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Homogenization of Nonvariational Viscosity Solutions

ABSTRACT. — In this paper, we are going to consider the possibility of multiple solutions for minimal surfaces and free boundaries in "oscillating media". We give simple one or two dimensional examples to point out the possibility of hysteresis phenomena in, for instance, movement by mean curavture or flame propagation. These solutions are not variational solutions which can be obtained by minimizing corresponding energies. We can show that these elliptic nonvariational solutions can be barriers for the corresponding parabolic flows, for example mean curvature flow or one phase flame propagation with a suitable initial data, and that it can be also the limit of the flows as the time goes to infinity.

1. - MEAN CURVATURE SURFACE

In this section, we will consider graphs with oscillating mean curvature f_ε and show the existence of many nonvariational solutions. The importance of these nonvariational solutions is due to the fact that they can be barriers and ultimately become the limit of a mean curvature flow for an initial data trapped between two of these solutions. We are going to consider the one dimensional problem in order to demonstrate the issue above in as simple as possible setting. It can be also generalized easily to the multi-dimensional problem.

First, let us consider a nonnegative function: Set $f(z,x)$ be a nonnegative smooth function on $\mathbb{R}\times\mathbb{R}$ such that

$$f(z+k_1, x+k_2) = f(z,x) \quad \text{for} \quad (k_1,k_2)\in\mathbb{Z}\times\mathbb{Z}$$

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and

$$f(0,x) = f(k,x) = 0 \quad \text{for} \quad k \in \mathbb{Z}.$$

The surface $u^\varepsilon(x)$ having a oscillating mean curvature with ε-periodicity satisfies the following equation:

$$(MCE_\varepsilon) \qquad \mathcal{M}u^\varepsilon = \left(\frac{u^\varepsilon_x}{\sqrt{1+|u^\varepsilon_x|^2}} \right)_x = f_\varepsilon(u^\varepsilon, x) = f\left(\frac{u^\varepsilon}{\varepsilon}, \frac{x}{\varepsilon}\right) \quad \text{in } \Omega$$

with a boundary condition $u^\varepsilon = 0$ on $\partial\Omega$. The solutions of (MCE_ε) can be founded as critical points of the following variational problem: find a minimizer $u^\varepsilon \in BV(\Omega)$ of the energy

$$(VMCE_\varepsilon) \qquad I_\varepsilon(v) = \int_\Omega \sqrt{1+v_x^2} + F_\varepsilon(v,x)dx$$

among $v \in \mathrm{BV}(\Omega)$ such that

$$v = 0 \quad \text{on } \partial\,\Omega$$

where Ω is a bounded domain $[-2,2]$ in $\mathbb{R}$ and $F_\varepsilon(\cdot,x)$ is the anti-derivative of $f_\varepsilon(\cdot,x)$.

1.1. *The variational solutions*

Let u_ε be the minimizer in $(VMCE_\varepsilon)$. From the standard regularity theory, u^ε will be bounded convex and have a uniform $C^{1,1}$-estimate independent of ε. Hence we can extract a subsequence u^{ε_i} converging to u_o uniformly. Now we are going to show u be the minimizer of the homogenized variational problem.

Lemma 1.1: *The limit u will be the minimizer of the following homogenized energy*

$$(VHMC) \qquad I(v) = \int \sqrt{1+|v_x|^2} + \overline{f}v dx$$

where $\overline{f}$ *is the average of* $f(z,x)$ *in* $[0,1]^2$.

Proof: Due to the uniform $C^{1,1}$-estimate of u^ε, ∇u^ε converges uniformly to ∇u_o. Hence we have

$$\int \sqrt{1+|u^\varepsilon_x|^2}dx \to \int \sqrt{1+|u_x|^2}dx.$$

Let $\tilde{f} = f - \langle f \rangle_z$ where $\langle f \rangle_z$ is the average of $f(z,x)$ in z-variable. Then the average of $\tilde{f}$

will be zero. A anti-derivative F_ε of f_ε can be given by

$$\begin{aligned} F_\varepsilon(u,x) &= \int_0^u f_\varepsilon(z,x)dz = \int_0^u \overline{f} + \tilde{f}\Big(\frac{u}{\varepsilon}\Big)dz \\ &= \langle f\rangle_z u + \varepsilon \int_0^{\frac{u}{\varepsilon}} \tilde{f}(z,x)dz \\ &= \langle f\rangle_z u + \varepsilon \int_{[\frac{u}{\varepsilon}]}^{\frac{u}{\varepsilon}} \tilde{f}(z,x)dz \\ &= \langle f\rangle_z u + \varepsilon B(u,\varepsilon,x). \end{aligned}$$

where $B(u,\varepsilon)$ is a bounded function. Therefore we have

$$\begin{aligned} I_\varepsilon(u^\varepsilon) &= \int_\Omega \sqrt{1+|\nabla u^\varepsilon|^2} + F_\varepsilon\Big(u^\varepsilon, \frac{x}{\varepsilon}\Big)dx \\ &= \int_\Omega \sqrt{1+|u_x^\varepsilon|^2} + \langle f\rangle_z\Big(\frac{x}{\varepsilon}\Big)u^\varepsilon + \varepsilon B\Big(u,\varepsilon,\frac{x}{\varepsilon}\Big)du \\ &\to \int_\Omega \sqrt{1+|u_x|^2} + \overline{f}udx = I(u). \end{aligned}$$

as ε goes to zero. In addition, a similar argument tells us u is the minimizer of the energy $I(v)$. For any BV function v, we know $I_\varepsilon(u^\varepsilon) \le I_\varepsilon(v)$. By passing the limit similarly, we have $I(u) \le I(v)$. □

REMARK 1.2: u above satisfies the constant mean curvature equation $\mathcal{M}u = \overline{f}$ and the graph of u is a piece of a circle of the radius $\frac{1}{\overline{f}}$.

1.2. *Nonvariational solution*

We now show the existence of many nonvariational solutions.

EXAMPLE 1.3: For the simplicity, let us choose f such that $\overline{f} = \frac{1}{2}$ and set $\Omega = [-2,2]$. Then the variational solution u is a half circle of radius 2 centered at the zero. Since $f(k) = 0$ for any $k \in \mathbb{Z}$, $h = -\varepsilon\left[\frac{1}{\varepsilon}\right]$ will be a solution (MCE_ε). The variational solution u^ε is close to u with an error of $O(\varepsilon)$. Now we want to find a solution of (MCE_ε) bigger than $-\varepsilon\left[\frac{1}{\varepsilon}\right] \approx -1$ and having the boundary value 0 on $\partial\Omega$. Let Let $\mathcal{A}_\varepsilon$ be the collection super-solutions v of (MCE_ε) satisfying $v \ge h$, and $v \ge 0$ on $\partial\Omega$.

First of all $\mathcal{A}_\varepsilon$ is not empty because a vertical translation of the variational solution $u^\varepsilon + 1 + C\varepsilon$ satisfies the condition of $\mathcal{A}_\varepsilon$ for a uniform constant C. Now we choose the infimum h^*_ε of v in $\mathcal{A}_\varepsilon$ i.e.

$$h^*_\varepsilon(x) = \inf_{v \in \mathcal{A}_\varepsilon} v(x).$$

From the uniform $C^{1,1}$-estimate and Harnack, we can follow Perron's method to show h^*_ε satisfy (MCE_ε) and belongs to $\mathcal{A}_\varepsilon$. Now we claim $h^*_\varepsilon(x) = 0$ on $\partial\Omega$. First we can notice any constant linear function is a super-solution since f_ε is nonnegative. If the claim fails, we can find a super solution $\min(h^*_\varepsilon, 0)$ which is smaller than h^*_ε which implies there is a solution smaller than h^*_ε and satisfying the condition of $\mathcal{A}_\varepsilon$. it gives a contradiction. In addition, from $\min(u^\varepsilon + 1 + C\varepsilon, 0) \geq h^*_\varepsilon > h$, we have $|\min h^*_\varepsilon + 1| < C\varepsilon$. Therefore

Lemma 1.4: *For $\overline{f} = \dfrac{1}{2}$, there is a solution h^*_ε of (MCE_ε) such that its limit h^* is not a minimizer of $I(v)$ in* (1.1) *and $h^*(0) = -1$.*

Example 1.5: Let $e_n = \dfrac{1}{2^n}$. We can pick a support of $f(z,x)$ so that the lines $z = x - 2 = l_1(x)$ and $z = \dfrac{1}{2}(x-2) = l_2(x)$ don't meet the support of $f_\varepsilon(z,x) = f\left(\dfrac{z}{\varepsilon_n}, \dfrac{x}{\varepsilon_n}\right)$. Clearly $f_\varepsilon(-1,x) = f_\varepsilon\left(-\dfrac{1}{2}, x\right) = 0$. Hence the affine functions $L_1(x) = \max\left(l_1(-x), -\dfrac{1}{2}, l_1(x)\right)$ and $L_2(x) = \max(l_2(-x), -1, l_2(x))$ do not meet the supports of $f_\varepsilon(z,x)$. Therefore the maximums of linear functions lying outside of the support of $f_\varepsilon(z,x)$ will be sub-solutions.

Now we consider the minimizing problem of $I_\varepsilon(v)$ on the condition that v is above the $L_1(x)$ or $L_2(x)$,and let $h^+_\varepsilon(x)$ and $h^-_\varepsilon(x)$ be the corresponding minimizers respectively. Then clearly $h^-_\varepsilon(x) < h^+_\varepsilon(x)$. The uniform $C^{1,1}$-estimates of $h^+_\varepsilon(x)$ and $h^-_\varepsilon(x)$ make them converges to the limit $h^+(x)$ and $h^-(x)$ respectively.

Lemma 1.6: *$h^\pm(x)$ are the minimizers of the homogenized energy $I(v)$ on the condition $v \geq L_1(x)$ or $v \geq L_2(x)$ respectively. In addition $h^\pm(x)$ may coincide with $L_1(x)$ or $L_2(x)$ for x in some segments. Otherwise $h^\pm(x)$ will be an arc of the circle with radius $\dfrac{1}{\overline{f}}$.*

Proof: From an argument similar to Lemma (1.1), the uniform limit $h^\pm$ are the minimizers of $I(v)$ on the condition $v \geq L_1(x)$ or $v \geq L_2(x)$ respectively. If $h^\pm$ doesn't coincide with the obstacle $L_1(x)$ and $L_2(x)$ respectively, then $h^\pm$ satisfies the equation $\mathcal{M}h^\pm = \overline{f}$, which implies that the graph of $h^\pm$ are a piece of circle of radius $\dfrac{1}{\overline{f}}$ □

1.3. *Mean curvature flows*

If we start a mean curvature flow with an initial data trapped between $h^\pm$, it alway stay between them since $h^\pm$ also satisfy the mean curvature flow equation. This flow also has a

limit as the time goes to infinity due to the $C^{1,\alpha}$-estimate and the limit will be a stationary solution, still between these two barriers $h^{\pm}$. Hence the limit stationary solution will not be the variational solution.

2. - Flame Propagation

In this section, we consider an one-phase free boundary problem for the heat equation, describing the laminar flames as an asymptotic limit for the high activation energy model [CV]. We will consider the case that the different reaction materials are distributed periodically forming layers, and find the homogenized limit of the flame flows when the periodicity layers goes to zero. Set $x = (x_1, x_2) \in \mathbb{R}^2$ and let $f(x_2)$ be a periodic function with a periodicity 1 and $\lambda < f < \Lambda$. The classical formulation of the problem is the following: for a given initial data u_o whose positive region is $\Omega_o = \{ u_o > 0 \}$, find a domain $\Omega \subset Q_T = \mathbb{R}^2 \times (0, T)$ (the unburnt area) and a function u which is strictly positive in Ω and smooth on Ω, up to the *interface* $\Gamma = \partial\Omega \cap Q_T$, such that

$$(FB_\varepsilon) \qquad \begin{cases} u_t = \triangle u & \text{in } \Omega \\ u = 0, |\nabla u|^2 = f\left(\dfrac{x_2}{\varepsilon}\right) & \text{at } \Gamma \\ u = 1 & \text{on } B_1 \\ u(x,0) = u_o(x) & \text{on } \Omega_o \end{cases}$$

Such an equation naturally arises as the asymptotic limit (δ goes to zero) of the following reaction-diffusion equation:

$$(FB_{\varepsilon,\delta}) \qquad \begin{cases} u_t = \triangle u - f_\varepsilon(x_2)\beta_\delta(u) & \text{in } \mathbb{R}^2 \setminus B_1 \\ u = 1 & \text{on } B_1 \\ u(x,0) = u_o(x) & \text{in } \mathbb{R}^2 \setminus B_1 \end{cases}$$

where the reaction term is defined by $\beta_\delta(s) = \dfrac{1}{\delta}\beta\left(\dfrac{s}{\delta}\right)$, with $\beta(s)$ a Lipschitz function satisfying:

$$(2.1) \qquad \begin{cases} \beta(s) > 0 \text{ in } (0,1), \text{ and } \beta(s) = 0 \text{ otherwise.} \\ 1 = \displaystyle\int_0^1 \beta(s)ds. \end{cases}$$

The stationary solution of (FB_ε) will satisfies the following elliptic problem:

$$(EFB_\varepsilon) \qquad \begin{cases} \triangle u = 0 & \text{in } \Omega \\ u = 0, \ |\nabla u|^2 = f\left(\dfrac{x_2}{\varepsilon}\right) & \text{at } \Gamma \\ u = 1 & \text{on } B_1 \end{cases}$$

The solution of (EFB_ε) can be approximated by the solutions of the following singular

perturbation problems:

$$(EFB_{\varepsilon,\delta}) \qquad \begin{cases} \triangle u = f_\varepsilon(x_2)\beta_\delta(u) & \text{in } \mathbb{R}^2 \setminus B_1 \\ u = 1 & \text{on } B_1 \end{cases}$$

The solutions of (EFB_ε) and $(EFB_{\varepsilon,\delta})$ can be the critical points of the following energies:

$$(VEFB_\varepsilon) \qquad I_{\varepsilon,\delta}(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 + f_\varepsilon(x_2)\chi_{\{v>0\}}dx$$

and

$$(VEFB_{\varepsilon,\delta}) \qquad I_{\varepsilon,\delta}(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 + f_\varepsilon(x_2)B_\delta(v)dx$$

respectively where $B_\delta(s)' = \beta_\delta(s)$.

2.1. *Variational minimizers*

The variational solution was studied in [ACF]. Now we are going to summarize the known results.

LEMMA 2.1: *(ACF)*

$H^{n-1}(D \cap \partial\{u > 0\}) < \infty$ for every $D \subset\subset \Omega$.
$|\nabla u| \leq C$ and $|\nabla u|_{L^2_{loc}(\Omega)} < C$.

For any $D \subset\subset \Omega$, there exist positive constants c, C such that if $B_r(x)$ is a ball in $D \cap \{u > 0\}$, then

$$cr \leq u(x) \leq Cr.$$

LEMMA 2.2: *Let u^ε be the minimizer of $I_\varepsilon(v)$ in $(VEFB_\varepsilon)$. Then the limit u of u^ε will be the minimizer of the homogenized energy*

$$I(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 + \overline{f}\chi_{\{v>0\}}dx.$$

PROOF: From the estimates in 2.1, we know

$$\nabla u^\varepsilon \to u \quad \text{weakly in } L^2_{loc}(\Omega)$$

and

$$u^\varepsilon \to u \quad \text{uniformly in } C^a_{loc}, \text{ for all } a < 1.$$

In addition from the finite (n-1)-Hausdorff measure of the Free boundary, it follows, [ACF],that the set $\{u^\varepsilon > 0\}$ has a finite perimeter uniformly in any compact subset in Ω

which means $-\nabla\chi_{\{u^\varepsilon>0\}}$ is a Borel measure. We also have

$$\begin{aligned} \partial\{u^\varepsilon > 0\} &\to \partial\{u > 0\} && \text{locally in Hausdorff distance,} \\ \chi_{\{u^\varepsilon>0\}} &\to \chi_{\{u>0\}} && \text{in } L^1_{loc}, \\ \nabla u^\varepsilon &\to \nabla u && \text{a.e.} \end{aligned}$$

Hence

$$\begin{aligned} I_\varepsilon(u^\varepsilon) &= \int_\Omega \frac{1}{2}|\nabla u^\varepsilon|^2 + f_\varepsilon(x_2)\chi_{\{u^\varepsilon>0\}}dx \\ &= \int_\Omega \frac{1}{2}|\nabla u^\varepsilon|^2 + f_\varepsilon(x_2)(\chi_{\{u^\varepsilon>0\}} - \chi_{\{u>0\}}) + f_\varepsilon(x_2)\chi_{\{u>0\}}dx \\ &\to \int_\Omega \frac{1}{2}|\nabla u|^2 + \overline{f}\chi_{\{u>0\}}dx = I(u) \end{aligned}$$

since f_ε is bounded. □

Hence the limit u will satisfies the following free boundary problems:

$$(HEFB) \qquad \begin{cases} \triangle u = 0 & \text{in } \Omega \\ u = 0,\ |\nabla u|^2 = \overline{f} & \text{at } \Gamma = \partial\Omega(u) \\ u = 1 & \text{on } B_1 \end{cases}$$

2.2. *Nonvariational solution*

We are going to consider the least viscosity super-solutions u^ε of (EFB_ε) and $u^{\varepsilon\delta}$ of $(EFB_{\varepsilon,\delta})$ as they are done in [CLW1]. Then we have the following uniform estimates obtained in [CLW1].

Lemma 2.3: *Let $u^{\varepsilon,\delta}$ be a solution of $(EFB_{\varepsilon,\delta})$. Then*

$$|\nabla u^{\varepsilon,\delta}| \le C < \infty$$

for a uniform constant.

And we have a nondegeneracy of $u^{\varepsilon,\delta}(x)$.

Lemma 2.4: *There is a uniform constant $c_o >$ such that, for $x_o \in \overline{\Omega(u)}$,*

$$\sup_{B_r(x_o)} u^{\varepsilon,\delta}(x) \ge c_o r.$$

Proof: First let's scale the function $\frac{1}{r}u(x_o + r(x - x_o))$. Then we may assume $r = 1$ without loss of generality. Let us assume $\sup_{B_1(x_o)} u^{\varepsilon,\delta}(x) = \eta$ for small $\eta > 0$. We can choose *a*

and h such that $h = a\left(2^{n-2} - \frac{1}{r^{n-2}}\right)_+$ and $|\nabla h|^2 < 2\lambda$ on $\partial\{h > 0\}$. Then h will be a super-solution of (EFB_ε) and an approximation h_δ of h will be also a super-solution of $(EFB_{\varepsilon,\delta})$ like an Appendix in [CLM2]. Since $u^{\varepsilon,\delta}(x)$ is the least super-solution, the existence of smaller super-solution $\min(u^{\varepsilon,\delta}(x), h_\delta(x)) < u^{\varepsilon,\delta}(x)$, for small $\eta > 0$ will be a contradiction. Therefore η should be bounded below by a uniform positive constant c_o. □

The uniform gradient estimate and nondegeneracy of the solution gives us a nontrivial limits of $u^{\varepsilon\delta}$.

LEMMA 2.5: *Let $u^{\varepsilon,\delta}$ be solutions of $EFB_{\varepsilon,\delta}$.*

(1) *$u^{\varepsilon,\delta}$ are even functions of x_1.*

(2) *$u^{\varepsilon,\delta}$ are monotonically decreasing for positive x_1 i.e. for $0 \le a_1 \le a_2$, $u^{\varepsilon,\delta}(a_1, x_2) \le$ $\le u^{\varepsilon,\delta}(a_2, x_2)$.*

(3) *$\partial\Omega(u^\varepsilon)$ (or $\partial\Omega(u^{\varepsilon,\delta})$) is a graph of x_2 variables if*

$$\min_{(y_1,y_2)\in\Omega(u^{\varepsilon,\delta})} y_2 < x_2 < \max_{(y_1,y_2)\in\Omega(u^{\varepsilon,\delta})} y_2.$$

PROOF: For simplicity, we use $u(x)$ for $u^{\varepsilon,\delta}(x)$ in this proof.

(1) From the symmetry of the ball $B_1(0)$ and the independence of $f(x_2)$ in x_1-variable, we know $u(-x_1, x_2)$ is a solution of $(EFB_{\varepsilon,\delta})$ .Hence $\min(u(x_1, x_2), u^\varepsilon(-x_1, x_2))$ is a super-solution. On the other hand u^ε touches this super-solution. Hence $u(x_1, x_2) =$ $= \min(u(x_1, x_2), u(-x_1, x_2))$, which implies $u(x_1, x_2)$ is an even function in x_1-variable.

(2) Let $u^\lambda(x_1, x_2) = u(2\lambda - x_1, x_2)$ and $v(x_1, x_2) = u(x_1, x_2) - u^\lambda(x_1, x_2)$. Clearly $v < 0$ for a large positive λ. Let us decrease λ until v hit the first zero at $x_1 > \lambda$. Then u and u^λ will touch each other on the interior or on the free boundary, which will give us a contradiction unless they coincid. Hence $u(x_1, x_2) > u^\lambda(x_1, x_2)$ for $\lambda > 0$, which will implies the conclusion. The proof of (3) follows from (2). □

COROLLARY 2.6: Lemma (2.5) holds for u^ε, a limit of $u^{\varepsilon,\delta}$ as $\delta \to 0$.

Let us assume that , on each domain $D \in \mathbb{R}^n$, u^{ε,δ_j} converges uniformly to u^ε and then u^{e_i} also converges uniformly to u. Let x_o be a point on $\partial\Omega(u) \cap D$ and let $x_m \in \partial\Omega(u)$ be such that $x_m \to x_o$ as $m \to \infty$. For $\lambda_m \to 0$, let $u_{\lambda_m}(x) = \frac{1}{\lambda_m} u(x_m + \lambda_m x)$ and $u^{\varepsilon_i,\delta_j}_{\lambda_m}(x) = \frac{1}{\lambda_m} u^{\varepsilon_i,\delta_j}(x_m + \lambda_m x)$. A simple modification of Lemma (3.2) in [CLW1] will give us the following lemma.

LEMMA 2.7: *Assume that $u_{\lambda_m} \to U$ as $m \to \infty$ uniformly on compact sets of R^n. There exist $i_k, j_k \to \infty$ such that $\delta_{j_k}/\varepsilon_{i_k}, \varepsilon_{i_k}/\lambda_k \to 0$ and*

(1) *$u^{\varepsilon_i,\delta_j}_{\lambda_m}(x) \to U$ uniformly on compact sets of $\mathbb{R}^n$,*

(2) $\nabla u_{\lambda_m}^{\varepsilon_i,\delta_j}(x) \to \nabla U$ *in* $L^2_{loc}(\mathbb{R}^n)$,
(3) $\nabla u_{\lambda_k} \to \nabla U$ *in* $L^2_{loc}(\mathbb{R}^n)$.

To understand a possible free boundary condition for u, let us consider the case that the limit u is a hyperplane. A modification of Lemma (5.1) will tell us that the homogenization in the nontransversal direction to x_2 is a simple averaging out as the variational solution.

LEMMA 2.8: *Let x^o be a point on $\partial\Omega(u) \cap D$.*
If u^{i_k,j_k} converges to $u(x) = a\langle x - x^o, \upsilon\rangle_+ + o(|x - x^o|)$ for a unit vector υ and $a \geq 0$, then we have $a^2 = 2\overline{f}$ when $\langle \upsilon, e_2\rangle > 0$.

PROOF: Let us multiply $u_\upsilon\phi$ to $(EFB_{\varepsilon,\delta})$ and assume $\upsilon = \cos(\theta)e_1 + \sin(\theta)e_2$. Then we have

$$\iint_D \frac{1}{2}|\nabla u|^2\phi_\upsilon - \iint_D u_\upsilon \nabla u \cdot \nabla\phi + \iint B_\delta f\left(\frac{x_2}{\varepsilon}\right)\phi_\upsilon = 0. \tag{2.2}$$

By Lemma (2.7), we know that

$$\nabla u^{\varepsilon_k,\delta_k} \to a\chi_{\{x\cdot\upsilon>0\}}\upsilon$$

and

$$B_{\delta_k}(u^{\varepsilon_k,\delta_k}) \to \chi_{\{x\cdot\upsilon>0\}}.$$

In addition we have

$$\begin{aligned}
&\iint_{\cos\theta x_1+\sin\theta x_2>0} f\left(\frac{x_2}{\varepsilon}\right)\phi_\upsilon dx_2 dx_1 \\
&\quad = \iint_{y_1>0} f(\frac{y_2}{\varepsilon})\frac{1}{\cos\theta}\phi_{nu} dy_2 dy_1 \\
&\quad \to \iint_{y_1>0} \overline{f}\frac{1}{\cos\theta}\phi_{nu} dy_2 dy_1 \\
&\quad = \iint_{x\cdot\upsilon>0} \overline{f}\phi_\upsilon dx_2 dx_1,
\end{aligned}$$

when $\cos\theta x_1 + \sin\theta x_2 = y_1, x_2 = y_2$. Therefore (2.2) converges to

$$-\frac{a^2}{2}\iint_{x\cdot\upsilon>0} \phi_\upsilon + \iint_{x\cdot\upsilon>0} \overline{f}\phi_\upsilon$$

□

By Corollary A.1 in [CLW1], they show $U(x) = \lim_{\lambda\to 0} u_\lambda$ is $U(x) = a\langle x - x_o, \upsilon\rangle_+ +$

$+o(|x-x_o|)$ for some direction υ. The elliptic monotonicity formula in [CK] and Theorem 3.1 in [CLW2] imply the following lemma.

DEFINITION 2.9: *We say that υ is the inward unit spatial normal to the free boundary $\partial\Omega(u)$, at a point $x_o \in \partial\Omega(u)$, in the elliptic measure theoretic sense, if $\upsilon \in \mathbb{R}, |\upsilon| = 1$ and*

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B_r(x_o)} |\chi_{u>0} - \chi_{\{\langle x-x_o,\upsilon\rangle>0\}}| dx = 0.$$

LEMMA 2.10: *Let u^{e_k,δ_k} converges uniformly to u on any compact subset of D in $\mathbb{R}^n$ and let $x_o \in D \cap \Omega(u)$ such that*

(1) *$\Omega(u)$ has a unit spatial normal υ in the elliptic measure sense such that $\langle \upsilon, e_1\rangle > 0$.*
(2) *And*

$$\lim_{r\to 0} \frac{|\{u=0\} \cap B_r(x_o)|}{|B_r(x_o)|}.$$

Then there is $a > 0$ such that

$$u(x) = a\langle x - x_o, \upsilon\rangle_+ + o(|x-x_o|).$$

LEMMA 2.11: *Under the same conditions as in Lemma* (2.8),
When $\upsilon = e_2$.
In addition

(1) *If $\max f > \overline{f}$, there is a line segment in $\partial\Omega(u)$ which is parallel to the x_1-axis and $|\nabla u| = \max f$ at least on one point on the line segment.*

(2) *Set $[-a^*, a^*] \times \{x_2^0\}$ be the maximal interval among such line segment. Then we also have*

(a) *$|\nabla f(x_1, x_2^o)|$ is an even function in $[-a^*, a^*]$ and monotonically decreasing in $[0, a^*]$.*

(b) *$\partial\Omega(u)$ is C^1 at $(\pm a^*, x_2^o)$.*

(c) $\lim_{x_1 \to \pm a^*} |\nabla f(x_1, x_2^o)| = \overline{f}$.

(3) $a \in [\overline{f}, \max f]$.

PROOF: (1). If there is no line segment containing x_o,then a will be $\overline{f}$ from the Lipschiz continuity of u^ε. Then we are able to cut the graph of u with a plane with the slope $\max f$ i.e. $L(x) = \max f\langle x - x_0 - \varepsilon e_2, \upsilon\rangle_+$ so that we have a smaller super solution $\min(u^\varepsilon, u)$ which is a contradiction. From there same argument there should be at least one point on the line segment ,whose gradient is $\max f$, to avoid further cutting by the plane with the slope $\max f$.

(2a). A simple reflection argument gives us the result. To compare the gradient at the two points on the line segment, (a_1, x_2^0) and (a_2, x_2^o) such that $0 \le a_1 \le a_2$, we reflect the graph of u^ε with respect to $x_1 = \dfrac{a_1 + a_2}{2}$ and compare $u^\varepsilon(x_1, x_2)$ with $u_1^\varepsilon(x_1, x_2) =$

$= u^\varepsilon(a_1 + a_2 - x_1, x_2)$. We know that u_1^ε is equal to u^ε on the line $x_1 = \dfrac{a_1 + a_2}{2}$ and that u_1^ε is nonnegative on $\partial\Omega(u^\varepsilon) \cap \left\{x_2 \geq \dfrac{y_1 + y_2}{2}\right\}$. By a comparison principle, $u^\varepsilon(a_1, x_2) = = u_1^\varepsilon(a_2, x_2) \geq u^\varepsilon(a_2, x_2)$. Since a positive value $u^\varepsilon(x_1, x_2)$ goes to zero as $(x_i, x_2) \in \Omega(u^\varepsilon)$ approaches to (a_i, x_2^0), for $i = 1, 2$, we will have $|\nabla u^\varepsilon(a_1, x_2^o)| \geq |\nabla u^\varepsilon(a_2, x_2^o)|$.

(2b). We can blow up our solution by a linear scale i.e. $u_\eta = \dfrac{1}{\eta} u(\eta x)$. From the uniform gradient estimate of u and the nondegeneracy, there is a nontrivial limit w of u_η. If u is not C^1 at $(\pm a^*, x_2^0)$, there will be a cone with an angle less than $\dfrac{\pi}{2}$ containing the support $\Omega(w)$ of w, which implies $w = O(|x|^{1+a})$ for a small positive constant a. On the hand , $|\nabla w(x)| = |\nabla u(\eta x)| > c_o > 0$ from the nondegeneracy. It is a contradiction.

(2c). First of all $\lim\limits_{x_1 \to \pm(a^*)^\pm} |\nabla f(x_1, x_2^o)|$ exists due to the monotone behavior of $|\nabla u|$ on $[-a^*, a^*] \times \{x_2^o\}$. From (2b), $\partial\Omega(w)$ will be a half plane due the C^1-regularity of $\partial\Omega(w)$ at $(\pm a^*, x_2^o)$. Let us assume $x_2^o > 0$ and without loss of generality, $\Omega(w) = = \{(x_1, x_2) | x_2 \leq x_2^0\}$. We can extend w as an add function with respect to $x_2 = x_2^o$. The extension of w, $\overline{w}$, will be harmonic function and so is $\overline{w}_{x_1}$. On the other hand $\overline{w}_{x_1} = 0$ at an interior line segment. Hence $\overline{w}_{x_1} = w_{x_1} = 0$ in the $\Omega(w)$. Therefore $\lim\limits_{x_1 \to \pm a^*} |\nabla f(x_1, x_2^o)| = \overline{f}$.

Part (3) is clear from (2). □

3. - Flame flow

As a consequence, suppose that we start with an initial data below the least super-solution $u^{\varepsilon,\delta}(x)$ of $(EFB_{\varepsilon,\delta})$ with the boundary condition $u^{\varepsilon,\delta}(x) = 1$ on $B_1(0)$.Then the flow will stay below $u^{\varepsilon,\delta}(x)$ for all t since $u^{\varepsilon,\delta}$ is a stationary solution of the parabolic problem(2). On the other hand, uniform Lipschitz estimate in space and Hölder estimate in time will give us a stationary solution as t goes to infinity. This limit will be a non-variational stationary solution $u^{\varepsilon,\delta}$ since $u^{\varepsilon,\delta}$ is the smallest. On the other hand, the smallest super-solution of (EFB_ε) may not be approximated by solutions of singular perturbation problems $(EFB_{\varepsilon,\delta})$. Therefore it is not clear that the solution of (FB_ε) converges to $u^\varepsilon = \liminf\limits_{\delta \to 0} u^{\varepsilon,\delta}$ even though its initial data is smaller than u^ε. In [CL], we study the the least viscosity of super-solution of (EFB_ε) with full periodicity on x_1 and x_2 directions and its homogenization. The homogenization of pulsating waves in one-phase flame flows is studied in [CLM1],[CLM2].

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