Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
123° (2005), Vol. XXIX, fasc. 1, pagg. 101-110

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Some Estimates for Non-Linear Homogenization

ABSTRACT. — Under minimal assumptions of regularity we discuss estimates for zero and first approximation to the solution of non-linear monotonic equation of the second order with highly oscillating symbol. They are analogues of operator estimates for resolvent in linear case obtained before by several authors.

1. We study the homogenization problem for the nonlinear equation in the whole space $\mathbb{R}^n$:

$$u_\varepsilon \in H^1(\mathbb{R}^n),\ A_\varepsilon u_\varepsilon + u_\varepsilon \equiv -div\, a\Big(\frac{x}{\varepsilon}, \nabla u_\varepsilon\Big) + u_\varepsilon = f(x) \in L^2(\mathbb{R}^n), \tag{1}$$

where $a(y,\xi)$ is a vector-valued monotonic function that verifies suitable growth condition in ξ and is 1-periodic in y, $\varepsilon > 0$ and tends to zero.

The homogenized equation associated to (1) is the following (see [JKO])

$$u_0 \in H^1(\mathbb{R}^n),\ A_0 u_0 + u_0 \equiv -div\, a^0(\nabla u_0) + u_0 = f(x), \tag{2}$$

where a_0 is a function depending only on ξ that will be defined later.

The simplest result of homogenization of the equation (1) consists in L^2-convergence

$$\|u_\varepsilon - u_0\|_{L^2(\mathbb{R}^n)} \to 0 \ \ \forall f \in L^2(\mathbb{R}^n). \tag{3}$$

We are interested in operator type estimate in L^2-space

$$\|u_\varepsilon - u_0\|_{L^2(\mathbb{R}^n)} \leq c\varepsilon \|f\|_{L^2(\mathbb{R}^n)}. \tag{4}$$

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First we recall linear case. By applying spectral or Bloch method M.S. Birman and T.A. Suslina in [BS] have recently established the estimate (4) for a broad class of linear elliptic problems in $\mathbb{R}^n$. In particular this class includes the so called acoustic equation

$$-div\, a\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon + u_\varepsilon = f(x), \tag{5}$$

where $a(y)$ is a measurable periodic symmetric matrix satisfying the boundedness and ellipticity condition

$$\lambda\xi\cdot\xi \le a(y)\xi\xi \le \lambda^{-1}\xi\cdot\xi.$$

For this equation the constant c in the estimate (4) depends only on the dimension n and the ellipticity constant λ.

On the other hand, by special analysis of the first approximation in method of asymptotic expansions there were proved in [Zh] some new type estimates for the solution of (5) and its gradient, which imply (4). These new type estimates involve additional averaging in auxiliary "shift parameter" ω.

Here we extend the approach of the paper [Zh] to non-linear homogenization problem (1). First we investigate some regularity properties of cell problem solution. Then we prove the estimate (4) under smoothness assumptions. After this in general case L^2-estimate is obtained through some "uniform approximations". Finally we derive H^1-estimate for the difference between the solution u_ε of (1) and the first approximation with smoothed corrector.

2. Assume that the vector-valued function $a(y,\xi)$ verifies the following structure conditions:

$$\begin{array}{ll} i) & a(y,\xi) \text{ is } 1-\text{periodic and measurable in } y; \\ ii) & \text{for any } y \text{ a.e. and any } \xi_1,\xi_2\in\mathbb{R}^n \\ & \qquad (a(y,\xi_1)-a(y,\xi_2))\cdot(\xi_1-\xi_2) > c_0|\xi_1-\xi_2|^2, \\ & \qquad |a(y,\xi_1)-a(y,\xi_2)| \le c_1|\xi_1-\xi_2|, \\ & \qquad a(y,0)=0. \end{array} \tag{6}$$

The function u_0 can be considered as zero approximation of the solution u_ε (see (3)). Let us recall the construction of the first approximation.

Denote $\Box = \left[-\frac{1}{2},\frac{1}{2}\right)^n$ and $\langle\cdot\rangle = \int_\Box \cdot\, dy$. Introduce the periodic problem depending on parameter $\xi\in\mathbb{R}^n$

$$N(\cdot,\xi)\in H^1_{per}(\Box),\ div_y a\big(y,\xi+\nabla_y N(y,\xi)\big)=0,\ \langle N(\cdot,\xi)\rangle = 0. \tag{7}$$

It is known that function a_0 in (2) is defined by formula

$$a^0(\xi) = \big\langle a\big(\cdot,\xi+\nabla_y N(\cdot,\xi)\big)\big\rangle. \tag{8}$$

Let us pose

$$v_\varepsilon(x,\omega) = u_0(x) + \varepsilon N(y+\omega,\nabla u_0),\ \ y=\frac{x}{\varepsilon},\ \omega\in\mathbb{R}^n, \tag{9}$$

where u_0 is the solution of the homogenized problem (2). Then function $v_\varepsilon(x) = v_\varepsilon(x, 0)$ is usually called the first approximation of the solution of (1) and its second term is known as a corrector.

Our main goal in the following result.

THEOREM 1: *The estimate* (4) *holds with constant c depending only on dimension n and* c_0, c_1 *from* (6).

To this end we prove

THEOREM 2: *The following inequality holds:*

$$\int_{\square}\int_{\mathbb{R}^n}\Big(|u_\varepsilon(x+\varepsilon\omega) - v_\varepsilon(x,\omega)|^2 + |\nabla u_\varepsilon(x+\varepsilon\omega) - \nabla v_\varepsilon(x,\omega)|^2\Big)dx d\omega \leq c\varepsilon^2 \int_{\mathbb{R}^n} f^2 dx, \tag{10}$$

where constant c depends only on dimension n and c_0, c_1 from (6).

As in [Zh] the estimate (4) is deduced from (10) in the following way. Recall that $\int_{\square} u_\varepsilon(x+\varepsilon\omega)d\omega$ is the Steklov smoothing of the solution u_ε and $\int_{\square} v_\varepsilon(x,\omega)d\omega = u_0(x)$, (see (9)). It follows from (10) and the Cauchy-Schwarz inequality that

$$\int_{\mathbb{R}^n}\left|\int_{\square} u_\varepsilon(x+\varepsilon\omega)d\omega - u_0(x)\right|^2 dx \leq c\varepsilon^2 \int_{\mathbb{R}^n} f^2 dx,$$

whence the estimate (4) is straightforward consequence if we apply the following property of Steklov smoothing (see Lemma 2 in [Zh]):

$$\int_{\mathbb{R}^n}\left|\int_{\square} \varphi(x+\varepsilon\omega)d\omega - \varphi(x)\right|^2 dx \leq \frac{\varepsilon^2 n}{4}\int_{\mathbb{R}^n} |\nabla\varphi|^2 dx, \quad \varphi \in H^1(\mathbb{R}^n). \tag{11}$$

We can clarify the role of shift parameter ω. Generally, under the condition $f \in L^2(\mathbb{R}^n)$ the first approximation $v_\varepsilon(x) = v_\varepsilon(x, 0)$ does not belong to $H^1(\mathbb{R}^n)$ or even to $L^2(\mathbb{R}^n)$. Besides, it is a problem how to give sense to the corrector $N(y, \nabla u_0)$. On the contrary, we will see that $N(y+\omega, \nabla u_0(x))$ exists as an element of $L^2(\mathbb{R}^n \times \square)$ and it is possible to consider function $N\Big(\frac{x}{\varepsilon}+\omega, \nabla u_0(x)\Big)$. Moreover $N\Big(\frac{x}{\varepsilon}+\omega, \nabla u_0(x)\Big)$ has a generalized gradient belonging to the same space. So function $N\Big(\frac{x}{\varepsilon}+\omega, \nabla u_0\Big)$ is an element of $H^1(\mathbb{R}^n)$ (for details see section 6). Parameter ω becomes useful also by estimating the residual in asymptotic expansion method (see section 5).

As we have seen, under our assumptions it is impossible to obtain usual H^1-estimate (see, e.g. [JKO], chapter 1) for the difference between the exact solution $u_\varepsilon(x)$ of the original problem and its first approximation $v_\varepsilon(x)$. But H^1-estimate is true if we take instead of $v_\varepsilon(x)$ the first approximation with so called smoothed corrector. For details see section 7. Some analogous approach to regularise a corrector was made in [G] for H^1-estimate in linear case for the elliptic problem in bounded domain.

3. In this section we establish some usefull properties of the cell problem solution $N(y,\xi)$.

LEMMA 3: *The function $\nabla_y N(y,\xi)$ is Lipschitz in ξ with values in $L^2(\square) = L^2_{per}(\square)$ and its Lipschitz constant depends only on c_0, c_1 from* (6).

PROOF: Denote $N_\xi(y) = N(y,\xi)$ and

$$I = \int_\square \big(a\big(y, \xi + \nabla_y N_\xi\big) - a\big(y, \eta + \nabla_y N_\eta\big)\big) \cdot \big(\xi + \nabla_y N_\xi - \big(\eta + \nabla_y N_\eta\big)\big) dy.$$

By monotonicity of $a(y,\cdot)$ (see (6ii)),

$$I \geq c_0 \int_\square \big|\xi + \nabla_y N_\xi - \big(\eta + \nabla_y N_\eta\big)\big|^2 dy.$$

Since $div_y a\big(y, \xi + \nabla_y N_\xi\big) = 0$ and $div_y a\big(y, \eta + \nabla_y N_\eta\big) = 0$ we have

$$I = \int_\square \big(a\big(y, \xi + \nabla_y N_\xi\big) - a\big(y, \eta + \nabla_y N_\eta\big)\big) \cdot (\xi - \eta) dy.$$

Since $a(y,\cdot)$ is Lipschitz, then

$$I \leq c_1 |\xi - \eta| \int_\square \big|\xi + \nabla_y N_\xi - \big(\eta + \nabla_y N_\eta\big)\big| dy \leq c_1 |\xi - \eta| \left(\int_\square \big|\xi + \nabla_y N_\xi - \big(\eta + \nabla_y N_\eta\big)\big|^2 dy \right)^{\frac{1}{2}}.$$

Hence

$$\left(\int_\square \big|\xi + \nabla_y N_\xi - \big(\eta + \nabla_y N_\eta\big)\big|^2 dy \right)^{\frac{1}{2}} \leq \frac{c_1}{c_0} |\xi - \eta|$$

and then $\xi + \nabla_y N_\xi$ is a Lipschitz functions in ξ with values in $L^2(\square)$ (with Lipschitz constant given explicitly above). The same is true for $\nabla_y N_\xi$. ■

The last lemma and the lipschitzianity of $a(y,\cdot)$ imply, by composition, the following results.

COROLLARY 4: *The function $a\big(y, \xi + \nabla_y N_\xi\big)$ is Lipschitz in ξ with values in $L^2(\square)$ and its Lipschitz constant depends only on c_0, c_1 from* (6).

COROLLARY 5: *The function $a^0(\xi)$ is Lipschitz and its Lipschitz constant depends only on c_0, c_1 from* (6).

The proof of the last assertion is also given in [FM].

LEMMA 6: *The function* $N(y,\xi)$ *is Lipschitz in* ξ *with values in* $L^2(\square)$ *and its Lipschitz constant depends only on dimension* n, *and on* c_0, c_1 *from* (6).

PROOF: Since $\langle N\rangle = 0$, by Poincaré inequality

$$\int_{\square} |N(y,\xi) - N(y,\eta)|^2 dy \le c_p \int_{\square} \left|\nabla_y N(y,\xi) - \nabla_y N(y,\eta)\right|^2 dy \le C|\xi-\eta|^2,$$

where constant C is controlled by c_0, c_1 from (6) and Poincaré constant c_p. ■

From Lemma 6 we have the following

COROLLARY 7: *There exists* $\nabla_\xi N(y,\xi)$ *as a bounded function of* ξ *with values in* $L^2(\square)$ *such that*

$$\int_{\square} |\nabla_\xi N(y,\xi)|^2 dy \le C,$$

where the constant C depends only on dimension n *and* c_0, c_1 *from* (6).

4. We prove Theorem 9 in several steps.

First we assume that $a(y,\xi)$ is infinitely differentiable. From the elliptic theory we have that the solution of cell problem $N(y,\xi)$ is also infinitely differentiable together with the limit function $a^0(\xi)$. We also assume that $f \in C_0^\infty(\mathbb{R}^n)$.

In this case the solution u_0 of the homogenized problem is differentiable and exponentially decays to zero with all its derivatives.

Now consider the first approximation $v_\varepsilon(x) = v_\varepsilon(x,0)$ (see (9)). Its full gradient is

$$\nabla v_\varepsilon(x) = \nabla u_0(x) + \nabla_y N(y, \nabla u_0) + \varepsilon \nabla^2 u_0 \nabla_\xi N.$$

We pose

$$a(y, \nabla v_\varepsilon) = a\big(y, \nabla u_0 + \nabla_y N(y, \nabla u_0)\big) + r_\varepsilon^1, \quad y = \frac{x}{\varepsilon},$$

where

$$r_\varepsilon^1 = a(y,\nabla v_\varepsilon) - a\big(y, \nabla u_0 + \nabla_y N\big). \tag{12}$$

Since $a(y,\cdot)$ is Lipschitz function, we have

$$\left|r_\varepsilon^1\right| \le c_1 \left|\nabla v_\varepsilon - \big(\nabla u_0 + \nabla_y N\big)\right| = c_1 \varepsilon \left|\nabla^2 u_0 \nabla_\xi N\right|. \tag{13}$$

Denote

$$g(y,\xi) = a\big(y, \xi + \nabla_y N\big) - a^0(\xi).$$

By (7) and (9), we have

$$div_y g(y,\xi) = 0 \text{ and } \langle g(\cdot,\xi)\rangle = 0.$$

By representation of solenoidal vectors (see [JKO], §1, chapter 1) there exists a matrix $G(y,\xi)$ such that

$$\begin{aligned} &1)\quad G(y,\xi) \text{ is differentiable in } y,\xi,\\ &2)\quad G(y,\xi) \text{ is skew-symmetric},\\ &3)\quad g(y,\xi) = div_y G(y,\xi). \end{aligned}$$

Moreover $G(y,\xi)$ is a Lipschitz function in ξ with values in $H^1(\Box)$ and with Lipschitz constant depending on dimension n and c_0, c_1. In fact

$$g(y,\xi) = a\big(y,\xi+\nabla_y N_\xi\big) - a^0(\xi) = a\big(y,\xi+\nabla_y N_\xi\big) - \big\langle a\big(y,\xi+\nabla_y N_\xi\big)\big\rangle$$

is Lipschitz with values in $L^2(\Box)$ by corollaries 4, 5.

According to construction given in [JKO], the transformation of vectors into matrices $g(y,\xi)\to G(y,\xi)$ is a linear operator and so $G(y,\cdot)$ is Lipschitz. Then we can deduce that there exists $\nabla_\xi G(y,\xi)$ such that

$$\int_\Box |\nabla_\xi G(y,\xi)|^2 dy \le C,$$

where C is a constant dependent on n, c_0, c_1.

We have

$$(14)\quad a\big(y,\nabla u_0+\nabla_y N\big) - a^0(\nabla u_0) = g(y,\nabla u_0) = div_y G(y,\nabla u_0) = \\ = \varepsilon div G(y,\nabla u_0) - \varepsilon div_x G(y,\nabla u_0) \equiv r_\varepsilon^2, \quad y=\frac{x}{\varepsilon}.$$

We observe that $divG(y,\nabla u_0)$ is a solenoidal vector. In fact, since G is a skew-symmetric matrix and $\nabla^2\varphi$ is a symmetric one, we have

$$\int_{\mathbb{R}^n} divG(y,\nabla u_0)\cdot\nabla\varphi dx = -\int_{\mathbb{R}^n} G(y,\nabla u_0)\cdot\nabla^2\varphi dx = 0\quad \forall\varphi\in C_0^\infty(\mathbb{R}^n).$$

So

$$(15)\quad div\, r_\varepsilon^2 = -\,\varepsilon div[div_x G(y,\nabla u_0)] = -\varepsilon div\Big[(\nabla^2 u_0)_{ij}\frac{\partial}{\partial\xi_i}G_{pj}(y,\nabla u_0)\Big], y=\frac{x}{\varepsilon},\ p=1,..,n.$$

Then

$$div\Big[a\Big(\frac{x}{\varepsilon},\nabla v_\varepsilon\Big) - a^0(\nabla u_0)\Big] = div\big(r_\varepsilon^1 + r_\varepsilon^2\big).$$

According to previous section, we have

$$\begin{aligned} A_\varepsilon(v_\varepsilon-u_\varepsilon)+(v_\varepsilon-u_\varepsilon) &= -div\, a\Big(\frac{x}{\varepsilon},\nabla v_\varepsilon\Big) + v_\varepsilon + div\, a\Big(\frac{x}{\varepsilon},\nabla u_\varepsilon\Big) - u_\varepsilon = \\ &= -div\, a\Big(\frac{x}{\varepsilon},\nabla v_\varepsilon\Big) + v_\varepsilon - f = \\ &= -div\, a\Big(\frac{x}{\varepsilon},\nabla v_\varepsilon\Big) + v_\varepsilon + div\, a^0(\nabla u_0) - u_0 = \\ &= -div\Big[a\Big(\frac{x}{\varepsilon},\nabla v_\varepsilon\Big) - a^0(\nabla u_0)\Big] + v_\varepsilon - u_0 = \\ &= -div\big(r_\varepsilon^1 + r_\varepsilon^2\big) + \varepsilon r_\varepsilon^0, \end{aligned}$$

where $r_\varepsilon^0 = N(y,\nabla u_0)$, $r_\varepsilon^1, r_\varepsilon^2$ are given in (12), (14) and (15).

For the equation $A_\varepsilon z^\varepsilon + z^\varepsilon = f_0 + divF$ the following estimate holds

$$\|z^\varepsilon\|^2_{H^1(\mathbb{R}^n)} \leq C\Big(\|f_0\|^2_{L^2(\mathbb{R}^n)} + \|F\|^2_{L^2(\mathbb{R}^n)}\Big). \tag{16}$$

Here and hereafter we denote by C a constant that depends only on the dimension n and c_0, c_1 from (6).

So

$$\|u_\varepsilon - v_\varepsilon\|^2_{H^1(\mathbb{R}^n)} \leq C\varepsilon^2 \int\limits_{\mathbb{R}^n} |r^0_\varepsilon|^2 dx + C \int\limits_{\mathbb{R}^n} |r^1_\varepsilon + r^2_\varepsilon|^2 dx.$$

By (13) and (15), we obtain

$$\|u_\varepsilon - v_\varepsilon\|^2_{H^1(\mathbb{R}^n)} \leq C\varepsilon^2 \int\limits_{\mathbb{R}^n} \left(N^2\Big(\frac{x}{\varepsilon}, \nabla u_0\Big) + \left|b\Big(\frac{x}{\varepsilon}, \nabla u_0\Big)\right|^2 |\nabla^2 u_0|^2 \right) dx, \tag{17}$$

where $b(y, \xi)$ are functions of the form $\nabla_\xi N(y, \xi)$, $\nabla_\xi G_{\cdot j}(y, \xi)$. In general case functions $b\Big(\dfrac{x}{\varepsilon}, \nabla u_0\Big)$ do not belong to $L^\infty(\mathbb{R}^n)$ and, hence, cannot be excluded from the above estimate. The appropriate estimate of $N\Big(\dfrac{x}{\varepsilon}, \nabla u_0\Big)$ is also problematic. Now we shall cope with these difficulties in the same way as in [Zh].

5. Along with original problem (1), we consider the problems corresponding to the "shifted" function $a(y + \omega, \xi)$, where $\omega \in \Box$,

$$-div\, a\Big(\frac{x}{\varepsilon} + \omega, \nabla u_\varepsilon(x, \omega)\Big) + u_\varepsilon(x, \omega) = f(x) \text{ in } \mathbb{R}^n, \tag{18}$$

with the same right-hand side $f(x)$. Equation (1) is obtained if $\omega = 0$, i.e. $u_\varepsilon(x, 0) = u_\varepsilon(x)$. Note that homogenized function $a^0(\xi)$ does not depend on ω, because the function $N(y, \omega, \xi)$ as well as the remaining functions $b(y, \omega, \xi)$ are obtained from the initial ones by means of a shift: $b(y, \omega, \xi) = b(y + \omega, \xi)$. In particular, the first approximation is given by (9).

For every $\omega \in \Box$ the estimate of the type (17) holds, that is

$$\int\limits_{\mathbb{R}^n} \Big(|u_\varepsilon(x, \omega) - v_\varepsilon(x, \omega)|^2 + |\nabla u_\varepsilon(x, \omega) - \nabla v_\varepsilon(x, \omega)|^2 \Big) dx \leq$$
$$\leq C\varepsilon^2 \int\limits_{\mathbb{R}^n} \left(N^2\Big(\frac{x}{\varepsilon} + \omega, \nabla u_0\Big) + \left|b\Big(\frac{x}{\varepsilon} + \omega, \nabla u_0\Big)\right|^2 |\nabla^2 u_0|^2 \right) dx.$$

Integrating this estimate with respect to $\omega \in \Box$ and using properties of the functions $N(y, \xi)$, $b(y, \xi)$ (see sections 3,4), we obtain

$$\begin{aligned} (19)\quad & \int\limits_{\Box}\int\limits_{\mathbb{R}^n} \Big(|u_\varepsilon(x, \omega) - v_\varepsilon(x, \omega)|^2 + |\nabla u_\varepsilon(x, \omega) - \nabla v_\varepsilon(x, \omega)|^2 \Big) dx d\omega \leq \\ & \leq C\varepsilon^2 \int\limits_{\Box}\int\limits_{\mathbb{R}^n} \left(N^2\Big(\frac{x}{\varepsilon} + \omega, \nabla u_0\Big) + \left|b\Big(\frac{x}{\varepsilon} + \omega, \nabla u_0\Big)\right|^2 |\nabla^2 u_0|^2 \right) dx d\omega \leq \\ & \leq C\varepsilon^2 \int\limits_{\mathbb{R}^n} \Big(|\nabla u_0|^2 + |\nabla^2 u_0|^2 \Big) dx \leq C\varepsilon^2 \int\limits_{\mathbb{R}^n} f^2 dx, \end{aligned}$$

since $\|u_0\|_{H^2(\mathbb{R}^n)}$ can be estimated by $\|f\|_{L^2(\mathbb{R}^n)}$ in virtue of equation (2).

It remains to compare solution $u_\varepsilon(x,\omega)$ of the shifted problem (18) with shifted function $u_\varepsilon(x+\varepsilon\omega)$, where u_ε is solution of the initial problem (1). To this end we observe that $u_\varepsilon(x+\varepsilon\omega)$ is solution of (18) with $f(x+\varepsilon\omega)$ instead of f. So it is sufficient to compare the right terms $f(x)$ and $f(x+\varepsilon\omega)$ and apply the energy estimate.

By Lemma 3 in [Zh], we have

$$\|f(\cdot+\varepsilon\omega)-f(\cdot)\|_{H^{-1}(\mathbb{R}^n)}\leq \varepsilon|\omega|\|f\|_{L^2(\mathbb{R}^n)}\ \forall f\in L^2(\mathbb{R}^n).$$

Then by the energy estimate we obtain

$$\int\limits_{\mathbb{R}^n}\Big(|u_\varepsilon(x,\omega)-u_\varepsilon(x+\varepsilon\omega)|^2+|\nabla u_\varepsilon(x,\omega)-\nabla u_\varepsilon(x+\varepsilon\omega)|^2\Big)dx\leq C\varepsilon^2\int\limits_{\mathbb{R}^n}f^2dx\ \ \forall\omega\in\Box,$$

and so we can replace in (19) $u_\varepsilon(x,\omega)$ with $u_\varepsilon(x+\varepsilon\omega)$.

6. We have proved estimate (10) under the assumption that $a(y,\xi)$ was infinitely differentiable and $f\in C_0^\infty(\mathbb{R}^n)$. In general case it is possible to find sequences of infinitely differentiable functions $a^\delta(y,\xi)$ and $f^\delta\in C_0^\infty(\mathbb{R}^n)$ such that

$$a^\delta(y,\xi)\to a(y,\xi)\text{ for a.e. } y\in\Box,\ \forall\xi\in\mathbb{R}^n,$$
$$f^\delta\to f\text{ in } L^2(\mathbb{R}^n).$$

For every $\delta>0$, let us consider the problem (1) with a^δ and f^δ instead of a and f

$$A_\varepsilon^\delta u_\varepsilon^\delta+u_\varepsilon^\delta\equiv -div\, a^\delta\Big(\frac{x}{\varepsilon},\nabla u_\varepsilon^\delta\Big)+u_\varepsilon^\delta=f^\delta(x) \tag{20}$$

with its set of homogenization attributes. That is homogenized equation

$$A_0^\delta u_0^\delta+u_0^\delta\equiv -div\, a_0^\delta\big(\nabla u_0^\delta\big)+u_0^\delta=f^\delta(x), \tag{21}$$

cell problem with solution $N^\delta(y,\xi)$ and the corresponding shifted family of first approximations

$$v_\varepsilon^\delta(x,\omega)=u_0^\delta(x)+\varepsilon N^\delta\big(y,\nabla u_0^\delta(x)\big),\ y=\frac{x}{\varepsilon}.$$

Lemma 8: *The solution u_ε^δ of the problem* (20) *converges in $L^2(\mathbb{R}^n)$ to the solution u_ε of the problem* (1) *when $\delta\to 0$.*

Lemma 9: *Let u_0 and u_0^δ be the solutions of the problems* (2) *and* (21). *Then*

$$u_0^\delta\in H^2(\mathbb{R}^n),\ \|u_0^\delta\|_{H^2(\mathbb{R}^n)}\leq c\|f\|_{L^2(\mathbb{R}^n)},$$

where constant c depends on dimension n and on c_0, c_1, and

$$u_0^\delta\to u_0\text{ in } H^2(\mathbb{R}^n).$$

We observe that the constant c in estimate of the type (10) for the equation (20) does not depend on δ. Therefore it remains only to pass to the limit in it in terms $v_\varepsilon^\delta(x,\omega)$ and $\nabla v_\varepsilon^\delta(x,\omega)$ to obtain (10) itself. To this end, the following lemma is available.

LEMMA 10: *Let us assume that*

i) $M^{\delta}(y,\xi)$ is infinitely differentiable function such that:

(22) $$\int_{\square}\left|M^{\delta}(\cdot,\xi)\right|^2 dy \le c|\xi|^2,\ \int_{\square}\left|\nabla_y M^{\delta}(\cdot,\xi)\right|^2 dy \le c|\xi|^2,\ \int_{\square}\left|\nabla_\xi M^{\delta}(\cdot,\xi)\right|^2 dy \le c,$$

where c does not depend on δ;

ii) $M^{\delta}(\cdot,\xi) \to M(\cdot,\xi)$ in $H^1(\square)$ $\forall \xi \in \mathbb{R}^n$;
iii) $\Phi^{\delta} \to \Phi$ in $H^1(\mathbb{R}^n)^n$.

Then for a.e. $\omega \in \square$ *the function*

$$P(x,\omega) = M(x+\omega, \Phi(x))$$

belongs to $H^1(\mathbb{R}^n)$ *and its generalized gradient is defined by formula*

$$\nabla_x P(x,\omega) = \nabla_y M(x+\omega,\Phi(x)) + \nabla\Phi \cdot \nabla_\xi M(x+\omega,\Phi(x)).$$

Moreover if $P^{\delta}(x,\omega) = M^{\delta}\left(x+\omega, \Phi^{\delta}(x)\right)$, *then*

$$P^{\delta}(\cdot,\omega) \to P(\cdot,\omega) \text{ in } H^1(\mathbb{R}^n) \ \textit{for} \text{ a.e. } \omega \in \square.$$

This lemma can be applied to the solutions of cell problems $N^{\delta}(y,\xi)$ $(= M^{\delta}(y,\xi))$ and $\nabla u_0^{\delta}(x)$ $(= \varphi^{\delta}(x))$. Here we also take into account the following

LEMMA 11: *For each* $\xi \in \mathbb{R}^n$ *we have*

$$N^{\delta}(\cdot,\xi) \to N(\cdot,\xi) \quad \text{in } H^1(\mathbb{R}^n)$$

and conditions (22) *are satisfied uniformly in* δ.

7. After the change of variables $x + \varepsilon\omega \to x$ and order of integration due to the Cauchy-Schwarz inequality it follows from the estimate (10) that

(23) $$\int_{\mathbb{R}^n}\left(|u_\varepsilon(x) - \widetilde{v}_\varepsilon(x)|^2 + |\nabla u_\varepsilon(x) - \nabla\widetilde{v}_\varepsilon(x)|^2\right) dx d\omega \le c\varepsilon^2 \int_{\mathbb{R}^n} f^2 dx,$$

where the function

(24) $$\widetilde{v}_\varepsilon(x) = \int_{\square} u_0(x-\varepsilon\omega)d\omega + \varepsilon\int_{\square} N\Big(\frac{x}{\varepsilon}, \nabla u_0(x-\varepsilon\omega)\Big)d\omega$$

can be called as a smoothed first approximation. Applying inequality (11) to functions $u_0(x)$ and $\nabla u_0(x)$ we eliminate in (24) smoothing of the zero approximation $u_0(x)$ and, hence, replace in (23) $\widetilde{v}_\varepsilon(x)$ with

(25) $$\widehat{v}_\varepsilon(x) = u_0(x) + \varepsilon\int_{\square} N\Big(\frac{x}{\varepsilon}, \nabla u_0(x-\varepsilon\omega)\Big)d\omega,$$

that is first approximation with smoothed corrector. So the following result is valid.

THEOREM 12: *It is true that*

$$\int\limits_{\mathbb{R}^n} \Big(|u_\varepsilon(x) - \widehat{v}_\varepsilon(x)|^2 + |\nabla u_\varepsilon(x) - \nabla \widehat{v}_\varepsilon(x)|^2 \Big) dx d\omega \leq c\varepsilon^2 \int\limits_{\mathbb{R}^n} f^2 dx,$$

where $\widehat{v}_\varepsilon(x)$ *is defined in* (25)*, c is a constant depending only on dimension n and* c_0, c_1 *from* (6).

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Direttore responsabile: Prof. A. BALLIO - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
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