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# A Polynomial one-side Liouville Theorem for a Class of Real Second Order Hypoelliptic Operators (**)

ABSTRACT. — In this note we investigate the notion of *one-side polynomial Liouville property* for a class of linear second order hypoelliptic partial differential operators whit real coefficients. Our results apply in particular to some classes of Kolmogorov and degenerate Ornstein-Uhlenbeck operators, extending recent results in references [KL1], [KL2] and [PZ].

## 1. - INTRODUCTION

Let $\Delta$ be the Laplace operator in $\mathbb{R}^n$. The classical Liouville Theorem states that any *bounded* solution to

$$\Delta u = 0 \quad \text{in} \quad \mathbb{R}^N \tag{1.1}$$

is constant in $\mathbb{R}^N$. More generally, if $u$ solves (1.1) and $u(x) = O(|x|^m)$, as $x$ goes to infinity, for a suitable $m > 0$, then $u$ is a polynomial function. In the 1983 paper [G], Geller pointed out the following very short proof of this theorem.

Let $u$ be a tempered distribution in $\mathbb{R}^N$ satisfying (1.1) in the weak sense of distributions. Then, denoting by $\mathcal{F}$ the Fourier transform,

$$0 = \mathcal{F}(\Delta u) = -|\xi|^2 \mathcal{F}u,$$

so that $\mathcal{F}u$ is a tempered distribution supported at the origin. Hence $\mathcal{F}u$ is a linear combination of derivatives of the Dirac measure. Thus, $u$ is a polynomial function.

It is a trivial fact that the Laplace operator is invariant with respect to the Euclidean translations and homogeneous of degree two with respect to the isotropic dilations $x \to \lambda x, \ \lambda > 0$. Moreover, it is also well known that $\Delta$ is *hypoelliptic*.

These three properties of the Laplacian are deeply related to its polynomial Liouville property. Indeed, the following sharp generalization of the previous theorem holds true.

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**(GR)**. Let $L$ be a partial differential operator in $\mathbb{R}^N$ with smooth coefficients. Assume there exists a homogeneous Lie group $\mathbb{G} = (\mathbb{R}^N, \circ, d_\lambda)$ such that $L$ is left translations invariant on $\mathbb{G}$ and $d_\lambda$-homogeneous of degree $m \geq 1$. Then $L$ has the polynomial Liouville property if and only if $L$ is hypoelliptic.

Here, and in what follows, we agree to say that $L$ has the *polynomial Liouville property* if every tempered distributional solution to the equation $Lu = 0$ in $\mathbb{R}^N$ is a polynomial function.

The *if* part of Theorem **(GR)** was proved by Geller in [G]. The *only if* part was proved by Rothschild in [R] soon after Geller's result appeared.

Before proceeding we would like to clarify some of the notion we used before. A group of *dilation* in $\mathbb{R}^N$ is a family $(d_\lambda)_{\lambda>0}$ of linear transformations of the following type

$$d_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \;\; d_\lambda(x_1, \ldots, x_N) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N), \tag{1.2}$$

where $\sigma_1, \ldots, \sigma_N$ are suitable positive integers. Given a Lie group $(\mathbb{R}^N, \circ)$ and a family $(d_\lambda)_{\lambda>0}$, we say that $\mathbb{G} = (\mathbb{R}^N, \circ, d_\lambda)$ is a *homogeneous group* if $d_\lambda$ is a automorphism of $(\mathbb{R}^N, \circ)$.

We also recall that a linear partial differential operator is *hypoelliptic* if every distributional solution to the equation

$$Lu = f, \quad \text{in} \quad \Omega \subset \mathbb{R}^n$$

is of class $C^\infty$ whenever $f$ is of class $C^\infty$. The operator $L$ is left *translations invariant* on $\mathbb{G}$ and $d_\lambda$-homogeneous of degree $m$ if, respectively,

$$L(u(a \circ x)) = (Lu)(a \circ x), \quad \text{for every} \quad a,\, x \in \mathbb{R}^N, \tag{1.3}$$

$$L(u(d_\lambda x)) = \lambda^m (Lu)(d_\lambda x), \quad \text{for every} \quad \lambda > 0,\, x \in \mathbb{R}^N, \tag{1.4}$$

and for every function $u \in C^\infty(\mathbb{R}^N, \mathbb{R})$

In 1996 Luo Xuebo extended Geller's Theorem by proving the polynomial Liouville property for every hypoelliptic operator $L$ just satisfying (1.4) for a suitable group of dilations $(d_\lambda)_{\lambda>0}$, [L].

As Luo Xuebo stressed in his paper, the Egorov's operator

$$E = \partial^2_{x_1} + i x_1 \partial^2_{x_2} \quad \text{in} \quad \mathbb{R}^2$$

is hypoelliptic and $d_\lambda$-homogeneous of degree $m = 4$ with respect to the dilations

$$d_\lambda(x_1, x_2) = (\lambda^2 x_1, \lambda^3 x_2).$$

It follows that $E$ has the polynomial Liouville property since it satisfies the hypotheses of [L]. However, there is no Lie group in $\mathbb{R}^2$ leaving $E$ left translations invariant. Thus, Luo Xuebo's result is a true extension of Geller's Theorem. It has to be noticed that Geller-Rothschild and Luo Xuebo Theorems apply to operators with real and complex coefficients as well. We would also like to remark that the assumption, in these theorems, that $u$ is a tempered distribution appears to be very mild. However, it implicitly requires a bound for the absolute value of u, so that, if $u$ is real valued, a bound from above as well as from below.

In dealing with operators with *real* coefficients, it seems quite natural to look for Liouville properties of solutions just bounded from one side, and to give the following definition.

DEFINITION 1.1: *A linear Partial Differential Operator L in $\mathbb{R}^n$ with real valued coefficients will be said to have the one-side polynomial Liouville property if every global solution to $Lu = 0$ in $\mathbb{R}^n$ is a polynomial function whenever bounded from below by a polynomial function.*

The aim of this paper is to investigate this property for the class of linear second order hypoelliptic ultraparabolic operators introduced in [KL2]. More explicitly, the operators we shall deal with are of the following type

$$\mathcal{L} = \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x)\partial_{x_j}) + \sum_{i=1}^{N} b_i(x)\partial_{x_i} - \partial_t \qquad \text{in } \mathbb{R}^{N+1}, \tag{1.5}$$

where the coefficients $a_{ij}$ and $b_i$ are smooth functions defined in $\mathbb{R}^N$. The matrix $A = (a_{ij})_{i,j=1,\dots,N}$ is supposed to be symmetric and nonnegative definite at any point of $\mathbb{R}^N$.

We shall denote by $z = (x, t)$, $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, the point of $\mathbb{R}^{N+1}$ and by $Y$ the vector field in $\mathbb{R}^{N+1}$

$$Y := \sum_{i=1}^{N} b_i(x)\partial_{x_i} - \partial_t. \tag{1.6}$$

Moreover, we shall denote by $\mathcal{L}_0$ the *stationary* part of $\mathcal{L}$, i. e.

$$\mathcal{L}_0 = \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x)\partial_{x_j}) + \sum_{i=1}^{N} b_i(x)\partial_{x_i}. \tag{1.7}$$

We assume the following hypotheses are satisfied.

(H1) $\mathcal{L}$ is hypoelliptic in $\mathbb{R}^{N+1}$ and homogeneous of degree two with respect to the group of dilations $(D_\lambda)_{\lambda>0}$ given by

$$D_\lambda(x, t) = (d_\lambda x, \lambda^2 t) \tag{1.8}$$

where $d_\lambda$ is a dilation as in (1.2)

(H2) For every $(x, t), (y, \tau) \in \mathbb{R}^{N+1}$, $t > \tau$, there exists an $\mathcal{L}$- admissible path $\eta : [0, T] \longrightarrow \mathbb{R}^{N+1}$ such that $\eta(0) = (x, t)$, $\eta(T) = (y, \tau)$.

An *$\mathcal{L}$-admissible* path is any continuous path $\eta$ which is the sum of a finite number of diffusion and drift trajectories.

A *diffusion trajectory* is a curve $\eta$ satisfying, at any points of its domain, the inequality

$$(\langle \eta'(s), \xi \rangle)^2 \leq \langle \hat{A}(\eta(s))\xi, \xi \rangle \qquad \forall \xi \in \mathbb{R}^{N+1}.$$

Here $\langle , \rangle$ denotes the inner product in $\mathbb{R}^{N+1}$ and $\hat{A}(z) = \hat{A}(x, t) = \hat{A}(x)$ stands for the

$(N+1) \times (N+1)$ matrix

$$\hat{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

A *drift trajectory* is a positively oriented integral curve of $Y$.

Throughout the paper we shall denote by $Q$ the homogeneous dimension of $\mathbb{R}^{N+1}$ with respect to the dilations (1.8), i.e.

$$Q = \sigma_1 + \ldots + \sigma_N + 2$$

and we assume

$$Q \geq 5.$$

Then, the $d_\lambda$-homogeneous dimension of $\mathbb{R}^N$ is $Q - 2 \geq 3$.

We explicitly remark that the smoothness of the coefficients of $\mathcal{L}$ and the homogeneity assumption in (H1) imply that the $a_{ij}$' s and the $b_i$' s are polynomial functions (see [L], Lemma 2).

For any $z = (x, t) \in \mathbb{R}^{N+1}$ we define the $D_\lambda$-homogeneous norm $|\cdot|$ by

$$|z| = |(x, t)| := (|x|^4 + t^2)^{\frac{1}{4}}$$

where

$$|x| = |(x_1, \ldots, x_N)| = \left( \sum_{j=1}^{N} (x_j^2)^{\frac{\sigma}{\sigma_j}} \right)^{\frac{1}{2\sigma}}, \quad \sigma = \prod_{j=1}^{N} \sigma_j.$$

Some explicit examples of operators satisfying (H1) and (H2) will be given in Section 2.

The main result of this paper is the following theorem.

THEOREM 1.2: *Let* $u : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ *be a (smooth) solution to* $\mathcal{L}u = 0$ *in* $\mathbb{R}^{N+1}$. *Suppose* $u \geq p$, *were* $p$ *is a polynomial function, and*

$$u(0, t) = O(t^m) \qquad \text{as} \quad t \longrightarrow \infty \tag{1.9}$$

*for some* $m \geq 0$. *Then*

$$u \quad \text{is a polynomial function} \tag{1.10}$$

We remind that condition (1.9) cannot be removed in order to get (1.10). Indeed, for example, the function

$$u(x, t) = \exp(x_1 + x_2 + \ldots + x_N + Nt), \qquad x \in \mathbb{R}^N, \ t \in \mathbb{R},$$

is a nonnegative non-polynomial solution to the heat equation

$$\Delta u - \partial_t u = 0 \qquad \text{in} \quad \mathbb{R}^{N+1}, \qquad \Delta = \sum_{j=1}^{N} \partial_{x_j}^2.$$

We stress that $u$ does not satisfy condition (1.9) since $u(0, t) = \exp(Nt)$.

From Theorem 1.2 we obtain a Liouville-type theorem for nonnegative solution to $\mathcal{L}u = 0$, first proved in [KL2], Theorem 1.1.

COROLLARY 1.3: *Let* $u : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ *be a (smooth) solution to* $\mathcal{L}u = 0$ *in* $\mathbb{R}^{N+1}$*. Suppose* $u \geq 0$ *and*

$$u(0,t) = O(t^m) \qquad as \quad t \longrightarrow \infty \tag{1.11}$$

*for some* $m \geq 0$*. Then*

$$u = \text{const.} \qquad in \quad \mathbb{R}^{N+1}. \tag{1.12}$$

From this corollary, one easily gets a Liouville property for the *stationary* operator $\mathcal{L}_0$ analogous to the one for the classical Laplace operator (see [KL2], Corollary 1.2)

COROLLARY 1.4: *Let* $v : \mathbb{R}^N \longrightarrow \mathbb{R}$ *be a (smooth) solution* (1) *to* $\mathcal{L}_0 v = 0$ *in* $\mathbb{R}^N$*. Then, if* $v \geq 0$,

$$v = \text{const.} \qquad in \quad \mathbb{R}^N.$$

This result extends to our class of *stationary* operators a recent important Liouville-type property for *bounded* solutions to degenerate Ornstein-Uhlenbeck operators, proved by Priola and Zabczyk in [PZ]. In Section 3, Remark 2.2, we will make more precise this statement.

Directly from Theorem 1.2 we obtain the following polynomial Liouville theorem for $\mathcal{L}_0$, in which we do not require any a-priori asymptotic behavior for the solutions.

THEOREM 1.5: *Let* $v : \mathbb{R}^N \longrightarrow \mathbb{R}$ *be a solution to* $\mathcal{L}_0 v = 0$ *in* $\mathbb{R}^N$*. If there exists a polynomial function* $q$ *such that* $v \geq q$ *in* $\mathbb{R}^N$*, then* $v$ *is a polynomial function.*

The proofs of Theorem 1.2 and of its consequences, are postponed to Section 4. In next Section 2 we show some explicit examples of operators to which our results apply. Section 3 is devoted to some recalls and preliminary results needed in Section 4 for the proof of Theorem 1.2

## 2. - SOME EXAMPLES

In this section we show some explicit examples of operators satisfying hypotheses (H1)and (H2). They are mainly taken from [KL1].

(1) Obviously, $\mathcal{L}_0$ is hypoelliptic in $\mathbb{R}^N$ since $\mathcal{L}$ is hypoelliptic in $\mathbb{R}^{N+1}$. Then, every distributional solution to $\mathcal{L}_0 v = 0$ is smooth.

EXAMPLE: [Heat operators on Carnot groups] Let $(\mathbb{R}^N, \circ)$ be a Lie group in $\mathbb{R}^N$. Assume that $\mathbb{R}^N$ can be split as follows

$$\mathbb{R}^N = \mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_m}$$

and that the dilations

$$d_\lambda : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad d_\lambda(x^{(N_1)}, \ldots, x^{(N_m)}) = (\lambda x^{(N_1)}, \ldots, \lambda^m x^{(N_m)})$$

$$x^{(N_i)} \in \mathbb{R}^{N_i}, \quad i = 1, \ldots, m, \quad \lambda > 0,$$

are automorphisms of $(\mathbb{R}^N, \circ)$.

We also assume

$$\text{rank Lie}\{X_1, \ldots, X_{N_1}\}(x) = N \quad \forall x \in \mathbb{R}^N \tag{2.1}$$

where the $X_j$'s are left invariant on $(\mathbb{R}^N, \circ)$ and

$$X_j(0) = \frac{\partial}{\partial x_j^{(N_1)}}, \qquad j = 1, \ldots, N_1.$$

Then $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ is a *Carnot group* whose *homogeneous dimension* $Q_0$ is the natural number

$$Q_0 : = N_1 + 2N_2 + mN_m.$$

The vector fields $X_1, \ldots, X_{N_1}$ are the *generators* of $\mathbb{G}$,

$$\Delta_{\mathbb{G}} := \sum_{j=1}^{N_1} X_j^2$$

is the *canonical sub-Laplacian* on $\mathbb{G}$ and the parabolic operator

$$\mathcal{L} = \Delta_{\mathbb{G}} - \partial_t \qquad \text{in } \mathbb{R}^{N+1} \tag{2.2}$$

is called the *canonical heat operator* on $\mathbb{G}$. Obviously $\mathcal{L}$ can be written as in (1.5). Moreover, if we define

$$\mathbb{L} = (\mathbb{R}^{N+1}, \circ, D_\lambda)$$

with $D_\lambda(x, t) = (d_\lambda x, \lambda^2 t)$ and the composition law $\circ$ given by

$$(x, t) \circ (x', t') = (x \circ x', t + t')$$

then $\mathbb{L}$ is a homogeneous group, and the operator $\mathcal{L}$ in (2.2) satisfies condition (H1) in the Introduction. We explicitly remark that the homogeneous dimension of $\mathbb{L}$ is $Q := Q_0 + 2$.

In [KL1], page 70, it is proved that $\mathcal{L}$ also satisfies (H2).

EXAMPLE: [Kolmogorov operators.] Let us split $\mathbb{R}^N$ as follows

$$\mathbb{R}^N = \mathbb{R}^p \times \mathbb{R}^r$$

and denote by $x = (x^{(p)}, x^{(r)})$ its points. Let $B$ a $N \times N$ real matrix taking the following block form

$$B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ B_1 & 0 & 0 & \dots & 0 \\ 0 & B_2 & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & B_k & 0 \end{pmatrix}$$

where $B_j$ is a $r_j \times r_{j-1}$ matrix with rank $r_j$, and $r_0 = p \geq r_1 \geq \ldots \geq r_k \geq 1$, $r_0 + r_1 + \ldots + r_k = N$. Denote

$$E(t) = \exp\left(-tB\right)$$

and introduce in $\mathbb{R}^{N+1}$ the following composition law

$$(x, t) \circ (y, \tau) := (y + E(\tau)x, t + \tau). \tag{2.3}$$

The triplet

$$\mathbb{K} = (\mathbb{R}^{N+1}, \circ, d_\lambda)$$

is a homogeneous Lie group with respect to the dilations

$$\begin{aligned} D_\lambda(x, t) =& D_\lambda(x^{(p)}, x^{(r_1)}, \ldots, x^{(r_k)}, t) \\ =& (\lambda x^{(p)}, \lambda^3 x^{(r_1)}, \ldots, \lambda^{2k+1} x^{(r_k)}, \lambda^2 t) \end{aligned}$$

(see [LP]) The homogeneous dimension of $\mathbb{K}$ is

$$Q = p + 3r_1 + \ldots + (2k+1)r_k + 2.$$

We call $\mathbb{K}$ a *Kolmogorov-type group*.

Let us now consider the operator

$$\mathcal{K} = \Delta_{\mathbb{R}_p} + \langle Bx, D\rangle - \partial_t,$$

where $\Delta_{\mathbb{R}_p}$ denotes the usual Laplace operator in $\mathbb{R}^p$, $\langle , \rangle$ is the inner product in $\mathbb{R}^N$ and $D = (\partial_{x_1}, \ldots, \partial_{x_N})$. It is easy to see that $\mathcal{K}$ can be written as in (1.5).The first order partial differential operator

$$Y = \langle Bx, D\rangle - \partial_t$$

is called the *total derivative operator* on $\mathbb{K}$. By Proposition 2.2 in [LP], $Y$ is $D_\lambda$-homogeneous of degree two.

The operator $\mathcal{K}$ satisfies (H1)and (H2), and it is left translation invariant on $\mathbb{K}$( see [KL1] and [LP]).

REMARK 2.1: *The matrix $E(t)$ in* (2.3) *takes the following triangular form*

$$E(t) = \begin{pmatrix} I_p & 0 \\ E_1(t) & I_r + E_2(t) \end{pmatrix}$$

*where $I_p$ and $I_r$ are the identity matrix in $\mathbb{R}^p$ and $\mathbb{R}^r$, respectively. Then, the composition* *law in $\mathbb{K}$ has the following structure:*

$$(x^{(p)}, x^{(r)}, t) \circ (y^{(p)}, y^{(r)}, \tau) = (x^{(p)} + y^{(p)}, x^{(r)} + y^{(r)} + E_1(\tau)x^{(p)} + E_2(\tau)x^{(r)},\ t + \tau)$$

REMARK 2.2: *The stationary part of $\mathcal{K}$:*

$$\mathcal{K}_0 = \Delta_{\mathbb{R}_p} + \langle Bx, D\rangle,$$

*is contained in the class of degenerate Ornstein-Uhlenbeck operators studied in [PZ]. In [PZ] a Liouville Theorem for bounded solutions is proved.*

EXAMPLE: [A non-translations invariant operator.] The operator

$$\mathcal{L} = \partial_{x_1}^2 + x_1^{2m+1}\partial_{x_2} - \partial_t \quad in \quad \mathbb{R}^3$$

$m \in \mathbb{N}$, is hypoelliptic since, if we let $X = \partial_{x_1}$ and $Y = x_1^{2m+1}\partial_{x_2} - \partial_t$, we have $\mathcal{L} = X^2 + Y$ and rank $\mathrm{Lie}(X.Y) = 3$ at any point of $\mathbb{R}^3$. Then $\mathcal{L}$ satisfies the hypoellipticity-Hörmander rank condition. Moreover, $\mathcal{L}$ is homogeneous of degree 2 with respect to the dilations

$$D_\lambda(x_1, x_2, t) = (\lambda x_1, \lambda^{2m+3}x_2, \lambda^2)$$

Then, $\mathcal{L}$ satisfies hypothesis (H1). It can be also proved, just proceeding as in [KL1], pages 72-74, that it also satisfies hypothesis (H2). Finally, it is easy to recognize that there is no Lie group structure in $\mathbb{R}^3$ leaving left translation invariant the operator $\mathcal{L}$.

## 3. - SOME PRELIMINARY RESULTS

Hypotheses (H1) and (H2)imply the existence of a global fundamental solution $\Gamma(z, \zeta)$ of $\mathcal{L}$ smooth out of $\{z = \zeta\}$. In particular, from (H2) it follows that $\Gamma((x, t), (\xi, \tau)) > 0$ *iff* $t > \tau$. (see [KL1],[KL2],[LP1],[LP2]). We will also use the following property of $\Gamma$, showed in [KL1], Theorem 2.7 and Proposition 2.8.

(i) For any fixed $z \in \mathbb{R}^{N+1}$, $\Gamma(\cdot, z)$ and $\Gamma(z, \cdot)$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^{N+1})$.

(ii) For every $\varphi \in C^\infty(\mathbb{R}^{N+1})$ and $z \in \mathbb{R}^{N+1}$,

$$\mathcal{L} \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta)\varphi(\zeta)\, d\zeta \;=\; \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta)\mathcal{L}\varphi(\zeta)\, d\zeta \;=\; -\varphi(z).$$

(iii) There exists $C > 0$ such that

$$0 \le \Gamma(z, \zeta) \le \frac{C}{|z|^{Q-2}} \qquad \text{if} \quad |z| \ge 2|\zeta| \tag{3.1}$$

From these properties of $\Gamma$ we easily obtain the following lemma

LEMMA 3.1: *Let $\varphi$ a smooth function with compact support in $\mathbb{R}^{N+1}$. Then*

$$\sup_{\mathbb{R}^{N+1}} |\Gamma_\varphi| < \infty.$$

*Here we have set*

$$\Gamma_\varphi(z) := \int_{\mathbb{R}^{N+1}} \Gamma(z,\zeta)\varphi(\zeta)\, d\zeta.$$

PROOF: We know that $\Gamma_\varphi \in C^\infty(\mathbb{R}^{N+1})$. Moreover, denoting by $K$ the support of $\varphi$, we have

$$|\Gamma_\varphi(z)| \leq C|z|^{-Q} \int_K |\varphi| d\zeta$$

for $|z|$ sufficiently large. This inequality implies that $\Gamma_\varphi$ vanishes at infinity. Thus, the lemma follows. □

In [KL2], Theorem 2.1, the following Harnack inequality is proved.

There exist two positive constants $C = C(\mathcal{L})$ and $\theta = \theta(\mathcal{L}) < 1$ such that, if $u$ is a nonnegative solution to $\mathcal{L}u = 0$ in $C_{\frac{1}{\theta}}$, then

$$\sup_{C_\theta} u \leq C\, u(0,1) \tag{3.2}$$

where, for $\rho > 0$, $C_\rho$ denotes the $D_\lambda$-symmetric ball

$$C_\rho := \{z \in \mathbb{R}^{N+1} |\ |z| < \rho\}.$$

Next lemma is crucial for our purposes.

LEMMA 3.2: *Let $f$ be a smooth function in $\mathbb{R}^{N+1}$ and let $u$ be a nonnegative solution to*

$$\mathcal{L}u = f \qquad \text{in} \quad \mathbb{R}^{N+1}$$

*There exists a positive constant $C$ independent of $u$ and $f$ such that*

$$\sup_{C_\theta} u \leq C(u(0,1) + \sup_{C_{\frac{1}{\theta}}} |f|) \tag{3.3}$$

PROOF: Let $\psi$ be a nonnegative smooth and compactly supported function such that $\psi \equiv 1$ in $C_{\frac{1}{\theta}}$. Define

$$v(z) := \int_{\mathbb{R}^{N+1}} \Gamma(z,\zeta)\psi(\zeta) f(\zeta) d\zeta$$

This function is smooth, satisfies $\mathcal{L}u = -f$ in $C_{\frac{1}{\theta}}$ and

$$|v| \leq \sup_{C_{\frac{1}{\theta}}} |f| \sup_{\mathbb{R}^{N+1}} \Gamma_\psi \;=: \alpha.$$

It follows that

$$u + v + a \geq 0, \quad \text{and} \quad \mathcal{L}(u + v + a) = 0 \quad \text{in} \quad C_{\frac{1}{\theta}}.$$

Then, from the previously recalled Harnack inequality, we have

$$\begin{aligned}\sup_{C_\theta} (u + v + a) &\leq C\,(u(0,1) + v(0,1) + a)\\ &\leq C\,(u(0,1) + 2a)\end{aligned}$$

so that

$$\begin{aligned}\sup_{C_\theta} u \leq \sup_{C_\theta} (u + v + a) &+ \sup_{\mathbb{R}^{N+1}} |v + a|\\ \leq C\,(u(0,1) + 2a) &+ 2a\end{aligned}$$

Keeping in mind the definition of $a$, this completes the proof of the lemma. □

By using the homogeneity of $\mathcal{L}$, from the previous lemma we obtain the following result.

Corollary 3.3: *Assume the hypotheses of Lemma 3.2 are satisfied. Then*

$$u(z) \leq C\left(u\left(0, \left(\frac{|z|}{\theta}\right)^2\right) + |z|^2 \sup_{|\zeta| \leq \frac{|z|}{\theta^2}} |f(\zeta)|\right)$$

*for every $z \in \mathbb{R}^{N+1}$. The positive constant $C$ is independent of $u$ and $f$*

Proof: Fore every $r > 0$ define

$$u_r(z) = u(D_r z), \quad \text{and} \quad f_r(z) = r^2 f(D_r z).$$

Then, since $\mathcal{L}$ is $D_\lambda$-homogeneous of degree two,

$$\mathcal{L}u_r = r^2 (\mathcal{L}u) \circ D_r = f_r$$

Therefore, by the previous lemma,

$$\sup_{C_\theta} u_r \;\leq\; C\,(u_r(0,1) + \sup_{C_{\frac{1}{\theta}}} |f_r|)$$

This inequality can be written as follows

$$\sup_{|z| \leq \theta r} u(z) \leq C(u(0, r^2) + r^2 \sup_{|\zeta| \leq \frac{r}{\theta}} |f(\zeta)|)$$

from which the assertion immediately follows. □

For our purposes, it is convenient to specialize the above corollary as follows.

Corollary 3.4: *Let $p$ be a polynomial function and let $u$ be a nonnegative solution to*

$$\mathcal{L}u = p \qquad \text{in} \quad \mathbb{R}^{N+1}.$$

*Assume*

$$u(0,t) = O(t^m) \quad as \quad t \to \infty \tag{3.4}$$

*for a suitable* $m > 0$. *Then*

$$u(z) = O(|z|^n) \quad as \quad |z| \to \infty$$

*where* $n = \max\{2m, 2 + deg_{D_\lambda} p\}$.

Note. If $\alpha = (\alpha_1, \ldots, \alpha_{N+1})$ is a multi-index with nonnegative integer components, we let $|\alpha|_{D_\lambda} := \sum_{j=1}^{N} \sigma_j \alpha_j + 2\alpha_{N+1}$.

If $p(z) = \sum_{\alpha \in \mathcal{A}} c_\alpha z^\alpha$, we define

$$deg_{D_\lambda} p = max\{|\alpha|_{D_\lambda} : \alpha \in \mathcal{A}\}.$$

Proof: . Condition (3.4) implies

$$u\left(0, \left(\frac{|z|}{\theta}\right)^2\right) = O(|z|^{2m}) \quad \text{as} \quad |z| \to \infty$$

On the other hand, if

$$p(z) = \sum_{k=0}^{s} \sum_{|\alpha|=k} c_\alpha z^\alpha,$$

we have

$$|z|^2 \sup_{|\zeta| \le \frac{|z|}{\theta^2}} |f(\zeta)| = O(|z|^{2+s}) \quad \text{as} \quad |z| \to \infty$$

From this estimate and Corollary 3.3, the assertion follows. □

We close this Section by giving the proof of a strong maximum principle, a direct consequence of hypothesis (H2).

Proposition 3.5: *Let u be a nonnegative solution to the equation* $\mathcal{L}u = 0$ *in the half-space*

$$S := \mathbb{R}^N \times ]-\infty, t_0[, \quad t_0 \in \mathbb{R}.$$

*Suppose there exists a point* $z_1 = (x_1, t_1) \in S$ *such that*

$$u(x_1, t_1) = 0.$$

*Then* $u = 0$ *in* $\mathbb{R}^N \times ]-\infty, t_1[$.

Proof: Let us denote by $P_{z_1}(S)$ the propagation set of $z_1$ in $S$, i.e. the set

$$P_{z_1}(S) = \{z \in S : \text{there exists an } \mathcal{L}\text{-admissible path } \eta : [0,T] \longrightarrow S \text{ s. t. } \eta(0) = z_1,\ \eta(T) = z\}.$$

The hypothesis (H2) implies $P_{z_1}(S) = \mathbb{R}^N \times ]-\infty, t_1[$. On the other hand since $z_1$ is a minimum point of $u$ and the minimum spreads all over $P_{z_1}$ (see [A]), we get

$$u(z) = u(z_1) \qquad \forall\, z \in \mathbb{R}^N \times ]-\infty, t_1[.$$

Then, the assertion follows since $u(z_1) = 0$. □

## 4. - Proof of Theorem 1.2 and of its consequences

We begin with the proof of our main theorem.

Proof of Theorem 1.2. Defining $v := u - p$ and $q = -\mathcal{L}(p)$, we have

$$v \geq 0 \quad \text{and} \quad \mathcal{L}v = q$$

Moreover, $q$ is a polynomial function, since $\mathcal{L}$ has polynomial coefficients, and $v(0,t) = O(t^s)$ as $t \to \infty$, for a suitable integer $s > 0$ Then, by Corollary 3.4 we have

$$v(z) = O(|z|^\beta) \quad \text{as} \quad |z| \to \infty$$

for a suitable integer $\beta > 0$. It follows that $u = v + p$ is a tempered distributional solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{N+1}$. From Luo Xuebo's Theorem it follows that $u$ is a polynomial function.

Proof of Corollary 1.3 By Theorem 1.2 we know that $u$ is a polynomial function. Then, the proof can be completed just proceeding as in [KL2], Section 1, Proof of Theorem 1.1. For reading convenience, we would like to give the details of the proof. Since $u$ is a polynomial function, we have $u = u_0 + \ldots + u_m$, for a suitable integer $m \geq 0$ and polynomial functions $u_k$ $D_\lambda$-homogeneous of degree $k$, $k = 0, 1, \ldots, m$. Since $u \geq 0$, we have $u_m \geq 0$. On the other hand, being $\mathcal{L}u = 0$ and $\mathcal{L}u_k$ $D_\lambda$-homogeneous of degree $k-2$, if $k \geq 2$, we have $\mathcal{L}u_k = 0$ for every $k = 0, 1, \ldots, m$. In particular $\mathcal{L}u_m = 0$. Since $u_m$ is nonnegative and $D_\lambda$-homogeneous, there exists $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$ such that

$$u_m(z_0) = \inf_{\mathbb{R}^{N+1}} u_m, \quad (z_0 = (0,0)).$$

By the strong Maximum Principle recalled in the previous Section (see Proposition 3.5), we then have

$$u_m(x,t) = u_m(x_0, t_0) \qquad \forall\, (x,t) \in \mathbb{R}^N \times ]-\infty, t_0[.$$

Since $u_m$ is a polynomial function, this obviously implies

$$u_m(x,t) = u_m(x_0, t_0) \qquad \forall\, (x,t) \in \mathbb{R}^{N+1}.$$

Then $m = 0$ and $u \equiv u_0$, i.e. $u$ is a constant function.

Now, we can go into the proof of Corollary 1.4.

PROOF OF COROLLARY 1.4. The function

$$V : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}, \qquad V(x,t) = v(x)$$

satisfies $\mathcal{L}V = 0$ in $\mathbb{R}^{N+1}$. Moreover, $V \geq 0$ and

$$V(0,t) = v(0) \qquad \forall t \in \mathbb{R}.$$

By Corollary 1.3, $V =$ const. in $\mathbb{R}^{N+1}$ so that $v =$ const. in $\mathbb{R}^N$.
Finally, we prove Theorem 1.5.

PROOF OF THEOREM 1.5. We argue as in the previous proof and define

$$V, Q : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}, \qquad V(x,t) = v(x),\ \ Q(x,t) = q(x).$$

We have

$$V(0,t) = v(0) = O(1) \ \text{ as} |z| \to \infty, \quad V \geq Q$$

and $\mathcal{L}V = \mathcal{L}_0 v = 0$. By Theorem 1.2, $V$ is a polynomial function, and the assertion follows.

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