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Liu's Markov Chains Generalized (**)

ABSTRACT. — In this paper, the Markov chain model proposed by Liu [2] is extended and analyzed in detail. Recurrence conditions and invariant measures are discussed, the mean passage times between states are worked out in explicit form.

Una generalizzazione delle catene di Markov del tipo di Liu

SUNTO. — In questa nota viene esposta e analizzata una generalizzazione delle catene di Markov proposte da Liu [2]. Si stabiliscono condizioni di ricorrenza, esistenza e forma delle misure invarianti; si perviene infine al calcolo in forma esplicita dei tempi medi di passaggio tra gli stati.

1. - INTRODUCTION AND PRELIMINARY RESULTS

We consider in this paper a general model of Markov chains on the positive integers, with transition probabilities p(m, n) given by

p(m, n) = 0 for n > m + 1 $p(n, n + 1) = p_n$ with $0 < p_n < 1$

$$p(n, k) = \frac{r_k}{r_0 + \ldots + r_n} (1 - p_n) \text{ for } k \le n \quad (r_n \ge 0)$$

We suppose $r_0 = 1$, and put $s_n = r_0 + \ldots + r_n$ for short.

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Markov chains X_n of this kind are irreducible; they are skip-free to the right, in that the probability $P(X_{n+1} - X_n > 1)$ vanishes for every n.

Letting $r_n = 1$ for every *n*, we get the model of Liu [2]. The renewal chain is obtained when $r_n = 0$ for every n > 0.

Some noticeable applications of the former case are mentioned in Liu [2]; the model of this paper is concerned with the following replacement situation: an electronic component is replaced (upon failure at age n) with another component, still operating though not necessarily fresh, aged according to a distribution which is proportional to the numbers $r_0, ..., r_n$.

In the present section we collect some preliminaries and establish a technical lemma; the invariant measures are treated in Section 2. In Section 3 some hitting probabilities are determined and the recurrence condition is found. Finally, Section 4 is devoted to the explicit computation of the mean passage times between states.

We first introduce two positive functions defined on the state space; they will play a major role in all developments concerning the quantitative behaviour of our Markov chains.

Define

(1)
$$g(0) = 0$$
 $g(1) = \frac{1}{p_0}$ $g(n+1) = g(n) + \frac{1-p_n}{s_n p_0 \dots p_n}$ for $n > 0$

(2)
$$\sigma(0) = 1 \quad \sigma(n) = s_n p_0 p_1 \dots p_{n-1} \quad \text{for } n > 0$$

Clearly, g is increasing, σ is strictly positive. By σ we also mean the measure $\sigma(A) = \sum_{k \in A} \sigma(k)$.

LEMMA 1: For n > 0 the following equality holds:

$$g(n) = \sum_{k=1}^{n} \frac{r_k}{s_{k-1}\sigma(k)} + \frac{1}{\sigma(n)} .$$

PROOF: The proof is a trivial verification for n = 1. For n > 1, a simple computation gives

$$\frac{1}{\sigma(n+1)} - \frac{1}{\sigma(n)} + \frac{r_{n+1}}{s_n \sigma(n+1)} =$$

$$= \frac{1}{s_{n+1} p_0 \dots p_n} - \frac{1}{s_n p_0 \dots p_{n-1}} + \frac{r_{n+1}}{s_n s_{n+1} p_0 \dots p_n} =$$

$$= \frac{1 - p_n}{s_n p_0 \dots p_n} = g(n+1) - g(n)$$

and the lemma is proved.

For a given function f, we write

$$\overline{f}(n) = \sum_{k=0}^{n} \frac{r_k}{s_n} f(k) \quad S_n(f) = s_n f(n) - \sum_{k=0}^{n} r_k f(k) = s_n (f(n) - \overline{f}(n))$$

whence the identity

(3)
$$\frac{S_{n+1}(f)}{s_n} = f(n+1) - \overline{f}(n)$$

2. - INVARIANT MEASURES

The existence of invariant measures is completely settled by the following proposition.

PROPOSITION 1: An invariant non zero measure exists if and only if the products $(p_0 \dots p_n)$ tend to zero for $n \to \infty$. In this case the measure σ , defined as in (2), is the only invariant measure up to multiplication by a positive constant.

PROOF: Let *m* be a (positive) non zero invariant measure. The invariance equation $m(k) = \sum_{j} p(j, k) m(j)$ for k = 0 reads

(4)
$$m(0) = \sum_{j=0}^{\infty} \frac{1-p_j}{s_j} m(j).$$

Then m(0) > 0. Denoting m(0) by *c*, the other invariance equations are

(5)
$$m(k+1) = p_k m(k) + r_{k+1} \sum_{j=k+1}^{\infty} \frac{1-p_j}{s_j} m(j) =$$

$$= p_k m(k) + r_{k+1} \left(c - \sum_{j=0}^k \frac{1-p_j}{s_j} m(j) \right).$$

For k = 0, we immediately get

$$m(1) = p_0 c + r_1 c - r_1 (1 - p_0) c = cs_1 p_0 = c\sigma(1)$$

If the equality $m(k) = c\sigma(k)$, just checked for k = 0 and k = 1, is supposed to hold for j = 0, ..., k, equation (5) implies $m(k + 1) = c\sigma(k + 1)$, as seen by a tedious but straightforward verification.

In conclusion, we have $m = c\sigma$, and the equality (4) implies

$$c = c(1 - p_0) + \lim_{n} \sum_{j=1}^{n} c(1 - p_j) p_0 \dots p_{j-1} = c - c \lim_{n} (p_0 \dots p_n)$$

that is $(p_0 \dots p_n) \rightarrow 0$.

Conversely, suppose $(p_0 \dots p_n) \rightarrow 0$. Then $m(k) = \sigma(k)$ is easily shown to satisfy the invariance relations (4) and (5).

The proposition is proved.

Note that the condition occurring in Proposition 1 can also be expressed under the equivalent form $\sum_{n} (1 - p_n) = \infty$.

3. - HARMONIC FUNCTIONS, HITTING PROBABILITIES AND RECURRENCE

A function f satisfying the equality $f(k) = \sum_{j} p(k, j) f(j)$ is usually said to be *harmonic* in k.

The probability $u^{(n)}(k)$ of hitting the state *n* before entering 0 as a function of the starting point *k* is a well known example of a function which vanishes in 0 and is harmonic in the interval]0, *n*[. Due to the irreducibility, $u^{(n)}(1) \neq 0$.

We prove the following

PROPOSITION 2: Let f be harmonic in the interval]m, n[and g be defined as in (1). Then the ratio

$$\frac{f(k) - \overline{f}(m)}{g(k) - \overline{g}(m)}$$

does not depend on k for $m < k \le n$.

PROOF: For m < k < n, harmonicity in k entails

$$f(k) = p_k f(k+1) + (1-p_k) \,\overline{f}(k)$$

that is

(6)
$$p_k(f(k+1) - f(k)) = (1 - p_k)(f(k) - \overline{f}(k)) = \frac{S_k(f)}{s_k}.$$

Taking (3) into account, the following simple recursive relation for $S_k(f)$ is obtained

$$S_{k+1}(f) = S_k(f) + s_k(f(k+1) - f(k)) = S_k(f) + s_k \frac{1 - p_k}{p_k}(f(k) - \overline{f}(k)) =$$

$$- 163 - S_k(f) + S_k(f) \frac{1 - p_k}{p_k} = \frac{S_k(f)}{p_k}$$

leading by iteration to

(7)
$$S_k(f) = \frac{p_0 \dots p_m}{p_0 \dots p_{k-1}} S_{m+1}(f).$$

Letting $C_m = p_0 \dots p_m$, substitution in (6) yields

$$f(k+1) - f(k) = C_m S_{m+1}(f) \frac{1 - p_k}{s_k p_0 \dots p_k} = C_m S_{m+1}(f) \left(g(k+1) - g(k) \right)$$

for m < k < n. Summing over k

(8)
$$f(k) - f(m+1) = C_m S_{m+1}(f) (g(k) - g(m+1))$$

for $m < k \le n$.

Take a function u, vanishing in 0, harmonic in]0, N[, with $u(1) \neq 0$ (we saw an example at the beginning of this Section). Since clearly $S_1(u) = u(1)$, equation (8) for f = u and m = 0 gives

$$u(k) - u(1) = C_1 u(1) (g(k) - g(1))$$
 for $0 < k < N$.

Thus $g(k) = \frac{1}{C_1 u(1)} u(k) + constant$ is also harmonic in]0, *N*[, for any *N*. Relation (7) applies to *g*, m = 0 and k > 0; remarking that $p_0 S_1(g) = 1$ we

have

(9)
$$S_k(g) = \frac{p_0 S_1(g)}{p_0 \dots p_{k-1}} = \frac{1}{C_{k-1}} = \frac{s_k}{\sigma(k)} \quad (k > 0)$$

The identity (3) allows us to write

$$f(m+1) - \overline{f}(m) = \frac{S_{m+1}(f)}{s_m} = C_m S_{m+1}(f) \frac{S_{m+1}(g)}{s_m} =$$
$$= C_m S_{m+1}(f) \left(g(m+1) - \overline{g}(m) \right).$$

Adding this to (8), we finally obtain

$$f(k) - \overline{f}(m) = C_m S_{m+1}(f) \left(g(k) - \overline{g}(m) \right)$$

for $m < k \le n$. The proof of Proposition 2 is accomplished.

The next proposition deals with the hitting probabilities.

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PROPOSITION 3: Starting from m < n, the probability w(m, n) of hitting n before returning to m is given by

$$w(m, n)^{-1} = \sigma(m) \left(\frac{1}{\sigma(n)} + \sum_{k=m+1}^{n} \frac{r_k}{s_{k-1}\sigma(k)} \right).$$

PROOF: Let u(k) be the probability of hitting *n* before entering *m*, as a function of the starting point *k*; obviously $w(m, n) = p_m u(m + 1)$.

Since *u* is harmonic in the interval]m, n[and $\overline{u}(m) = 0$, Proposition 2 implies

$$u(k) = C(g(k) - \overline{g}(m))$$

Observing that $u(n) = C(g(n) - \overline{g}(m)) = 1$, we find

$$u(k) = \frac{g(k) - \overline{g}(m)}{g(n) - \overline{g}(m)}$$

whence

$$w(m, n) = \frac{p_m(g(m+1) - \overline{g}(m))}{g(n) - \overline{g}(m)}$$

Thanks to (9), $\overline{g}(m)$ is easily computed:

$$\bar{g}(m) = g(m) - (g(m) - \bar{g}(m)) = g(m) - \frac{S_m(g)}{s_m} = g(m) - \frac{1}{\sigma(m)}$$

Taking Lemma 1 into account, Proposition 3 is proved.

COROLLARY 1: The chain is transient if and only if the function g is bounded.

PROOF: The sequence w(0, n) tends to the probability of no return to 0; so transience is equivalent to boundedness of $w(0, n)^{-1}$. On the other hand, $w(0, n)^{-1} = g(n)$ by Proposition 3.

Recurrent chains do have an invariant measure. But also transient chains may have one, as shown in the following simple example: take $r_n = 1$ for every n, $p_n = \sqrt{(n+1)/(n+2)}$ and verify that $\sigma(n) = \sqrt{n+1}$ is indeed an invariant measure, though the chain is transient.

Positive recurrence, in turn, is easily characterized, being equivalent to $\sigma(N) = \sum_{k \ge 0}^{\infty} \sigma(k) < +\infty$.

In the positive recurrent case, the invariant measure σ can be normalized to the (unique) invariant probability distribution π ; the quantity $1/\sigma(N)$ equals the expected return time to 0.

4. - Mean passage times

This section deals with the calculation of the mean passage times between states. Let t(m, n) denote the expected time to hit the state *n* starting from *m*. The results we prove are summarized in two propositions.

PROPOSITION: 4: For m < n the expected time to reach n starting from m is given by t(m, n) = t(0, n) - t(0, m), where

$$t(0, n) = \tilde{\sigma}(n) + \sum_{k=1}^{n} \frac{r_k \tilde{\sigma}(k)}{s_{k-1}} \quad \text{with } \tilde{\sigma}(k) = \frac{1}{\sigma(k)} \sum_{j=0}^{k-1} \sigma(j)$$

for any n > 0, and σ defined as in (2).

PROOF: The chain being skip-free to the right, the additivity t(0, m) + t(m, n) = t(0, n) holds for $0 \le m \le n$, which allows us to treat t(0, n) for n > 0 as the sum $t(0, n) = t(0) + \ldots + t(n-1)$ of *one stair* passage times t(k) = t(k, k+1).

A simple first step analysis yields

$$\begin{aligned} t(k) &= t(k, \, k+1) = 1 + \sum_{i} p(k, \, i) \, t(i, \, k+1) = 1 + (1-p_k) \sum_{i=0}^{k} \frac{r_i}{s_k} \, t(i, \, k+1) = \\ &= 1 + \frac{1-p_k}{s_k} \sum_{i=0}^{k} r_i t(i, \, k+1) \, . \end{aligned}$$

In particular, we immediately get $t(0) = 1 + (1 - p_0)t(0)$, $t(0) = 1/p_0$. As $\tilde{\sigma}(1) = 1/\sigma(1) = 1/(s_1 p_0)$, the claim of Proposition 4 is true for n = 1.

The equations above are best written putting $A_k = \sum_{i=0}^k r_i t(i, k+1)$ and read $s_k t(k) = s_k + (1 - p_k) A_k$.

Remarking that

$$A_{k} = \sum_{i=0}^{n} \sum_{j=i}^{k} t(j) = \sum_{j=0}^{k} \sum_{i=0}^{j} r_{i}t(j) = \sum_{j=0}^{k} s_{j}t(j) \qquad A_{k} - A_{k-1} = s_{k}t(k)$$

the recurrence equation becomes

$$p_k A_k = s_k + A_{k-1}.$$

A very simple relation is obtained by multiplying both sides by $p_0 \dots p_{k-1}$:

$$p_0 \dots p_k A_k = p_0 \dots p_{k-1} A_{k-1} + \sigma(k)$$

leading to

$$p_0 \dots p_k A_k = p_0 A_0 + \sigma(1) + \dots + \sigma(k) = \sigma[0, k].$$

Adding up the one stair passage times gives, for n > 1

$$t(0, n) = \sum_{k=0}^{n-1} t(k) = A_0 + \sum_{k=1}^{n-1} \frac{A_k - A_{k-1}}{s_k} =$$
$$= A_0 + \sum_{k=1}^{n-1} \left(\frac{A_k}{s_k} - \frac{A_{k-1}}{s_{k-1}} \right) + \sum_{k=1}^{n-1} A_{k-1} \left(\frac{1}{s_{k-1}} - \frac{1}{s_k} \right) =$$
$$= \frac{A_{n-1}}{s_{n-1}} + \sum_{k=1}^{n-1} \frac{r_k A_{k-1}}{s_k s_{k-1}} = \frac{A_{n-1}}{s_n} + \sum_{k=1}^n \frac{r_k A_{k-1}}{s_k s_{k-1}}.$$

The proof is complete, because

$$\frac{A_{k-1}}{s_k} = \frac{p_0 \dots p_{k-1}}{\sigma(k)} A_{k-1} = \frac{\sigma[0, k[}{\sigma(k)} = \tilde{\sigma}(k).$$

Since t(m, n) is finite for m < n, the passage times t(m, n) with m > n have finite expectation if and only if the chain is positive recurrent (see [1], Th.1, p. 62), in which case the following proposition holds.

PROPOSITION 5: In the positive recurrent case, let π be the invariant probability distribution and m < n. The expected time to reach the state m starting from n is given by

$$t(n, m) = t(n, 0) - t(m, 0) + \frac{1}{\pi(m)}$$

where

$$t(n, 0) = \mu(n) + \sum_{k=1}^{n} \frac{r_k \mu(k)}{s_{k-1}}$$
 with $\mu(k) = \frac{1}{\sigma(k)} \sum_{j \ge k} \sigma(j)$

for any n > 0 and σ defined as in (2).

PROOF: Recall that π is related to σ by the relation $\pi(n) = \sigma(n)/\sigma(N)$.

According to a well known result about positive recurrent Markov chains (see [1], Cor. 1, p. 65), the mean *commute time* comm(m, n) = t(m, n) + t(n, m) is given by

(10)
$$\operatorname{comm}(m, n) = \frac{\sigma(N)}{\sigma(m) w(m, n)} = \frac{1}{\pi(n)} + \sum_{k=m+1}^{n} \frac{r_k}{s_{k-1}\pi(k)}$$

For m = 0 one obtains

$$t(n, 0) = \operatorname{comm}(0, n) - t(0, n) = \sigma(N) \left(\frac{1}{\sigma(n)} + \sum_{k=1}^{n} \frac{r_k}{s_{k-1}\sigma(k)} \right) - t(0, n).$$

Observing that $\tilde{\sigma}(k) + \mu(k) = \sigma(N)/\sigma(k)$, the second half of Proposition 5 is proved.

For the first half, we have after (10)

$$comm(0, m) + comm(m, n) - comm(0, n) = \frac{1}{\pi(m)}$$

.

As t(0, m) + t(m, n) - t(0, n) vanishes, the conclusion follows.

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