^n, e) \tag{13}$$

and

$$\boldsymbol{h}_0[\omega^{(n)} - \omega^{(m)}] \to 0 \tag{14}$$

as $n, m \to +\infty$.

Analogously, $Q(H_1)$ is the set of those $\omega \in L^2_1(\boldsymbol{R}^n, g)$ such that there exists $\{\psi^{(n)}\} \subset \Lambda^1_c(\boldsymbol{R}^n)$ for which

$$\psi^{(n)} \to \omega \quad \textit{in } L^2(\boldsymbol{R}^n, g) \tag{15}$$

and

$$\boldsymbol{h}_1[\psi^{(n)} - \psi^{(m)}] \to 0 \tag{16}$$

as $n, m \to +\infty$.

We prove now that

$$Q(H_0) \subseteq Q(H_1)\,. \tag{17}$$

For $\omega \in Q(H_0)$, there exists a sequence $\{\omega^{(n)}\} \subset \Lambda^1_c(\boldsymbol{R}^n)$ satisfying (13) and (14). Due to the equivalence of the norms (1) and (3),

$$\omega^{(n)} \to \omega \text{ in } L^2_1(\boldsymbol{R}^n, g)\,;$$

hence, in order to see that $\omega \in Q(H_1)$ it suffices to prove that

$$\boldsymbol{h}_1[\omega^{(n)} - \omega^{(m)}] \to 0$$

as $m, n \to +\infty$.

To establish this fact, we consider first the curvature part of $\boldsymbol{b}_1[\omega^{(n)} - \omega^{(m)}]$,

(18)
$$\int_{\boldsymbol{R}^n} \langle R(\omega^{(n)} - \omega^{(m)}), (\omega^{(n)} - \omega^{(m)})\rangle_g \sqrt{g}\, dx \,.$$

The following Lemma holds:

LEMMA 3.5: *There exists $C > 0$ such that*

(19)
$$\left| \int_{\boldsymbol{R}^n} \langle R\omega, \omega\rangle_g \sqrt{g}\, dx \right| \leq C \|\omega\|^2_{L_1^2(\boldsymbol{R}^n,\, e)}$$

for every $\omega \in L_1^2(\boldsymbol{R}^n, e)$.

PROOF: Consider for every $x \in \boldsymbol{R}^n$ the quadratic form on $T_x^*(\boldsymbol{R}^n)$

$$\omega \mapsto g^{\alpha\beta}(x)\, R_\alpha^i(x)\, \omega_i \omega_\beta = R^{i\beta}(x)\, \omega_i \omega_\beta .$$

Since the matrix $R^{i\beta}(x)$ depends continuously on x, its eigenvalues $\lambda_1(x), \dots, \lambda_n(x)$ are continuous functions of x. Hence the function

$$f(x) := \sup_i \lambda_i(x) \,,$$

is continuous. Moreover, since the metric g is asymptotically euclidean, $f(x) \to 0$ as $|x| \to +\infty$. As a consequence, there exists $C > 0$ such that $|f(x)| \leq C$ for every $x \in \boldsymbol{R}^n$. This in turn implies

$$|R^{i\beta}(x)\, \omega_i \omega_\beta| \leq C \|\omega\|^2_{T_x^*(\boldsymbol{R}^n)}$$

for every $x \in \boldsymbol{R}^n$ and for every $\omega \in T_x^*(\boldsymbol{R}^n)$, which yields (19). ■

Since $\{\omega^{(n)}\}$ is a Cauchy sequence, the preceding Lemma implies that

(20)
$$\int_{\boldsymbol{R}^n} \langle R(\omega^{(n)} - \omega^{(m)}), (\omega^{(n)} - \omega^{(m)})\rangle_g \sqrt{g}\, dx \mapsto 0$$

as $n, m \to +\infty$.

As for the gradient part of $\boldsymbol{b}_1[\omega^{(n)} - \omega^{(m)}]$,

(21)
$$\int_{\boldsymbol{R}^n} |\nabla(\omega^{(n)} - \omega^{(m)})|_g^2 \sqrt{g}\, dx \,,$$

we begin by proving

LEMMA 3.6: *There exist $C, D > 0$ such that*

(22)
$$C \int_{\boldsymbol{R}^n} |\eta|_e^2\, dx \leq \int_{\boldsymbol{R}^n} |\eta|_g^2 \sqrt{g} dx \leq D \int_{\boldsymbol{R}^n} |\eta|_e^2\, dx$$

for every smooth, compactly supported tensor $\eta = \eta_{ij}$ *of rank* 2, *where*

$$|\eta|_e^2 = \sum_{i,j=1}^{n} \eta_{ij}^2$$

and

$$|\eta|_g^2 = g^{ij}(x)\, g^{kl}(x)\, \eta_{ik}\eta_{jl}.$$

Proof: Consider, for every $x \in \boldsymbol{R}^n$, the quadratic form

$$\boldsymbol{R}^{n^2} \times \boldsymbol{R}^{n^2} \to \boldsymbol{R}$$

defined by

$$\eta = \eta_{ij} \mapsto a(x)[\eta] = g^{ij}(x)\, g^{kl}(x)\, \eta_{ik}\eta_{jl}.$$

Since this quadratic form is positive and depends continuously on x, its eigenvalues $\lambda_k(x)$, for $k = 1, \ldots, n^2$, are continuous, positive functions of x. Moreover,

$$a(x)[\eta] \to \sum_{i,j=1}^{n} \eta_{ij}^2$$

as $|x| \to +\infty$, implying that $\lambda_k(x) \to 1$, for every $k = 1, \ldots, n^2$, as $|x| \to +\infty$. As a consequence, there exist $C, D > 0$ such that

$$C|\eta|_e^2 \leqslant a(x)[\eta] \leqslant D|\eta|_e^2$$

for every $x \in \boldsymbol{R}^n$, $\eta \in \boldsymbol{R}^{n^2}$, which implies (22). ■

Setting

$$\eta_{ik} = \nabla_i(\omega_k^{(n)} - \omega_k^{(m)}),$$

(22) yields

$$\int_{\boldsymbol{R}^n} |\nabla(\omega^{(n)} - \omega^{(m)})|_e^2\, dx \leqslant \frac{1}{C} \int_{\boldsymbol{R}^n} |\nabla(\omega^{(n)} - \omega^{(m)})|_g^2 \sqrt{g}\, dx \tag{23}$$

and

$$\int_{\boldsymbol{R}^n} |\nabla(\omega^{(n)} - \omega^{(m)})|_g^2 \sqrt{g}\, dx \leqslant D \int_{\boldsymbol{R}^n} |\nabla(\omega^{(n)} - \omega^{(m)})|_e^2\, dx\,. \tag{24}$$

Now,

$$\nabla_i \omega_k = \frac{\partial \omega_k}{\partial x_i} - \Gamma_{ik}^{\alpha} \omega_\alpha,$$

whence an easy computation shows that for every $i, k = 1, \ldots, n$

$$\|\nabla_i(\omega_k^{(n)} - \omega_k^{(m)})\|_{L^2(\boldsymbol{R}^n, e)} \leq \left\| \frac{\partial(\omega_k^{(n)} - \omega_k^{(m)})}{\partial x_i} \right\|_{L^2(\boldsymbol{R}^n, e)} + K\|\omega^{(n)} - \omega^{(m)}\|_{L^2_1(\boldsymbol{R}^n, e)}.$$

As a consequence,

$$\boldsymbol{h}_1[\omega^{(n)} - \omega^{(m)}] \to 0$$

as $n, m \to 0$. Thus, $Q(H_0) \subseteq Q(H_1)$.

We complete the proof of Theorem 2.1 showing that $Q(H_1) \subseteq Q(H_0)$.

For any $\omega \in Q(H_1)$ there exists a sequence $\{\psi^{(n)}\} \subset \Lambda_c^1(\boldsymbol{R}^n)$ such that (15) and (16) hold.

Thanks to the equivalence of the norms (1) and (3),

$$\psi^{(n)} \to \omega \quad \text{in } L_1^2(\boldsymbol{R}^n, e).$$

Thus, in order to see that $\omega \in Q(H_0)$ it suffices to prove that

$$\boldsymbol{h}_0[\psi^{(n)} - \psi^{(m)}] \to 0$$

as $m, n \to +\infty$.

Now, (15) and (16), together with (18), imply that

$$\int_{\boldsymbol{R}^n} |\nabla(\psi^{(n)} - \psi^{(m)})|_g^2 \sqrt{g}\, dx \to 0$$

as $n, m \to +\infty$.

For every $i, k = 1, \ldots, n$

$$\left\| \frac{\partial(\psi_k^{(n)} - \psi_k^{(m)})}{\partial x_i} \right\|_{L^2(\boldsymbol{R}^n, e)} \leq \|\nabla_i(\psi_k^{(n)} - \psi_k^{(m)})\|_{L^2(\boldsymbol{R}^n, e)} + \|\Gamma_{ik}^\alpha(\psi_\alpha^{(n)} - \psi_\alpha^{(m)})\|_{L^2(\boldsymbol{R}^n, e)} \leq$$

$$\leq \|\nabla_i(\psi_k^{(n)} - \psi_k^{(m)})\|_{L^2(\boldsymbol{R}^n, e)} + C\|\psi^{(n)} - \psi^{(m)}\|_{L_1^2(\boldsymbol{R}^n, e)},$$

Then, in view of (23),

$$\boldsymbol{h}_0[\psi^{(n)} - \psi^{(m)}] \to 0$$

as $n, m \to +\infty$, and $Q(H_1) \subseteq Q(H_0)$.

Therefore,

$$J(Q(H_0)) = Q(H_1).$$

REMARK 3.7: Theorem 2.1 holds, more in general, for p-forms, with $p = 1, \ldots n$, with arguments following the same patterns of the ones developed for $p = 1$. Indeed, estimates like (9), (10), (22) hold for p-forms, showing that the identification $J = I : L_p^2(\boldsymbol{R}^n, g) \to L_p^2(\boldsymbol{R}^n, e)$ is continuous with two-sided bounded inverse. To establish the validity of conditions 2, 3 of Belopol'ski-Birman theorem requires replacing Δ_g with the Laplace-Beltrami operator on p-forms, given by

$$(\Delta_{g(p)}\,\omega)_{i_1 \dots i_p} = -\sum_{\alpha,\,\beta} g^{\alpha\beta}\,\nabla_\alpha \nabla_\beta\,\omega_{i_1 \dots i_p} + \sum_{j,\,\alpha} R^{\alpha}_{i_j}\,\omega_{i_1 \dots \alpha,\,\hat{i}_j \dots i_p} +$$

$$-\sum_{j,\,l \neq j,\,\alpha,\,\beta} R^{\alpha\ \ \beta}_{\ \ i_j\ \ i_l}\,\omega_{\alpha i_1 \dots \beta \hat{i}_l \dots \hat{i}_j \dots i_p},$$

where $R^{i\ j}_{\ k\ l}$ is the Riemann curvature tensor, which satisfies the condition $|R^{i\ j}_{\ k\ l}| < \dfrac{C}{|x|^k}$ for $|x| \gg 0$.

The quadratic forms $\boldsymbol{h}_0$ and $\boldsymbol{h}_1$ have now to be replaced by

$$\boldsymbol{h}_{0(p)} = \int_{\boldsymbol{R}^n} \sum_{i_1,\,\dots,\,i_p,\,j=1}^{n} \left(\frac{\partial \omega_{i_1 \dots i_p}}{\partial x_j} \right)^2 dx$$

and by $\boldsymbol{h}_{1(p)}$ expressed by

$$\boldsymbol{h}_{1(p)}[\omega] = \int_{\boldsymbol{R}^n} |\nabla\omega|_g^2 \sqrt{g}\,dx + \int_{\boldsymbol{R}^n} \langle \tilde{R}\omega,\,\omega\rangle_g \sqrt{g}\,dx\,,$$

where

$$|\nabla\omega|_g^2 = g^{\alpha\beta}\,g^{i_1 j_1} \dots g^{i_p j_p}\,\nabla_\alpha\,\omega_{i_1 \dots i_p}\,\nabla_\beta\,\omega_{j_1 \dots j_p}$$

and

$$\langle \tilde{R}\omega,\,\omega\rangle_g = g^{i_1 j_1} \dots g^{i_p j_p} R^{\alpha}_{i_j}\,\omega_{i_1 \dots \alpha \dots i_p}\,\omega_{j_1 \dots j_p} + g^{i_1 j_1} \dots g^{i_p j_p} R^{\alpha\ \ \beta}_{\ \ i_j\ \ i_l}\,\omega_{\alpha i_1 \dots \beta \dots i_p}\,\omega_{j_1 \dots j_p}.$$

Then from the fact that a condition similar to (19) holds for $\langle \tilde{R}\omega,\,\omega\rangle_g$, the proof follows.

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Direttore responsabile: Prof. A. BALLIO - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
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