Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
123° (2005), Vol. XXIX, fasc. 1, pagg. 35-54

BOLIS BASIT (*) - HANS GÜNZLER (**)

Harmonic Analysis for Generalized Vector Valued Almost Periodic and Ergodic Distributions

ABSTRACT. — For a given class $\mathcal{A}$ of Banach space valued functions on $\mathbb{R}$, a corresponding class of $\mathcal{A}$-distributions $\mathcal{D}'_{\mathcal{A}}(\mathbb{R}, X)$ can be defined, analogous to almost periodic (= ap) Schwartz's distributions $\mathcal{B}'_{pp}$. For $\mathcal{A} =$ ergodic functions $\mathcal{E}$ or more generally $Av = \left\{ f \in L^1_{\text{loc}} \text{ with only } \lim (1/2T) \int_{-T}^{T} f(t)\, dt \text{ exists} \right\}$, the mean there can be extended to $\mathcal{D}'_{Av}$, and with this Fourier coefficients c_ω, the Bohr spectrum σ_B and a Fourier series is definable for $S \in \mathcal{D}'_{\mathcal{T}Av}$, $\mathcal{T}Av = \{f \in Av : e^{i\omega t} f \in Av \text{ for } \omega \in \mathbb{R}\}$. The classes $\mathcal{A}$ of ap, asymptotic ap, Eberlein weakly ap, Weyl W^p-ap, Besicovitch B^p- ap and pseudo ap functions are all $\subset \mathcal{T}Av$, the distributions $S \in$ corresponding $\mathcal{D}'_{\mathcal{A}}$ are tempered and have countable spectrum σ_B, with $c_{\omega_n}(S) = O(1 + |\omega_n|^q)$ with some $q \in \mathbb{N}$ if for example $\mathcal{A} \subset W^pAP$; conversely to such $c_{\omega_n} \in X$ and $\omega_n \to \infty$ not too slowly always an $S \in \mathcal{D}'_{AP}$ with this given Fourier series exists. For $\mathcal{A} =$ any of the above classes and $S \in \mathcal{D}'_{\mathcal{A}}$, Bohr-spectrum $\sigma_B(S) \subset \text{supp}\, \widehat{S}$.

0. - INTRODUCTION

In an earlier paper [8] we extended the almost periodic (= ap) distributions $\mathcal{B}'_{pp}$ of Schwartz in two directions: First for X a Banach space X-valued distributions were admitted, second instead of starting with the class AP of ap functions, a quite arbitrary class $\mathcal{A} \subset L^1_{\text{loc}}(\mathbb{R}, X)$ was admissible, the class of $\mathcal{A}$-distributions $\mathcal{D}'_{\mathcal{A}}$ being defined as the set of those X-valued distributions $T \in$ Schwartz's $\mathcal{D}'(\mathbb{R}, X)$ for which the convolution $T * \varphi \in \mathcal{A}$ for all complex-valued test functions $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{C})$. So various extensions of the class of ap functions AP, for example asymptotic ap AAP, Eberlein weakly ap EAP, Zhang's pseudo ap PAP and generalized ap $GPAP$, Besicovitch ap B^pAP and various

(*) Indirizzo dell'Autore: School of Math. Sci., P.O. Box No. 28M, Monash Univ., Vic. 3800. E-mail: bolis.basit@sci.monash.edu.au.
(**) Indirizzo dell'Autore: Math. Seminar, Univ. Kiel, Ludewig-Meyn-Str., 24098 Kiel, Germany. E-mail: guenzler@math.uni-kiel.de.

ergodic classes can now be extended to $\mathcal{A}$-distributions $\mathcal{D}'_{\mathcal{A}}$, with applications to the study of the asymptotic behaviour of solutions of differential-difference equations and systems [8, §5].

Here we show that for such distributions harmonic analysis still is possible : For the class $\mathcal{E}$ of (uniformly) ergodic functions and even the class Av of functions for which only the average $m(f) := \lim_{T\to\infty} (1/2T) \int\limits_{-T}^{T} f(s+t)\, ds$ exists uniformly with respect to $t \in [0,\delta]$ for some $\delta > 0$, it can be shown (§ 4) that this mean m can be uniquely extended to the corresponding distributions $\mathcal{D}'_{Av}(\mathbb{R}, X)$, so that one can define Fourier coefficients $c_\omega(S)$, the Bohr spectrum $\sigma_B(S)$ and Fourier series for distributions $S \in \mathcal{D}'_{\mathcal{T}Av}$, fortunately still tempered $\subset S'(\mathbb{R}, X)$, where $\mathcal{T}\mathcal{A} := \{f \in \mathcal{A} : e^{i\omega t} f \in \mathcal{A}\,, \text{for all } \omega \in \mathbb{R}\}$. Since $AP \subset AAP \subset EAP \subset \mathcal{T}\mathcal{E} \subset \mathcal{T}Av$, $AP \subset$ Weyl $W^pAP \subset B^pAP \subset \mathcal{T}Av$, $PAP \subset$ $\subset B^pAP$, all these extensions of the Bohr ap functions are included.

In § 2 we further show that Bohr's, Bochner's and von Neumann's characterizations of ap functions still hold for vector valued distributions (see Theorem 2.1).

In § 3 various classes of ergodic (see (3.1)), totally ergodic ($= \mathcal{T}\mathcal{E}$) and functions with non-uniform mean (see (3.4), (3.13)) are introduced and their relations discussed (see (3.2) and also (3.3), (3.8)-(3.10), (3.13).

In § 5 we give the usual formulas for Fourier coefficient for $S \in \mathcal{D}'_{\mathcal{T}Av}(\mathbb{R}, X)$, the Bohr spectrum $\sigma_B(S)$ of $S \in \mathcal{D}'_{\mathcal{T}Av}$ is countable if it is for $f \in \mathcal{T}Av$; if $|\omega_n| \to \infty$ not too slowly and $c_n = O(n^q)$, there is $S \in \mathcal{D}'_{AP}(\mathbb{R}, X)$ with $\sum c_n e^{i\omega t}$ as Fourier series. For all the $\mathcal{A}$ considered above and $S \in \mathcal{D}'_{\mathcal{A}}$ one has $\sigma_B(S) \subset \operatorname{supp} \widehat{S}$. We also indicate how ap distributions of $\mathcal{D}'_{AP}(\mathbb{R}, X)$ can be subsumed by the Bochner-von Neumann theory of ap functions with values in a suitable locally convex topologically complete vector space [13], giving thus summation methods for the Fourier series of ap distributions.

1. - Notation, Definitions and Preliminaries

In the following $\mathbb{J}$ will always be an interval of the form $\mathbb{R}, (a,\infty), [a,\infty)$ for some $a \in \mathbb{R}$, $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}^+ = (0,\infty)$, $\mathbb{N} = \{1, 2, \cdots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Denote by X a real or complex Banach space, with scalar field $\mathbb{K} = \mathbb{K}(X)$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

If f is a function defined on $\mathbb{J} \to X$, then f_s, $\Delta_s f$ will stand for the functions defined on $\mathbb{J}$ by $f_s(t) = f(t+s)$, $\Delta_s f(t) = f_s(t) - f(t)$ for all $s \in \mathbb{R}$ with $s + \mathbb{J} \subset \mathbb{J}$, $|f|$ will denote the function $|f|(t) := \|f(t)\|$ for all $t \in \mathbb{J}$ and $\|f\|_\infty := \sup_{x\in\mathbb{J}} \|f(x)\|$.

For $U, V \subset X$ or $\subset X^{\mathbb{J}}$, $U - V := \{u - v : u \in U, v \in V\}$; similarly for $U + V$.

If $f \in L^1_{loc}(\mathbb{J}, X)$, then Pf will denote the indefinite integral defined by $Pf(t) = \int\limits_{a_0}^{t} f(s)\, ds$ (where $a_0 = a + 1$ respectively 0 if $\mathbb{J} = \mathbb{R}$, all integrals are Lebesgue-Bochner integrals (see [2, pp. 6-15], [23, p. 79], [16, p. 232], [21, p. 50, 97], [35, p. 132]), similarly for measurable).

In $X^{\mathbb{J}}$ the spaces of all constants, continuous, bounded continuous, uniformly continuous, bounded uniformly continuous, continuous with relatively weakly compact range, vanishing at infinity continuous and continuous with compact support will respectively be denoted by X, $C(\mathbb{J},X)$, $C_b(\mathbb{J},X)$, $C_u(\mathbb{J},X)$, $C_{ub}(\mathbb{J},X)$, $C_{rwc}(\mathbb{J},X)$, $C_0(\mathbb{R},X)$ and $C_{cs}(\mathbb{J},X)$.

$$AP,\ S^pAP,\ W^pAP,\ B^pAP,\ AAP,\ EAP_{rc},\ EAP,\ PAP \text{ and } GPAP$$

will respectively stand for the spaces of Bohr-Bochner almost periodic (= ap), Stepanoff S^p-ap, Weyl W^p-ap, Besicovitch B^p-ap, asymptotically ap, Eberlein weakly ap with relatively compact range, Eberlein weakly ap, Zhang's pseudo ap and generalized pseudo ap X-valued functions. For ap see [1], [2, p. 285], [13], [20], [26], for asymptotically ap [2, p. 304], [3], [18], [29], [31], for S^p-ap, W^p-ap, B^p-ap [12], [14], [27], for Eberlein weakly ap [3], [11], [15, p. 80], [17], [19], [28], [29], [30], for pseudo ap [10], [36], [37] and for generalized pseudo ap [37, p. 67] and below after (3.5).

Here $\phi \in B^pAP(\mathbb{R},X)$ means $\phi \in L^1_{loc}(\mathbb{R},X)$ and it can be approximated in the $\|\cdot\|_{B^p}$-norm by X-valued trigonometric polynomials, with

$$\|\phi\|_{B^p} := \overline{\lim}_{T\to\infty}\left[\frac{1}{2T}\int_{-T}^{T}\|\phi(t)\|^p\,dt\right]^{1/p}.$$

Various classes of ergodic functions $\mathcal{E}$, $\mathcal{E}_0$, $\mathcal{TE}$ and Av are introduced in § 3.

For general $\mathcal{A}(\mathbb{J},X)$, with $\gamma_\omega(t) := e^{i\omega t}$,

(1.1) $$\mathcal{TA}(\mathbb{J},X) := \{f \in A(\mathbb{J},X) : \gamma_\omega \cdot f \in A(\mathbb{J},X) \text{ for all } \omega \in \mathbb{R}\}.$$

$\mathcal{A}$ has (Γ) means $\mathcal{A} = \mathcal{TA}$.

$\mathcal{D}(\mathbb{J},\mathbb{K})$ denotes the Schwartz test functions (infinitely differentiable $\mathbb{K}$-valued functions with compact support in $\mathbb{J}$) (see [33, pp. 21, 24]).

$\mathcal{D}'(\mathbb{J},X)$ denotes the set of linear continuous $T : \mathcal{D}(\mathbb{J},\mathbb{K}) \to X$ as in [33, pp. 24, 30] or [32, p. 49]. Here $\mathbb{J}$ in $\mathcal{D}(\mathbb{J},\mathbb{K})$, $\mathcal{D}'(\mathbb{J},X)$ is always open.

Similarly, $\mathcal{S}(\mathbb{R},\mathbb{K})$ will stand for the Schwartz space of all rapidly decreasing infinitely differentiable $\mathbb{K}$-valued functions defined on $\mathbb{R}$ (see [35, p. 146]) and $\mathcal{S}'(\mathbb{R},X)$ is the space of Banach valued tempered distributions of linear continuous $T : \mathcal{S}(\mathbb{R},\mathbb{K}) \to X$ (see [33, p. 234], [35, p. 149]).

Translates T_a for distributions T are defined in accordance with the above definition of translates f_a for functions f by $T_a(\varphi) := T(\varphi_{-a})$ (contrary to the definition in [33, (II, 5; 2), p. 55]).

Let $\mathcal{A} \subset L^1_{loc}(\mathbb{J},X)$ or $\mathcal{A} \subset \mathcal{D}'(\mathbb{J},X)$. We use the following assumptions for $\mathcal{A}$:

Real-linear: $sF + tG \in \mathcal{A}$ if $F, G \in \mathcal{A}$ and $s, t \in \mathbb{R}$.

Positive-invariant: translate $F_a \in \mathcal{A}$ if $F \in \mathcal{A}$ and $a \in [0,\infty)$.

Invariant: $F_a \in \mathcal{A}$ if $F \in \mathcal{A}$ for all $a \in \mathbb{R}$.

Uniformly closed : $(\phi_n) \subset \mathcal{A}$ and $\phi_n \to \phi$ uniformly on $\mathbb{J}$ implies $\phi \in \mathcal{A}$.

We say that $\mathcal{A} \subset L^1_{loc}(\mathbb{J}, X)$ or $\subset \mathcal{D}'(\mathbb{J}, X)$ *satisfies* (Δ) (respectively (Δ')) if for any $f \in L^1_{loc}(\mathbb{J}, X)$ for which all differences $\Delta_s f \in \mathcal{A}$ for $0 < s \in \mathbb{R}$ one has $f - M_h f \in \mathcal{A}$ for all $h > 0$, with M_h of (1.2); $\mathcal{A}$ satisfies (Δ_1) if $\Delta_s f \in \mathcal{A}$ for $s > 0$ implies only $f - M_1 f \in \mathcal{A}$. In (Δ') the f is in $\mathcal{D}'(\mathbb{J}, X)$, M_h is replaced by $\widetilde{M}_h$.

Mean classes have been introduced and found useful in [4, pp. 120, 122], [6], [7], [8], they will be needed here too: If $\mathcal{A} \subset L^1_{loc}(\mathbb{J}, X)$ or $\mathcal{D}'(\mathbb{J}, X)$,

$$(1.2) \qquad \mathcal{M}\mathcal{A} := \{f \in L^1_{loc} : M_h f(\cdot) := (1/h)\int_0^h f(\cdot + s)ds \in \mathcal{A},\ \text{all } h > 0\},$$

$$\mathcal{M}^0\mathcal{A} := A \cap L^1_{loc}, \qquad \mathcal{M}^k\mathcal{A} := \mathcal{M}(\mathcal{M}^{k-1}\mathcal{A}).$$

$$(1.3) \qquad \widetilde{\mathcal{M}}\mathcal{A} := \{T \in \mathcal{D}'(\mathbb{J}, X) : \widetilde{M}_h T \in \mathcal{A},\ \text{all } h > 0\},$$

with $(\widetilde{M}_h T)(\varphi) := T(M_{-h}\varphi)$ for $\varphi \in \mathcal{D}(\mathbb{J}, X)$, $\varphi := 0$ on $\mathbb{R} \setminus \mathbb{J}$.

$\widetilde{M}_h | D'(\mathbb{J}, X)$ extends $M_h | L^1_{loc}$, one has $M_h f = f * s_h$ with $s_h := 1$ on $(-h, 0)$, 0 else in $\mathbb{R}$, $f \in L^1_{loc}(\mathbb{R}, X)$.

A further extension of a given class $\mathcal{A}$ of functions or distributions, to "$\mathcal{A}$-distributions", has been introduced and discussed in [8], for $\mathbb{J} = \mathbb{R}$:

$$(1.4) \qquad \mathcal{D}'_{\mathcal{A}}(\mathbb{R}, X) := \{T \in \mathcal{D}'(\mathbb{R}, X) : T * \varphi \in \mathcal{A} \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}, \mathbb{K})\}.$$

For $\mathcal{A} = AP$ and $X = \mathbb{C}$ this gives the ap distributions $\mathcal{B}'_{pp}$ of Schwartz [33, p. 206].

For any $\mathcal{A}$, if $T \in \mathcal{D}'_{\mathcal{A}}(\mathbb{R}, X)$, then $T^{(n)} \in \mathcal{D}'_{\mathcal{A}}$ since $T^{(n)} * \varphi = T * (\varphi^{(n)})$. By [8, Theorem 2.10], if $\mathcal{A}$ is linear and $T \in \mathcal{D}'(\mathbb{R}, X)$, then

$$(1.5). \qquad T * \mathcal{D}(\mathbb{R}, \mathbb{K}) \subset \mathcal{D}'_{\mathcal{A}}(\mathbb{R}, X) \text{ implies } T \in \mathcal{D}'_{\mathcal{A}}.$$

For the definition and properties, especially associativity, of the convolution with X-valued distributions see [8, §2].

Proposition 1.1: *(i) If $\mathcal{A} \subset \mathcal{D}'(\mathbb{J}, X)$ is linear positive-invariant, $f \in \mathcal{A}$ and distributional derivative $f^{(n)} \in L^1_{loc}(\mathbb{J}, X)$ for some $n \in \mathbb{N}$, then $f \in C^{n-1}(\mathbb{J}, X)$ and $f^{(n)} \in \mathcal{M}^n\mathcal{A}$.*

(ii) If $\mathcal{A} \subset L^1_{loc}(\mathbb{J}, X)$ satisfies (Δ_1) and $r \cdot \mathcal{A} \subset \mathcal{A}$ for real $r > 0$, then $\mathcal{M}\mathcal{A} \subset \mathcal{A} + \mathcal{A}'_{Loc}$.

Here for open $\mathbb{J}$ (needed here only)

$$\mathcal{A}'_{Loc} := \{g \in L^1_{loc}(\mathbb{J}, X) : \text{to } g \text{ exists } f \in \mathcal{A},\ g = \text{distribution derivative of } f\}.$$

Proof: (i), $n = 1$: Then $f = P(f') + c$ and $hM_h(f') = \Delta_h f \in \mathcal{A}$ by [23, Theorem 3.8.6, p. 88].

(ii) For open $\mathbb{J}$: Let $\phi \in \mathcal{M}\mathcal{A}, \subset L^1_{loc}$. Then $\phi = (\phi - M_1\phi) + M_1\phi$ with $M_1\phi \in \mathcal{A}$. By the assumption, $(P\phi)_h - P\phi = hM_h\phi \in \mathcal{A}$ for all $0 < h \in \mathbb{R}$. Hence by (Δ_1),

$(P\phi - M_1 P\phi) \in \mathcal{A}$. Now for $\psi \in L^1_{loc}(\mathbb{J}, X)$, $h > 0$,

$$M_h P\psi - PM_h\psi = c_h = (M_h P\psi)(a_0) \qquad \text{on } \mathbb{J} \tag{1.6}$$

(proof by differentiation), so $\psi := P(\phi - M_1\phi) - c_1 \in \mathcal{A}$; by [23, Theorem 3.8.6, p. 88], $\phi - M_1\phi =$ distribution derivative of ψ. This means $\psi' = \phi - M_1\phi \in \mathcal{A}'_{Loc}$.

For closed $\mathbb{J}$ see [8, Lemma 2.2 (c), (2.5)]. □

2. - Vector-valued almost periodic distributions

For the classical ap and Stepanoff ap functions one has by [4, (3.8)] (even strictly and for any $\mathbb{J}$, X, $1 \le p < \infty$)

$$AP(\mathbb{J}, X) \subset S^p AP(\mathbb{J}, X) \subset \mathcal{M}AP(\mathbb{J}, X). \tag{2.1}$$

Since $AP(\mathbb{R}, X)$ obviously satisfies $AP(\mathbb{R}, X) * D(\mathbb{R}, \mathbb{K}) \subset AP(\mathbb{R}, X)$, and since it also satisfies (Δ) by [7, Example 3.2], all the results of [8, §2] can be applied to $\mathcal{A} = AP$; so for example Corollaries 2.13/2.14 and (2.19) of [8] yield, for any X and $1 \le p < \infty$, $\mathbb{J} = \mathbb{R}$,

$$\mathcal{D}'_{AP} = \bigcup_{n=0}^{\infty} \widetilde{\mathcal{M}}^n AP(\mathbb{R}, X) = \bigcup_{n=0}^{\infty} \widetilde{\mathcal{M}}^n S^p AP = \mathcal{D}'_{S^p AP}, \tag{2.2}$$

$$\mathcal{D}'_{AP} \cap L^1_{loc} = \bigcup_{n=0}^{\infty} \mathcal{M}^n AP.$$

So if, for fixed $T \in \mathcal{D}'(\mathbb{R}, X)$, $T * \varphi$ is Stepanoff S^p -ap for all $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{K})$, then $T * \varphi$ is already Bohr-ap; this seems to be new even for the scalar case $X = \mathbb{C}$.

Almost periodic distributions can also be characterized by translation or compactness properties, that is Bohr's, Bochner's and von Neumann's definition all give $\mathcal{D}'_{AP}$:

Theorem 2.1: *For $T \in \mathcal{D}'_{L^\infty}(\mathbb{R}, X)$ and $\Phi_T(l) := T_l$ for $l \in \mathbb{R}$, the following statements are equivalent:*

(a) $T \in \mathcal{D}'_{AP}(\mathbb{R}, X)$;

(b) $T \in \mathcal{D}'_{S^p AP}(\mathbb{R}, X)$ for some $p \in [1, \infty)$ [or equivalently for all $p \in [1, \infty)$];

(c) For any $V =$ neighborhood of 0 in $\mathcal{D}'_{L^\infty}(\mathbb{R}, X)$, the set $\mathcal{T}(T, V)$ of V-periods of T is relatively dense in $\mathbb{R}$;

(d) To each sequence $(a_m)_{m\in\mathbb{N}} \subset \mathbb{R}$ there exists a subsequence $(a_{m_n})_{n\in\mathbb{N}}$ and $S \in \mathcal{D}'_{L^\infty}(\mathbb{R}, X)$ with $T_{a_{m_n}} \to S$ in $\mathcal{D}'_{L^\infty}(\mathbb{R}, X)$ [or in $(\mathcal{D}_{L^1})'(\mathbb{R}, X)$];

(e) $=$ (d), with "subsequence $(a_{m_n}))_{n\in\mathbb{N}}$" replaced by "subnet $(a_{n(i)})_{i\in I}$";

(f) $=$ (e), with "sequence" replaced by "net";

(g) $\{T_l : l \in \mathbb{R}\}$ is totally bounded [= relatively compact] in $\mathcal{D}'_{L^\infty}(\mathbb{R}, X)$;

(h) $\Phi_T \in AP(\mathbb{R}, \mathcal{D}'_{L^\infty}(\mathbb{R}, X))$ [or $\in AP(\mathbb{R}, (\mathcal{D}_{L^1})'(\mathbb{R}, X))$];

(i) there exist $f, g \in AP(\mathbb{R}, X)$ and $m \in \mathbb{N}_0$ such that $T = f + g^{(m)}$ on $\mathcal{D}(\mathbb{R}, \mathbb{K})$;

(j) there exists a sequence $(f_n) \subset AP(\mathbb{R}, X)$ [or equivalently a net $(T_i)_{i\in I} \subset \mathcal{D}'_{AP}(\mathbb{R}, X)$] with f_n [respectively T_i] $\to T$ in $\mathcal{D}'_{AP}(\mathbb{R}, X)$.

Here $\mathcal{T}(T, V) := \{\tau \in \mathbb{R} : T_\tau - T \in V\}$; the topology of $\mathcal{D}'_{AP}(\mathbb{R}, X)$ and $\mathcal{D}'_{L^\infty}(\mathbb{R}, X)$ is given by the seminorms $\|T\|_U :=$ sup $\{\|T(\varphi)\| : \varphi \in U\}$, with $U \subset \mathcal{D}(\mathbb{R}, \mathbb{K})$, U bounded in $\mathcal{D}_{L^1}(\mathbb{R}, \mathbb{K})$ (see [8, §1]).

Proof: This follows mostly from Proposition 2.9 and Theorem 2.11 of [8] and the properties of X-valued ap functions. $T_i \to S$ in $(\mathcal{D}_{L^1})'(\mathbb{R}, X)$ is equivalent with $\Phi_{T_i} \to \Phi_S$ in $(\mathcal{D}_{L^1})'(\mathbb{R}, X)$ uniformly on $\mathbb{R}$, and also equivalent with $T_i \to S$ in $C_b(\mathbb{R}, (\mathcal{D}_{L^1})'(\mathbb{R}, X))$, that is uniformly only on $\mathcal{D}_{L^1}$-bounded $U \subset \mathcal{D}(\mathbb{R}, \mathbb{K})$.

$(\mathcal{D}_{L^1})'(\mathbb{R}, X)$ is complete and therefore topologically complete, and satisfies von Neumann's countability axiom (A_0) (there exist countably many neighbourhoods V_n with $\cap_{n=1}^{\infty} V_n = \{0\}$), since $\mathcal{D}(\mathbb{R}, \mathbb{K})$ and then $\mathcal{D}_{L^1}(\mathbb{R}, \mathbb{K})$ are separable. So, $\mathcal{D}'_{L^\infty}(\mathbb{R}, X)$ is complete and satisfies (A_0). The equivalence of (d)-(g) holds for suitable totally bounded uniform spaces with (A_0). For more details see [5, p. 41-45]. $\square$

For a discussion of the relations between the $\mathcal{D}'_{\mathcal{A}}$ for $\mathcal{A} = AP, AAP, EAP, W^pAP$ and B^pAP see [8, before Example 3.7], not even $AAP \subset \mathcal{D}'_{AP}$, $\mathcal{M}AP \not\subset B^pAP$, there are Weyl-ap functions f with $[f]_{B^p} \cap \mathcal{D}'_{AP} = \emptyset$, with $[f]_{B^p} =$ Besicovitch equivalence class. See also Corollary 5.6.

3. - Ergodic classes and distributions

In this section we study the classes of functions with uniform mean or non-uniform mean given respectively by (3.1), (3.4). We obtain here new properties of theses classes.

We recall (see [8], [37, p. 203], [7, § 1])

$$\text{(3.1)} \qquad \mathcal{E}(\mathbb{J}, X) := \{f \in L^1_{loc}(\mathbb{J}, X) : \text{ to } f \text{ exists } m \in X \text{ with } M_T f \to m \text{ uniformly on } \mathbb{J} \text{ if } T \to \infty\};$$

(Bohr-) mean $m(f) :=$ this unique m. Contrary to [3], [4] or [37] however the f need not be in C_{ub} or $C_b(\mathbb{J}, X)$.

$$\mathcal{E}_0(\mathbb{J}, X) := \{f \in \mathcal{E}(\mathbb{J}, X) : m(f) = 0\},$$
$$\mathcal{E}_n(\mathbb{J}, X) := \{f \in L^1_{loc}(\mathbb{J}, X) : |f| \in \mathcal{E}_0\}, \subset \mathcal{E}_0.$$
$\mathcal{T}\mathcal{E}$ and $\mathcal{T}\mathcal{E}_0$ are given by (1.1), $\mathcal{T}\mathcal{E}_n = \mathcal{E}_n$.

The elements of $\mathcal{T}\mathcal{E}$ are also called almost periodic in the sense of *Ryll-Nardzewski* (see [25, p. 231], [22, p. 348]).

The above $\mathcal{E}$-spaces and the $\mathcal{E}_{ub} := \mathcal{E} \cap C_{ub}$, $\mathcal{T}\mathcal{E}_{ub} = \mathcal{T}(\mathcal{E}_{ub}) = (\mathcal{T}\mathcal{E}) \cap C_{ub}$ considered earlier (for example [3], [4]) and their mean extensions $\mathcal{M}^n\mathcal{E}$ are fortunately all linearly ordered by

Theorem 3.1: *For any* $\mathbb{J}$, X, *the following inclusions hold and are strict*

$$\mathcal{TE}_{ub} \subset \mathcal{E}_{ub} = \mathcal{E}_u \subset \mathcal{ME}_{ub} \subset \mathcal{E} \subset \mathcal{M}^2\mathcal{E}_{ub} \subset \tag{3.2}$$

$$\subset \mathcal{ME} \subset \mathcal{M}^3\mathcal{E}_{ub} \cdots \subset \mathcal{M}^n\mathcal{E} \subset \mathcal{M}^{n+2}\mathcal{E}_{ub} \subset \cdots,$$

$$\subset \mathcal{D}'_{\mathcal{E}_{ub}}(\mathbb{R}, X) = \mathcal{D}'_{\mathcal{E}}(\mathbb{R}, X) \textit{ for } \mathbb{J} = \mathbb{R}.$$

(3.2) holds also, if there everywhere $\mathcal{E}$ *is replaced respectively by* $\mathcal{E}_0$, $\mathcal{TE}_0$ *or* $\mathcal{TE}$. *Furthermore,* $C_u(\mathbb{R}, X) \cap \mathcal{D}'_{\mathcal{TE}}(\mathbb{R}, X) = \mathcal{TE}_{ub}(\mathbb{R}, X)$ *and* $C_u(\mathbb{R}, X) \cap \mathcal{D}'_{\mathcal{TE}_0}(\mathbb{R}, X) = = \mathcal{TE}_0(\mathbb{R}, X) \cap C_{ub}(\mathbb{R}, X)$*; but* $\mathcal{E}_{ub} \not\subset \cup_0^\infty \mathcal{M}^n\mathcal{TE} = \cup_0^\infty \mathcal{M}^n\mathcal{TE}_{ub}$*, so* $\mathcal{E}_{ub} \not\subset \mathcal{D}'_{\mathcal{TE}}$ *if* $\mathbb{J} = \mathbb{R}$.

Proof: Case $\mathcal{E}$: If $\phi \in \mathcal{ME}_{ub}(\mathbb{J}, X)$, $\phi = \psi + \xi$ with $\psi \in \mathcal{E}_{ub}(\mathbb{J}, X)$ and $\xi \in (\mathcal{E}_{ub})'_{Loc}(\mathbb{J}, X)$ by Propositions 1.1 and 3.2. This means that $\xi \in \mathcal{E}(\mathbb{J}, X)$ and proves $\phi \in \mathcal{E}(\mathbb{J}, X)$.

If $\phi \in \mathcal{E}(\mathbb{J}, X)$, $M_h\phi \in \mathcal{E}(\mathbb{J}, X)$ for each $h > 0$ since $M_T M_h = M_h M_T$. Now

$$\mathcal{E}(\mathbb{J}, X) \subset \mathcal{M}C_b(\mathbb{J}, X), \tag{3.3}$$

since $M_T\phi$ and $M_{T+h}\phi$ are bounded for suitable T, also for open $\mathbb{J} \neq \mathbb{R}$.

This implies that $M_\tau M_h\phi \in \mathcal{E}_{ub}(\mathbb{J}, X)$ for all $\tau, h > 0$ or $\mathcal{E} \subset \mathcal{M}^2\mathcal{E}_{ub}$. (3.3) and $C_u \cap \mathcal{M}C_b \subset C_b$ of [7, Proposition 2.9] gives $\mathcal{E}_{ub} = \mathcal{E}_u := \mathcal{E} \cap C_u$.

If $f \in C_{ub}(\mathbb{J}, \mathbb{R})$ is defined by $f = 1$ on I_{2n} and $f = 0$ on I_{2n+1}, where $I_n = [10^n + 1, 10^{n+1} - 1]$ then $\gamma_\omega f \in \mathcal{E}_{ub}(\mathbb{J}, \mathbb{R})$ for all $\omega \neq 0$, but $f \notin \mathcal{E}$, not even $\in Av$ of (3.4). This gives $\mathcal{TE}_{ub}(\mathbb{J}, X) \neq \mathcal{E}_{ub}(\mathbb{J}, X)$.

If $f(t) = \sin t^2$, $f' \in \mathcal{E}(\mathbb{J}, \mathbb{R})$; Proposition 1.1 (i) gives $f^{(n)} \in \mathcal{M}^{n-1}\mathcal{E}(\mathbb{J}, \mathbb{R})$. If $f^{(n)} \in \mathcal{M}^n\mathcal{E}_{ub}(\mathbb{J}, \mathbb{R})$, then

$$h_n \cdots h_1 M_{h_n} \cdots M_{h_1} f^{(n)} = \Delta_{h_n} \cdots \Delta_{h_1} f \in \mathcal{E}_{ub}(\mathbb{J}, \mathbb{R}) \subset C_{ub}(\mathbb{J}, \mathbb{R});$$

then [20, p. 281] yields inductively $f \in C_{ub}(\mathbb{J}, \mathbb{R}) \subset C_u(\mathbb{J}, \mathbb{R})$, a contradiction. This means that the inclusions $\mathcal{M}^n\mathcal{E}_{ub}(\mathbb{J}, X) \subset \mathcal{M}^{n-1}\mathcal{E}(\mathbb{J}, X)$ are strict for all $n \in \mathbb{N}$.

We omit the examples for $\mathcal{M}^{n-1}\mathcal{E}(\mathbb{J}, X) \neq \mathcal{M}^{n+1}\mathcal{E}_{ub}(\mathbb{J}, X)$.

$\mathcal{M}^n\mathcal{E}_{ub} \subset \mathcal{D}'_{\mathcal{E}_{ub}}$ by [8, Corollary 2.5] and Proposition 3.2, $\mathcal{D}'_{\mathcal{E}_{ub}} = \mathcal{D}'_{\mathcal{E}}$ then by [8, Theorem 2.10] and $\mathcal{E}_{ub} \subset \mathcal{E} \subset \mathcal{D}'_{\mathcal{E}_{ub}}$ of (3.2) (see also (3.7) below).

Case $\mathcal{E}_0$ can be proved similarly.

For the $\mathcal{TE}$, $\mathcal{TE}_0$-cases we need first $\mathcal{TE} \subset \mathcal{MTE}$, $\mathcal{TE}_0 \subset \mathcal{MTE}_0$:

If $\phi \in \mathcal{TE}(\mathbb{J}, X)$ (respectively $\mathcal{TE}_0(\mathbb{J}, X)$), $M_h\phi \in C_b(\mathbb{J}, X)$ by (3.3). This implies $\gamma_\omega M_h\phi \in C_b(\mathbb{J}, X)$ for all γ_ω. Therefore $(\gamma_\omega M_h\phi)' \in \mathcal{E}_0(\mathbb{J}, X)$. Since $\gamma'_\omega M_h\phi = = (\gamma_\omega M_h\phi)' - \gamma_\omega(\Delta_h\phi)/h$, one gets $\gamma'_\omega M_h\phi \in \mathcal{E}(\mathbb{J}, X)$ (respectively $\mathcal{E}_0(\mathbb{J}, X)$). This gives $(\gamma_\omega M_h\phi) \in \mathcal{E}(\mathbb{J}, X)$(respectively $\mathcal{E}_0(\mathbb{J}, X)$) for all $\omega \neq 0$ and hence $M_h\phi \in \mathcal{TE}(\mathbb{J}, X)$ (respectively $\mathcal{TE}_0(\mathbb{J}, X)$).

(3.2) for $\mathcal{TE}$ (respectively $\mathcal{TE}_0(\mathbb{J}, X)$) follows then as for $\mathcal{E}$, especially $\mathcal{D}'_{\mathcal{TE}_{ub}} = \mathcal{D}'_{\mathcal{TE}}$ (respectively $\mathcal{D}'_{\mathcal{TE}_{0,ub}} = \mathcal{D}'_{\mathcal{TE}_0}$).

$C_u \cap \mathcal{D}'_{\mathcal{TE}} = \mathcal{TE}_{ub}$ etc. follows with $C_u \cap \mathcal{D}'_{\mathcal{A}} \subset \mathcal{A}$ of [8, Proposition 4.8] and $C_u \cap \mathcal{TE} \subset \mathcal{E}_u = \mathcal{E}_{ub} \subset C_{nb}$.

Similarly $\mathcal{E}_{ub} \subset \cup_{n=0}^{\infty} \mathcal{M}^n \mathcal{T}\mathcal{E} \subset \mathcal{D}'_{\mathcal{T}\mathcal{E}}$ would give $\mathcal{E}_{ub} \subset \mathcal{T}\mathcal{E}_{ub}$ contradicting the above. $\square$

In some situations, for example for Besicovitch ap functions (see [34, p. 93]) and especially Zhang's pseudo ap functions, a non-uniform mean is needed. Therefore we introduce for any X (for $\mathbb{J} \neq \mathbb{R}$ see [7, §1], then no local t-uniform convergence is needed)

(3.4) $Av(\mathbb{R}, X) := \{f \in L^1_{loc}(\mathbb{R}, X) :$ to f exist $m \in X$ and $\delta > 0$ so that

$$(1/2T) \int_{-T}^{T} f(t+s)ds \to m \text{ as } T \to \infty, \text{ uniformly in } t \in [0, \delta]\}.$$

For $f \in Av(\mathbb{R}, X)$, *mean* $m(f) :=$ this unique m;

(3.5) the limit exists then uniformly in $t \in [-k, k]$ for any $k > 0$.

For the case $\mathbb{J} = \mathbb{R}$, even uniform existence of the limit in (3.4) alone does not imply invariance of the mean: For $f(t) = t$, one has $m(f_a) = a$. See also (3.13).

We set

$Av_0(\mathbb{R}, X) := \{f \in Av(\mathbb{R}, X) : m(f) = 0\}$,

$Av_n(\mathbb{R}, X) := \{f \in L^1_{loc}(\mathbb{R}, X) : |f| \in Av_0(\mathbb{R}, \mathbb{R})\} = \{f \in B^1AP : \|f\|_{B^1} = 0\}$.

Also, for Av_n, if the limit in (3.4) exists only for t=0 (Zhang's definition), it automatically exists locally uniformly in t, so

$PAP = AP \oplus Av_n \cap C_b \subset GPAP = AP \oplus Av_n \subset \mathcal{T}Av$.

$\mathcal{T}Av(\mathbb{R}, X)$ and $\mathcal{T}Av_0(\mathbb{R}, X)$ are again given by (1.1), $\mathcal{T}Av_n = Av_n$.

The elements of $\mathcal{T}Av(\mathbb{R}, X)$ are also called almost periodic functions in the sense of *Hartman* (see [25, p. 231], [22, p. 348]).

Proposition 3.2: *For any $\mathbb{J}$ and X, $U \in \{\mathcal{E}_{ub}, \mathcal{E}, \mathcal{E}_0, \mathcal{E}_n, Av, Av_0, Av_n\}$, $\mathcal{A} = U$ or $\mathcal{T}U$, all these $\mathcal{A}$ (if defined) are linear, positive-invariant, uniformly closed, with $\mathcal{A} \subset \mathcal{M}\mathcal{A}$ and (Δ), for $\mathbb{J} = \mathbb{R}$ they are affine-invariant with $\mathcal{A} * \mathcal{D}(\mathbb{R}, \mathbb{K}) \subset \mathcal{A}$; the $m|Av$ is linear and positive-invariant, for $\mathbb{J} = \mathbb{R}$ affine-invariant, with $m(M_h f) = m(f)$ if $f \in Av$, $h > 0$.*

Here "affine-invariant" means $f_{(r,a)} \in \mathcal{A}$ if $f \in \mathcal{A}$, $0 \neq r \in \mathbb{R}$, $a \in \mathbb{R}$, with $f_{(r,a)}(t) = f(rt + a)$; similarly for $m|Av$.

Proof: This follows mostly from the definitions, with (3.5); $\mathcal{A} \subset \mathcal{M}\mathcal{A}$ and $\mathcal{A} * \mathcal{D} \subset \mathcal{A}$ follow from Lemma 4.1 if $\mathbb{J} = \mathbb{R}$. (Δ) has been shown in [7, Proposition 3.1, Proposition 3.8 and Proposition 3.10] except for $\mathcal{T}Av$ and $\mathcal{T}Av_0$.

For these, since $\mathcal{T}\mathcal{A} = \cap\{\gamma_\omega \mathcal{A} : \omega \in \mathbb{R}\}$, it is enough to show ($\Delta$) for $\gamma_\omega Av$ and $\gamma_\omega Av_0$.

Case $\gamma_\omega Av$, $\mathbb{J} = \mathbb{R}$, with $g := \gamma_{-\omega}$: As in the proof of (Δ) for $\mathcal{E}$ in Proposition 3.8 in [7], with separable $U := \{(1/2T) \int_{-T}^{T} g(s)f(t+s)ds : t \in \mathbb{R}, T > 0\}$,

$\Gamma = \{a_n : n \in N\}$:= countable $\mathbb{Q}$-vector space dense in the closed linear hull of U, ε fixed > 0, define

$$A^{\varepsilon}_{m,n} := \{v \in L^1_{loc}(\mathbb{R}, X) : \|(1/2T) \int_{-T}^{T} g(s)v(t+s)ds - a_m\| \leq \varepsilon \text{ for } |t| \leq 1,\ T \geq n\},$$

$$A^{\varepsilon} := \bigcup_{m,n} A^{\varepsilon}_{m,n}.$$

Then as in [7] with Lemma 3.6 there, if all $\Delta_h f \in \gamma_\omega Av$, one gets a $\delta > 0$ with $f - M_h f \in A^{2\varepsilon}$ if $0 < h \leq \delta$, and then $f - M_h f \in A^{3\varepsilon}$ for all $h > 0$, yielding (Δ) for $\gamma_\omega Av$.

Case $\gamma_\omega Av_0$: Only $A^{\varepsilon}_{1,n}$ is used with $a_1 = 0$.

(Δ) for linear positive-invariant $\mathcal{A}$ implies $\mathcal{A} \subset \mathcal{M}\mathcal{A}$. □

The Av and $\mathcal{E}$-spaces are however not lattices, with the exception of Av_n, $\mathcal{E}_n$. (3.3), (3.2), Proposition 3.2 and [8, Corollary 2.14] yield

$$\mathcal{E} \subset \mathcal{M}C_b \subset \cup_0^\infty \widetilde{\mathcal{M}}^n L^\infty = \mathcal{D}'_{L^\infty}, \tag{3.6}$$

so all the assumptions needed in [8, §2] are fulfilled for the various $\mathcal{E}$-spaces and therefore ap-spaces, the results in [8] can be applied here, so for example with [8, Corollary 2.14]

$$\mathcal{D}'_{\mathcal{A}} = \cup_0^\infty \widetilde{\mathcal{M}}^n \mathcal{A},\ \ \mathcal{D}'_{\mathcal{A}} \cap L^1_{loc} = \cup_{n=0}^\infty \mathcal{M}^n \mathcal{A}, \tag{3.7}$$

$$\mathcal{A} = \mathcal{E},\ \mathcal{E}_0,\ \mathcal{E}_n,\ \mathcal{T}\mathcal{E},\ \mathcal{T}\mathcal{E}_0,\ AP,\ AAP,\ EAP_{rc},\ EAP,\ PAP.\quad (^*)$$

For Av, instead of (3.3), one only has, with (3.5) and $w_1(t) := 1 + |t|$,

$$Av(\mathbb{R}, X) \subset \mathcal{M}(o(w_1)), \tag{3.8}$$

$$o(w_1) := \{f \in C(\mathbb{R}, X) : f(t)/w_1(t) \to 0 \text{ as } |t| \to \infty\} :$$

$f \in Av(\mathbb{R}, X)$ implies

$$\|(1/(2T)) \int_{-T-a+t}^{T+a+t} f(s)\, ds - m\| \leq \varepsilon \text{ if } 0 \leq t \leq \delta,\ |a| \leq \delta,\ T \geq T_{\varepsilon,\delta},$$

since $(T+a)/T = 1 + o(1)$ for large T; $t = a$ gives

$$\|1/(2T) \int_{T}^{T+2a} f(s)\, ds\| \leq 2\varepsilon, \text{ or } \|M_{2a} f(T)\| \leq (2/a)\varepsilon w_1(T),$$

$$|M_{4a} f(T)\| \leq (1/2)(\|M_{2a} f(T)\| + (|M_{2a} f)_{2a}(T))\| \leq (2/a)\varepsilon w_1(T) \text{ for large } T,$$

etc., similarly for negative T.

(*) $\mathbb{J} = \mathbb{R}$; for *PAP* with [9, example 5.5].

Now $o(w_1) \subset O(w_1) \cap C(\mathbb{R}, X) =: U$; $O(w_1)$ has (Δ) by [7, Example 3.13], $C(\mathbb{R}, X)$ by [7, Proposition 1.3], implying (Δ) for U ; so $Av \subset \mathcal{M}U \subset \mathcal{S}'(\mathbb{R}, X)$ by Proposition 1.1(ii). Since $T \in \mathcal{D}'(\mathbb{R}, X)$, $\varphi \in \mathcal{D}$, $T * \varphi \in \mathcal{D}'(\mathbb{R}, X)$ implies $T \in \mathcal{S}'(\mathbb{R}, X)$ by [8, Theorem 2.15], one gets at least

$$Av \subset \mathcal{D}'_{Av} \subset \text{ tempered distributions } \mathcal{S}'(\mathbb{R}, X). \tag{3.9}$$

An analogue of (3.2) is also no longer true, one only has (proof as for Theorem 3.1)

$$Av_{ub} \subset \mathcal{M}(Av_{ub}) \subset Av, \; Av_{ub} := Av \cap C_{ub}. \tag{3.10}$$

$Av \subset \mathcal{M}^2(Av_{ub})$ is no longer true, since even $Av_n \not\subset \mathcal{D}'_{L^\infty}$, $= \mathcal{D}'_{C_{ub}} = \cup_{n=0}^{\infty} \widetilde{\mathcal{M}}^n C_{ub}$, $\supset \cup_0^\infty \mathcal{M}^n C_{ub}$, by [8, Corollary 2.14, Theorem 3.1], which implies $\mathcal{E} \neq Av$:

$f = n - |t - n^4|$ on $[n^4 - n, n^4 + n]$, else 0, $n \in \mathbb{N}$ is well defined, uniformly continuous on $\mathbb{R}$, [8, Proposition 4.8].

One has also $Av_n \cap C_{ub} \not\subset \mathcal{E}$, so $B^p AP \not\subset \mathcal{D}'_{\mathcal{E}}$, $1 \leq p < \infty$.

So for example [8, Theorem 2.11 and Corollary 2.14] do not apply to Av, we can only show for $\mathcal{A} = Av$, Av_0, Av_n, $\mathcal{T}Av$, $\mathcal{T}Av_0$, with [8, Corollary 2.14] and Proposition 3.2,

$$\bigcup_{n=0}^{\infty} \mathcal{M}^n \mathcal{A} \subset \bigcup_{n=0}^{\infty} \widetilde{\mathcal{M}}^n \mathcal{A} \subset \mathcal{D}'_{\mathcal{A}}. \tag{3.11}$$

The $\mathcal{E}_{ub} \not\subset \mathcal{D}'_{\mathcal{T}\mathcal{E}}$ of Theorem 3.1, which implies $\mathcal{T}\mathcal{E}$ strictly $\subset \mathcal{E}$, can be generalized to (see the end of the proof of Theorem 3.1)

$$\mathcal{E}_{ub} \not\subset \mathcal{D}'_{\mathcal{T}Av}, \qquad \text{so } \mathcal{T}Av \text{ strictly } \subset Av. \tag{3.12}$$

Let us finally remark that with (3.8), Proposition 1.1(ii) and (Δ) for $o(w_1)$ (see Example 3.13 of [7, p. 1015]) one can show, for any X

$$Av(\mathbb{R}, X) = \{f \in \mathcal{M}(o(w_1)) : \text{ to } f \text{ exists } a_0 \in \mathbb{R} \text{ with } \lim_{T \to \infty} (1/2T) \int_{-T}^{T} f_{a_0}(s)\, ds \text{ exists, } \in X\} \tag{3.13}$$

4. - The mean for uniform and non-uniform ergodic distribution classes

In this section we prove the existence of a generalized Bohr mean for $\mathcal{D}'_{\mathcal{E}}(\mathbb{R}, X)$ and $\mathcal{D}'_{Av}(\mathbb{R}, X)$ extending the mean for $\mathcal{E}(\mathbb{R}, X)$ respectively $Av(\mathbb{R}, X)$.

Lemma 4.1: *For $\mathcal{A} = Av$, Av_0, Av_n, $\mathcal{E}$, $\mathcal{E}_0$, $\mathcal{E}_n$ of § 3, with $\mathbb{J} = \mathbb{R}$ and any X, one has $\mathcal{A} * L^\infty_{cs}(\mathbb{R}, \mathbb{K}) \subset \mathcal{A}$, $(\mathcal{T}\mathcal{A}) * L^\infty_{cs}(\mathbb{R}, \mathbb{K}) \subset \mathcal{T}\mathcal{A}$, with*

$$m(f * \phi) = m(f) \int_{\mathbb{R}} \phi(t)\, dt, \; f \in \mathcal{A}, \; \phi \in L^\infty_{cs}(\mathbb{R}, \mathbb{K}). \tag{4.1}$$

Here $L^\infty_{cs}(\mathbb{J},\mathbb{K})$ contains all measurable $\phi : \mathbb{J} \to \mathbb{K}$ which are bounded a.e. and vanish outside some compact interval $\subset \mathbb{J}$.

Proof: By (3.5), one has $(1/2T)\int\limits_{-T}^{T} f(t+s)\,ds \to m(f)$ as $T \to \infty$ locally uniformly in $t \in \mathbb{R}$. Replacing t by $x-t$, multiplying by $\phi(t)$ with $\phi \in L^\infty_{cs}$, $\phi = 0$ outside $[-k,k]$, and integrating over $\mathbb{R}$, with Fubini one gets $(1/2T)\int\limits_{-T}^{T} (f*\phi)(x+s)ds \to$ $\to m(f)\int\limits_{\mathbb{R}} \phi(t)\,dt$, locally uniformly in x, i.e. $f*\phi \in Av$ and (4.1).

If $f \in \mathcal{E}(\mathbb{R},X)$, the convergence is uniform in t respectively $x \in \mathbb{R}$, so $f*\phi \in \mathcal{E}$. (4.1) gives then $\mathcal{A} * L^\infty_{cs} \subset \mathcal{A}$ for $\mathcal{A} = Av_0$, E_0; for Av_n, $\mathcal{E}_n$ it follows with $|f*\phi| \le |f|*|\phi|$. Then $(\mathcal{T}\mathcal{A}) * L^\infty_{cs} \subset \mathcal{T}\mathcal{A}$ since for $\omega \in \mathbb{R}$, $f \in L^1_{loc}(\mathbb{R},X)$, $\phi \in L^\infty_{cs}$

$$\gamma_\omega(f*\phi) = (\gamma_\omega f)*(\gamma_\omega \phi) \tag{4.2}$$ □

Especially one has

$$\mathcal{A} \subset \mathcal{D}'_{\mathcal{A}}(\mathbb{R},X) \text{ for } \mathcal{A} = Av,\ Av_0,\ Av_n,\ \mathcal{E},\ \mathcal{E}_0,\ \mathcal{E}_n,\ \mathcal{T}Av,\ \cdots,\ \mathcal{T}\mathcal{E}_0. \tag{4.3}$$

We are now in a position to extend the mean $m : Av(\mathbb{R},X) \to X$ of (3.4) to distributions:

Definition 4.2: *For* $T \in \mathcal{D}'_{Av}(\mathbb{R},X)$, $m(T) := m(T*\varphi)$ *with* $\varphi \in \mathcal{D}(\mathbb{R},\mathbb{K})$ *with* $\int\limits_{\mathbb{R}} \varphi(t)\,dt = 1$.

Theorem 4.3: *The mean* $m : \mathcal{D}'_{Av}(\mathbb{R},X) \to X$ *is well defined,* $\mathcal{D}'_{Av}$ *and* m *are linear, affine-invariant, with*

$$m(T*\varphi) = m(T)\int\limits_{\mathbb{R}} \varphi(x)\,dx,\ T \in \mathcal{D}'_{Av}(\mathbb{R},X), \varphi \in \mathcal{D}(\mathbb{R},\mathbb{K}), \tag{4.4}$$

extending the $m|Av(\mathbb{R},X)$ *of* (3.4), *"continuous" with respect to* (w_1)*-convergence. This* m *is uniquely determined by* $m|Av(\mathbb{R},X)$ *and* (4.4).

Here "continuous" means $m(T_n) \to 0$ if $T_n \to 0$ (w_1), which in turn means $T_n(\varphi) \to 0$ uniformly in $\varphi \in U$, whenever $U \subset \mathcal{D}(\mathbb{R},\mathbb{K})$ w_1-bounded, that is

$$\sup_{\varphi \in U} \|\varphi\, w_1\|_\infty < \infty, \qquad \text{with } w_1(t) := 1 + |t|.$$

Proof: Uniqueness : (4.4) for $\int\limits_{\mathbb{R}} \varphi(x)\,dx = 1$, since $T*\varphi \in Av$.

Existence: Since $T*\varphi \in Av$, the $m(T)$ of Definition 4.2 makes sense; it is independent of such φ since $m|Av$ is linear and , with $\rho \in \mathcal{D}(\mathbb{R},\mathbb{R})$ with $\int\limits_{\mathbb{R}} \rho(x)\,dx = 1$, by (4.1) and associativity [8, (2.2)] one has $m(T*\varphi) = m((T*\varphi)*\rho) =$ $= m((T*\rho)*\varphi) = m(T*\rho)\int\limits_{\mathbb{R}} \varphi(t)\,dt = 0$ if $\int\limits_{\mathbb{R}} \varphi(x)\,dx = 0$. Linearity and invariance follow from those for convolution and $m|Av$, (4.4) then from Definition 4.2 respectively the above if $\int_{\mathbb{R}} \varphi(t)\,dt = 0$. Definition 4.2 and (4.1) give $m|Av =$ that of (3.4).

Continuity: With $f_n = T_n * \varphi$ this can be reduced to $m(f_n) \to 0$ if $f_n \to 0$ (w_1), since with U also $U * \psi$ is w_1-bounded. $m(f_n) \to 0$ follows with $U := \{\varphi_T * \rho_m : T > 1, m \in \mathbb{N}\}$, $\varphi_T := 1/2T$ on $[-T, T]$, else 0, $0 \le \rho_m \in \mathcal{D}$, $\int_{\mathbb{R}} \rho_m(x)\,dx = 1$, supp $\rho_m \subset [-1/m, 1/m]$. Since we do not need this here, we omit the details. □

REMARK 4.4: The $m|\mathcal{E}(\mathbb{R}, X)$ is even $(\mathcal{D}_{L^1})'$ -continuous, i.e. $m(T_n) \to 0$ if $T_n(\varphi) \to 0$ uniformly in $\varphi \in U$ for any $\mathcal{D}_{L^1}$-bounded $U \subset D_{L^1}$, meaning $\sup_{\varphi \in U} \int_{\mathbb{R}} |\varphi^{(j)}(x)|\,dx < \infty$ for each $j \in \mathbb{N}_0$: [5, Proposition 8.4].

COROLLARY 4.5: *If $T \in \mathcal{D}'_{Av}(\mathbb{R}, X)$ and $n \in \mathbb{N}$, then $T^{(n)} \in \mathcal{D}'_{Av}$ and $m(T^{(n)}) = 0$.*

PROOF: $T^{(n)} * \varphi = T * (\varphi^{(n)})$, Definition 4.2 and (4.4). □

5. - FOURIER ANALYSIS FOR ERGODIC DISTRIBUTIONS

In this section X will be a complex Banach space, $\mathbb{K} = \mathbb{C}$ (if $\mathbb{K} = \mathbb{R}$, everything works with $\sin \omega t$, $\cos \omega t$ instead of $e^{i\omega t}$).

To get Fourier coefficients and a formal Fourier series for elements of a class $\mathcal{A}$, two properties are sufficient: $\mathcal{A}$ is closed with respect to multiplication by characters, and there is a linear invariant and continuous mean on $\mathcal{A}$; that is $\mathcal{A} \subset Av$ of §4:

PROPOSITION 5.1: *(i) If $\mathcal{A} \subset \mathcal{D}'(\mathbb{R}, X)$ satisfies (Γ) that is $\mathcal{A} = \mathcal{T}\mathcal{A}$, so does $\mathcal{D}'_{\mathcal{A}}(\mathbb{R}, X)$.*
(ii) If $\mathcal{A} \subset L^1_{loc}(\mathbb{J}, X)$ (respectively $\mathcal{D}'(\mathbb{J}, X)$) is linear, positive-invariant, satisfies (Γ) and (Δ) (respectively (Δ')), then $\mathcal{M}^n\mathcal{A}$ (respectively $\widetilde{\mathcal{M}}^n\mathcal{A}$) satisfies (Γ), $n \in \mathbb{N}$.

PROOF: (i) If $T \in \mathcal{D}'_{\mathcal{A}}(\mathbb{R}, X)$ and $\gamma_\omega(t) = e^{i\omega t}$, $\gamma_\omega T \in \mathcal{D}'(\mathbb{R}, X)$ and $(\gamma_\omega T) * \varphi(x) = (\gamma_\omega T)(\check{\varphi}_{-x}) = T(\gamma_\omega \check{\varphi}_{-x}) = T(e^{i\omega x}\check{\psi}_{-x})$, where $\check{\varphi}(t) = \varphi(-t)$, $\psi := \gamma_{-\omega}\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{C})$, so $(\gamma_\omega T) * \varphi = \gamma_\omega \cdot (T * \psi), \in \mathcal{A}$ for $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{C})$.

(ii) follows by induction : With Proposition 1.1 (ii) one has $\mathcal{M}^n\mathcal{A} \subset U + U'_{Loc}$, $U := \mathcal{M}^{n-1}\mathcal{A}$, satisfying (Δ) by [8, Lemma 2.3]; if $f = u + v'$ with $u, v \in U$, $v' \in L^1_{loc}$, then $\gamma_\omega f = \gamma_\omega u - \gamma'_\omega v + (\gamma_\omega v)' \in U + U'_{Loc}$ if U satisfies (Γ), since also $(\gamma_\omega v)' \in L^1_{loc}$. $U'_{Loc} \subset \mathcal{M}U$ by Proposition 1.1 (i), $U \subset \mathcal{M}U$ with (Δ).
Similarly for $\widetilde{\mathcal{M}}^n\mathcal{A}$. □

Proposition 3.8 of [4] is the special case $n = 1$, $\mathcal{A} \subset C_{ub}(\mathbb{R}, X)$.

EXAMPLES 5.2: *$\mathcal{A} = AP$, S^pAP, W^pAP, B^pAP, AAP, EAP_{rc}, EAP, $\mathcal{T}\mathcal{E}_{ub}$, $\mathcal{E}_n$, Av_n and Zhang's PAP all satisfy even $AP(\mathbb{J}, \mathbb{C}) \cdot \mathcal{A} \subset \mathcal{A}$; $\mathcal{T}\mathcal{E}$, $\mathcal{T}\mathcal{E}_0$, $\mathcal{T}Av$, $\mathcal{T}Av_0$ satisfy (Γ) by definition.*

This is well known respectively follows from the definitions (also for EAP).
With the mean m of Theorem 4.3 one can now define Fourier coefficients etc.:

Definition 5.3: *For $S \in \mathcal{D}'_{\mathcal{T}Av}(\mathbb{R}, X)$ we define ($\gamma_\omega(t) = e^{i\omega t}$)*

$$c_\omega(S) := m(\gamma_{-\omega} S),\ \omega \in \mathbb{R} \qquad (\textit{Fourier coefficients of } S). \tag{5.1}$$

$$\sigma_B(S) := \{\omega \in \mathbb{R} : c_\omega(S) \neq 0\} \qquad (\textit{Bohr} - \textit{spectrum of } S). \tag{5.2}$$

and the formal Fourier series

$$\sum_{\omega \in \sigma_B(S)} c_\omega(S)\, \gamma_\omega.$$

By Proposition 5.1, $\gamma_{-\omega} S \in \mathcal{D}'_{Av}(\mathbb{R}, X)$, so everything is well defined.

If $f \in B^p AP(\mathbb{R}, X) \subset \mathcal{T}Av(\mathbb{R}, X)$, one gets the usual Fourier coefficients, series and spectrum, with $AP \subset S^p AP \subset\ W^p AP$, even $\subset \mathcal{T}\mathcal{E}(\mathbb{R}, X)$ for $1 \leq p < \infty$ ([27, Theorem 5.6.2]), $W^p AP \subset B^p AP$.

$\mathcal{D}'_{\mathcal{T}Av_0}$ contains by (4.4) exactly the $S \in \mathcal{D}'_{\mathcal{T}Av}$ with Fourier series 0.

We say that $U \subset \mathcal{D}'_{\mathcal{T}Av}(\mathbb{R}, X)$ has *countable spectra*, if for each $S \in U$ the Bohr spectrum $\sigma_B(S)$ is at most countable.

Set

$$Av_+(\mathbb{R}, X) := \{f \in Av(\mathbb{R}, X) : f|\mathbb{R}_+ \in Av(\mathbb{R}_+, X)\}, \tag{5.3}$$

where $Av(\mathbb{R}_+, X) := \{f \in L^1_{loc}(\mathbb{R}_+, X) : lim_{T\to\infty}(1/T) \int_0^T f\, dt$ exists$\}$ [7, p.1007].

One has

$$\mathcal{T}\mathcal{E}(\mathbb{R}, X) \subset \mathcal{T}(Av_+(\mathbb{R}, X)) \subset \mathcal{T}Av(\mathbb{R}, X), \tag{5.4}$$

if $f \in \mathcal{T}(Av_+(\mathbb{R}, X))$, then $f(-\cdot) \in \mathcal{T}(Av_+(\mathbb{R}, X))$.

Note that if $f(0) = 0$, $f \in C_{ub}(\mathbb{R}_+, X)$, $\gamma_\omega f \in \mathcal{E}_{ub}(\mathbb{R}_+, X)$ for all $\omega \neq 0$ and $f \notin Av(\mathbb{R}_+, X)$ (see the example after (3.3)), then F defined by $F = f$ on $\mathbb{R}_+$ and $F(t) = -f(-t)$, $t \leq 0$ does not belong to $Av_+(\mathbb{R}, X)$ but $F \in \mathcal{T}Av(\mathbb{R}, X)$.

This implies that the inclusion $\mathcal{T}(Av_+(\mathbb{R}, X)) \subset \mathcal{T}Av(\mathbb{R}, X)$ is strict.

Proposition 5.4: *If $S \in \mathcal{D}'_{\mathcal{T}Av}(\mathbb{R}, X)$, $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{K})$ and $n \in \mathbb{N}$, then $S^{(n)} \in \mathcal{D}'_{\mathcal{T}Av}$, $S * \varphi \in \mathcal{T}Av(\mathbb{R}, X)$ and, with $\hat{\varphi}$ = Fourier transform of φ,*

$$c_\omega(S^{(n)}) = (i\omega)^n c_\omega(S),\ \ c_\omega(S * \varphi) = c_\omega(S)\hat{\varphi}(\omega),\ \omega \in \mathbb{R}. \tag{5.5}$$

So $\mathcal{D}'_{\mathcal{A}}(\mathbb{R}, X)$ has countable spectra if $\mathcal{A}$ has, $\mathcal{A} \subset \mathcal{D}'_{\mathcal{T}Av}(\mathbb{R}, X)$. The classes $\mathcal{T}Av_+(\mathbb{R}, X)$ of (5.3) and so $\mathcal{T}\mathcal{E}(\mathbb{R}, X)$ have countable spectra.

Proof: By Proposition 5.1 and Corollary 4.5, $\gamma_{-\omega} S$ and $S^{(n)} \in \mathcal{D}'_{\mathcal{T}Av}$ with $0 = m((\gamma_{-\omega} S)') = m(-\, i\omega\gamma_{-\omega} S + \gamma_{-\omega} S')$; this and induction gives the first part of (5.5).

The second part follows with (4.1) and

(5.6) $$\gamma_{-\omega}(S * \varphi) = (\gamma_{-\omega}S) * (\gamma_{-\omega}\varphi), \ \varphi \in S.$$

Since $\hat{\varphi}$ is entire, (5.5) gives the countability of $\sigma_B(S)$ if $S \in \mathcal{D}'_{\mathcal{A}}$.

If $g \in \mathcal{T}(Av(\mathbb{R}_+, X))$, then $\lim_{T\to\infty} (1/T) \int_0^T f(t)e^{i\omega t}\, dt = 0$ except for a countable set of ω's by Théorème 1 of Kahane [24] (see also Urbanik [34]), the proof there works also for X-valued g instead of complex-valued ones. With the line after (5.4) Kahane's result gives countable spectra for $\mathcal{T}Av_+(\mathbb{R}, X)$ and so for $\mathcal{T}\mathcal{E}(\mathbb{R}, X)$, by (5.4). □

REMARK 5.5: *Proposition* 5.4 *holds also if* $\mathcal{T}Av$ *is replaced everywhere by* $\mathcal{T}\mathcal{E}$, *with* $\varphi \in \mathcal{D}_{L^1}$ ([5, *Proposition* 8.7]).

COROLLARY 5.6: *(i) If* $\mathcal{A}$ *is linear uniformly closed* $\subset W^pAP \oplus (\mathcal{T}Av_0 \cap \mathcal{D}'_{L^\infty})$ *and with* $\mathcal{A} * \mathcal{D} \subset \mathcal{A}$, $1 \le p < \infty$, *then* $S \in \mathcal{D}'_{\mathcal{A}}$ *has countable spectrum* $\sigma_B(S) = \{\omega_n : n \in \mathbb{N}\}$ *and there is* $q \in \mathbb{N}$ *with* $c_{\omega_n}(S) = O(max(1, |\omega_n|^q))$.

(ii) Conversely, for given $\sigma = \{\omega_n : n \in \mathbb{N}\} \subset \mathbb{R}$ *and* $c_n \in X$ *with* $|\omega_n| \ge \varepsilon n^\delta$ *if* $n \ge 1/\varepsilon$ *with positive* ε, δ *and* $c_n = O(n^q)$ *for some* $q \in \mathbb{N}$, *there is a (unique)* $S \in \mathcal{D}'_{AP}(\mathbb{R}, X)$ *with Fourier series* $\Sigma c_n \gamma_{\omega_n}$.

So if $T \in \mathcal{D}'_{\mathcal{T}Av}(\mathbb{R}, X)$ *has spectrum and Fourier coefficients as in (ii),* $T \in \mathcal{D}'_{AP} \oplus \mathcal{D}'_{\mathcal{T}Av_0}$.

PROOF: (i) Since $W^pAP \subset \mathcal{T}\mathcal{E} \subset \mathcal{D}'_{L^\infty}$ (see after Definition 5.3 and (3.6)), by [8, Theorem 2.11] one has $S = f + g^{(q)}$ with $f, g \in \mathcal{A}$, $q \in \mathbb{N}$, $f, g \in W^pAP \oplus TAv_0$ have countable spectra and bounded Fourier coefficients, (5.5) gives (i).

(ii) By the assumptions, $g := \sum_{\omega_n \ne 0} (i\omega_n)^{-q} c_n \gamma_{\omega_n} \in AP(\mathbb{R}, X)$ for large q, so $c + g^{(q)} \in \mathcal{D}'_{AP}$ has Fourier series $\sum c_n \gamma_{\omega_n}$ by (5.5), with $c = c_{n_0}$ if $\omega_{n_0} = 0$, else $c = 0$. □

EXAMPLES 5.7: *All of the above can be applied to* $\mathcal{A} = AP$, *AAP*, S^pAP, W^pAP, B^pAP, EAP_{rc}, *EAP, PAP, GPAP and the corresponding* $\mathcal{D}'_{\mathcal{A}}$, *with* $\mathcal{D}'_{AP} = \mathcal{D}'_{S^pAP} \subset \mathcal{D}'_{W^pAP} \subset \subset \mathcal{D}'_{B^pAP} \subset \mathcal{D}'_{\mathcal{T}Av} \subset \mathcal{S}'$, *since all these* $\mathcal{A}$ *are* $\subset \mathcal{T}(Av_+)$ *of Proposition* 5.4 *and so have countable spectra; except PAP, GPAP and* B^pAP *they are even* $\subset \mathcal{T}\mathcal{E}$ *(see* [27, *Theorem* 5.6.2] *for* $W^pAP \subset \mathcal{T}\mathcal{E}$, [17, *Theorem* 3.1], [3, *Theorem* 2.4.7], [37, *Theorem* 1.3.12, *p.* 36] *for EAP;* $B^pAP \not\subset \mathcal{E}$ *since* $B^pAP \not\subset \mathcal{D}'_{L^\infty}$ *by the remarks after* (3.10); *for* (G)PAP *see after* (3.5).

Definition 5.3 is thus also meaningful for all mean classes $\mathcal{M}^n\mathcal{A}$, $\widetilde{\mathcal{M}}^n\mathcal{A}$, $\mathcal{A}$ as in Examples 5.7, since $\mathcal{A} \subset \mathcal{T}Av$ implies $\mathcal{M}^n\mathcal{A} \subset \widetilde{\mathcal{M}}^n\mathcal{A} \subset \widetilde{\mathcal{M}}^n\mathcal{T}Av \subset \mathcal{D}'_{\mathcal{T}Av}$ by (3.11).

We need the following restriction of $\mathcal{T}Av$:

(5.7) $$Av_*(\mathbb{R}, X) := \{f \in Av(\mathbb{R}, X) : \text{ to } f \text{ exists } k \in \mathbb{N}_0 \text{ with } (*)\},$$

$(*)$ $$\sup\left\{ \|1/(2T(1+|x|)^k) \int_{-T}^{T} f(x+t)\, dt\| : |x| \ge k, T \ge k \right\} < \infty.$$

Proposition 5.8: *For* $S \in \mathcal{D}'_{TAv_*}(\mathbb{R}, X)$ *one has* $\sigma_B(S) \subset$ *supp* $\widehat{S}$ *(see* (5.14)).

Proof: $\mathcal{D}'_{TAv}(\mathbb{R}, X) \subset \mathcal{S}'(\mathbb{R}, X)$ by (3.9), so $\widehat{S}$ is defined. If $\omega \notin$ supp $\widehat{S}$, there is $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{C})$ with $\varphi(\omega) \neq 0$ and $\varphi \cdot \widehat{S} = 0$. To φ exists $\psi \in \mathcal{S}(\mathbb{R}, \mathbb{K}) \subset \mathcal{D}_{L^1}$ with $\varphi = \hat{\psi}$. Since $\widehat{S * \psi} = \hat{\psi} \cdot \widehat{S}$ also for $S \in \mathcal{S}'(\mathbb{R}, X)$, $\psi \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ one gets $S * \psi = 0$.

Since in general the $\psi \notin \mathcal{D}$, we cannot use (5.5) for $c_\omega(S)$.

We first treat the case $S = f \in Av_*$ and $\omega = 0$; by the above there is $\psi \in S$ with $f * \psi = 0$ and $\hat{\psi}(0) = 1$, we have to show $m(f) = 0$.

By (3.8) $f \in \mathcal{MA}$, $\mathcal{A} = O(w_1) \cap C(\mathbb{R}, X)$; now this $\mathcal{A}$ satisfies (Δ) by [7, Proposition 1.3 and Example 3.13], so with Proposition 1.1(ii) one has $f = u + v'$, $u, v \in \mathcal{A}$, or $\|Pf(t)\| \leq c_1 \cdot (1 + t^2)$ for $t \in \mathbb{R}$. With $\rho(t) := \psi(-t)$ then for $x \in \mathbb{R}$

$$0 = (f * \psi)(x) = ((Pf)')(\rho_{-x}) = -Pf(\rho'_{-x}) = +\int_{\mathbb{R}} (Pf)(x-s)\psi'(s)\, ds.$$

With integration by parts one gets $lim_{c \to \infty} \int_{-c}^{c} f(x-s)\psi(s)\, ds = 0$, locally uniformly in $x \in \mathbb{R}$. So for fixed $T > 0$ and $\varepsilon > 0$ there is $c_{\varepsilon,T}$ with

$$\| \int_{-c}^{c} (1/2T) \int_{-T}^{T} f(x-s)\, dx \psi(s) ds \| \leq \varepsilon \textit{ if } c \geq c_{\varepsilon,T}. \tag{5.8}$$

Now (*) and (3.5) give c_2 and $T_2 \in \mathbb{R}_+$ with

$$\|(1(2T) \int_{-T}^{T} f(x+t)\, dt\| \leq c_2(1+|x|)^k \text{ for all } x \in \mathbb{R},\ T \geq T_2 \geq 1. \tag{5.9}$$

Choose now $\varphi_n \in \mathcal{D}$ with $\varphi_n \to \psi$ in $\mathcal{S}$ ([33, Théorème III, p. 273]). With (5.9) there is n_ε such for $n \geq n_\varepsilon$, uniformly in $c \in \mathbb{R}_+$ and $T \geq T_2$

$$\| \int_{-c}^{c} ((1/2T) \int_{-T}^{T} f(x-s)\, dx)(\varphi_n(s) - \psi(s))\, ds\| \leq \varepsilon. \tag{5.10}$$

Therefore (5.8) and Fubini gives, for $n \geq n_\varepsilon$, $T \geq T_2$, $c \geq c_{\varepsilon,T}$

$$\|(1/2T) \int_{-T}^{T} \int_{-c}^{c} f(x-s)\varphi_n(s) ds\, dx\| \leq 2\varepsilon, \tag{5.11}$$

$c \to \infty$ yields $\|(1/2T) \int_{-T}^{+T} (f * \varphi_n)(x)\, dx\| \leq 2\varepsilon$, $n \geq n_\varepsilon$, $T \geq T_2$. Since $f * \varphi_n \in Av$ by Proposition 3.2, $T \to \infty$ yields $\|m(f * \varphi_n)\| \leq 2\varepsilon$, $n \geq n_\varepsilon$, then Lemma 4.1 $m(f)\hat{\varphi}_n(0) \to 0$; since $\hat{\varphi}_n(0) \to \hat{\psi}(0) = 1$, one gets $m(f) = 0$ as desired.

Case general ω, $\gamma_{-\omega} f \in Av_*$, $\hat{\psi}(\omega) = 1, f * \psi = 0$:

With (5.6) the case $\omega = 0$ can be applied to $g = \gamma_{-\omega} f$, $\chi = \gamma_{-\omega}\psi$, yielding $c_\omega(f) = m(g) = 0$.

So if $f \in TAv_*$, $\sigma_B(f) \subset \operatorname{supp} \hat{f}$, defined since $f \in \mathcal{S}'$ by (3.9).

Assume now $S \in \mathcal{D}'_{TAv_*}$, $\omega \notin \operatorname{supp} \widehat{S}$ and ψ as in the beginning. For $\varphi \in \mathcal{D}$ one has $f = S * \varphi \in TAv_*$ or $\gamma_{-\omega} f \in Av_*$, with $f * \psi = (S * \varphi) * \psi = (S * \psi) * \varphi = 0$ by associativity ([33, Théorème XI, p. 247/248] for $X = \mathbb{C}$, for general X as in [8, §2]). The case general ω above yields therefore $c_\omega(S * \varphi) = 0$. Choosing φ with $\hat{\varphi}(\omega) = 1$, (5.5) gives $c_\omega(S) = 0$. This proves $\sigma_B(S) \subset \operatorname{supp} \widehat{S}$. □

Proposition 5.8 *can be applied to* $\mathcal{D}'_{\mathcal{A}}$ *for all the* $\mathcal{A}$ *of Examples* 5.7:

Except PAP and B^pAP they are $\subset \mathcal{TE}$, and for $f \in \mathcal{E}$ the condition (*) holds even without $(1+|x|)^k$; the same holds for bounded $f \in Av$, especially PAP; $f \in B^pAP(\mathbb{R}, X) \subset B^1AP$ implies $|f| \in B^1AP(\mathbb{R}, \mathbb{C})$ and thus $m(|f|) < \infty$, this alone gives (*) with $k = 1$.

For the $\mathcal{A}$ as in Examples 5.7, in the case $\mathcal{A} = AP$ or $\mathcal{A} = S^pAP$ one has:

$S \in \mathcal{D}'_{\mathcal{A}}$ *is uniquely defined by its Fourier series*:

If $S_1, S_2 \in \mathcal{D}'_{AP}(\mathbb{R}, X) = \mathcal{D}'_{S^pAP}(\mathbb{R}, X)$ with $c_\omega(S_1) = c_\omega(S_2)$ for $\omega \in \mathbb{R}$, with $S = S_1 - S_2$ and Proposition 5.8 one gets $c_\omega(S * \varphi) = 0$ for all ω and $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{K})$; since $S * \varphi \in AP(\mathbb{R}, X)$, this implies $S * \varphi = 0$ by [1, p. 25, VI]. Since this holds for all $\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{K})$, one gets $S_1 = S_2$.

If $\mathcal{A} = W^pAP$ or B^pAP one gets only $\|S_1 * \varphi - S_2 * \varphi\|_{\mathcal{A}} = 0$, $\varphi \in \mathcal{D}$, using $y(S * \varphi) \in A(\mathbb{R}, \mathbb{K})$, $c_\omega(y(S * \varphi)) = y(c_\omega(S * \varphi))$ for $y \in$ dual X^* and the scalar uniqueness theorem of [14, p. 46/47] .

This determination of S by its Fourier series for the case $\mathcal{D}'_{AP}$ (and a fortiori for $\mathcal{D}'_{S^pAP}$) is nothing new however: By Theorem 2.1, $(a) \Leftrightarrow (b)$, every $S \in \mathcal{D}'_{AP}$ can be considered as an almost periodic function $\Phi_S \in AP(\mathbb{R}, Y)$ with $Y =$ locally convex topologically complete vector space $(\mathcal{D}_{L^1})'$, so by the general Bochner-von Neumann theory [13], there exist even summation methods for the Fourier series of Φ_S, converging uniformly on $\mathbb{R}$ to Φ_S. Using the uniqueness of a linear, invariant, normalized and continuous (with respect to uniform convergence) map: $AP(\mathbb{R}, X) \to X$, one can show

$$m_N(\Phi_S) = m(S), \text{ (constant distribution) for } S \in \mathcal{D}'_{AP}(\mathbb{R}, X). \tag{5.12}$$

where m_N denotes the Bochner-von Neumann mean [13, pp. 28-29] on $AP(\mathbb{R}, Y)$. With this one gets

$$c_\omega(\Phi_S) = m_N(\gamma_{-\omega}\Phi_S) = \gamma_\omega m_N(\Phi_{\gamma_{-\omega}S}) = \gamma_\omega c_\omega(S).$$

So if $\sum_\omega s_{n,\omega}\, c_\omega(\Phi_S)\, \gamma_\omega \to \Phi_S$ uniformly on $\mathbb{R}$ for some summation method $(s_{n,\omega})$ (see [4, Theorem 3.10]), $t = 0$ gives

$$\sum_\omega s_{n,\omega}\; c_\omega(S)\; \gamma_\omega \to S \text{ as } n \to \infty \text{ in } (\mathcal{D}_{L^1})',\;\; S \in \mathcal{D}'_{AP}(\mathbb{R}, X), \tag{5.13}$$

where $(s_{n,\omega})$ depends only on $\sigma_B(S)$.

This holds especially for $f \in \mathcal{M}^n AP$ or $S \in \widetilde{\mathcal{M}}^n AP$, but here a stronger convergence holds with Proposition 1.1:

For $f \in \mathcal{M}^n AP(\mathbb{J}, X)$, any $\mathbb{J}$,

$$\sum_{\omega} s_{n,\omega}\, c_\omega(f)\, \gamma_\omega \to f$$

in the locally convex topology defined on $\mathcal{M}^n L^\infty(\mathbb{J}, X)$ by the seminorms $\|g\|_h :=$ $:= \|M_{h_n} \cdots M_{h_1} g\|_\infty$, $h = (h_1, \dots, h_n)$, $h_j > 0$. Here also the ergodic mean on $\mathcal{E}(\mathbb{J}, X)$ can be extended uniquely and continuously to $\bigcup_{n=0}^{\infty} \mathcal{M}^n \mathcal{E}(\mathbb{J}, X)$ by [6, §2]. This generalizes Theorem 3.10 of [4].

For $S \in \widetilde{\mathcal{M}}^n AP(\mathbb{J}, X)$ similar results hold, with $\widetilde{\mathcal{M}}_{h_j}$ in the definition of $\|g\|_h$.

With Proposition 5.8, (5.13) and $\hat{\gamma}_\omega = 2\pi\delta_{-\omega}$ one gets

$$\text{If } S \in \mathcal{D}'_{AP}(\mathbb{R}, X), \text{ then closure } \sigma_B(S) = \operatorname{supp} \widehat{S}. \tag{5.14}$$

This is in general false already for AAP or $W^p AP$: If $\phi \in C_0(\mathbb{R}, \mathbb{C}) \cap L^1(\mathbb{R}, \mathbb{C})$, $\phi \neq 0$, then *closure* $\sigma_B(\phi) = \emptyset \neq$ *supp* $\widehat{\phi}$.

Furthermore with [8, Theorem 2.10] one can show

$$\mathcal{D}'_{\mathcal{A}} = \mathcal{D}'_{AP} \oplus \mathcal{D}'_{\mathcal{A}_0} \tag{5.15}$$

for $\mathcal{A} = AAP$, EAP_{rc} and EAP, where $AAP_0 = C_0$, $EAP_0 =$ null functions in EAP (see [29, p. 18], [30, $W_0(\mathbb{R}_+, X)$, p. 424]), similarly for $(EAP_{rc})_0$. Therefore to $S \in \mathcal{D}'_{\mathcal{A}}$ there is exactly one $U \in \mathcal{D}'_{AP}$ with $c_\omega(S) = c_\omega(U)$, namely the U in $S = U + V$ of (5.15).

Let us finally remark that the ergodic mean m can also be extended to

$$\bigcup_{0}^{\infty} \mathcal{M}^n Av(\mathbb{J}, X) \text{ for } \mathbb{J} \neq \mathbb{R},$$

with corresponding consequences; this can be done as in [6, § 2].

QUESTION: Has $\mathcal{T}Av(\mathbb{R}, X)$ countable spectra?

(By the example after (5.4), $\mathcal{T}Av(\mathbb{R}, \mathbb{C}) \cap C_{ub}(\mathbb{R}, \mathbb{C})$ is not a subset of $Av_+(\mathbb{R}, \mathbb{C})$ of (5.3); another such example would be Kahane's unbounded $\sum_1^\infty \sin(10^{-n} t)$ ([24, p. 105, Remarque 1]); see also (3.13) and the question of Hartman in [22, (1) and (2)]).

REFERENCES

[1] L. AMERIO - G. PROUSE, *Almost-Periodic Functions and Functional Equations*, Van Nostrand, 1971.

[2] W. ARENDT - C.J.K. BATTY - M. HIEBER - F. NEUBRANDER, *Vector-valued Laplace Transforms and Cauchy problems*, Monographs in Math., Vol. 96, Basel, Boston, Berlin: Birkhäuser, 2001.

[3] B. BASIT, *Some problems concerning different types of vector valued almost periodic functions*, Dissertationes Math., 338 (1995), 26 pages.

[4] B. Basit - H. Günzler, *Asymptotic behavior of solutions of systems of neutral and convolution equations*, J. Differential Equations, 149 (1998), 115-142.

[5] B. Basit - H. Günzler, *Generalized vector valued almost periodic and ergodic distributions*, Analysis Paper 113, (September 2002, 65 pp), Monash University.

[6] B. Basit - H. Günzler, *Generalized almost periodic and ergodic solutions of linear differential equations on the half line in Banach spaces*, J. Math. Anal. Appl. 282 (2003), 673-697.

[7] B. Basit - H. Günzler, *Generalized Esclangon-Landau results and applications to linear difference-differential systems in Banach spaces*, J. Difference Equations and Applications, Vol. 10, No. 11 (2004), 1005-1023.

[8] B. Basit - H. Günzler, *Generalized vector valued almost periodic and ergodic distributions*, J. Math. Anal. Appl. 314 (2006), 363-381.

[9] B. Basit - H. Günzler, *A difference property for perturbations of vector valued Levitan almost periodic functions and analogues*, Russ. J. of Math. Phys., 12 (4) (2005), 424-438.

[10] B. Basit - C. Zhang, *New almost periodic type functions and solutions of differential equations*, Can. J. Math. 48 (1996), 1138-1153.

[11] J. F. Berglund - H. D. Junghenn - P. Milnes, *Analysis on Semigroups*, John Wiley, 1989.

[12] A. S. Besicovitch, *Almost Periodic Functions*, Dover Publ., 1954.

[13] S. Bochner - J. von Neumann, *Almost periodic functions on groups, II*, Trans. Amer. Math. Soc. 37 (1935), 21-50.

[14] H. Bohr - E. Følner, *On some types of functional spaces*, Acta. Math. 76 (1944), 31-155.

[15] K. Deleeuw - I. Glicksberg, *Applications of almost periodic compactifications*, 105 (1961), 63-97.

[16] N. Dunford - J. T. Schwartz, *Linear Operators, Part I*, Interscience, New York, 1963.

[17] W. F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. 67 (1949), 217-240.

[18] A. M. Fink, *Almost Periodic Differential Equations*, Lecture Notes in Math., 377, Springer-Verlag 1974.

[19] S. Goldberg - P. Irwin, *Weakly almost periodic vector-valued functions*, Dissertationes Math. 157 (1979), 42 pages.

[20] H. Günzler, *Integration of almost periodic functions*, Math. Zeitschr. 102 (1967), 253-287.

[21] H. Günzler, *Integration, Bibliographisches Institut*, Wissenschaftsverlag, 1985.

[22] S. Hartman, *Verallgemeinerte harmonische Analysis*, Ann. Pol. Math. XVI (1965), 341-352.

[23] E. Hille - R. S. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloquim Publications, Providence, 1957.

[24] J.-P. Kahane, *Sur les coefficients de Fourier-Bohr*, Studia Math. T. 21 (1961), 103-106.

[25] J.-P. Kahane, *Sur les fonctions présque-périodiques généralisées dont le spectre est vide*, Studia Math. 21 (1962), 231-236.

[26] B. M. Levitan - V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, 1982.

[27] B. M. Levitan, *Almost Periodic Functions* (Russian) Gos. Izd., Moscow, 1953.

[28] P. Milnes, *On vector-valued weakly almost periodic functions*, J. London Math. Soc. (2) 22 (1980), 467-472.

[29] W. M. Ruess - W. H. Summers, *Integration of asymptotically almost periodic functions and weak almost periodicity*, Dissertationes Math. 279 (1989), 35 pages.

[30] W. M. Ruess - W. H. Summers, *Ergodic theorems for semigroups of operators*, Proc. Amer. Math. Soc. 114 (1992), 423-432.

[31] W. M. Ruess - V. Q. Phóng, *Asymptotically almost periodic solutions of evolution equations in Banach spaces*, J. Differential Equations 122 (1995), 282-301.

[32] L. Schwartz, *Distributions á valeures vectorielles*, Annales de l'Institut Fourier 7 (1957), 1-139

[33] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, 1966.

[34] K. Urbanik, *Fourier analysis in Marcinkiewicz spaces*, Studia Math. T. 21 (1961), 93-102.
[35] K. Yosida, *Functional Analysis*, Springer Verlag, 1976
[36] C. Zhang, *Integration of vector-valued pseudo-almost periodic functions*, Proc. Amer. Math. Soc. 121 (1994), 167-174
[37] C. Zhang, *Almost Periodic Type Functions and Ergodicity*, Science Press/Kluwer Acad. Publ., 2003

Direttore responsabile: Prof. A. BALLIO - Autorizz. Trib. di Roma n. 7269 dell'8-12-1959
«Monograf» - Via Collamarini, 5 - Bologna