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## Some Recent Quantitative Unique Continuation Theorems

In this note I will survey some recent work on quantitative unique continuation problems which have had some interesting applications. These results have their origin in recent work with J. Bourgain [B-K] on Anderson localization for the continuous Bernoulli model, a well-known problem in the theory of disordered media. I will start out by describing this work. The problem of localization originates in a seminal 1958 paper by Anderson [A], who argued that, for a simple Schrödinger operator in a disordered medium, “at sufficiently low densities, transport does not take place; the exact wave functions are localized in a small region of space.” In this work we have concentrated on continuous models; the corresponding issues for discrete problems remain open. Thus, consider a random Schrödinger operator on  $\mathbb{R}^n$  of the form

$$H_\varepsilon = -\Delta + V_\varepsilon,$$

where the potential  $V_\varepsilon(x) = \sum_{j \in \mathbb{Z}^n} \varepsilon_j \varphi(x-j)$ , where  $\varepsilon_j \in \{0, 1\}$  are independent, and  $0 \leq \varphi \leq 1$ ,  $\varphi \in C_0^\infty(B(0, 1/10))$ . This is commonly referred to as the Anderson-Bernoulli model. It is not hard to see ([P-F]) that under these assumptions

$$\inf \text{spec } H_\varepsilon = 0 \quad \text{a.s.}$$

In this context, Anderson localization means that, near the bottom of the spectrum (i.e. for energies  $E > 0$ ,  $E < \delta$ ,  $\delta = \delta(n)$  small),  $H_\varepsilon$  has pure point spectrum with exponentially decaying eigenfunctions, a.s. This phenomenon is by now well-understood in the case when the random potential  $V_\varepsilon$  has a continuous site distribution (i.e. the  $\varepsilon_j$  take their values in  $[0, 1]$ ). When  $n = 1$ , this was first proved, for all energies and potentials with a continuous site distribution, by Gol'dsheid-Molchanov-Pastur ([G-M-P]). The extensions to  $n > 1$ , for the same potentials, were achieved by the use of a method, called “multi-scale analysis”, developed by Fröhlich-Spencer ([F-S]) and Fröhlich-Martinelli-

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Scoppola-Spencer ([F-M-S-S]). When the random variables  $\varepsilon_j$  are discrete valued (i.e. the Anderson-Bernoulli model), the result was established for  $n = 1$ , using the Furstenberg-Lepage approach, by Carmona-Klein-Martinelli ([C-K-M]) and by Shubin-Vakilian-Wolff ([S-V-W]), using the so-called supersymmetric formalism. Neither one of these methods extended to  $n > 1$ . We now have:

**THEOREM 1** (Bourgain-Kenig [B-K]): *For energies near the bottom of the spectrum ( $0 < E < \delta$ ),  $H_\varepsilon$  displays Anderson localization a.s. in  $\varepsilon$  for  $n \geq 1$ .*

The only previous result when  $n > 1$  was due to Bourgain ([B]), who considered  $V_\varepsilon(x) = \sum_{j \in \mathbb{Z}^n} \varepsilon_j \varphi(x - j)$ , where now  $\varphi(x) \sim \exp(-|x|)$ , instead of  $\varphi \in C_0^\infty$ . The non-vanishing of  $\varphi$  as  $|x| \rightarrow \infty$  was essential in Bourgain's argument (which also applied to the corresponding discrete problem). In our work, on the true Bernoulli model, we overcome this by the use of a quantitative unique continuation result.

The proof of the above Theorem proceeds by an “induction on scales” argument. We consider the restriction of our operator to a cube of size  $\ell$  (under Dirichlet boundary conditions) and establish our estimates by induction in  $\ell$ . The estimates that we establish are weak versions of the so-called “Wegner estimates” ([We]), which roughly speaking show that, for a large set of  $\varepsilon$ , at scale  $\ell$ , we have “good resolvent estimates” depending favorably on  $\ell$ . The difficulty in proving such an estimate in the Bernoulli case, as opposed to the case when we have a continuous site distribution, is that we cannot obtain the estimate by varying a single  $j$  at a time. Here, ‘rare event’ bounds must be obtained by considering the dependence of eigenvalues on a large collection of variables,  $\{\varepsilon_j\}_{j \in S}$ . In doing this, one of our key tools is a probabilistic lemma on Boolean functions, used by Bourgain in his work [B].

**LEMMA** (Lemma 3.1 in [B-K]): *Let  $f = f(\varepsilon_1, \dots, \varepsilon_d)$  be a bounded function on  $\{0, 1\}^d$  and denote  $I_j = f|_{\varepsilon_j=1} - f|_{\varepsilon_j=0}$ , the  $j^{\text{th}}$  influence, which is a function of  $\varepsilon_{j'}$ ,  $j' \neq j$ . Let  $J \subset \{1, \dots, d\}$  be a subset with  $|J| \leq \delta^{-1/4}$ , so that  $k < |I_j| < \delta$  for all  $j \in J$ . Then, for all  $E$ ,*

$$\text{meas} \{|f - E| < k/4\} \leq |J|^{-1/2}.$$

(Here  $\text{meas}$  refers to normalized counting measure on  $\{0, 1\}^d$ ). The proof of this Lemma relies on Sperner's Lemma in the theory of partially ordered sets (see [Bo], p. 10).

The function to which this Lemma is applied to is the eigenvalue. It then becomes critical to find bounds for the  $j^{\text{th}}$  influence of eigenvalues. One can see that if  $\xi$  is a normalized eigenstate ( $\|\xi\|_{L^2} = 1$ ), with eigenvalue  $E$ , by first order eigenvalue variation one has that  $I_j = \int |\xi(x)|^2 \varphi(x - j) dx$ . Upper bounds for this are more or less standard and what is at issue is lower bounds for  $I_j$ . Recalling that  $(-\Delta + V_\varepsilon)(\xi) = E\xi$ ,  $\|\xi\|_{L^2} = 1$ , we are led to the following quantitative unique continuation problem at infinity:

Suppose that  $u$  is a solution to

$$\Delta u + Vu = 0 \text{ in } \mathbb{R}^n, \text{ with } \|V\|_\infty \leq 1,$$

so that  $\|u\|_\infty \leq C_0$  and  $u(0) = 1$ . Note that by Carleman's unique continuation principle ([H]) we know that, for each  $x_0 \in \mathbb{R}^n$ ,  $\sup_{x \in B(x_0, 1)} |u(x)| > 0$ . For  $R$  large, define

$$M(R) = \inf_{x_0=R} \sup_{B(x_0, 1)} |u(x)|.$$

The question that we need to address is:

How small can  $M(R)$  be?

**THEOREM 2** (Bourgain-Kenig [B-K]): *Under the above conditions, we have*

$$M(R) \geq C \exp(-CR^{4/3} \log R).$$

**REMARK** (See [B-K], Section 5): In order for our induction on scales argument to work to prove the weak Wegner estimate, we need an estimate of the form  $M(R) \geq C \exp(-CR^\beta)$ , with  $\beta < \frac{1+\sqrt{3}}{2} = 1.35 \dots$ . Note that  $4/3 = 1.33 \dots$

It turns out that the problem just described in Theorem 2 is a quantitative version of a conjecture of E. M. Landis ([K-L]): Landis conjectured that if  $\Delta u + Vu = 0$  in  $\mathbb{R}^n$ , with  $\|V\|_\infty \leq 1$ ,  $\|u\|_\infty \leq C_0$ , and  $|u(x)| \leq C \exp(-C|x|^{1+})$ , then  $u \equiv 0$ . This was disproved in 1992 by Meshkov [M] who constructed such  $V$ ,  $u$ ,  $u \not\equiv 0$ , with

$$|u(x)| \leq C \exp(-C|x|^{4/3}).$$

(Meshkov also showed that if  $|u(x)| \leq C \exp(-C|x|^{4/3+})$ , then  $u \equiv 0$ ). Meshkov's example clearly shows the sharpness of the lower bound on  $M(R)$  in Theorem 2. Nevertheless, in the Meshkov example,  $u$ ,  $V$  are complex valued, while for many applications, we are only interested in real  $u$ ,  $V$ . We thus pose:

**QUESTION 1:** *Can  $4/3$  in Theorem 2 be improved to 1 for real-valued  $u$ ,  $V$ ?*

We turn to a sketch of the proof of Theorem 2. Our starting point is the following well-known Carleman inequality (see [H]).

**LEMMA:** *There are dimensional constants  $C_1$ ,  $C_2$ ,  $C_3$  and an increasing function  $w(r)$ , defined for  $0 < r < 10$ , so that*

$$\frac{1}{C_1} \leq \frac{w(r)}{r} \leq C_1$$

*and such that, for all  $f \in C_0^\infty(B(0, 10) \setminus \{0\})$ ,  $a > C_2$  we have*

$$a^3 \int w^{-1-2a} |f|^2 \leq C_3 \int w^{2-2a} |\Delta f|^2.$$

The classical application of this lemma (see [H]) is to the following unique continuation result, due to Carleman ([C]).

PROPOSITION: Assume that  $\Delta u = Vu$  in  $B(0, 10)$  and that  $\|u\|_{L^\infty} \leq C_0$ ,  $\|V\|_{L^\infty} \leq M$ . Suppose that  $|u(x)| \leq C_N |x|^N$  for each  $N > 0$ . Then  $u \equiv 0$  in  $B(0, 10)$ .

It turns out that to prove the Proposition, the power  $a^3$  on the left-hand side of the inequality in the Lemma is not crucial; in fact, any  $b(a)$  with  $b(a) \rightarrow \infty$  as  $a \rightarrow \infty$  would do. On the other hand, for Theorem 2 the exact power is crucial and as we will see from the sketch of the proof, the Meshkov example implies that no higher power than 3 can be used, no matter what the choice of  $w$  is. To sketch the proof of Theorem 2, pick  $x_0$ ,  $|x_0| = R$  so that  $M(R) = \sup_{B(x_0, 1)} |u(x)|$ . We now “interchange 0 and  $x_0$ ” and “rescale to

$R = 1$ ” by setting  $u_R(x) = u\left(AR\left(x + \frac{x_0}{AR}\right)\right)$ , where  $A$  is a large dimensional constant to be fixed later. We have  $|\Delta u_R(x)| \leq A^2 R^2 |u_R(x)|$  and if  $\tilde{x}_0 = -x_0/AR$ , then  $u_R(\tilde{x}_0) = 1$ ,  $|\tilde{x}_0| = \frac{1}{A}$ . Moreover,  $M(R) = \sup_{|x| \leq 2r_0} |u_R(x)|$ , where  $r_0 = \frac{1}{2AR}$ . Pick now  $\rho$  a cut-off function,  $\rho \equiv 0$  on  $|x| < \frac{r_0}{2}$ ,  $|x| > 4$  and  $\rho \equiv 1$  on  $\frac{2}{3}r_0 < |x| < 3$  and apply the lemma to  $f = (u_R \rho)$ . We obtain:

$$\begin{aligned} a^3 \int w^{-1-2a} u_R^2 \leq C_3 \int w^{2-2a} \rho^2 |\Delta u_R|^2 + \\ + C_3 \left[ \int_{\frac{1}{2}r_0 < |x| < \frac{2}{3}r_0} w^{2-2a} \{ |\Delta \rho|^2 |u_R|^2 + 2|\nabla \rho|^2 |\nabla u_R|^2 \} \right] + \\ + C_3 \left[ \int_{3 < |x| < 4} w^{2-2a} \{ |\Delta \rho|^2 |u_R|^2 + 2|\nabla \rho|^2 |\nabla u_R|^2 \} \right]. \end{aligned}$$

Note that  $|\Delta u_R|^2 \leq A^2 R^4 |u_R|^2$  and hence, if  $a^4 \simeq R^3$  we can absorb the first term on the right-hand side into the left-hand side. The left-hand side can be seen to be greater than or equal to  $C a^3 A^{-n} R^{-n} w^{-1-2a} (2/A)$ , using that  $u_R(\tilde{x}_0) = 1$  and interior estimates. The last two terms of the right-hand side are bounded from above by  $(CR)^{2a-n+2} M(R)^2$  and by  $CC_0^2 A^2 w(3)^{-2-2a}$ , respectively, using interior estimates. Hence, taking  $A$  so large that  $w\left(\frac{2}{A}\right) \leq \frac{1}{10} w(3)$  and  $R$  large, depending on  $n$ ,  $C_0$ , we obtain

$$C a^3 R^{-n} w^{-1-2a} \left(\frac{2}{A}\right) \leq (CR)^{2a+2-n} M(R)^2,$$

which, since  $a^4 \simeq R^3$ , gives the desired lower bound.

The arguments we have just sketched can be sharpened to address the following question: What is the sharp lower bound on the possible rate of vanishing in Carleman’s proposition? More precisely, suppose that we are in the following normalized situation:

Assume  $\Delta u + Vu = 0$  in  $B(0, 10)$ , with  $\|V\|_\infty \leq M$ ,  $\|u\|_\infty \leq C_0$ . Assume also that  $\sup_{|x| \leq 1} |u(x)| \geq 1$ . Then, what is the best lower bound for  $m(r) = \max_{|x| \leq r} |u|$ , of the form  $m(r) \geq a_1 r^{a_2 \beta}$ , as  $r \rightarrow 0$ , with  $a_i = a_i(n, C_0)$  and  $\beta = \beta(M)$ ,  $M \gg 1$ . When  $V \equiv -M$ , i.e. we consider eigenvalues, in the setting of Riemannian manifolds, H. Donnelly and C. Fefferman ([D-F]) showed in 1988 that  $\beta = M^{1/2}$ . Our arguments show that, for general  $V$ ,  $\beta = M^{2/3}$  and moreover, the Meshkov example can be used to show that for complex valued  $u$ ,  $V$ , this is sharp. (These observations were made jointly with D. Jerison).

QUESTION 2: *Can one take  $\beta = M^{1/2}$  for real-valued  $u$ ,  $V$ ?*

We next turn our attention to parabolic equations. Thus, let us consider solutions to  $\partial_t u - \Delta u + W(x, t) \cdot \nabla u + V(x, t)u = 0$  in  $\mathbb{R}^n \times (0, 1)$ , with  $\|V\|_\infty \leq M$ ,  $\|W\|_\infty \leq N$ . We will also restrict ourselves to considering bounded solutions, i.e.  $\|u\|_{L^\infty} \leq C_0$ . Then, as is well known (see [E-S-S 1]) for references), we have the following backward uniqueness result: if  $u(x, 1) \equiv 0$ , then  $u(x, t) \equiv 0$ ,  $x \in \mathbb{R}^n$ ,  $t \in (0, 1)$ . Recently (see [E-S-S 2], [E-S-S 3]), Escauriaza-Seregin-Sverak have shown that in fact, it suffices to have solutions  $u$  defined in  $\mathbb{R}_+^n \times (0, 1) = \{(x, t) : x = (x_1, \dots, x_n), x_1 > 0, 0 < t < 1\}$  so that  $u(x, 1) \equiv 0$ ,  $x \in \mathbb{R}_+^n$ , to reach the same conclusion. This was a crucial ingredient in their proof (see [E-S-S 3]) that if  $\vec{u}$  solves the Navier-Stokes system in  $\mathbb{R}^3 \times (0, T)$ , in the weak sense (Leray-Hopf solution) and  $\vec{u} \in L^\infty([0, T]; L^3(\mathbb{R}^3))$ , then  $\vec{u}$  is smooth in  $\mathbb{R}^3 \times (0, T]$  and unique. This is an end-point result which generalizes well-known ones due to Ladyzhenskaya, Leray, Prodi and Serrin. (See [E-S-S 3] for details.) On the other hand, in 1974, in [L-O], Landis and Oleinik, in parallel with Landis' elliptic conjecture, formulated the following:

CONJECTURE: *Assume that  $u$  is as in the backward uniqueness situation, but at  $t = 1$  we only assume  $|u(x, 1)| \leq C \exp(-C|x|^{2+\varepsilon})$ , for some  $\varepsilon > 0$ . Show that  $u \equiv 0$ .*

Note that in this evolution situation, the growth rate is clearly optimal (for both real- and complex-valued solutions). We now have:

THEOREM 3 (Escauriaza-Kenig-Ponce-Vega 2005 [E-K-P-V 1]): *In the above situation, if  $|u(x, 1)| \leq C_k \exp(-k|x|^2)$  for each  $k$ , then  $u \equiv 0$ . The same conclusion holds for  $u$  defined in  $\mathbb{R}_+^n \times (0, 1)$ . Moreover, there are quantitative versions of these results: for instance, if  $\|u(-, 1)\|_{L^2(B_1)} > 0$ , there exists  $N$  such that, for  $|x| > N$ , we have:*

$$\|u(-, 1)\|_{L^2(B(x, |x|/2))} \geq \exp(-N|x|^2) \quad \text{and}$$

$$\|u(-, 1)\|_{L^2(B(x, 1))} \geq \exp(-N|x|^2 \log |x|).$$

*Corresponding quantitative results hold in the case of  $\mathbb{R}_+^n \times (0, 1)$ .*

The proof of this result is inspired by the one of the elliptic one described before. The main tools are a rescaling argument and a quantification of the size of the constants involved in the two-sphere and one-cylinder inequalities (see [E-F-V]) satisfied by solutions of certain parabolic equations, in terms of the  $L^\infty$  norm of the lower order coefficients and of the time of existence of solutions. (See [E-K-P-V 1].)

QUESTION 3: Landis and Oleinik ([L-O]) in fact were interested in variable coefficient top order terms, i.e. operators of the form  $\partial_t u - \sum \partial_{x_i} a_{ij}(x, t) \partial_{x_j} u + W(x, t) \nabla u + Vu$ , where  $\{a_{ij}(x, t)\}$  is uniformly elliptic and symmetric. They ask for conditions on the local smoothness and the behavior of the coefficients at infinity for the validity of their conjecture. We conjecture that smoothness and growth conditions of the type  $|\nabla_{(x,t)} a_{ij}(x, t)| \leq \frac{C}{(1 + |x|)^{1+\varepsilon}}$  suffice for this.

The last topic that we want to discuss here is the possible existence of analogous results for dispersive equations. Let us consider for example, non-linear Schrödinger equations, i.e. equations of the form

$$i\partial_t u + \Delta u + F(u, \bar{u})u = 0 \quad \text{in } \mathbb{R}^n \times [0, 1],$$

where  $F$  is a suitable non-linearity. The first thing we would like to discuss is what is the analog of the backward uniqueness result for parabolic equations which we have just discussed. The first obstacle in doing this is that Schrödinger equations are time reversible and so “backward in time” makes no sense. As is usual in the study of uniqueness questions, we consider first linear Schrödinger equations of the form  $i\partial_t u + \Delta u + V(x, t)u = 0$  in  $\mathbb{R}^n \times [0, 1]$ , for suitable  $V(x, t)$ , so that in the end we can let  $V(x, t) = F(u(x, t))$ . We first recall the following well-known version of the uncertainty principle, due to Hardy (see [S-S]): Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be such that  $f(x) = O(\exp(-\pi A x^2))$  and such that its Fourier transform  $\hat{f}(\xi) = O(\exp(-\pi B \xi^2))$  with  $A, B > 0$ . Then, if  $A \cdot B > 1$ , we must have  $f \equiv 0$ . For instance, if  $|f(x)| \leq C_\varepsilon \exp(-C_\varepsilon |x|^{2+\varepsilon})$  and  $|\hat{f}(\xi)| \leq C_\varepsilon \exp(-C_\varepsilon |\xi|^{2+\varepsilon})$ , for some  $\varepsilon > 0$ , then  $f \equiv 0$ . (The usual proof of this result uses the theory of analytic functions of exponential type.) It turns out that this version of the uncertainty principle can be easily translated into an equivalent formulation for the free Schrödinger equation.

If  $v$  solves  $i\partial_t v + \partial_x^2 v = 0$  in  $\mathbb{R} \times [0, 1]$ , with  $v(x, 0) = v_0(x)$ , then

$$v(x, t) = \frac{c}{\sqrt{t}} \int e^{i|x-y|^2/4t} v_0(y) dy$$

so that  $v(x, 1) = c e^{i|x|^2/4} \int e^{-ixy/2} e^{i|y|^2/4} v_0(y) dy$ . If we then apply the corollary to Hardy’s uncertainty principle to  $f(y) = e^{i|y|^2/4} v_0(y)$ , we see that if

$$\begin{aligned} |v(x, 0)| &\leq C_\varepsilon \exp(-C_\varepsilon |x|^{2+\varepsilon}) & \text{and} \\ |v(x, 1)| &\leq C_\varepsilon \exp(-C_\varepsilon |x|^{2+\varepsilon}) & \text{for some } \varepsilon > 0, \end{aligned}$$

we must have  $v(x, t) \equiv 0$ . Thus, for time-reversible dispersive equations, the analog of “backward in time uniqueness” should be “uniqueness from behavior at two different

times”. We are thus interested in such results with “data which is 0 at infinity” or with “rapidly decaying data” and even in quantitative versions, where we obtain “lower bounds for all non-zero solutions”.

It turns out that, for the case of “data which is 0 at infinity”, this question has been studied for some time.

For the one-dimensional cubic Schrödinger equation,

$$i\partial_t u + \partial_x^2 u \mp |u|^2 u = 0 \quad \text{in } \mathbb{R} \times [0, 1],$$

B. Y. Zhang ([Z]) showed that if  $u \equiv 0$  on  $(-\infty, a) \times \{0, 1\}$ , or on  $[a, +\infty) \times \{0, 1\}$ , for some  $a \in \mathbb{R}$ , then  $u \equiv 0$  on  $\mathbb{R} \times [0, 1]$ . His proof used inverse scattering (thus making it only applicable to the one-dimensional cubic Schrödinger equation), exploiting a non-linear Fourier transform and analyticity. In 2002, Kenig-Ponce-Vega ([K-P-V]) introduced a general method which allowed them to prove the corresponding results for solutions to  $i\partial_t u + \Delta u + V(x, t)u = 0$  in  $\mathbb{R}^n \times [0, 1]$ ,  $n \geq 1$ , for a large class of potentials  $V$ . We thus have:

**THEOREM 4** (Kenig-Ponce-Vega [K-P-V]): *If  $V \in L_t^1 L_x^\infty \cap L_{\text{loc}}^\infty$  and  $\lim_{R \rightarrow \infty} \|V\|_{L_t^1 L^\infty(|x| > R)} = 0$  and there exists a strictly convex cone  $\Gamma \subset \mathbb{R}^n$  and a  $y_0 \in \mathbb{R}^n$  so that*

$$\text{supp } u(-, 0) \subset y_0 + \Gamma$$

$$\text{supp } u(-, 1) \subset y_0 + \Gamma,$$

*then we must have  $u \equiv 0$  on  $\mathbb{R}^n \times [0, 1]$ .*

This work was extended by Ionescu-Kenig ([I-K 1]) who considered more general potentials  $V$  and the case when  $\Gamma$  is a half-space. For instance, if  $V \in L_{xt}^{\frac{n+2}{2}}(\mathbb{R}^n \times \mathbb{R})$  or more generally,  $V \in L_t^p L_x^q(\mathbb{R}^n \times [0, 1])$  with  $\frac{2}{p} + \frac{n}{q} \leq 2$ ,  $1 < p < \infty$  (for  $n = 1$ ,  $1 < p \leq 2$ ) or  $V \in C([0, 1]; L^{n/2}(\mathbb{R}^n))$ ,  $n \geq 3$ , the result holds, with  $\Gamma$  a half-plane. This work involves some delicate constructions of parametrices and is quite involved technically.

We next turn to our extension of Hardy’s uncertainty principle to this context, i.e. the case of “rapidly decaying data”. Here there seems to be no previous literature on the problem.

**THEOREM 5** (Escariz-Kenig-Ponce-Vega [E-K-P-V 2]): *Let  $u$  be a solution to  $i\partial_t u + \Delta u + V(x, t)u = 0$  in  $\mathbb{R}^n \times [0, 1]$ , with  $u \in C([0, 1]; H^2(\mathbb{R}^n))$ . Assume that  $V \in L^\infty(\mathbb{R}^n \times [0, 1])$ ,  $\nabla_x V \in L^1([0, 1]; L^\infty(\mathbb{R}^n))$  and  $\lim_{R \rightarrow \infty} \|V\|_{L_t^1 L^\infty(|x| > R)} = 0$ . If  $u_0 = u(x, 0)$  and  $u_1 = u(x, 1)$  belong to  $H^1(e^{k|x|^2} dx)$ , for each  $k > 1$ , then  $u \equiv 0$ .*

As we will see soon, there actually even is a quantitative version of this result. The rest of this note will be devoted to a sketch of the proof of Theorem 5. Our starting point is:

LEMMA (Kenig-Ponce-Vega [K-P-V]): Suppose that  $u \in C([0, 1]; L^2(\mathbb{R}^n))$  solves  $i\partial_t u + \Delta u + V(x, t)u = H$ ,  $H \in L_t^1 L_x^2$  and  $\|V\|_{L_t^1 L_x^\infty} \leq \varepsilon$ , where  $\varepsilon = \varepsilon(n)$  is small enough. Suppose that  $u_0(x) = u(x, 0)$ ,  $u_1(x) = u(x, 1)$  both belong to  $L^2(\mathbb{R}^n; e^{2\beta x_1} dx)$  and  $H \in L^1([0, 1]; L^2(e^{2\beta x_1} dx))$ . Then  $u \in C([0, 1]; L^2(e^{2\beta x_1} dx))$  and

$$\begin{aligned} \sup_{0 \leq t \leq 1} \|u(-, t)\|_{L^2(e^{2\beta x_1} dx)} &\leq C\{\|u_0\|_{L^2(e^{2\beta x_1} dx)} + \\ &+ \|u_1\|_{L^2(e^{2\beta x_1} dx)} + \|H\|_{L^1([0, 1]; L^2(e^{2\beta x_1} dx))}\}, \end{aligned}$$

with  $C$  independent of  $\beta$ .

The proof of this lemma is quite subtle. If we know *a priori* that  $u \in C([0, 1]; L^2(e^{2\beta x_1} dx))$ , the proof could be carried out by a variant of the energy method (after conjugation with the weight  $e^{2\beta x_1}$ ) where we split into frequencies  $\xi_1 > 0$  and  $\xi_1 < 0$ , performing the time integral from 0 to  $t$  or from  $t$  to 1, according to each case. However, since we are not free to prescribe both  $u_0$  and  $u_1$ , we cannot use *a priori* estimates. We thus introduce a fixed smooth function  $\varphi$ , with  $\varphi(0) = 0$ ,  $\varphi'$  non-increasing,  $\varphi'(r) \equiv 1$  for  $r \leq 1$ ,  $\varphi'(r) = 0$  for  $r \geq 2$ . We then let, for  $\lambda$  large,  $\varphi_\lambda(r) = \lambda\varphi(r/\lambda)$ , so that  $\varphi_\lambda(r) \uparrow r$  as  $\lambda \rightarrow \infty$ . We replace the weight  $e^{2\beta x_1}$  ( $\beta > 0$ ) with  $e^{2\beta\varphi_\lambda(x_1)}$  and prove the analogous estimate for these weights, uniformly in  $\lambda$ , for  $\lambda \geq C(1 + \beta^6)$ . The point is that all the quantities involved are now *a priori* finite.

The price one pays is that, after conjugation with the weight  $e^{2\beta\varphi_\lambda(x_1)}$ , the resulting operators are no longer constant coefficient (as is the case for  $e^{2\beta x_1}$ ) and their study presents complications. At this point there are two approaches: in [K-P-V] one adapts the use of the energy estimates, combined with commutator estimates and the standard pseudo-differential calculus. The second approach, in [I-K 1], constructs parametrices for the resulting operators and proves bounds for them.

With this Lemma as our point of departure, our first step is to deduce from it further weighted estimates.

COROLLARY ([E-K-P-V 2]): If we are under the hypothesis of the previous Lemma and in addition, for some  $a > 0$ ,  $a > 1$ ,  $u_0, u_1 \in L^2(e^{a|x|^a} dx)$ ,  $H \in L^1([0, 1]; L^2(e^{a|x|^a} dx))$ , then there exist  $C_{a,n}, C_n > 0$  such that

$$\sup_{0 \leq t \leq 1} \int_{|x| > C_{a,n}} |u(x, t)|^2 e^{C_n a |x|^a} dx < \infty.$$

The idea used for the proof of the corollary is as follows: let  $u_R(x, t) = u(x, t)\eta_R(x)$ , where  $\eta_R(x) = \eta(x/R)$  and  $\eta \equiv 0$  for  $|x| \leq 1$ ,  $\eta \equiv 1$  for  $|x| \geq 2$ . We apply the Lemma to  $u_R$  and a choice of  $\beta = bR^{a-1}$ , for suitable  $b$ , in each direction  $x_1, x_2, \dots, x_n$ . The corollary then follows readily.

The next step in the proof is to deduce lower bounds for  $L^2$  space-time integrals, in analogy with the elliptic and parabolic situations. These are our “quantitative lower bounds”.



THEOREM 6 ([E-K-P-V 2]): Let  $u \in C([0, 1]; H^2(\mathbb{R}^n))$  solve  $i\partial_t u + \Delta u + Vu = 0$  in  $\mathbb{R}^n \times [0, 1]$ . Assume that  $\int_0^1 \int_{\mathbb{R}^n} |u|^2 + |\nabla u|^2 dx dt \leq A^2$  and that  $\int_{\frac{1}{2}-\frac{1}{8} \leq |x| < 1} |u|^2 dx dt \geq 1$ , with  $\|V\|_{L^\infty} \leq L$ . Then there exists  $R_0 = R_0(n, A, L) > 0$  and  $C_n > 0$  such that if  $R \geq R_0$ , we have  $\delta(R) \geq C_n \exp(-C_n R^2)$ , where  $\delta(R) = \left( \int_0^1 \int_{R-1 \leq |x| \leq R} (|u|^2 + |\nabla u|^2) dx dt \right)^{1/2}$ .

Once Theorem 6 is proved, applying the Corollary to  $u$  and  $\nabla u$  (which verifies a similar equation to the one  $u$  does) we see that Theorem 6 yields a contradiction.

In order to prove Theorem 6, a key tool is the following Carleman estimate, which is a variant of the one due to V. Isakov ([I] and also [I-K 2]).

LEMMA: Assume that  $R > 0$  and  $\varphi : [0, 1] \rightarrow \mathbb{R}$  is a smooth compactly supported function. Then there exists  $C = C(n, \|\varphi'\|_\infty, \|\varphi''\|_\infty) > 0$  such that, for all  $g \in C_0^\infty(\mathbb{R}^{n+1})$  with  $\text{supp } g \subset \left\{ (x, t) : \left| \frac{x}{R} + \varphi(t)e_1 \right| \geq 1 \right\}$  and  $a \geq CR^2$ , we have

$$\frac{a^{3/2}}{R^2} \|e^{a|\frac{x}{R} + \varphi(t)e_1|^2} g\|_{L^2} \leq C \|e^{a|\frac{x}{R} + \varphi(t)e_1|^2} (i\partial_t + \Delta)(g)\|_{L^2}.$$

(Here  $e_1 = (1, 0, \dots, 0)$ .)

PROOF: We conjugate  $(i\partial_t + \Delta)$  by the weight  $e^{a|\frac{x}{R} + \varphi(t)e_1|^2}$  and split the resulting operator into its Hermitian and its anti-Hermitian parts. Thus, let  $f = e^{a|\frac{x}{R} + \varphi(t)e_1|^2} g$ , so that  $e^{a|\frac{x}{R} + \varphi(t)e_1|^2} (i\partial_t + \Delta)g = S_a f - 4a A_a f$ , where  $S_a = i\partial_t + \Delta + \frac{4a^2}{R^2} \left| \frac{x}{R} + \varphi(t)e_1 \right|^2$  and  $A_a = \frac{1}{R} \left( \frac{x}{R} + \varphi e_1 \right) \cdot \nabla + \frac{n}{2R^2} + i \frac{\varphi'}{2} \left( \frac{x_1}{R} + \varphi \right)$ . Thus,  $S_a^* = S_a$ ,  $A_a^* = -A_a$  and

$$\begin{aligned} \|e^{a|\frac{x}{R} + \varphi(t)e_1|^2} (i\partial_t + \Delta)(g)\|_{L^2}^2 &= \langle S_a f - 4a A_a f, S_a f - 4a A_a f \rangle \geq \\ &\geq -4a \langle (S_a A_a - A_a S_a) f, f \rangle = -4a \langle [S_a, A_a] f, f \rangle. \end{aligned}$$

A calculation shows that

$$[S_a, A_a] = \frac{2}{R^2} \Delta - \frac{4a^2}{R^4} \left| \frac{x}{R} + \varphi e_1 \right|^2 - \frac{1}{2} \left[ \left( \frac{x_1}{R} + \varphi \right) \varphi'' + (\varphi')^2 \right] + 2i \frac{\varphi'}{R} \partial_{x_1}$$

and

$$\begin{aligned} \|e^{a|\frac{x}{R} + \varphi(t)e_1|^2} (i\partial_t + \Delta)g\|_{L^2}^2 &\geq \frac{16a^3}{R^4} \int \left| \frac{x}{R} + \varphi(t)e_1 \right|^2 |f|^2 + \\ &+ \frac{8a}{R^2} \int |\nabla f|^2 + 2a \int \left[ \left( \frac{x_1}{R} + \varphi \right) \varphi'' + (\varphi')^2 \right] |f|^2 - \\ &- \frac{8ai}{R} \int \varphi' \partial_{x_1} f \bar{f}. \end{aligned}$$

Using our support hypothesis on  $g$ , and taking  $a > CR^2$ , with  $C = C(n, \|\varphi'\|_\infty, \|\varphi''\|_\infty)$  yields our estimate.

In order to use the Lemma to prove Theorem 6, we choose  $\theta_R$ ,  $\theta \in C_0^\infty(\mathbb{R}^n)$ ,  $\varphi \in C_0^\infty((0, 1))$  so that  $\theta_R(x) \equiv 1$  if  $|x| < R - 1$ ,  $\theta_R(x) = 0$ ,  $|x| \geq R$ ,  $\theta(x) \equiv 0$  when  $|x| \leq 1$ ,  $\theta(x) \equiv 1$  when  $|x| \geq 2$ ,  $0 \leq \varphi \leq 3$ , with  $\varphi \equiv 1$  on  $\left[\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}\right]$ ,  $\varphi \equiv 0$  on  $\left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$ . We let  $g(x, t) = \theta_R(x) \theta\left(\frac{x}{R} + \varphi(t)e_1\right) u(x, t)$ . Note that  $\text{supp } g \subset \left\{\left|\frac{x}{R} + \varphi(t)e_1\right| \geq 1\right\}$ ,  $g(x, t) \equiv 0$  if  $|x| > R$  and if  $t \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$ ,  $|x| \leq R$ ,  $g(x, t) \equiv 0$ , so that the Lemma applies. Note that  $g \equiv u$  in  $B_{R-1} \times \left[\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}\right]$  where  $\left|\frac{x}{R} + \varphi(t)e_1\right| \geq 3 - 1 = 2$ . We have:

$$\begin{aligned} (i\partial_t + \Delta + V)(g) &= \theta\left(\frac{x}{R} + \varphi e_1\right) \{2\nabla\theta_R \cdot \nabla u + u\Delta\theta_R\} + \\ &\quad + \theta_R(x) \left\{2R^{-1}\nabla\theta\left(\frac{x}{R} + \varphi e_1\right) \cdot \nabla u + R^{-2}u\Delta\theta\left(\frac{x}{R} + \varphi e_1\right) + \right. \\ &\quad \left. + i\varphi'(t)\partial_{x_1}\theta\left(\frac{x}{R} + \varphi e_1\right)u\right\}. \end{aligned}$$

The first term on the right-hand side is supported in  $(B_R \setminus B_{R-1}) \times [0, 1]$ , where  $\left|\frac{x}{R} + \varphi e_1\right| \leq 4$ . Then second one is supported in  $\{(x, t) : 1 \leq \left|\frac{x}{R} + \varphi e_1\right| \leq 2\}$ . Thus,

$$\begin{aligned} \|e^{a|\frac{x}{R} + \varphi(t)e_1|^2} g\|_{L^2}^2 &\geq e^{4a} \|u\|_{L^2(B_1 \times [\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}])}^2 \geq \\ &\geq e^{4a}, \end{aligned}$$

and by the Lemma,

$$\begin{aligned} \frac{a^{3/2}}{R^2} \|e^{a|\frac{x}{R} + \varphi(t)e_1|^2} g\|_{L^2(dx dt)} &\leq C_n L \|e^{a|\frac{x}{R} + \varphi(t)e_1|^2} g\|_{L^2(dx dt)} + \\ &+ C_n e^{16a} \delta(R) + C_n e^{4a} A, \end{aligned}$$

provided  $a \geq C_n R^2$ . If we choose  $a = C_n R^2$ , for  $R$  large we can hide the first term on the right-hand side in the left-hand side to obtain

$$R e^{4a} \leq \tilde{C}_n e^{16a} \delta(R) + \tilde{C}_n e^{4a} A,$$

so that  $R \leq \tilde{C}_n e^{12a} \delta(R) + \tilde{C}_n A$ , and for  $R$  large, depending on  $A$ , we obtain  $R \leq 2\tilde{C}_n e^{12a} \delta(R)$ , which, since  $a = C_n R^2$ , is the desired result.  $\square$

**QUESTION 4:** *Can one obtain sharper versions of Theorem 6, in the spirit of the uncertainty principle of Hardy. For instance, assume  $u_0 \in H^1(e^{-a_0|x|^2} dx)$  for a fixed  $a_0 > 0$  and  $u_1 \in H^1(e^{-k|x|^2} dx)$  for all  $k > 0$ . Prove, for the class of  $V$  as in Theorem 6, that  $u \equiv 0$ .*

QUESTION 5: *Extend the results of Theorem 6 to a more general class of potentials as in [I-K 1] and add gradient terms as in [I-K 2].*

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