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## Liu's Markov Chains Generalized (**)

Abstract. - In this paper, the Markov chain model proposed by Liu [2] is extended and analyzed in detail. Recurrence conditions and invariant measures are discussed, the mean passage times between states are worked out in explicit form.

## Una generalizzazione delle catene di Markov del tipo di Liu

Sunto. - In questa nota viene esposta e analizzata una generalizzazione delle catene di Markov proposte da Liu [2]. Si stabiliscono condizioni di ricorrenza, esistenza e forma delle misure invarianti; si perviene infine al calcolo in forma esplicita dei tempi medi di passaggio tra gli stati.

## 1. - Introduction and preliminary results

We consider in this paper a general model of Markov chains on the positive integers, with transition probabilities $p(m, n)$ given by

$$
\begin{gathered}
p(m, n)=0 \text { for } n>m+1 \quad p(n, n+1)=p_{n} \text { with } 0<p_{n}<1 \\
p(n, k)=\frac{r_{k}}{r_{0}+\ldots+r_{n}}\left(1-p_{n}\right) \quad \text { for } k \leqslant n \quad\left(r_{n} \geqslant 0\right)
\end{gathered}
$$

We suppose $r_{0}=1$, and put $s_{n}=r_{0}+\ldots+r_{n}$ for short.
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Markov chains $X_{n}$ of this kind are irreducible; they are skip-free to the right, in that the probability $P\left(X_{n+1}-X_{n}>1\right)$ vanishes for every $n$.

Letting $r_{n}=1$ for every $n$, we get the model of Liu [2]. The renewal chain is obtained when $r_{n}=0$ for every $n>0$.

Some noticeable applications of the former case are mentioned in Liu [2]; the model of this paper is concerned with the following replacement situation: an electronic component is replaced (upon failure at age $n$ ) with another component, still operating though not necessarily fresh, aged according to a distribution which is proportional to the numbers $r_{0}, \ldots, r_{n}$.

In the present section we collect some preliminaries and establish a technical lemma; the invariant measures are treated in Section 2. In Section 3 some hitting probabilities are determined and the recurrence condition is found. Finally, Section 4 is devoted to the explicit computation of the mean passage times between states.

We first introduce two positive functions defined on the state space; they will play a major role in all developments concerning the quantitative behaviour of our Markov chains.

Define

$$
\begin{gather*}
g(0)=0 \quad g(1)=\frac{1}{p_{0}} \quad g(n+1)=g(n)+\frac{1-p_{n}}{s_{n} p_{0} \ldots p_{n}} \text { for } n>0  \tag{1}\\
\sigma(0)=1 \quad \sigma(n)=s_{n} p_{0} p_{1} \ldots p_{n-1} \quad \text { for } n>0 \tag{2}
\end{gather*}
$$

Clearly, $g$ is increasing, $\sigma$ is strictly positive. By $\sigma$ we also mean the measure $\sigma(A)=\sum_{k \in A} \sigma(k)$.

Lemma 1: For $n>0$ the following equality holds:

$$
g(n)=\sum_{k=1}^{n} \frac{r_{k}}{s_{k-1} \sigma(k)}+\frac{1}{\sigma(n)} .
$$

Proof: The proof is a trivial verification for $n=1$. For $n>1$, a simple computation gives

$$
\begin{gathered}
\frac{1}{\sigma(n+1)}-\frac{1}{\sigma(n)}+\frac{r_{n+1}}{s_{n} \sigma(n+1)}= \\
=\frac{1}{s_{n+1} p_{0} \cdots p_{n}}-\frac{1}{s_{n} p_{0} \cdots p_{n-1}}+\frac{r_{n+1}}{s_{n} s_{n+1} p_{0} \cdots p_{n}}= \\
=\frac{1-p_{n}}{s_{n} p_{0} \cdots p_{n}}=g(n+1)-g(n)
\end{gathered}
$$

and the lemma is proved.

For a given function $f$, we write

$$
\bar{f}(n)=\sum_{k=0}^{n} \frac{r_{k}}{s_{n}} f(k) \quad S_{n}(f)=s_{n} f(n)-\sum_{k=0}^{n} r_{k} f(k)=s_{n}(f(n)-\bar{f}(n))
$$

whence the identity
(3)

$$
\frac{S_{n+1}(f)}{s_{n}}=f(n+1)-\bar{f}(n) .
$$

## 2. - Invariant measures

The existence of invariant measures is completely settled by the following proposition.

Proposition 1: An invariant non zero measure exists if and only if the products $\left(p_{0} \ldots p_{n}\right)$ tend to zero for $n \rightarrow \infty$. In this case the measure $\sigma$, defined as in (2), is the only invariant measure up to multiplication by a positive constant.

Proof: Let $m$ be a (positive) non zero invariant measure. The invariance equation $m(k)=\sum_{j} p(j, k) m(j)$ for $k=0$ reads

$$
\begin{equation*}
m(0)=\sum_{j=0}^{\infty} \frac{1-p_{j}}{s_{j}} m(j) \tag{4}
\end{equation*}
$$

Then $m(0)>0$. Denoting $m(0)$ by $c$, the other invariance equations are

$$
\begin{align*}
m(k & +1)=p_{k} m(k)+r_{k+1} \sum_{j=k+1}^{\infty} \frac{1-p_{j}}{s_{j}} m(j)=  \tag{5}\\
& =p_{k} m(k)+r_{k+1}\left(c-\sum_{j=0}^{k} \frac{1-p_{j}}{s_{j}} m(j)\right)
\end{align*}
$$

For $k=0$, we immediately get

$$
m(1)=p_{0} c+r_{1} c-r_{1}\left(1-p_{0}\right) c=c s_{1} p_{0}=c \sigma(1)
$$

If the equality $m(k)=c \sigma(k)$, just checked for $k=0$ and $k=1$, is supposed to hold for $j=0, \ldots, k$, equation (5) implies $m(k+1)=c \sigma(k+1)$, as seen by a tedious but straightforward verification.

In conclusion, we have $m=c \sigma$, and the equality (4) implies

$$
c=c\left(1-p_{0}\right)+\lim _{n} \sum_{j=1}^{n} c\left(1-p_{j}\right) p_{0} \ldots p_{j-1}=c-c \lim _{n}\left(p_{0} \cdots p_{n}\right)
$$

that is $\left(p_{0} \ldots p_{n}\right) \rightarrow 0$.
Conversely, suppose $\left(p_{0} \ldots p_{n}\right) \rightarrow 0$. Then $m(k)=\sigma(k)$ is easily shown to satisfy the invariance relations (4) and (5).

The proposition is proved.
Note that the condition occurring in Proposition 1 can also be expressed under the equivalent form $\sum_{n}\left(1-p_{n}\right)=\infty$.

## 3. - Harmonic functions, hitting probabilities and recurrence

A function $f$ satisfying the equality $f(k)=\sum_{j} p(k, j) f(j)$ is usually said to be harmonic in $k$.

The probability $u^{(n)}(k)$ of hitting the state $n$ before entering 0 as a function of the starting point $k$ is a well known example of a function which vanishes in 0 and is harmonic in the interval $] 0, n$ [. Due to the irreducibility, $u^{(n)}(1) \neq 0$.

We prove the following
Proposition 2: Let $f$ be harmonic in the interval $] m, n[$ and $g$ be defined as in (1). Then the ratio

$$
\frac{f(k)-\bar{f}(m)}{g(k)-\bar{g}(m)}
$$

does not depend on $k$ for $m<k \leqslant n$.
Proof: For $m<k<n$, harmonicity in $k$ entails

$$
f(k)=p_{k} f(k+1)+\left(1-p_{k}\right) \bar{f}(k)
$$

that is

$$
\begin{equation*}
p_{k}(f(k+1)-f(k))=\left(1-p_{k}\right)(f(k)-\bar{f}(k))=\frac{S_{k}(f)}{s_{k}} \tag{6}
\end{equation*}
$$

Taking (3) into account, the following simple recursive relation for $S_{k}(f)$ is obtained

$$
S_{k+1}(f)=S_{k}(f)+s_{k}(f(k+1)-f(k))=S_{k}(f)+s_{k} \frac{1-p_{k}}{p_{k}}(f(k)-\bar{f}(k))=
$$

$$
=S_{k}(f)+S_{k}(f) \frac{1-p_{k}}{p_{k}}=\frac{S_{k}(f)}{p_{k}}
$$

leading by iteration to

$$
\begin{equation*}
S_{k}(f)=\frac{p_{0} \cdots p_{m}}{p_{0} \cdots p_{k-1}} S_{m+1}(f) \tag{7}
\end{equation*}
$$

Letting $C_{m}=p_{0} \ldots p_{m}$, substitution in (6) yields

$$
f(k+1)-f(k)=C_{m} S_{m+1}(f) \frac{1-p_{k}}{s_{k} p_{0} \cdots p_{k}}=C_{m} S_{m+1}(f)(g(k+1)-g(k))
$$

for $m<k<n$. Summing over $k$

$$
\begin{equation*}
f(k)-f(m+1)=C_{m} S_{m+1}(f)(g(k)-g(m+1)) \tag{8}
\end{equation*}
$$

for $m<k \leqslant n$.
Take a function $u$, vanishing in 0 , harmonic in $] 0, N[$, with $u(1) \neq 0$ (we saw an example at the beginning of this Section). Since clearly $S_{1}(u)=u(1)$, equation (8) for $f=u$ and $m=0$ gives

$$
u(k)-u(1)=C_{1} u(1)(g(k)-g(1)) \quad \text { for } 0<k<N .
$$

Thus $g(k)=\frac{1}{C_{1} u(1)} u(k)+$ constant is also harmonic in $] 0, N[$, for any $N$.
Relation (7) applies to $g, m=0$ and $k>0$; remarking that $p_{0} S_{1}(g)=1$ we have

$$
\begin{equation*}
S_{k}(g)=\frac{p_{0} S_{1}(g)}{p_{0} \cdots p_{k-1}}=\frac{1}{C_{k-1}}=\frac{s_{k}}{\sigma(k)} \quad(k>0) \tag{9}
\end{equation*}
$$

The identity (3) allows us to write

$$
\begin{gathered}
f(m+1)-\bar{f}(m)=\frac{S_{m+1}(f)}{s_{m}}=C_{m} S_{m+1}(f) \frac{S_{m+1}(g)}{s_{m}}= \\
=C_{m} S_{m+1}(f)(g(m+1)-\bar{g}(m))
\end{gathered}
$$

Adding this to (8), we finally obtain

$$
f(k)-\bar{f}(m)=C_{m} S_{m+1}(f)(g(k)-\bar{g}(m))
$$

for $m<k \leqslant n$. The proof of Proposition 2 is accomplished.
The next proposition deals with the hitting probabilities.

Proposition 3: Starting from $m<n$, the probability $w(m, n)$ of bitting $n$ before returning to $m$ is given by

$$
w(m, n)^{-1}=\sigma(m)\left(\frac{1}{\sigma(n)}+\sum_{k=m+1}^{n} \frac{r_{k}}{s_{k-1} \sigma(k)}\right)
$$

Proof: Let $u(k)$ be the probability of hitting $n$ before entering $m$, as a function of the starting point $k$; obviously $w(m, n)=p_{m} u(m+1)$.

Since $u$ is harmonic in the interval ] $m, n$ [ and $\bar{u}(m)=0$, Proposition 2 implies

$$
u(k)=C(g(k)-\bar{g}(m))
$$

Observing that $u(n)=C(g(n)-\bar{g}(m))=1$, we find

$$
u(k)=\frac{g(k)-\bar{g}(m)}{g(n)-\bar{g}(m)}
$$

whence

$$
w(m, n)=\frac{p_{m}(g(m+1)-\bar{g}(m))}{g(n)-\bar{g}(m)}
$$

Thanks to (9), $\bar{g}(m)$ is easily computed:

$$
\bar{g}(m)=g(m)-(g(m)-\bar{g}(m))=g(m)-\frac{S_{m}(g)}{s_{m}}=g(m)-\frac{1}{\sigma(m)}
$$

Taking Lemma 1 into account, Proposition 3 is proved.
Corollary 1: The chain is transient if and only if the function $g$ is bounded.
Proof: The sequence $w(0, n)$ tends to the probability of no return to 0 ; so transience is equivalent to boundedness of $w(0, n)^{-1}$. On the other hand, $w(0, n)^{-1}=$ $=g(n)$ by Proposition 3 .

Recurrent chains do have an invariant measure. But also transient chains may have one, as shown in the following simple example: take $r_{n}=1$ for every $n, p_{n}=$ $=\sqrt{(n+1) /(n+2)}$ and verify that $\sigma(n)=\sqrt{n+1}$ is indeed an invariant measure, though the chain is transient.

Positive recurrence, in turn, is easily characterized, being equivalent to $\sigma(N)=$ $=\sum_{k \geqslant 0} \sigma(k)<+\infty$.

In the positive recurrent case, the invariant measure $\sigma$ can be normalized to the (unique) invariant probability distribution $\pi$; the quantity $1 / \sigma(N)$ equals the expected return time to 0 .

## 4. - Mean passage times

This section deals with the calculation of the mean passage times between states. Let $t(m, n)$ denote the expected time to hit the state $n$ starting from $m$. The results we prove are summarized in two propositions.

Proposition: 4: For $m<n$ the expected time to reach $n$ starting from $m$ is given by $t(m, n)=t(0, n)-t(0, m)$, where

$$
t(0, n)=\tilde{\sigma}(n)+\sum_{k=1}^{n} \frac{r_{k} \tilde{\sigma}(k)}{s_{k-1}} \quad \text { with } \quad \tilde{\sigma}(k)=\frac{1}{\sigma(k)} \sum_{j=0}^{k-1} \sigma(j)
$$

for any $n>0$, and $\sigma$ defined as in (2).
Proof: The chain being skip-free to the right, the additivity $t(0, m)+t(m, n)=$ $=t(0, n)$ holds for $0 \leqslant m \leqslant n$, which allows us to treat $t(0, n)$ for $n>0$ as the sum $t(0, n)=t(0)+\ldots+t(n-1)$ of one stair passage times $t(k)=t(k, k+1)$.

A simple first step analysis yields

$$
\begin{aligned}
t(k)=t(k, k+1)=1+ & \sum_{i} p(k, i) t(i, k+1)=1+\left(1-p_{k}\right) \sum_{i=0}^{k} \frac{r_{i}}{s_{k}} t(i, k+1)= \\
& =1+\frac{1-p_{k}}{s_{k}} \sum_{i=0}^{k} r_{i} t(i, k+1) .
\end{aligned}
$$

In particular, we immediately get $t(0)=1+\left(1-p_{0}\right) t(0), t(0)=1 / p_{0}$.
As $\tilde{\sigma}(1)=1 / \sigma(1)=1 /\left(s_{1} p_{0}\right)$, the claim of Proposition 4 is true for $n=1$.
The equations above are best written putting $A_{k}=\sum_{i=0}^{k} r_{i} t(i, k+1)$ and read

$$
s_{k} t(k)=s_{k}+\left(1-p_{k}\right) A_{k} .
$$

Remarking that

$$
A_{k}=\sum_{i=0}^{n} \sum_{j=i}^{k} t(j)=\sum_{j=0}^{k} \sum_{i=0}^{j} r_{i} t(j)=\sum_{j=0}^{k} s_{j} t(j) \quad A_{k}-A_{k-1}=s_{k} t(k)
$$

the recurrence equation becomes

$$
p_{k} A_{k}=s_{k}+A_{k-1}
$$

A very simple relation is obtained by multiplying both sides by $p_{0} \ldots p_{k-1}$ :

$$
p_{0} \ldots p_{k} A_{k}=p_{0} \ldots p_{k-1} A_{k-1}+\sigma(k)
$$

leading to

$$
p_{0} \cdots p_{k} A_{k}=p_{0} A_{0}+\sigma(1)+\ldots+\sigma(k)=\sigma[0, k] .
$$

Adding up the one stair passage times gives, for $n>1$

$$
\begin{gathered}
t(0, n)=\sum_{k=0}^{n-1} t(k)=A_{0}+\sum_{k=1}^{n-1} \frac{A_{k}-A_{k-1}}{s_{k}}= \\
=A_{0}+\sum_{k=1}^{n-1}\left(\frac{A_{k}}{s_{k}}-\frac{A_{k-1}}{s_{k-1}}\right)+\sum_{k=1}^{n-1} A_{k-1}\left(\frac{1}{s_{k-1}}-\frac{1}{s_{k}}\right)= \\
=\frac{A_{n-1}}{s_{n-1}}+\sum_{k=1}^{n-1} \frac{r_{k} A_{k-1}}{s_{k} s_{k-1}}=\frac{A_{n-1}}{s_{n}}+\sum_{k=1}^{n} \frac{r_{k} A_{k-1}}{s_{k} s_{k-1}} .
\end{gathered}
$$

The proof is complete, because

$$
\frac{A_{k-1}}{s_{k}}=\frac{p_{0} \ldots p_{k-1}}{\sigma(k)} A_{k-1}=\frac{\sigma[0, k[ }{\sigma(k)}=\tilde{\sigma}(k) .
$$

Since $t(m, n)$ is finite for $m<n$, the passage times $t(m, n)$ with $m>n$ have finite expectation if and only if the chain is positive recurrent (see [1], Th.1, p. 62), in which case the following proposition holds.

Proposition 5: In the positive recurrent case, let $\pi$ be the invariant probability distribution and $m<n$. The expected time to reach the state $m$ starting from $n$ is given by

$$
t(n, m)=t(n, 0)-t(m, 0)+\frac{1}{\pi(m)}
$$

where

$$
t(n, 0)=\mu(n)+\sum_{k=1}^{n} \frac{r_{k} \mu(k)}{s_{k-1}} \quad \text { with } \mu(k)=\frac{1}{\sigma(k)} \sum_{j \geqslant k} \sigma(j)
$$

for any $n>0$ and $\sigma$ defined as in (2).

Proof: Recall that $\pi$ is related to $\sigma$ by the relation $\pi(n)=\sigma(n) / \sigma(N)$.
According to a well known result about positive recurrent Markov chains (see [1], Cor. 1, p. 65), the mean commute time $\operatorname{comm}(m, n)=t(m, n)+t(n, m)$ is given by

$$
\begin{equation*}
\operatorname{comm}(m, n)=\frac{\sigma(N)}{\sigma(m) w(m, n)}=\frac{1}{\pi(n)}+\sum_{k=m+1}^{n} \frac{r_{k}}{s_{k-1} \pi(k)} . \tag{10}
\end{equation*}
$$

For $m=0$ one obtains

$$
t(n, 0)=\operatorname{comm}(0, n)-t(0, n)=\sigma(N)\left(\frac{1}{\sigma(n)}+\sum_{k=1}^{n} \frac{r_{k}}{s_{k-1} \sigma(k)}\right)-t(0, n)
$$

Observing that $\tilde{\sigma}(k)+\mu(k)=\sigma(N) / \sigma(k)$, the second half of Proposition 5 is proved.

For the first half, we have after (10)

$$
\operatorname{comm}(0, m)+\operatorname{comm}(m, n)-\operatorname{comm}(0, n)=\frac{1}{\pi(m)}
$$

As $t(0, m)+t(m, n)-t(0, n)$ vanishes, the conclusion follows.
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