Rendiconti
Accademia Nazionale delle Scienze detta dei XL
*Memorie di Matematica e Applicazioni*
123° (2005), Vol. XXIX, fasc. 1, pagg. 55-78

MARCO BIROLI (*)

# Nonlinear p-Homogeneous Dirichlet Forms on Nonreflexive Banach Spaces

ABSTRACT. — Our aim in this paper is an extension of the definition of nonlinear p-homogeneous Dirichlet form to the case where the domain of the form is a Banach space possibly non reflexive.

**Forme di Dirichlet nonlineari p-omogenee su spazi di Banach non riflessivi**

SUNTO. — Il nostro scopo in questo articolo è estendere la definizione di forme di Dirichlet nonlineari p-omogenee al caso in cui il dominio della forma sia uno spazio di Banach anche non riflessivo.

## 1. - INTRODUCTION

Our goal in this paper is an extension of the notion of strongly local (regular) Dirichlet form and of the related fundamental properties to the nonlinear case when the domain of the form is a general Banach space possibly nonreflexive . For the notion of (bilinear) Dirichlet form we refer to the book of Fukushima-Oshima-Takeda, [FOT]. In [FOT] a purely analytical proof of the fundamental properties of a (bilinear) Dirichlet form is given, this type of proof firstly appeared in [M]; we recall also the papers [BM1], [BM2], where an analytical investigation of the properties of the harmonic functions relative to a (bilinear) strongly local "Riemannian" Dirichlet forms is carried on. On the ground of the Beuerling-Deny representation formula, [Beu], a (bilinear) Dirichlet form is represented as the sum of a strongly local part, of a "killing" part and of a global part. The Beuerling-Deny representation theorem is the fundamental tool allowing to prove that in the (bilinear) strong local (regular) case same properties of Dirichlet forms (in particular the

(*) Indirizzo dell'Autore: Dipartimento di Matematica "F. Brioschi", Politecnico di Milano, Piazza Leonardo da Vinci, 32 - 20133 Milano, Italia; e-mail: marbir@mate.polimi.it

Markov property) hold again for the energy measures. Using the above mentioned properties of the energy measure it can be proved that for the energy measure of a (bilinear) strongly local (regular) Dirichlet form a chain rule and a Leibnitz rule hold; those properties are the starting point for an investigation of the local regularity of harmonic functions relative to a (bilinear) strongly local (regular) Dirichlet form, see in particular [BM1], [BM2]. The Beuerling-Deny representation theorem is proved using Riesz theorem on representation of measures, which is an essentially linear tool, then it seems that it is difficult find a nonlinear version of this result.

Previous work on a possible extension of the notion of Dirichlet form to the nonlinear case has been given by Benilan-Picard, [BeP], and Cipriani-Grillo, [CG1] [CG2]. In particular in [BeP] the relations between the maximum principle and the Markov property are investigated generalizing to the nonlinear monotone case previous results obtained in [Beu] and [Hirsch] in the linear case. In [CG2] a notion of nonlinear Dirichlet form is given and the relations with a class of nonlinear semigroups (the order preserving contractions semigroups with a cyclically monotone generator) are investigated. The above papers deal with the general global case and are interested in the properties of the corresponding nonlinear semigroup; then the existence of an energy measure is not ensured and there is no proof of chain or Leibnitz rule for the energy measure, when such a measure exists. The first paper dealing local forms was [MM], where a suitable chain rule for the energy measure connected with the form is assumed and the Sobolev-Morrey inequalities are proved as a consequence of a Poincaré inequality. In [C1], [C2], [C3], [CL] some nonlinear forms on fractals are explicitely given and it is proved that the assumptions in [MM] hold (see also the more recent papers [S], [HPS] on the p-Laplacian on the Sierpiski gasket).

In [BV1] a notion of nonlinear strongly local Dirichlet form is introduced; we give our assumptions (in particular the Markov property) directly on the energy measure of the form, whose existence is assumed. We are able to prove in this framework (by purely analytical methods in the line of [M]) suitable Leibnitz and chain rules, which are the starting point for an investigation of the local regularity of the harmonic functions relative to the form and in particular for a proof (under suitable assumptions) of a Harnack type inequality for positive harmonic functions given in [BV2] (we observe that the chain rule proved here is the same assumed in [MM] and that a Harnack inequality for positive harmonic functions in the bilinear case has been proved in [BM1], [BM2]).

In [BV1] the domain of the Dirichlet functional associated with the form is assumed to be a uniformly convex Banach space (under a suitable norm) and the functional is assumed to be locally uniformly convex. There are cases, as the p-energies on fractals considered in [S], [HPS] or the p-energies on metric spaces considered in [KM], where the above assumptions can not hold (conditions under which the reflexivity is assured are given in [Chee]). The goal of this paper is to prove that the results in [BV1] (then also the results in [BV2]) hold again without the above assumptions. Of course there is a cost to pay; in [BV1] the chain rule for the energy measure of a Dirichlet functional is proved without assumptions on the existence of a Dirichlet form associated to the functional (existence of a Gateaux derivative of the energy density of the functional in the weak$^\star$ topology) here we assume the existence of a Dirichlet form associated to the functional.

## 2. - The capacity

Concerning this section we observe that all the results in section 2 of [BV1] hold again and we recall in this section only the most interesting for the following.

We consider a locally compact separable Hausdorff space $X$ with a metrizable topology and a positive Radon measure $m$ on $X$ such that supp[$m$]= $X$. Let $\Phi: L^p(X,m) \to [0,+\infty]$, $1<p$, be a l.s.c. strictly convex functional with domain $D$, i.e. $D=\{v;\ \Phi(v)<+\infty\}$, such that $\Phi(0)=0$. We assume that $D$ is dense in $L^p(X,m)$ and that the following conditions hold:

$(H_1)$ $D$ is a dense linear subspace of $L^p(X,m)$, which can be endowed with a norm $\|.\|_D$; moreover $D$ has a structure of Banach space with respect to the norm $\|.\|_D$ and the following estimate holds

$$c_1\|v\|_D^p \le \Phi_1(v) = \Phi(v) + \int_X |v|^p dm \le c_2\|v\|_D^p$$

for every $v\in D$, where $c_1,\ c_2$ are positive constants.

$(H_2)$ We denote by $D_0$ the closure of $D\cap C_0(X)$ in $D$ (with respect to the norm $\|.\|_D$ and we assume that $D\cap C_0(X)$ is dense in $C_0(X)$ for the uniform convergence on $X$.

Remark 2.1: We observe that, since $\Phi$ is convex, $\Phi$ is l.s.c. also with respect to the weak topology of $L^p(X,m)$. Moreover from the assumption $(H_1)$ it follows that $\Phi$ is continuous on $D$ for the norm $\|.\|_D$, [PS] Ch.1 Sec.2 pg. 20, then from $(H_2)$ the restriction of $\Phi$ to $D_0$ coincides with the relaxation of $\Phi$ defined on $D\cap C_0(X)$.

$(H_3)$ For every $u,v\in D\cap C_0(X)$ we have $u\vee v\in D\cap C_0(X)$, $u\wedge v\in D\cap C_0(X)$ and

$$\Phi(u\vee v)+\Phi(u\wedge v)\le \Phi(u)+\Phi(v)\,.$$

We observe that from $(H_2)$, from Remark 2.1 and from the l.s.c. of our functional on $L^p(X,m)$ we have that the above inequality hold again for every $u,v\in D_0$.

Remark 2.2: We observe, [DS] pg. 15-19, that given an open set $O$, whose closure is contained in an open relatively compact open set $\Omega$, there exists a function $\tilde{u}\in C_0(X)$ such that $\tilde{u}\ge 1+\varepsilon$, $\varepsilon>0$, on $O$ and $\tilde{u}=0$ on $\Omega^c$, then from $(H_2)$ and $(H_3)$ there exists $u\in D\cap C_0(X)$ with $u\ge 1$ on $O$. Moreover we observe that, since $C_0(X)$ is dense in $L^p(X,m)$, we have that $D_0$ is dense in $L^p(X,m)$.

Remark 2.3: We observe that the assumption $(H_3)$ is connected with the assumptions in [CG2], moreover if $\Phi$ has a subdifferential $\partial\Phi$ with values in $D'$ (the dual space of $D$) at every point in $D$, then $(H_1)$ hold if $\partial\Phi$ is $T$-monotone.

The assumptions $(H_1)(H_2)$ and $(H_3)$ allow us to define a capacity relative to the functional $\Phi$ (and to the measure space$(X, m)$). The capacity of an open set $O$ is defined as

$$cap_{\Phi}(O) = \inf\{\Phi_1(v);\ v \in D_0,\ v \geq 1 \text{ a.e. on } O\}$$

if the set $\{v \in D_0\ , v \geq 1$ a.e. on $O\}$ is not empty and

$$cap_{\Phi}(O) = +\infty$$

if the set $\{v \in D_0\ , v \geq 1$ a.e. on $O\}$ is empty. Let $E$ be a subset of $X$ we define

$$cap_{\Phi}(E) = \inf\{cap(O); O \text{ open set with } E \subset O\}.$$

We observe that from Remark 2.2 it follows that given an open set $O$, whose closure is contained in an open relatively compact open set $\Omega$, we have $cap_{\Phi}(O) < +\infty$; moreover the infimum defining $cap_{\Phi}(O)$ is attained and the corresponding function is called the potential of $O$. As in [BV1] we can prove that $cap_{\Phi}$ defines a Choquet capacity on $X$. We say that a property holds quasi-everywhere (q.e) if the property holds up to sets of zero capacity. We observe that the potential of an open compact set $O$ is 1 q.e. on $O$. We recall that every function in $D_0$ is quasi-continuous for this capacity; moreover if we have that a sequence $u_n$ converges in $D_0$ to $u$ we have also that the sequence $u_n$ converges quasi-uniformly (for the capacity $cap_{\Phi}$) to $u$, [BV1].

Finally as in [BV1] we have that any function $u \in D_0$ is a measurable with respect to every positive Radon measure $\nu$, which does not charge sets of zero capacity.

## 3. - Strongly local Dirichlet functionals and forms

The assumptions $(H_1)(H_2)$ and $(H_3)$ have a global character; now we will define a *strongly local Dirichlet functional* with a homogeneity degree $p > 1$. Let $\Phi$ satisfy $(H_1)(H_2)$ and $(H_3)$; we say that $\Phi$ is a *strongly local Dirichlet functional* with a homogeneity degree $p > 1$ if the following conditions hold:

$(H_4)$ $\Phi$ has the following representation on $D_0$ $\Phi(u) = \int_X \alpha(u)(dx)$ where $\alpha$ is a non-negative bounded Radon measure depending on $u \in D_0$, which does not charge sets of zero capacity. We say that $\alpha(u)$ is the energy (measure) of our functional. The energy $\alpha(u)$ (of our functional) is convex with respect to $u$ in $D_0$ in the space of measures , i.e. let $u, v \in D_0$ and $t \in [0, 1]$ then $\alpha(tu + (1-t)v) \leq t\alpha(u) + (1-t)\alpha(v)$, and is homogeneous of degree $p > 1$, i.e. $\alpha(tu) = |t|^p \alpha(u),\ \forall u \in D_0,\ \forall t \in \mathbf{R}$.

Moreover the following closure property holds: if $u_n \to u$ in $D$ and $\alpha(u_n)$ converges to $\chi$ in the space of measures then $\chi \geq \alpha(u)$.

$(H_5)$ $\alpha$ is of strongly local type, i.e. if $u, v \in D$ and $u - v = constant$ on an open set $A$ we have $\alpha(u) = \alpha(v)$ on $A$.

$(H_6)$ $\alpha(u)$ is of Markov type, i.e. let $\beta \in C^1(\mathbf{R})$ such that $\beta'(t) \leq 1$ and $\beta(0) = 0$ and $u \in D \cap C_0(X)$ then $\beta(u) \in D \cap C_0(X)$ and $\alpha(\beta(u)) \leq \alpha(u)$ in the space of measures.

In the following we denote by $\mathcal{M}$ the space of Radon measures on $X$.

We now prove that $(H_1)(H_2)(H_4)$-$(H_6)$ imply $(H_3)$.

LEMMA 3.1: *Let $u_n$ be a sequence in $D_0$ weakly converging (in $D_0$) to $u$. Assume that $\alpha(u_n)$ weakly converges in $\mathcal{M}$ to $\chi$, then $\chi \geq \alpha(u)$.*

PROOF: From $(H_4)$ the epigraph of $\alpha(.)$ (i.e. the set $\{(\mu, v) \in \mathcal{M} \times D_0; \mu \geq \alpha(v)\}$) is convex and closed in $\mathcal{M} \times D_0$, then is closed also in the weak topology (see Th. 2.9.3 pg 36 [HP]). The result follows.

LEMMA 3.2: *Let $u_n$ be a sequence in $D_0$ converging (in $D_0$) to $u$; then $\alpha(u_n)$ weakly converges in $\mathcal{M}$ to $\alpha(u)$. Assume that $a_n$ are measures, Such that $|a_n|$ does not charge sets of zero capacity, and such that $|a_n|$ are weakly convergent in $\mathcal{M}$, so also $a_n$ are also weakly convergent in $\mathcal{M}$ and we denote by $a$ the weak limit. If $v_n$ is a uniformly bounded sequence of quasi-continuous functions, which converges quasi-uniformly to $v$, then $v_n a_n$ weakly converges in the measures to $va$.*

PROOF: We observe that $\alpha(u_n - u)$ converges to 0 in $\mathcal{M}$. Moreover we have

$$\alpha(u_n) \leq 2^{p-1}(\alpha(u_n - u) + \alpha(u)) \tag{3.1}$$

(where we use assumption $(H_4)$). From (3.1) and Th. 2 pg. 306 [DS] we have, that, at least after extraction of subsequences, $\alpha(u_n)$ weakly converges to $\chi$ in $\mathcal{M}$. From $(H_4)$ we have $\chi \geq \alpha(u)$; since $\Phi$ is continuous on $D_0$, we obtain

$$\int_X \chi(dx) = \int_X \alpha(u)(dx).$$

Since $\chi \geq \alpha(u)$, we have $\chi = \alpha(u)$.

Since $v$ is quasi-continuos and $a_n$ is a Radon measure that does not charge sets of zero capacity, we have that $v$ is measurable for all the measures $a_n$. Since $v$ is bounded $va_n$ are measures. Let $\sigma > 0$; since $v_n$ is quasi-uniformly convergent to $v$ (at least after extraction of subsequences), there exists an open set $A$ with $cap_\Phi(A) \leq \sigma$ such that $v_n$ converges uniformly to $v$ in $X - A$. Let $|v_n| \leq M$ and $\varepsilon > 0$. From (3.1) and Th. 2 pg. 306 [DS] there exists $\sigma_\varepsilon$ and $n_\varepsilon$ such that for $\sigma \leq \sigma_\varepsilon$ and $n \geq n_\varepsilon$ we have

$$|a_n|(A) \leq \varepsilon$$

$$|v - v_n| \leq \varepsilon \quad \text{on } X - A\,.$$

Then

$$|v - v_n||a_n| \leq \varepsilon|a_n| + 2M\mathbf{1}_A|a_n|$$

so $|v - v_n||a_n|$ converges to 0 in $\mathcal{M}$. Since $v$ is bounded, we have that the measures $va_n$ weakly converges in $\mathcal{M}$, at least after extraction of subsequences (see Th. 2 pg. 306 [DS]); using again Th. 2 pg. 306 [DS] we obtain that the weak limit of $va_n$ in $\mathcal{M}$ is $va$ and the result follows.

Proposition 3.3: *Let the assumptions* $(H_1)(H_2)(H_4)$-$(H_6)$ *hold, then also* $(H_3)$ *hold.*

Proof: Let $\beta(t) = t^+$ for $\varepsilon > 0$ there exists $\beta_\varepsilon(t)$ in $C^1$ such that

$$\beta_\varepsilon(t) = 0 \quad \text{if } t \leq \varepsilon, \quad 0 \leq \beta'_\varepsilon(t) \leq 1$$

$$\lim_{\varepsilon \to 0} \beta_\varepsilon(t) = \beta(t)$$

uniformly on $\mathbf{R}$. Let $u \in D \cap C_0(X)$, from $(H_6)$ we have that $\beta_\varepsilon(u) \in D \cap C_0(X)$ and $\alpha(\beta_\varepsilon(u)) \leq \alpha(u)$ in $\mathcal{M}$. Then $\beta(u) \in D \cap C_0(X)$ and

$$\lim_{\varepsilon \to 0} \beta_\varepsilon(u) = \beta(u)$$

weakly in $D_0$ and strongly in $L^q(X, m)$ for every finite $q > 1$. Let now $u, v \in D \cap C_0(X)$, we have $u \vee v = u + (v - u)^+$, $u \wedge v = v - (v - u)^+$. Consider the functions $r_\varepsilon = u + \beta_\varepsilon(v - u)$, $p_\varepsilon = v - \beta_\varepsilon(v - u)$, from assumptions $(H_4)(H_6)$ we have that $r_\varepsilon, p_\varepsilon \in D \cap C_0(X)$ and $\alpha(r_\varepsilon), \alpha(p_\varepsilon) \leq C(\alpha(u) + \alpha(v))$, moreover we have

$$\lim_{\varepsilon \to 0} r_\varepsilon = u \vee v$$

$$\lim_{\varepsilon \to 0} p_\varepsilon = u \wedge v$$

weakly in $D_0$ and strongly in $L^q(X, m)$ for every finite $q > 1$.

Th. 2 pg. 306 [DS] implies that we have, at least after extraction of subsequences,

$$\lim_{\varepsilon \to 0} \alpha(r_\varepsilon) = \chi$$

$$\lim_{\varepsilon \to 0} \alpha(p_\varepsilon) = \zeta$$

in the weak topology of $\mathcal{M}$, then by the lower semicontinuity of $\Phi$ on $L^p(X, m)$ we have that $u \vee v$, $u \wedge v$ are in $D_0 \cap C_0(X)$ and

$$\Phi(u \vee v) \leq \int_X \chi$$

$$\Phi(u \wedge v) \leq \int_X \zeta.$$

We observe that from $(H_5)$ $\alpha(r_\varepsilon)$ and then $\chi$ restricted to the set $\{v \leq u\}$ are equal to $\alpha(u)$ and $\alpha(p_\varepsilon)$ and then $\zeta$ restricted to the set $\{v \leq u\}$ is equal to $\alpha(v)$ then $\chi + \zeta \leq \alpha(u) + \alpha(v)$ on the set $\{v \leq u\}$. Interchanging the role of $v$ and $u$ we have also $\chi + \zeta \leq \alpha(u) + \alpha(v)$ on the set $\{u \leq v\}$ and the result follows.

Now we will prove that, as in the linear case, the Markov property $(H_6)$ has an equivalent form:

Proposition 3.4: *Let the assumptions* $(H_1)(H_2)(H_4)$-$(H_5)$ *hold, then the assumption* $(H_6)$ *is equivalent to the following one:*

$(H'_6)$ *Let* $u \in D \cap C_0(X)$ *then* $v = 0 \vee u \wedge 1$ *is in* $D \cap C_0(X)$ *and* $\alpha(v) \leq \alpha(u)$.

PROOF: Assume that $(H_6)$ holds. Then $(H_6')$ is a consequence of the Corollary 4.2 in the next section.

Assume that $(H_6')$ holds. Let at first $\beta$ be defined as

$$\beta(t) = 0 \quad \text{for } t < 0;\ \beta(t) = t \quad \text{for } 0 \leq t \leq 1;$$

$$\beta = k(t-1)+1 \quad \text{for } 1 < t \leq a;\ \beta(t) = k(a-1)+1 \quad \text{for } t \geq a\,.$$

We will prove that $\beta(u) \in D_0$

$$a(\beta(u)) \leq \mathbf{1}_{\{0<u<1\}}a(u) + k\mathbf{1}_{\{1<u<a\}}a(u)\,.$$

Let $\beta_\varepsilon$ be defined as

$$\beta_\varepsilon(t) = 0 \quad \text{for } t < \varepsilon;\ \beta_\varepsilon(t) = t-\varepsilon \quad \text{for } \varepsilon \leq t \leq 1-2\varepsilon;$$

$$\beta_\varepsilon(t)=1-3\varepsilon \ \text{ for } 1-2\varepsilon<t\leq 1+2\varepsilon;\ \beta_\varepsilon(t)=k(t-(1+2\varepsilon))+(1-3\varepsilon) \ \text{ for } 1+2\varepsilon<t\leq a-\varepsilon;$$

$$\beta_\varepsilon(t) = k((a-\varepsilon)-(1+2\varepsilon))+(1-3\varepsilon) \quad \text{for } a-\varepsilon < t$$

From $(H_6')$ we have that $\beta_\varepsilon(u) \in D_0$ and

$$a(\beta_\varepsilon(u)) \leq \mathbf{1}_{\{0<u<1\}}a(u) + k\mathbf{1}_{\{1<u<a\}}a(u)\,.$$

Moreover we have that $\lim\limits_{\varepsilon\to 0}\beta_\varepsilon(u) = \beta(u)$ uniformly on $X$.

We have for $\varepsilon < 2\varepsilon < \varepsilon'$

$$\beta_\varepsilon(t)-\beta_{\varepsilon'}(t) = 0 \quad \text{for } t < \varepsilon;\ \beta_\varepsilon(t)-\beta_{\varepsilon'}(t) = t-\varepsilon \quad \text{for } \varepsilon \leq t < \varepsilon';$$

$$\beta_\varepsilon(t)-\beta_{\varepsilon'}(t) = \varepsilon'-\varepsilon \quad \text{for } \varepsilon' \leq t < 1-2\varepsilon';$$

$$\beta_\varepsilon(t)-\beta_{\varepsilon'}(t) = (t-(1-2\varepsilon'))+(\varepsilon'-\varepsilon) \quad \text{for } 1-2\varepsilon' \leq t < 1-2\varepsilon;$$

$$\beta_\varepsilon(t)-\beta_{\varepsilon'}(t) = 3(\varepsilon'-\varepsilon) \quad \text{for } 1-2\varepsilon \leq t < 1+2\varepsilon;$$

$$\beta_\varepsilon(t)-\beta_{\varepsilon'}(t) = k(t-(1+2\varepsilon))+3(\varepsilon'-\varepsilon) \quad \text{for } 1+2\varepsilon \leq t < 1+2\varepsilon';$$

$$\beta_\varepsilon(t)-\beta_{\varepsilon'}(t) = (3+k)(\varepsilon'-\varepsilon) \quad \text{for } 1+2\varepsilon' \leq t < a-\varepsilon'$$

$$\beta_\varepsilon(t)-\beta_{\varepsilon'}(t) = k(t-(a-\varepsilon'))+(3+k)(\varepsilon'-\varepsilon) \quad \text{for } a-\varepsilon' \leq t < a-\varepsilon$$

$$\beta_\varepsilon(t)-\beta_{\varepsilon'}(t) = (3+2k)(\varepsilon'-\varepsilon) \quad \text{for } a-\varepsilon \leq t\,.$$

As in the first part of the proof we have

$$a(\beta_{\varepsilon'}(u)-\beta_\varepsilon(u)) \leq (k+1)^p(\mathbf{1}_{\{1-3\varepsilon'<u<1-\varepsilon\}}+\mathbf{1}_{\{1+\varepsilon<u<1+3\varepsilon'\}})a(u)+$$
$$+(\mathbf{1}_{\{2^{-1}\varepsilon<u<2\varepsilon'\}}+\mathbf{1}_{\{a-2\varepsilon'<u<a-2^{-1}\varepsilon\}})a(u)$$

So we have that $\beta_\varepsilon(u)$, $(\varepsilon = 3^{-n})$, is a Cauchy sequence in $D_0$ so

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(u) = \beta(u)$$

in $D_0$

$$\lim_{\varepsilon\to 0}a(\beta_\varepsilon(u)) = \chi$$

in $\mathcal{M}$, so

$$a(\beta(u)) \leq \chi \leq \mathbf{1}_{\{0<u<1\}} a(u) + \mathbf{1}_{\{1<u<a\}} a(u)$$

in $\mathcal{M}$. Consider now a set of number $a_i$ with $i = 0, 1, ..., m$ and the piecewise linear function

$$\beta(t) = \eta \text{ for } t \leq a_0, \quad \beta(t) = k_i(t - a_{i-1}) + \beta(a_{i-1}) \text{ for } a_{i-1} < t \leq a_i$$

with $\beta(0) = 0$.

From the above proof we have that $\beta(u) \in D_0$ and

$$a(\beta(u)) \leq \sum_{i=1}^{m} \mathbf{1}_{\{a_{i-1}<u<a_i\}} a(u) .$$

Consider now a function $\beta$ in $C^1$ with $\beta' \leq 1$ and $\beta(0) = 0$; there is a sequence $\beta_\varepsilon(t)$ of piecewise linear function such that $\beta_\varepsilon(0) = 0$ and $\beta_\varepsilon(t)$ converges to $\beta(t)$ uniformly on every bounded set moreover $\beta_\varepsilon(t)$ is derivable up to a finite number of points with $\beta'_\varepsilon(t) \leq 1$ and we have that for every $\sigma > 0$ there exists $\varepsilon_{\sigma,T}$ such that for $\varepsilon < \varepsilon_{\sigma,T}$

$$|\beta'(t) - \beta'_\varepsilon(t)| \leq \sigma$$

for $|t| \leq T$. We observe that from the above proof

$$a(\beta_\varepsilon(u) - \beta_{\varepsilon'}(u)) \leq \sigma a(u)$$

for $\varepsilon \leq \varepsilon_{\sigma,T}$ with $T = 2 \max |u|$; so

$$\lim_{\varepsilon \to 0} \beta_\varepsilon(u) = \beta(u)$$

in $D_0$ and uniformly on $X$.

We have also

$$a(\beta_\varepsilon(u)) \leq a(u).$$

Then, at least after extraction of a subsequence,

$$\lim_{\varepsilon \to 0} a(\beta_\varepsilon(u)) = \chi$$

in $\mathcal{M}$ and from $(H_4)$ we have

$$a(\beta(u)) \leq \chi \leq a(u)$$

So $(H_6)$ holds.

We observe that the assumptions $(H_4)(H_5)$ allow us to define the domain $D_0(\Omega)$ of our functional with respect to the open set $\Omega$ as the closure of $D \cap C_0(\Omega)$ in $D$. The result of Proposition 3.1 allows us to define the capacity $cap_\Phi(E, \Omega)$ of a set $E$ with closure contained in $\Omega$ with respect to the open set $\Omega$. We observe that all the results recalled in section 2 hold again for $cap_\Phi(E, \Omega)$ and for the functions in $D_0(\Omega)$. Moreover we can define the potential of a relatively compact open set $O \subset\subset \Omega$ with respect to the open set $\Omega$ and from $(H_6)$ the potential of $O$ with respect to $\Omega$ is equal 1 on the closure of $O$ q.e.

Finally we say that a function $v$ is locally in $D_0(\Omega)$ if for every fixed compact set $K$ in $\Omega$ there exists a function $w \in D_0(\Omega)$ such that $u = w$ in $K$ q.e.

Let $\Phi(u) = \int_X a(u)(dx)$ be a strongly local Dirichlet functional with domain $D_0$. Assume that for every $u, v \in D_0$ we have

$$\lim_{t\to 0} \frac{a(u+tv) - a(u)}{t} = \mu(u, v)$$

in the weak$^\star$ topology of $\mathcal{M}$ uniformly for $u, v$ in a compact set of $D_0$, where $\mu(u, v)$ is defined on $D_0 \times D_0$ and is linear in $v$. We say that $\Psi(u, v) = \int_X \mu(u, v)(dx)$ is a strongly local Dirichlet form.

Remark 3.1: From the above assumptions it follows that if $u_n$, $n = 1, 2, \ldots$, converges to $u$ in $D_0$ then

$$\lim_{n\to+\infty} \mu(u_n, v) = \mu(u, v)$$

in the weak$^\star$ topology of $\{\mathcal{M}\}$.

Theorem 3.5: *The following properties hold:*

(1) $\mu(\lambda u, v) = |\lambda|^{p-2}\lambda\mu(u, v)$ *for* $\lambda \neq 0$ *real and* $\mu(0, v) = 0$ *(here and in the following we assume* $|t|^{p-2}t = sign(t)|t|^{p-1}$*)*

(2) *the measure* $|\mu(u, v)|$ *does not charge sets of zero capacity*

(3) *If we have* $u_1 - u_2 = constant$ *on the open set* $A$, $u_1, u_2 \in D_0$, *then* $\mu(u_1, v) = = \mu(u_2, v)$ *on* $A$ *for every* $v \in D_0$

(4) *If we have* $v_1 - v_2 = constant$ *on an open set* $A$, $v_1, v_2 \in D_0$, *then* $\mu(u, v_1) = = \mu(u, v_2)$ *on* $A$ *for every* $u \in D_0$

(5) $pa(u) = \mu(u, u)$

(6) $|\mu(u, v)| \leq 2^{p-1}a^{-p}a(u) + 2^{p-1}a^{p(p-1)}a(v)$ *for every* $a > 0$ *where* $u, v \in D_0$

(7) *For any* $f \in L^{p'}(X, a(u))$ *and* $g \in L^p(X, a(v))$ *with* $1/p + 1/p' = 1$, $fg$ *is integrable with respect to the absolute variation of* $\mu(u, v)$ *and* $\forall\, a \in \mathbf{R}^+$

$$|fg|\ |\mu(u, v)|(dx) \leq 2^{p-1}a^{-p}|f|^{p'}\ a(u)(dx) + 2^{p-1}a^{p(p-1)}|g|^p\ a(v)(dx)$$

*where* $p' = \dfrac{p}{p-1}$ .

Proof: We have

$$\frac{a(\lambda u + tv) - a(\lambda u)}{t} = |\lambda|^p \frac{a\left(u + \dfrac{t}{\lambda}v\right) - a(u)}{t} = |\lambda|^{p-2}\lambda \frac{a\left(u + \dfrac{t}{\lambda}v\right) - a(\lambda u)}{\dfrac{t}{\lambda}}$$

Passing to the limit as $t \to 0$ we obtain (1).

The measures $a(v)$, $v \in D_0$, does not charge sets of zero capacity, and

$$\mu(u, v) = \lim_{t\to 0} \frac{a(u+tv) - a(u)}{t} \leq$$

$$\leq \lim_{t\to 0} \frac{(1-t)a(u) + ta(v+u) - a(u)}{t} = \leq 2^{p-1}(a(u) + a(v)) + a(u)\,.$$

Interchanching $v$ and $-v$ we obtain

$$|\mu(u,v)| \leq 2^{p-1}(\alpha(u)+\alpha(v))+\alpha(u)$$

and the result follows.

We observe that (3) and (4) follow easily from the definition of $\mu(u,v)$ and from the locality assumption on $\alpha(u)$.

We observe that

$$\mu(u,u)=\lim_{t\to 0}\frac{\alpha(u+tu)-\alpha(u)}{t}=\lim_{t\to 0}\frac{|1+t|^p-1}{t}\alpha(u)=p\alpha(u)\,.$$

Then (5) is proved.

We have

$$\mu(u,v)=\lim_{t\to 0}\frac{\alpha(u+tv)-\alpha(u)}{t}=\lim_{t\to 0+}\frac{\alpha((1-t)u+t(u+v))-\alpha(u)}{t}\,.$$

Since $\alpha$ is convex we obtain

$$\mu(u,v)\leq \lim_{t\to 0}\frac{(1-t)\alpha(u)+t\alpha(u+v)-\alpha(u)}{t}\leq \alpha(u+v)\,. \tag{3.2}$$

Using the convexity and the homogeneity of $\alpha$, we obtain for $a>0$

$$\mu(u,v)=\mu\left(\frac{u}{a},a^{p-1}v\right)\leq \alpha\left(\frac{u}{a}+a^{p-1}v\right)\leq$$

$$\leq 2^{p-1}\alpha\left(\frac{u}{a}\right)+2^{p-1}\alpha(a^{p-1}v)\leq 2^{p-1}a^{-p}\alpha(u)+2^{p-1}a^{p(p-1)}\alpha(v)\,.$$

Then (6) is proved.

We prove at first an integral version of the inequality in (7)

$$\int_X |fg|\;|\mu(u,v)|(dx)\leq 2^{p-1}a^{-p}\int_X |f|^{p'}\;\alpha(u)(dx)+2^{p-1}a^{p(p-1)}\int_X |g|^p\;\alpha(v)(dx)\,.$$

We firstly prove the inequality for $f$ and $g$ simple functions measurable with respect to $\alpha(v)$ and $\alpha(u)$. Let $f=\sum \alpha_i \mathbf{1}_{E_i}$ and $g=\sum \beta_i \mathbf{1}_{E_i}$

$$\int_X |fg|\,|\tilde{\mu}(u,v)|(dx)=\sum\int_{E_i}|\alpha_i\beta_i|\,|\tilde{\mu}(u,v)|(dx)=\sum\int_{E_i}|\tilde{\mu}(|\alpha_i|^{1/(p-1)}u,|\beta_i|v)(dx)\leq$$

$$\leq 2^{p-1}\sum\int_{E_i}a^{-p}\mu(|\alpha_i|^{1/(p-1)}u)(dx)+2^{p-1}\sum\int_{E_i}a^{p(p-1)}\mu(|\beta_i|v)(dx)=$$

$$=2^{p-1}\sum\int_{E_i}a^{-p}|\alpha_i|^{p/(p-1)}\mu(u)(dx)+2^{p-1}\sum\int_{E_i}a^{p(p-1)}|\beta_i|^p\mu(v)(dx)\leq$$

$$\leq 2^{p-1}a^{-p}\int_X|f|^{p'}\mu(u)(dx)+2^{p-1}a^{p(p-1)}\int_X|g|^p\mu(v)(dx)\,.$$

The integral inequality follows by approximation for all functions in $D \cap C_0(X)$ and then for all function in $C_0(X)$ and using again an approximation we obtain the stated inequality. Let now $\phi$ be positive in $C_0(X)$ and apply the integral inequality to $f\phi^{\frac{1}{p'}}$ and to $g\phi^{\frac{1}{p}}$; then

$$\int_X |fg|\phi \; |\mu(u,v)|(dx) \leq 2^{p-1}a^{-p} \int_X |f|^{p'}\phi \; \alpha(u)(dx) + 2^{p-1}a^{p(p-1)} \int_X |g|^p\phi \; \alpha(v)(dx)$$

and the result follows.

REMARK 3.2: From (6) it follows that if $v_n$, $n = 1, 2, \ldots$, converges to $v$ in $D_0$ then

$$\lim_{n \to +\infty} \mu(u, v_n) = \mu(u, v)$$

in the weak$^\star$ topology of $\mathcal{M}$.

## 4. - CHAIN RULES. THE CASE OF FUNCTIONS IN $D \cap C_0(X)$.

Let

$$\Phi(u) = \int_X \alpha(u)(dx)$$

be a strongly local Dirichlet functional with domain $D_0$. Firstly we prove a chain rule for functions in $D \cap C_0(\Omega)$.

PROPOSITION 4.1: *Let $u \in D \cap C_0(X)$ and $\beta \in C^1(\mathbf{R})$ with $\beta(0) = 0$; then*

$$\alpha(\beta(u)) = |\beta'(u)|^p \alpha(u).$$

PROOF: We observe that $|u| \leq M$ and $\beta'(t)$ is uniformly continuous on $[-M, M]$.

Let $\sigma > 0$ be arbitrary and let $\varepsilon > 0$ be such that $|\beta'(t_1) - \beta'(t_2)| < \sigma$ for $|t_1 - t_2| < \varepsilon$, $t_1, t_2 \in [-M, M]$.

We fix a point $x_0$, then $|\beta'(t) - \beta'(u(x_0)| < \sigma$, for $|u(x_0) - t| < \varepsilon$. Let $U_{x_0}$ be a neighborhood of $x_0$ such that $|u(x) - u(x_0)| < \varepsilon$ for $x \in U_{x_0}$. Let $\tilde{\beta}(t)$ be such that $\tilde{\beta}(0) = 0$, $\tilde{\beta}'(t) = \left(\dfrac{\beta'(t)}{|\beta'(u(x_0))| + \sigma} \wedge 1\right) \vee (-1)$. We observe that $\tilde{\beta}'(t) = \dfrac{\beta'(t)}{|\beta'(u(x_0))| + \sigma}$ on the set $|u(x_0) - t| < \varepsilon$. Then $\tilde{\beta}(t) - \dfrac{\beta(t)}{|\beta'(u(x_0))| + \sigma} = cst$ on the set $|u(x_0) - t| < \varepsilon$. As a consequence we have $\tilde{\beta}(u(x)) - \dfrac{\beta(u(x))}{|\beta'(u(x_0))| + \sigma} = cst$ on $U_{x_0}$. We have that $\tilde{\beta}'(t) \leq 1$. From $(H_6)$ we have

$$\alpha(\tilde{\beta}(u)) \leq \alpha(u)$$

then

$$\alpha(\beta(u)) \leq (|\beta'(u(x_0))| + \sigma)^p \alpha(u) \leq (|\beta'(u(x))| + 2\sigma)^p \alpha(u)$$

on $U_{x_0}$. By a covering argument we obtain

$$\alpha(\beta(u)) \leq (|\beta'(u(x))| + 2\sigma)^p \alpha(u)$$

on $X$. Since $\sigma > 0$ is arbitrary, we obtain

$$\alpha(\beta(u)) \leq |\beta'(u(x))|^p \alpha(u) \tag{4.1}$$

on $X$. From (4.1) we have in particular that

$$\alpha(\beta(u))(K) = 0 \tag{4.2}$$

where $K = \{\beta'(u(x)) = 0\}$.

Consider now a point $x_0$ and assume that $|\beta'(u(x_0))| > \sigma > 0$, then there exists $\varepsilon > 0$ such that $|\beta'(t)| \geq \dfrac{\sigma}{2}$ for $|u(x_0) - t| < 2\varepsilon$ and a neighborhood of $x_0$ in which we have $|u(x) - u(x_0)| < 2\varepsilon$. Since $\beta \in C^1$ then $\beta$ is invertible on $|u(x_0) - t| < 2\varepsilon$. The inverse function $\psi$ on $|u(x_0) - t| < \varepsilon$ may be extended to a function in $C^1(\mathbf{R})$ with $\psi(0) = 0$. Let $U_{x_0}$ be such that $|u(x) - u(x_0)| < \varepsilon$ in $U_{x_0}$. By (4.1) and the strong locality of $\alpha$ we have

$$\alpha(u) = \alpha(\psi(\beta(u))) \leq |\psi'(\beta(u))|^p \alpha(\beta(u))$$

on $U_{x_0}$. Then

$$|\beta'(u)|^p \alpha(u) \leq \alpha(\beta(u)) \tag{4.3}$$

on $U_{x_0}$. A covering argument prove that (4.3) holds on the open set $A = \{|\beta'(u)| > 0\}$. From (4.1) we have that

$$\alpha(\beta(u)) = |\beta'(u)|^p \alpha(u) \tag{4.4}$$

on $A$. From (4.2) and (4.4) we have the result.

Corollary 4.2: *Let the assumptions* $(H_1)(H_2)(H_4)(H_5)$ *and* $(H_6)$ *hold, then the assumption* $(H_6')$ *also holds.*

Proof: Let $\beta$ be defined as $\beta(t) = 0$ for $t \leq 0$, $\beta(t) = t$ for $0 < t < 1$ and $\beta(t) = 1$ for $t \geq 1$. Let $\beta_\varepsilon(t)$ be function in $C^1$ with $\beta_\varepsilon(t) = 0$ for $t \leq \varepsilon$, $\beta_\varepsilon(t) = 1$ for $t \geq (1 - \varepsilon)$ $(\varepsilon > 0)$ and $\beta'_\varepsilon \leq \delta_\varepsilon$ where $\delta_\varepsilon \to 1$ as $\varepsilon \to 0$. Moreover we can assume that

$$\lim_{\varepsilon \to 0} \beta_\varepsilon(t) = \beta(t) \tag{4.5}$$

uniformly on X and

$$\lim_{\varepsilon \to 0} \beta'_\varepsilon(t) = \mathbf{1}_{\{0<t<1\}} \tag{4.6}$$

for the pointwise convergence.

Let $u \in D \cap C_0(X)$. From (4.5) we have

$$\lim_{\varepsilon \to 0} \beta_\varepsilon(u) = \beta(u)$$

uniformly on $X$ and (from bounded convergence theorem)

$$\lim_{\varepsilon \to 0} \beta'_\varepsilon(u) = \mathbf{1}_{\{0<u<1\}} \tag{4.7}$$

in $L^1(X, \alpha(u))$. Then $\beta'_\varepsilon(u)$ is a Cauchy sequence in $L^1(X, \alpha(u))$. From Proposition 4.1 we obtain that $\beta_\varepsilon(u)$ is a Cauchy sequence in $D_0$, which converges to $\beta(u) \in D_0$. Moreover from (4.7) we obtain

$$\lim_{\varepsilon\to 0} \alpha(\beta_\varepsilon(u)) = \lim_{\varepsilon\to 0} \beta'_\varepsilon(u))\alpha(u) = \mathbf{1}_{\{0<u<1\}}\alpha(u)$$

in $\mathcal{M}$. Then

$$\alpha(\beta(u)) \leq \mathbf{1}_{\{0<u<1\}}\alpha(u) \leq \alpha(u)$$

in $\mathcal{M}$.

PROPOSITION 4.3: *Let $u, v \in D \cap C_0(X)$, then $uv$ is in $D \cap C_0(X)$. Moreover if $u_n, v_n \in$ $\in D \cap C_0(X)$ converges respectively to $u, v$ in $D_0$ and in $C_0(X)$ and $supp(u_n)$, $supp(v_n) \subset K$ where $K$ is a fixed compact set. Then $u, v \in D \cap C_0(X)$ and $u_n v_n$ converges to $uv$ in $D_0$ and in $C_0(X)$.*

PROOF: From polarization equality we have

$$uv = \frac{1}{4}[(u+v)^2 - (u-v)^2]$$

Since by Proposition 4.1 $(u+v)^2, (u-v)^2 \in D_0 \cap C_0(X)$ we have $uv \in D_0 \cap C_0(X)$. The sequences $u_n, v_n \in D_0 \cap L^\infty(X, m)$ converge respectively to $u, v$ in $D_0$ and in $L^\infty(X, m)$. Moreover we have

$$\lim_{n\to+\infty} (u_n + v_n) = (u+v)$$

$$\lim_{n\to+\infty} (u_n - v_n) = (u-v)$$

in $L^\infty(X, m)$ and in $D_0$. We observe that the proofs of Proposition 4.5-4.7 do not depend on the results in the present Proposition.

Then from Proposition 4.7 we have that $(u_n \pm v_n)^2$ converges to $(u \pm v)^2$ in $L^\infty(X, m)$ and in $D_0$. The result now follows.

PROPOSITION 4.4: *Let $u$ in $D_0$ and $v \in D \cap C_0(X)$ then we have*

$$\mu(u, v^2) = 2v\mu(u, v).$$

PROOF: Consider a relatively compact ball $B(x_0, r)$ we have

$$\mu(u, v^2) = \mu(u, v^2 - v(x_0)^2) = \mu(u, (v + v(x_0))(v - v(x_0))) =$$
$$= \mu(u, (v - v(x_0))^2) + 2v(x_0)\mu(u, v - v(x_0))$$

on $B(x_0, r)$. We have, for $\varepsilon > 0$ arbitrary,

$$|\mu(u, (v - v(x_0))^2)| \leq \varepsilon\alpha(u) + C_\varepsilon \alpha((v - v(x_0))^2) = \varepsilon\alpha(u) + 2^p C_\varepsilon |v(x) - v(x_0)|^p \alpha((v - v(x_0)))$$

on $B(x_0, r)$.

We choose $r$ such that for $x \in B(x_0, r)$ we have

$$|v(x) - v(x_0)|^p \leq \frac{\varepsilon}{2^p C_\varepsilon}$$

then

$$|\mu(u, (v - v(x_0))^2)| \leq \varepsilon(a(u) + a(v))$$

in $B(x_0, r)$. We obtain

$$|\mu(u, v^2) - 2v(x)\mu(u, v)| \leq \varepsilon(a(u) + a(v)) + 2|v(x) - v(x_0)||\mu(u, v)| \leq 3\varepsilon(a(u) + a(v))$$

in $B(x_0, r)$ (where we assume $C_\varepsilon \geq 1$). By a covering argument we obtain

$$|\mu(u, v^2) - 2v(x)\mu(u, v)| \leq 3\varepsilon(a(u) + a(v))$$

on $X$, then, since $\varepsilon > 0$ is arbitrary

$$\mu(u, v^2) = 2v(x)\mu(u, v)$$

on $X$.

Proposition 4.5: *Let $u \in D_0$ and $v, w \in D \cap C_0(X)$, then $vw \in D \cap C_0(X)$ and the following Leibnitz rule holds*

$$\mu(u, vw) = v\mu(u, w) + w\mu(u, v).$$

Proof: We have

$$vw = \frac{1}{4}[(v + w)^2 - (v - w)^2] .$$

From Proposition 4.4 we have $vw \in D_0$; moreover we have

$$\mu(u, vw) = \frac{1}{4}[\mu(u, (v + w)^2 - (v - w)^2)] = \frac{1}{4}[\mu(u, (v + w)^2) - \mu(u, (v - w)^2)] .$$

From Proposition 4.4 we obtain

$$\mu(u, vw) = \frac{1}{4}[2(v + w)\mu(u, v + w) - 2(v - w)\mu(u, v - w)] =$$

$$= \frac{1}{4}[2(v + w)(\mu(u, v) + \mu(u, w)) - 2(v - w)(\mu(u, v) - \mu(u, w))] = v\mu(u, w) + w\mu(u, v) .$$

Proposition 4.6: *Let $u \in D \cap C_0(X)$, $v \in D \cap C_0(X)$ and $\beta \in C^1(R)$ with $\beta(0) = 0$. Then we have $\beta(v) \in D \cap C_0(X)$ and*

$$\mu(u, \beta(v)) = \beta'(v)\mu(u, v).$$

Proof: By Propositions 4.4, 4.5 the result holds in the case $\beta(t)$ is a power of $t$, then in the case $\beta(t)$ is a polynome of any degree. In the general case there exists a sequence of polynomes $\beta_n(t)$ such that $\beta_n(t)$ and $\beta'_n(t)$ converges locally uniformly to $\beta(t)$ and $\beta'(t)$. We have that $\beta_n(v)$ converges to $\beta(v)$ uniformly on $X$. Moreover

$$\lim_{m,n \to +\infty} a(\beta_m(v) - \beta_n(v)) = \lim_{m,n \to +\infty} (\beta'_m(v) - \beta'_n(v))a(v) = 0$$

in $\mathcal{M}$. Then the sequence $\beta_n(v)$ converges to $\beta(v)$ in $D_0$ and $\beta(v) \in D_0$.

From (6) Theorem 3.4 and Remark 3.2 we have

$$\lim_{n\to+\infty} \mu(u, \beta_n(v)) = \mu(u, \beta(v)) \tag{4.5}$$

weakly* in $\mathcal{M}$. Moreover

$$\lim_{n\to+\infty} \mu(u, \beta_n(v)) = \lim_{n\to+\infty} \beta'_n(v)\mu(u, v) = \beta'(v)\mu(u, v) \tag{4.6}$$

in $\mathcal{M}$, where we use the dominated convergence Theorem. From (4.5) (4.6) the result follows.

PROPOSITION 4.7: *Let* $u_n, u \in D \cap C_0(X)$, $\beta \in C^1(R)$ *with* $\beta(0) = 0$; *assume that* $supp(u_n) \subset K$, *where* $K$ *is a fixed compact set and* $u_n$ *converges to* $u$ *uniformly on* $X$ *and in* $D_0$. *Then* $\beta(u_n)$ *converges to* $\beta(u)$ *uniformly on* $X$ *and in* $D_0$.

PROOF: We have

$$\begin{aligned}
a(\beta(u_n) - \beta(u)) &= \frac{1}{p}\mu(\beta(u_n) - \beta(u), \beta(u_n) - \beta(u)) = \\
&= \frac{1}{p}[\mu(\beta(u_n) - \beta(u), \beta(u_n)) - \mu(\beta(u_n) - \beta(u), \beta(u))] = \\
&= \frac{1}{p}[\beta'(u_n)\mu(\beta(u_n) - \beta(u), u_n) - \beta'(u)\mu(\beta(u_n) - \beta(u), u)] = \\
&= \frac{1}{p}[(\beta'(u_n)\mu(\beta(u_n) - \beta(u), u_n - u) + (\beta'(u_n) - \beta'(u))\mu(\beta(u_n) - \beta(u), u)]\,.
\end{aligned} \tag{4.7}$$

Consider the first term on the right hand side; we have that

$$\begin{aligned}
&|\beta'(u_n)||\mu(\beta(u_n) - \beta(u), u_n - u)| \leq \\
&\qquad \leq C_1\Phi(u_n - u)^{\frac{(p-1)}{2}}(a(\beta(u_n)) + a(\beta(u))) + C_2\Phi(u_n - u)^{-\frac{1}{2}}a(u_n - u) \leq \\
&\qquad \leq C_3[\Phi(u_n - u)^{\frac{(p-1)}{2}}(\beta'(u_n)a(u_n) + \beta'(u)a(u)) + \Phi(u_n - u)^{-\frac{1}{2}}a(u_n - u)] \leq \\
&\qquad \leq C_4[\Phi(u_n - u)^{\frac{(p-1)}{2}}(a(u_n) + a(u)) + \Phi(u_n - u)^{-\frac{1}{2}}a(u_n - u)]
\end{aligned}$$

where we take into account that the functions $u_n$ are uniformly bounded. Then the first term in the right hand side of (4.7) converges to 0 in $\mathcal{M}$.

By similar methods we have that $(\beta'(u_n) - \beta'(u))\mu(\beta(u_n) - \beta(u), u)$ converges to 0 in $\mathcal{M}$.

Then $a(\beta(u_n) - \beta(u))$ converges to 0 in $\mathcal{M}$ and so the result is proved.

PROPOSITION 4.8: *Let* $u \in D \cap C_0(X)$,$v \in D \cap C_0(X)$ *and* $\beta \in C^1(R)$ *with* $\beta(0) = 0$. *Then we have* $\beta(u) \in D \cap C_0(X)$ *and*

$$\mu(\beta(u), v) = |\beta'(u)|^{p-2}\beta'(u)\mu(u, v)\,.$$

Proof: Assume $\beta \in C^2(R)$. We have

$$\alpha(\beta(u+tv)) - \alpha(\beta(u)) = |\beta'(u+tv)|^p \alpha(u+tv) - |\beta'(u)|^p \alpha(u) = \tag{4.8}$$

$$= (|\beta'(u+tv)|^p - |\beta'(u)|^p)\alpha(u+tv) + |\beta'(u)|^p(\alpha(u+tv) - \alpha(u)) =$$

$$= \left( ptv \int_0^1 |\beta'(u+\zeta tv)|^{p-2}\beta'(u+\zeta tv)\beta''(u+\zeta tv)d\zeta \right) \alpha(u+tv) + |\beta'(u)|^p(\alpha(u+tv) - \alpha(u)).$$

We have that

$$\lim_{t\to 0} \alpha(u+tv) = \alpha(u)$$

weakly in $\mathcal{M}$. Dividing by $t$ and passing to the limit as $t \to 0$ in (4.8) we obtain

$$\lim_{t\to 0} \frac{\alpha(\beta(u+tv)) - \alpha(\beta(u))}{t} = pv|\beta'(u)|^{p-2}\beta'(u)\beta''(u)\alpha(u) + |\beta'(u)|^p \mu(u,v).$$

weakly$^\star$ in $\mathcal{M}$. We have also

$$\alpha(\beta(u+tv)) - \alpha(\beta(u)) = \alpha(\beta(u) + tv \int_0^1 \beta'(u+\zeta tv)d\zeta) - \alpha(\beta(u)) = \tag{4.9}$$

$$= \alpha\left( \beta(u) + tv\left( \int_0^1 (\beta'(u+\zeta tv) - \beta'(0))d\zeta + \beta'(0) \right)\right) - \alpha(\beta(u)\,.$$

We observe that by Proposition 4.1 we have $(\beta'(u+tv) - \beta'(0)) \in D_0$. Moreover

$$\alpha(\beta'(u+tv) - \beta'(0)) = \beta''(u+tv)\alpha(u+tv)\,.$$

By Proposition 4.6 we have

$$\lim_{t\to 0} (\beta'(u+tv) - \beta'(0)) = \beta'(u) - \beta'(0) \tag{4.10}$$

uniformly on $X$ and in $D_0$. Then

$$\lim_{t\to 0} (\beta'(u+\zeta tv) - \beta'(0)) = \beta'(u) - \beta'(0) \tag{4.11}$$

in $D_0$ uniformly for $\zeta \in [0,1]$. From (4.11) we obtain

$$\lim_{t\to 0} \int_0^1 (\beta'(u+\zeta tv) - \beta'(0))d\zeta = \beta'(u) - \beta'(0) \tag{4.12}$$

in $D_0$. From (4.12) we obtain

$$\lim_{t\to 0} v \int_0^1 \beta'(u+\zeta tv)d\zeta = v\beta'(u)$$

in $D_0$. Dividing by $t$ and passing to the limit as $t \to 0$ in (4.9) we obtain

$$\lim_{t\to 0} \frac{a(\beta(u+tv)) - a(\beta(u))}{t} = \mu(\beta(u), v\beta'(u))$$

weakly$^\star$ in $\mathcal{M}$. Then using the Leibnitz rule of Proposition 4.5 on the second argument

$$\lim_{t\to 0} \frac{a(\beta(u+tv)) - a(\beta(u))}{t} = v\mu(\beta(u), \beta'(u)) + \beta'(u)\mu(\beta(u), v) =$$
$$= v\beta''(u)\mu(\beta(u), u) + \beta'(u)\mu(\beta(u), v)\,.$$

By Proposition 4.1 we have

$$a(\beta(u)) = |\beta'(u)|^p a(u) = \frac{1}{p}|\beta'(u)|^p \mu(u,u) = \frac{1}{p}\mu(\beta(u), \beta(u)) = \frac{1}{p}\beta'(u)\mu(\beta(u), u)$$

weakly$^\star$ in $\mathcal{M}$. We have also proved that

$$\lim_{t\to 0} \frac{a(\beta(u+tv)) - a(\beta(u))}{t} = pv|\beta'(u)|^{p-2}\beta'(u)\beta''(u)a(u) + \beta'(u)\mu(\beta(u), v)$$

weakly$^\star$ in $\mathcal{M}$. Then we have

$$\mu(\beta(u), v) = |\beta'(u)|^{p-2}\beta'(u)\mu(u,v) \tag{4.13}$$

on the set $\{\beta'(u) \neq 0\}$. From (4) in Theorem 3.4 we have

$$\mu(\beta(u), v) \leq 2^{p-1}\varepsilon^{p(p-1)}a(v) + 2^{p-1}\varepsilon^{-p}a(\beta(u)) \leq 2^{p-1}\varepsilon^{p(p-1)}a(v) + 2^{p-1}\varepsilon^{-p}|\beta'(u)|^p a(u)$$

where $\varepsilon > 0$ is arbitrary. Then on the set $\{\beta'(u) = 0\}$ we have

$$\mu(\beta(u), v) \leq 2^{p-1}\varepsilon^{p(p-1)}a(v)$$

so on the set $\{\beta'(u) = 0\}$ we have

$$\mu(\beta(u), v) = 0\,. \tag{4.14}$$

Then we have

$$\mu(\beta(u), v) = |\beta'(u)|^{p-1}\beta'(u)\mu(u,v)\,.$$

Assume now $\beta \in C^1(R)$ and $u \in D \cap C_0(X)$ with $|u| \leq M$; there is a sequence $\beta_n \in C^2(R)$ such that $\beta_n$, $\beta'_n$ converges to $\beta$, $\beta'$ uniformly on $[-M, M]$.

We have

$$\mu(\beta_n(u), v) = |\beta'_n(u)|^{p-1}\beta'_n(u)\mu(u,v)$$

and

$$\lim_{n\to\infty} |\beta'_n(u)|^{p-1}\beta'_n(u) = |\beta'(u)|^{p-1}\beta'(u)$$

uniformly on $X$, then (at least after extraction of subsequences) in $L^\infty(X, |\mu(u,v)|)$ . Then

$$\lim_{n\to+\infty} \mu(\beta_n(u), v) = |\beta'(u)|^{p-1}\beta'(u)\mu(u,v) \tag{4.15}$$

in $\mathcal{M}$. Moreover we have

$$\lim_{n\to+\infty} a(\beta_n(u)-\beta(u)) = \lim_{n\to+\infty} |\beta'_n(u)-\beta'(u)|^p a(u) = 0\,.$$

Then

$$\lim_{n\to+\infty} \beta_n(u)) = \beta(u)$$

in $D_0$. Taking into account Remark 3.1 we obtain

$$\lim_{n\to+\infty} \mu(\beta_n(u),v) = \mu(\beta(u),v) \tag{4.16}$$

weakly$^\star$ in $\mathcal{M}$. From (4.15),(4.16) we obtain the result.

## 5. - Chain rules. The case of functions in $D_0 \cap L^\infty(X,m)$.

We begin this section by a generalization of Proposition 4.7.

Proposition 5.1: *Let $u_n, u \in D_0$, $\beta \in C^1(R)$ with $\beta(0)=0$; assume that the $u_n$ are uniformly bounded and converge to $u$ in $D_0$. Then $\beta(u_n)$ converges to $\beta(u)$ in $D_0$.*

Proof: We recall that, since $u_n$ converges to $u$ in $D_0$, $u_n$ converges to $u$ quasi-uniformly i.e. for every $\varepsilon > 0$ there exists a set $E_\varepsilon$ such that $u_n$ converges to $u$ uniformly on $X - E_\varepsilon$ and $cap_\Phi(E_\varepsilon) \le \varepsilon$. We have

$$\begin{aligned}
a(\beta(u_n)-\beta(u)) &= \frac{1}{p}\mu(\beta(u_n)-\beta(u),\beta(u_n)-\beta(u)) = \\
&= \frac{1}{p}[\mu(\beta(u_n)-\beta(u),\beta(u_n)) - \mu(\beta(u_n)-\beta(u),\beta(u))] = \\
&= \frac{1}{p}[\beta'(u_n)\mu(\beta(u_n)-\beta(u),u_n) - \beta'(u)\mu(\beta(u_n)-\beta(u),u)] = \\
&= \frac{1}{p}[(\beta'(u_n)\mu(\beta(u_n)-\beta(u),u_n-u) + (\beta'(u_n)-\beta'(u))\mu(\beta(u_n)-\beta(u),u)]\,.
\end{aligned} \tag{5.1}$$

Consider the first term on the right hand side; we have that

$$\begin{aligned}
&|\beta'(u_n)||\mu(\beta(u_n)-\beta(u),u_n-u)| \le \\
&\le C_1\Phi(u_n-u)^{\frac{(p-1)}{2}}(a(\beta(u_n))+a(\beta(u))) + C_2\Phi(u_n-u)^{-\frac{1}{2}}a(u_n-u) \le \\
&\le C_3[\Phi(u_n-u)^{\frac{(p-1)}{2}}(\beta'(u_n)a(u_n)+\beta'(u)a(u)) + \Phi(u_n-u)^{-\frac{1}{2}}a(u_n-u)] \le \\
&\le C_4[\Phi(u_n-u)^{\frac{(p-1)}{2}}(a(u_n)+a(u)) + \Phi(u_n-u)^{-\frac{1}{2}}a(u_n-u)]
\end{aligned}$$

where we take into account that the functions $u_n$ are uniformly bounded. Then the first term in the right hand side of (5.1) converges to 0 in $\mathcal{M}$. For the second term in the right hand side we observe that for every $\sigma > 0$ there exists an open set $E_\sigma$ such that $\int_{E_\sigma} a(u)(dx) \leq \sigma$ and the sequence $u_n$ converges uniformly to $u$ on $X - E_\sigma$.

By methods similar to the ones used in the first part of the proof we have that $|(\beta'(u_n) - \beta'(u))||\mu(\beta(u_n) - \beta(u), u)|$ restricted $X - E_\sigma$ converges to 0 in $\mathcal{M}$. We have also that

$$\mathbf{1}_{E_\sigma} a(u) \leq \sigma\,.$$

Then

$$\mathbf{1}_{E_\sigma}|(\beta'(u_n) - \beta'(u))||\mu(\beta(u_n) - \beta(u), u)| \leq C_5(\sigma^{\frac{(p-1)}{2}} a(u_n) + \sigma^{-\frac{1}{2}} a(u))$$

so

$$\int_{E_\sigma} |(\beta'(u_n) - \beta'(u))||\mu(\beta(u_n) - \beta(u), u)|(dx) \leq C_6(\sigma^{\frac{(p-1)}{2}} + \sigma^{-\frac{1}{2}})\,.$$

We obtain

$$\lim_{n\to+\infty} \int |(\beta'(u_n) - \beta'(u))||\mu(\beta(u_n) - \beta(u), u)|(dx) \leq C_6(\sigma^{\frac{(p-1)}{2}} + \sigma^{-\frac{1}{2}})$$

for every $\sigma$. Then $a(\beta(u_n) - \beta(u))$ converges to 0 in $\mathcal{M}$ and so the result is proved.

We will give now a result on approximation of functions in $D_0 \cap L^\infty(X, m)$.

PROPOSITION 5.2: *Let $u \in D_0 \cap L^\infty(X)$, there exists an uniformly bounded sequence of functions $u_n \in D \cap C_0(X)$ which converges in $D_0$ to $u$.*

PROOF: There exists a sequence $u_n \in D \cap C_0(X)$ such that $u_n$ converges to $u$ in $D_0$. Let $|u| \leq M$ and let $\beta$ be a $C^1$ function such that $\beta(t) = t$ for $|t| \leq M + 1$, $\beta(t) = C$ for $|t| \geq M + 2$ and $\beta'(t) \leq 1$. Define $v_n = \beta(u_n)$, we have that $v_n \in D \cap C_0(X)$ is uniformly bounded. From Proposition 5.1 we obtain that $v_n$ converges to $\beta(u) = u$ in $D_0$.

We are now ready to give the proof of the chain rule for Markov functionals corresponding to a p-homogeneous (strongly local) Dirichlet form.

THEOREM 5.3: *Let $u \in D_0 \cap L^\infty(X, m)$ and $\beta \in C^1(\mathbf{R})$ with $\beta(0) = 0$; then*

$$a(\beta(u)) = |\beta'(u)|^p a(u).$$

PROOF: From Proposition 5.2 there exists a uniformly bounded sequence $u_n$ such that $u_n$ converges to $u$ in $D_0$. From proposition 4.1 we have

$$a(\beta(u_n)) = |\beta'(u_n)|^p a(u_n)\,. \tag{5.2}$$

We observe that $\beta'(u_n)$ is an uniformly bounded sequence converging to $\beta'(u)$ quasi-uniformly; then by Lemma 3.2 the right hand side in (5.1) weakly converges in $\mathcal{M}$ to $|\beta'(u)|^p a(u)$.

From Proposition 5.1 we have that $\beta(u_n)$ converges in $D_0$ to $\beta(u)$ then from Lemma 3.2 we obtain that the left hand side in (5.2) weakly converges in $\mathcal{M}$ to $a(\beta(u))$. Then the result follows.

THEOREM 5.4: *Let $u \in D_0$ and $\beta_1(t) = \inf(t, M)$, $\beta_2(t) = \sup(t, -M)$, $\beta_3 = \inf(\sup(t, -M_1), M_2)$, $M, M_1, M_2 \geq 0$; then $\beta_i(u) \in D_0$ and*

$$\alpha(\beta_1(u)) = \mathbf{1}_{\{u<M\}}\alpha(u)$$

$$\alpha(\beta_2(u)) = \mathbf{1}_{\{u>-M\}}\alpha(u)$$

$$\alpha(\beta_3(u)) = \mathbf{1}_{\{-M_1<u<M_2\}}\alpha(vu)$$

*where $\mathbf{1}_E$ denotes the characteristic function of the set E (which is defined up to sets of capacity zero).*

PROOF: We prove the result for $\beta_1$, the proof in the other cases is the same.

Assume $u \in D_0$. There exists a sequence of functions $\beta_{1,n}(t)$ in $C^1(\mathbf{R})$ such that $\beta_{1,n}(0) = 0$, $\beta_{1,n}(t) = M$ for $t \geq M - \frac{1}{n}$, $\beta'_{1,n}(t) = 1 + \frac{2}{n}$ for $t \leq M - \frac{2}{n}$, $0 \leq \beta'_{1,n}(t) = 1 + \frac{2}{n}$, $\lim_{n\to+\infty} \beta_{1,n}(t) = \beta_1(t)$ uniformly on $\mathbf{R}$, $\lim_{n\to+\infty} \beta'_{1,n}(t) = \mathbf{1}_{\{t<M\}}$ pointwise on $\mathbf{R}$. From Theorem 5.3 we have

$$\alpha(\beta_{1,n}(u) - \beta_{1,m}(u)) = |\beta'_{1,n}(u) - \beta'_{1,m}(u)|^p \alpha(u)\,. \tag{5.3}$$

We have that $\beta'_{1,n}(u)$ is uniformly bounded and converges quasi-everywhere to $\beta'_1(u)$. Then $\beta'_{1,n}(u)$ converges to $\beta'_1(u)$ strongly in $L^p(X, \alpha(u))$ so $\beta_{1,n}(u)$ converges in $D_0$ to $\beta_1(u) \in D_0$ . Since $\beta'_{1,n}(u)$ converges to $\beta'_1(u)$ strongly in $L^p(X, \alpha(u))$ we have

$$\lim_{n\to+\infty} \alpha(\beta_{1,n}(u)) = \mathbf{1}_{\{u<M\}}\alpha(u) \tag{5.4}$$

in $\mathcal{M}$.

Since $\beta_{1,n}(u)$ converges in $D_0$ to $\beta_1(u) \in D_0$, from Lemma 3.2 we have

$$\lim_{n\to+\infty} \alpha(\beta_{1,n}(u)) = \alpha(\beta_1(u)) \tag{5.5}$$

weakly in $\mathcal{M}$. From (5.3)and (5.4) the result follows.

By Theorem 5.4. the result in Theorem 5.5 follows:

THEOREM 5.5: *Let $u \in D_0$, then $|u|$ is in $D_0$ and*

$$\alpha(|u|) = sign(u)\mathbf{1}_{\{u\neq 0\}}\alpha(u)$$

*where we define $sign(0) = 0$.*

THEOREM 5.6: *Let $u, v \in D_0 \cap L^\infty(X, m)$, then $uv$ is in $D_0 \cap L^\infty(X, m)$. Moreover if $u_n, v_n \in D_0 \cap L^\infty(X, m)$ converge respectively to $u, v$ in $D_0$ and in $L^\infty(X, m)$ then $u_n v_n$ converges to $uv$ in $D_0$ and in $L^\infty(X, m)$.*

PROOF: From polarization equality we have

$$uv = \frac{1}{4}[(u+v)^2 - (u-v)^2]\,.$$

Since by Theorem 4.3 $(u+v)^2, (u-v)^2 \in D_0 \cap L^\infty(X,m)$ we have $uv \in D_0 \cap L^\infty(X,m)$. Let now $u_n, v_n \in D_0 \cap L^\infty(X,m)$ converge respectively to $u, v$ in $D_0$ and in $L^\infty(X,m)$. From Lemma 3.2 we have

$$\lim_{n\to+\infty} \alpha(u_n + v_n) = \alpha(u+v)$$

$$\lim_{n\to+\infty} \alpha(u_n - v_n) = \alpha(u-v)$$

weakly in $\mathcal{M}$. Moreover we have

$$\lim_{n\to+\infty} (u_n + v_n) = (u+v)$$

$$\lim_{n\to+\infty} (u_n - v_n) = (u-v)$$

in $L^\infty(X,m)$ and in $D_0$.

Using the Proposition 5.1 we have that $(u_n \pm v_n)^2$ converges to $(u \pm v)^2$ in $L^\infty(X, m)$ and in $D_0$. Then the result follows.

By Theorem 5.6 we have that $D_0 \cap L^\infty(X,m)$ has the structure of a continuous Banach algebra.

Now we prove the chain rules concerning a p-homogeneous (strongly local) Dirichlet form. We begin by the following lemma:

LEMMA 5.7: *Let $u_n \in D_0$ and $v_n \in D_0$ converge respectively to $u$ and $v$ in $D_0$. Then*

$$\lim_{n\to\infty} \mu(u_n, v_n) = \mu(u,v)$$

*weakly$^\star$ in $\mathcal{M}$.*

PROOF: We have

$$\mu(u_n, v_n) = \mu(u_n, v) - \mu(u_n, v - v_n)\,.$$

Consider the second term in the right hand side; we have

$$|\mu(u_n, v - v_n)| \le 2^{p-1}\varepsilon^{-p}\alpha(v - v_n) + 2^{p-1}\varepsilon^{p(p-1)}\alpha(u_n)\,.$$

Then

$$\lim_{n\to\infty} \mu(u_n, v - v_n) = 0$$

in $\mathcal{M}$. Moreover from Remark 3.1 we have

$$\lim_{n\to\infty} \mu(u_n, v) = \mu(u,v)$$

in the weakly$^\star$ topology of $\mathcal{M}$, so we obtain the result.

THEOREM 5.8: *Let* $u \in D_0 \cap L^\infty(X,m)$, $v \in D_0 \cap L^\infty(X,m)$ *and* $\beta \in C^1(R)$ *with* $\beta(0) = 0$. *Then we have* $\beta(v) \in D_0 \cap L^\infty(X,m)$ *and*

$$\mu(u,\beta(v)) = \beta'(v)\mu(u,v).$$

PROOF: We can consider two sequences $u_n \in D_0 \cap C_0(X)$, $v_n \in D_0 \cap C_0(X)$ uniformly bounded and converging to $u$ and $v$ in $D_0$. By Proposition 5.1 we have that $\beta(v_n)$ converges to $\beta(v)$ in $D_0$; moreover $\beta'(v_n)$ converges to $\beta'(v)$ quasi-uniformly and $\beta'(v_n)$ is uniformly bounded. We observe that

$$|\mu(u_n,v_n)| \leq C(\alpha(u_n) + \alpha(v_n))$$

so from Th. 2 pg. 306 [DS] and lemma 5.7, we have that $\mu(u_n,v_n)$ weakly converges to $\mu(u,v)$ in $\mathcal{M}$. By Lemma 3.2 we obtain that

$$\lim_{n\to\infty} \beta'(v_n)\mu(u_n,v_n) = \beta'(v)\mu(u,v)$$

weakly in $\mathcal{M}$ and by Lemma 5.6 we obtain

$$\lim_{n\to\infty} \mu(u_n,\beta(v_n)) = \mu(u,\beta(v))$$

weakly* in $\mathcal{M}$. So

$$\mu(u,\beta(v)) = \beta'(v)\mu(u,v)$$

By the same methods we obtain also:

THEOREM 5.9: *Let* $u \in D_0 \cap L^\infty(X,m)$, $v \in D_0 \cap L^\infty(X,m)$ *and* $\beta \in C^1(R)$ *with* $\beta(0) = 0$. *Then we have* $\beta(u) \in D_0 \cap L^\infty(X,m)$ *and*

$$\mu(\beta(u),v) = |\beta'(u)|^{p-2}\beta'(u)\,\mu(u,v).$$

As in the bilinear case we can obtain a Leibnitz rule in the second term of the form:

THEOREM 5.10: *Let* $u \in D_0 \cap L^\infty(X,m)$, $v \in D_0 \cap L^\infty(X,m)$ *and* $w \in D_0$; *then* $uv \in D_0 \cap L^\infty(X,m)$ *and*

$$\mu(w,uv) = u\mu(w,v) + v\mu(w,u)\,.$$

We don't have a Leibnitz rule in the first term of the form but we have a Leibnitz inequality for the energy density $\alpha(.)$ of the functional associated to the form:

THEOREM 5.11: *Let* $u \in D_0 \cap L^\infty(X,m)$, $v \in D_0 \cap L^\infty(X,m)$; *then* $uv \in D_0 \cap$ $\cap L^\infty(X,m)$ *and*

$$\alpha(uv) \leq C(v^p\alpha(u) + u^p\alpha(v))\,.$$

Using Theorem 3.4, we have

$$p\alpha(uv) = \mu(uv,uv) = v\mu(uv,u) + u\mu(uv,v) \leq \frac{1}{4}\alpha(uv) + 2^{(p+3)(p-1)}(\alpha(u) + \alpha(v))$$

# REFERENCES


[BeP] P. Bénilan - C. Picard, *Quelques aspects non linéaires du principe du maximum*, Séminaire de Théorie du Potentiel, IV, Lectures Notes in Math. 713, Springer Verlag, Berlin - Heidelberg - New York (1979), 1-37.

[Beu] A. Beuerling - J. Deny, *Dirichlet spaces*, Proc. Nat. Ac. Sc., **45** (1959), 208-215.

[BM1] M. Biroli - U. Mosco, *Formes de Dirichlet et estimations structurelles dans les milieux discontinus*, Comptes Rendus Acad. Sc. Paris, **313** (1991), 593-598.

[BM2] M. Biroli - U. Mosco, *A Saint Venant type Principle for Dirichlet forms on discontinuous media*, Annali Mat. Pura Appl., **169(IV)** (1995), 125-181.

[BV1] M. Biroli - P.G. Vernole, *Strongly local nonlinear Dirichlet functionals and forms*, Adv. in Math. Sciences and Applications, **15** (2) (2005), 655-682.

[BV2] M. Biroli - P.G. Vernole, *A priori estimates for p–Laplacian associated with nonlinear Dirichlet forms*, Nonlinear Analysis Th. Meth. Appl., **64** (2005), 51-68.

[C1] R. Capitanelli, *Lagrangians on homogeneous spaces*, Tesi di Dottorato di Ricerca, Università di Roma "La Sapienza", 2001.

[C2] R. Capitanelli, *Nonlinear energy forms on certain fractal curves*, Journal Convex Anal., **3** (2002), 67-80.

[C3] R. Capitanelli, *Functional inequalities for measure valued lagrangians on homogeneous spaces*, Adv. Math. Sci. Appl., **13** (2003), 301-313.

[CL] R. Capitanelli - M.R. Lancia, *Nonlinear energy forms and Lipschitz spaces on the Koch curve*, Journal Convex Anal., **9** (2002), 245-257.

[Chee] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Func. Anal., **9** (1999), 428–517.

[CG1] F. Cipriani - G. Grillo, *The Markov property for classes of nonlinear evolutions*, Nonlinear Anal. **47** (2001), 3549-3554.

[CG2] F. Cipriani - G. Grillo, *Nonlinear Markov semigroups, nonlinear Dirichlet forms and applications to minimal surfaces*, J. Reine Ang. Math., 2003.

[DS] N. Dunford - J. Schwartz, *Linear Operators I*, Interscience Pub., New York, 1957.

[FOT] M. M. Fukushima - Y. Oshima - M. Takeda, *Dirichlet forms and Markov processes*, W. de Gruyter & Co., Berlin-Heidelberg-New York, 1994.

[Hajl] P. Hajlasz, *Sobolev spaces on an arbitrary metric space*, Pot. Anal., **5** (1996), 403-415.

[Hein] J. Heinonen, *Lectures on analysis in metric spaces*, Springer Verlag, Berlin - Heidelberg - New York, 2001.

[HPS] P.E. Herman - R. Peirone - R.S. Strichartz, *p-Energy and p-harmonic functions on Sierpinski gasket type fractals*, Potential Analysis, **20** (2004), 125-148.

[Hirsch] F. Hirsch, *Familles résolvantes, générateurs, cogénérateurs, potentiels*, Ann. Inst. Fourier, **22** (1972), 89-210.

[HP] E.H. Hille - R.S. Phillips, *Functional Analysis and Semigroups*, Colloqium Publications XXXI, American Mathematical Society, Providence, 1957.

[KM] J. Kinnunen - O. Martio, *Nonlinear Potential Theory on metric spaces*, Illinois J. of Math., **46** (2002), 857-883.

[MM] J. Malỳ - U. Mosco, *Remarks on measure valued Lagrangians on homogeneous spaces*, Ricerche di Mat. **48** (1999), 217-231.

[M] U. Mosco, *Composite media and asymptotic Dirichlet forms*, J. Funct. Anal., **123** (1994), 368-421.

[PS] D. Pascali - S. Sburlan, *Nonlinear mappings of monotone type*, Ed. Academiei, Bucarest, 1978.

[S] R.S. Strichartz - C. Wong, *The p-Laplacian on the Sierpinski gasket*, Nonlinearity, **17** (2004), 595-616.

*Direttore responsabile*: Prof. A. BALLIO - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
«Monograf» - Via Collamarini, 5 - Bologna