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Regularity of the Distance Function to the Boundary

Abstract. — Let Ω be a domain in a smooth complete Finsler manifold, and let G be the largest open subset of Ω such that for every x in G there is a unique closest point from $\partial\Omega$ to x (measured in the Finsler metric). We prove that the distance function from $\partial\Omega$ is in $C^{k,a}_{\mathrm{loc}}(G \cup \partial\Omega)$, $k \geq 2$ and $0 < a \leq 1$, if $\partial\Omega$ is in $C^{k,a}$.

1. - Introduction

In [1] we studied the singular set of viscosity solutions of some Hamilton-Jacobi equations. This was reduced to the study of the singular set of the distance function to the boundary of a domain Ω — for a Finsler metric. The singular set was defined as the complement of the following open set

$$G := \begin{array}{l}\text{the largest open subset of } \Omega \text{ such that for every } x \text{ in } G \text{ there is a} \\ \text{unique closest point from } \partial\Omega \text{ to } x \ \ (\text{ measured in the Finsler metric }).\end{array} \tag{1}$$

In [1] we stated that if $\partial\Omega$ is in $C^{k,a}$, $k \geq 2$ and $0 < a \leq 1$, then the distance function from the boundary belongs to $C^{k-1,a}(G \cup \partial\Omega)$. Recently Joel Spruck asked to see the proof for a Riemannian metric and pointed out that the result would imply that the distance function would be in $C^{k,a}(G \cup \partial\Omega)$. In this paper we provide a proof of that in the Finsler case. This paper can be regarded as an addendum to [1]. In the Riemannian case, the regularity of the distance function near $\partial\Omega$ can be found in [GT] and [F].

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We present two proofs of the $C^{k,a}$ result. We use the notation, as in [1],

$$\int_0^T \varphi(\xi(t); \dot{\xi}(t))dt$$

for the length of a curve $\xi(t)$. $\varphi(\xi; v)$ is homogeneous of degree one in v. For fixed ξ, the level surface $\varphi(\xi; v) = 1$ is smooth, closed, strictly convex, with positive principal curvatures.

The first proof uses very little of [1] and is essentially self-contained. The second proof uses some structure from [1], and may be of some interest to some readers.

We actually prove a more general result here, involving conjugate points from the boundary.

Definition: Conjugate Point: Consider a point y on $\partial\Omega$, and consider the geodesic $\xi(y, s)$ from y going inside Ω "normal" to $\partial\Omega$ (explained below) with s as arclength. The conjugate point to y is the first point $\bar{x}$ on the normal geodesic such that any point x'' on the geodesic beyond $\bar{x}$ has, in any neighborhood of the geodesic, a shorter join from $\partial\Omega$ to x'' than our normal geodesic to x''.

Normal: A geodesic Γ from a point $y \in \partial\Omega$ is "normal" to $\partial\Omega$ if for x on Γ close to y, the geodesic is the shortest join from $\partial\Omega$ to x.

To obtain the regularity in G we prove a slightly more general result which is local on $\partial\Omega$. Namely, suppose C is a neighborhood on $\partial\Omega$ of a point y and that the normal geodesic Γ from y to a point X in Ω is the unique shortest join from C to X. If the conjugate point to y is beyond X then there is a neighborhood A of X such that the distance from C to any point in A belongs to $C^{k,a}$. See Theorem 1 below.

We shall make use of special coordinates introduced in section 3 of [1] about a given normal geodesic Γ, going from a point $y \in \partial\Omega$ into Ω. In these coordinates y is the origin and the x_n−axis is normal to $\partial\Omega$ there and is the geodesic Γ. Furthermore, in these coordinates, φ has the following properties, see (4.1)-(4.6) in [1]. Here Greek letters α, β range from 1 to $n-1$, and Latin letters i, j range from 1 to n.

$$\varphi(te_n; e_n) \equiv 1, \tag{2}$$

$$\varphi_{\xi^j}(te_n; e_n) \equiv 0, \tag{3}$$

$$\varphi_{v^\alpha}(te_n; e_n) \equiv 0, \qquad \varphi_{v^n}(te_n; e_n) \equiv 1, \tag{4}$$

$$\varphi_{\xi^j v^k}(te_n; e_n) \equiv 0, \tag{5}$$

$$\varphi_{v^j v^n}(te_n; e_n) \equiv 0, \tag{6}$$

$$\varphi_{\xi^j \xi^n}(te_n; e_n) \equiv 0. \tag{7}$$

In these coordinates for $y \in \partial\Omega$ near the origin the geodesic $\xi(y, s)$ from y "normal" to

$\partial\Omega$ there satisfies

$$\dot{\xi}(y, 0) = V(y)$$

where $V(y)$ is the unique vector-valued function on $\partial\Omega$ satisfying (here $\upsilon(y)$ is the Euclidean interior unit normal to $\partial\Omega$ at y)

$$\begin{cases} V(y)\cdot \upsilon(y) > 0 \\ \varphi(y; V(y)) = 1 \\ \nabla_v \varphi(y; V(y)) \text{ is parallel to } \upsilon(y). \end{cases} \tag{8}$$

Using these special coordinates, near the origin, $\partial\Omega$ has the form

$$y = (x', f(x')), \quad f(0') = 0, \quad \nabla f(0') = 0'. \tag{9}$$

We assume that $f \in C^{k,a}$, $k \geq 2$, $0 < a \leq 1$.

The result we prove is

THEOREM 1: *Assume that the conjugate point of the origin on the geodesic $\Gamma = \{te_n\}$ is beyond e_n, and that there exists a neighborhood C of $0'$ on $\partial\Omega$ such that $\{te_n \mid 0 \leq t \leq 1\}$ is the unique shortest geodesic from C to e_n. Then there exist neighborhoods A of e_n and $\mathcal{A}$ of $0'$ on $\partial\Omega$ such that for any X in A there is a unique $y \in \mathcal{A}$ and geodesic from y to X which is the shortest join from $\mathcal{A}$ to A. Furthermore, if $d(X)$ is its length, then the Jacobian of the map $X \to (d, y)$ is nonsingular at e_n, and d lies in $C^{k,a}_{\mathrm{loc}}$ in A, and y lies in $C^{k-1,a}_{\mathrm{loc}}$ in A.*

2. - SECOND VARIATION

Consider one parameter family of curves $\tau(\varepsilon, t)$ from $\mathcal{A}$ to $\bar{t}e_n$, $\bar{t} > 0$, with $\tau(0, t) = te_n$. We look at the second variation of its length $I[\tau(\varepsilon, \cdot)]$. For $\bar{t}$ small, it is clearly positive definite. The first $\hat{t}$ for which it fails to be strictly positive definite is the conjugate point. For if $\tilde{t} = \hat{t} + \delta$, $\delta > 0$, then the second variation of curves to $\tilde{t}$ cannot be semipositive definite, and there would then be a shorter connection from $\mathcal{A}$ to $\tilde{t}e_n$ near Γ.

The standard computation of second variation yields

$$\frac{d^2}{d\varepsilon^2} I[\tau(\varepsilon, \cdot)]\bigg|_{\varepsilon=0} = J(\tau_\varepsilon|_{\varepsilon=0}) - f_{x_\alpha x_\beta}(0')\tau^\alpha_\varepsilon(0,0)\tau^\beta_\varepsilon(0,0).$$

Here J is the usual expression of the second variation if the bottom point were kept at the origin. Namely,

$$J(\tau_\varepsilon|_{\varepsilon=0}) = \int_0^{\bar{t}} \{\varphi_{\xi^\alpha \xi^\beta}(te_n; e_n)\tau^\alpha_\varepsilon(0,t)\tau^\beta_\varepsilon(0,t) + \varphi_{v^\alpha v^\beta}(te_n; e_n)\dot{\tau}^\alpha_\varepsilon(0,t)\dot{\tau}^\beta_\varepsilon(0,t)\}\, dt. \tag{10}$$

Note that τ^n_ε and $\dot{\tau}^n_\varepsilon$ do not occur in J.

3. - First proof of Theorem 1

3.1. Recalling (9) we shall denote the normal geodesic from $y = (x', f(x'))$ by $X = \xi(x', s)$; this is a slight change of notation. The geodesic ξ and ξ_s depend smoothly on s and their initial data, while the initial data depend $C^{k-1,a}$ on x'. To prove the theorem, it suffices to show that the Jacobian of the mapping (x', s) to X at $(0', 1)$ is nonsingular. It follows that d and y belong to $C^{k-1,a}$. Since $\nabla_X d = X_s$, it follows that $\nabla_X d$ is in $C^{k-1,a}$ and hence d is in $C^{k,a}$ — as Spruck pointed out to us.

We now prove the Jacobian is nonsingular.

Write $X = (X', X^n)$. Since $X_s(0', 1) = (0', 1)$, the Jacobian of the mapping (x', s) to X at $(0', 1)$ is simply

$$M := \frac{\partial X'}{\partial x'}(0', 1).$$

Assume M is singular, without loss of generality we may suppose that

$$X'_{x_1}(0', 1) = 0'. \tag{11}$$

We construct a perturbation $\tau(\varepsilon, t)$ of $\Gamma = \{te_n \mid 0 \le t \le 1\}$ such that $\zeta(t) := \tau_\varepsilon\big|_{\varepsilon=0}$ satisfies

$$J[\zeta] = f_{x_\alpha x_\beta}(0')\, \zeta^\alpha(0)\, \zeta^\beta(0). \tag{12}$$

3.2. Consider the geodesic $\xi(\delta e_1, t)$ of length 1 starting at $(\delta e_1, f(\delta e_1))$, $0 < \delta$ small and "normal" to $\partial\Omega$ there. Set

$$\zeta(t) = \frac{\partial}{\partial \delta}\xi(\delta e_1, t)\bigg|_{\delta=0}. \tag{13}$$

By (11),

$$\zeta(1) = 0. \tag{14}$$

We obtain an equation for $\zeta(t)$ by differentiating the geodesic equation

$$\varphi_{\xi^i} = \frac{d}{dt}\varphi_{v^i}(\xi; \dot{\xi})$$

with respect to δ, and setting $\delta = 0$. We find

$$\varphi_{\xi^i \xi^j}(te_n; e_n)\zeta^j = \frac{d}{dt}(\varphi_{v^i v^j}(te_n; e_n)\, \dot{\zeta}^j).$$

Here we have used property (5) of our special coordinates. By (7) and (6),

$$\varphi_{\xi^\alpha \xi^\beta}\zeta^\beta = \frac{d}{dt}\left(\varphi_{v^\alpha v^\beta}\dot{\zeta}^\beta\right). \tag{15}$$

We have

$$\zeta^\alpha(0) = \delta^\alpha_1. \tag{16}$$

In addition,

$$\dot{\zeta}(0) = \frac{\partial}{\partial\delta}\dot{\xi}(\delta e_1, 0)\bigg|_{\delta=0} = \frac{\partial}{\partial\delta}V(\delta e_1)\bigg|_{\delta=0} = V_{x_1}(0'). \tag{17}$$

By the last formula in (8) we have

$$\nabla_v\varphi((\delta e_1, f(\delta e_1)); V(\delta e_1)) \cdot (e_1 + f_{x_1}(\delta e_1)e_n) = 0.$$

Differentiating in δ and setting $\delta = 0$, we find, using properties of our special coordinates,

$$\varphi_{v^1 v^\beta}(0'; e_n)V^\beta_{x_1}(0') + f_{x_1x_1}(0') = 0. \tag{18}$$

Now we introduce the perturbation $\tau(\varepsilon, t)$ as follows

$$\begin{aligned} \tau^a(\varepsilon, t) &= \varepsilon\zeta^a(t), \\ \tau^n(\varepsilon, t) &= te_n + (1-t)f(\varepsilon e_1). \end{aligned}$$

The definition of τ^n is just to ensure that $\tau(\varepsilon, 0)$ lies on $\partial\Omega$.

According to (10),

$$J[\tau_\varepsilon\big|_{\varepsilon=0}] = \int_0^1 \{\varphi_{\xi^a\xi^\beta}(te_n; e_n)\,\zeta^a\zeta^\beta + \varphi_{v^a v^\beta}(te_n; e_n)\,\dot{\zeta}^a\dot{\zeta}^\beta\}dt.$$

Integrating the last expression by parts we find, using (15) , (16), (17) and (18),

$$\begin{aligned} J[\tau_\varepsilon\big|_{\varepsilon=0}] &= \int_0^1 \left\{\varphi_{\xi^a\xi^\beta}\zeta^a\zeta^\beta - \zeta^a\frac{d}{dt}(\varphi_{v^a v^\beta}\dot{\zeta}^\beta)\right\} - \zeta^a(0)\varphi_{v^a v^\beta}(0'; e_n)\dot{\zeta}^\beta(0) \\ &= -\,\varphi_{v^1 v^\beta}(0'; e_n)V^\beta_{x_1}(0') = f_{x_1x_1}(0') = f_{x_a x_\beta}(0')\tau^a_\varepsilon(0,0)\tau^\beta_\varepsilon(0,0). \end{aligned}$$

It follows from Section 2 that the second variation is zero. $\square$

4. - Second proof of Theorem 1

Second proof of Theorem 1: For X near e_n and for small $\sigma' = (\sigma_1, \cdots, \sigma_{n-1}) \in \mathbb{R}^{n-1}$, let $\tau = \tau(\sigma', X)$ be defined by $\varphi(X; (\sigma', \tau)) = 1$ and $\tau(0', e_n) = 1$. Since $\varphi_{v^n}(e_n; e_n) = 1$, by the implicit function theorem, τ exists as a smooth function of (σ', X) near $(0', e_n)$.

Let, as on page 111 of [1], $\eta = \eta(\sigma', X, t)$ be the unique smooth function of, with $\psi = \varphi^2$,

$$\psi_{\xi^i}(\eta; \dot{\eta}) = \frac{d}{dt}\psi_{v^i}(\eta; \dot{\eta}), \qquad t \le 1,$$

satisfying

$$\eta(\sigma', X, 1) = X, \qquad \dot{\eta}(\sigma', X, 1) = (\sigma', \tau(\sigma', X)).$$

As explained in the last two lines of page 108 in [1], $\eta(\sigma', X, t)$ is a geodesic with t the arclength.

In the special coordinates described in Section 1, $\partial\Omega$ has the form (9) near the origin with $f \in C^{k,a}$, $k \geq 2$, $0 < a \leq 1$. Since $\{te_n \mid 0 \leq t \leq 1\}$ is the unique shortest geodesic from C to e_n, we know that for X close to e_n, there exists x' close to $0'$ such that the "normal geodesic" starting from $(x', f(x'))$ will reach X as a shortest join from C to X. It follows that for some σ' close to $0'$ and t close to 0, we have

$$\tag{19} \begin{cases} \eta(\sigma', X, t) - (x', f(x')) &= 0, \\ \dot{\eta}^{\mu}(\sigma', X, t) - V^{\mu}(x') &= 0, \end{cases}$$

where $V(x') := V(x', f(x'))$ satisfies (8). Note that $1 - t$ is the distance from C to X.

To prove Theorem 1, we only need to show that the left hand side of (19), denoted as LHS, has nonsingular Jacobian $\dfrac{\partial(LHS)}{\partial(t, \sigma', x')}$ at $(t, \sigma', x', X) = (0, 0', 0', e_n)$. Indeed, this would allow the use of the implicit function theorem to show that for X close to e_n and in a neighborhood of $(0, 0', 0')$, there exists a unique $C^{k-1,a}$ solution $(t, \sigma', x') = (t(X), \sigma'(X), x'(X))$ of (19). Thus, Theorem 1 follows as explained at the beginning of Section 3.

Clearly,

$$\frac{\partial(LHS)}{\partial t}(0, 0', 0', e_n) = \begin{pmatrix} \dot{\eta}(0', e_n, 0) \\ (\ddot{\eta}^{\mu}(0', e_n, 0)) \end{pmatrix} = \begin{pmatrix} 0' \\ 1 \\ 0' \end{pmatrix},$$

a $(2n-1) \times 1$ column vector,

$$\tag{20} \frac{\partial(LHS)}{\partial \sigma'}(0, 0', 0', e_n) = \begin{pmatrix} (\eta_{\sigma_a}(0', e_n, 0)) \\ (\dot{\eta}^{\mu}_{\sigma_a}(0', e_n, 0)) \end{pmatrix} = \begin{pmatrix} (\eta^{\mu}_{\sigma_a}(0', e_n, 0)) \\ (\eta^{n}_{\sigma_a}(0', e_n, 0)) \\ (\dot{\eta}^{\mu}_{\sigma_a}(0', e_n, 0)) \end{pmatrix},$$

a $(2n-1) \times (n-1)$ matrix,

$$\tag{21} \frac{\partial(LHS)}{\partial x'}(0, 0', 0', e_n) = \begin{pmatrix} -I \\ 0 \\ -\nabla V'(0') \end{pmatrix},$$

a $(2n-1) \times (n-1)$ matrix, where I is the $(n-1) \times (n-1)$ identity matrix and $\nabla V' := (V^{\mu}_{x_\beta})$. Thus

$$\tag{22} \det\left(\frac{\partial(LHS)}{\partial(t, \sigma', x')}(0, 0', 0', e_n)\right) = (-1)^{n-1} \det \begin{pmatrix} \dfrac{\partial \eta'}{\partial \sigma'}(0', e_n, 0) & -I \\ \dfrac{\partial \dot{\eta}'}{\partial \sigma'}(0', e_n, 0) & -\nabla V'(0') \end{pmatrix},$$

where $\dfrac{\partial \eta'}{\partial \sigma'} := (\eta^{\mu}_{\sigma_a})$ and $\dfrac{\partial \dot{\eta}'}{\partial \sigma'} := (\dot{\eta}^{\mu}_{\sigma_a})$.

By the last line in (8),

$$\nabla_v \varphi((x', f(x')); V(x'))[e_\beta + f_{x_\beta}(x')e_n] = 0,$$

i.e.

$$\varphi_{v^\beta}((x', f(x')); V(x')) + \varphi_{v^n}((x', f(x')); V(x'))f_{x_\beta}(x') = 0.$$

Differentiating in x_α and setting $x' = 0'$ we find, using properties of our special coordinates (4), (5) and (6),

$$D^2_{v'}\varphi(0'; e_n) \cdot \nabla V'(0') + D^2 f(0') = 0, \tag{23}$$

where $D^2_{v'}\varphi := (\varphi_{v^\beta v^\mu})$.

We now evaluate $\dfrac{\partial \dot{\eta}'}{\partial \sigma'}(0', e_n, 0)$ and $\dfrac{\partial \eta'}{\partial \sigma'}(0', e_n, 0)$. It is proved in section 4.4 of [1] that there exists a $C^{2,1}$ function $\tilde{f}$ near $0'$ satisfying

$$\tilde{f}(0') = 0, \qquad \nabla\tilde{f}(0') = 0',$$

$$(D^2\tilde{f}(0') - D^2 f(0')) > 0, \tag{24}$$

$$\eta(\sigma', e_n, 0) = (y', \tilde{f}(y')), \tag{25}$$

$$\dot{\eta}^\mu(\sigma', e_n, 0) = \tilde{V}^\mu(y'), \tag{26}$$

where $\tilde{V}(y') := \tilde{V}((y', \tilde{f}(y')))$ is determined by (8) with f replaced by $\tilde{f}$, and $y' = y'(\sigma')$ satisfies

$$\det\left(\frac{\partial y'}{\partial \sigma'}(0')\right) \neq 0. \tag{27}$$

Note that (27) is given by (4.9) in [1], while (24) follows from corollary 4.15 in [1] together with the fact that e_n is not a conjugate point.

Differentiating (26) in σ_α and setting $\sigma' = 0'$ we find

$$\dot{\eta}^\mu_{\sigma_\alpha}(0', e_n, 0) = \tilde{V}^\mu_{y_\beta}(0')\frac{\partial y_\beta}{\partial \sigma_\alpha}(0'),$$

i.e.

$$\frac{\partial \dot{\eta}'}{\partial \sigma'}(0', e_n, 0) = \nabla\tilde{V}'(0')\frac{\partial y'}{\partial \sigma'}(0'). \tag{28}$$

Differentiating (25) in σ_α and setting $\sigma' = 0'$ we find

$$\frac{\partial \eta'}{\partial \sigma'}(0', e_n, 0) = \frac{\partial y'}{\partial \sigma'}(0'). \tag{29}$$

Since

$$\begin{pmatrix} \dfrac{\partial y'}{\partial \sigma'}(0') & -I \\ \nabla\tilde{V}'(0')\dfrac{\partial y'}{\partial \sigma'}(0') & -\nabla V'(0') \end{pmatrix} = \begin{pmatrix} I & -I \\ \nabla\tilde{V}'(0') & -\nabla V'(0') \end{pmatrix}\begin{pmatrix} \dfrac{\partial y'}{\partial \sigma'}(0') & \\ & I \end{pmatrix},$$

we have, by putting (28) and (29) into (22),

$$(30)\quad \det\left(\frac{\partial(LHS)}{\partial(t,\sigma',x')}(0,0',0',e_n)\right) = (-1)^{n-1}\det\left(\frac{\partial y'}{\partial\sigma'}(0')\right)\det\begin{pmatrix} I & -I \\ \nabla\tilde{V}'(0') & -\nabla V'(0') \end{pmatrix}.$$

The proof of (23), applied to $\tilde{f}$ instead of f, yields

$$(31)\qquad D^2_{v'}\varphi(0';e_n)\nabla\tilde{V}'(0') + D^2\tilde{f}(0') = 0.$$

Thus, by (23) and (31),

$$\begin{pmatrix} I & -I \\ -D^2\tilde{f}(0') & D^2 f(0') \end{pmatrix} = \begin{pmatrix} I & -I \\ D^2_{v'}\varphi(0';e_n)\nabla\tilde{V}'(0') & -D^2_{v'}\varphi(0';e_n)\nabla V'(0') \end{pmatrix} =$$

$$= \begin{pmatrix} I & \\ & D^2_{v'}\varphi(0';e_n) \end{pmatrix}\begin{pmatrix} I & -I \\ \nabla\tilde{V}'(0') & -\nabla V'(0') \end{pmatrix},$$

and therefore

$$(32)\qquad \det\left(D^2 f(0') - D^2\tilde{f}(0')\right) = \det D^2_{v'}\varphi(0';e_n)\det\begin{pmatrix} I & -I \\ \nabla\tilde{V}'(0') & -\nabla V'(0') \end{pmatrix}.$$

Since $D^2_{v'}\varphi(0';e_n)$ is positive definite, we deduce from (30), (27) and (32) that

$$\det\left(\frac{\partial(LHS)}{\partial(t,\sigma',x')}(0,0',0',e_n)\right) \neq 0. \qquad \square$$

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