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Second Order Elliptic Equations with Discontinuous Coefficients in Irregular Domains (**)

ABSTRACT. — In this paper we study the Dirichlet problem for a class of linear second order elliptic equations in non divergence form in an open subset Ω of R^n with $\partial\Omega$ irregular and coefficients discontinuous in Ω and singular on a subset of $\partial\Omega$.

Equazioni ellittiche del secondo ordine con coefficienti discontinui in domini non regolari

SUNTO. — In questo lavoro si studia il problema di Dirichlet per una classe di equazioni ellittiche del secondo ordine in forma non variazionale in un aperto Ω di R^n a coefficienti discontinui in Ω e singolari su un sottoinsieme di $\partial\Omega$.

INTRODUCTION

Let Ω be an open subset of R^n, $n \geqslant 2$.

Let us consider a weight function ϱ in the class $\mathcal{A}(\Omega)$ (see Section 1 for the definition) and denote by S_ϱ the subset of $\partial\Omega$ where ϱ goes to zero. We observe (see (1.1)) that, if $S_\varrho \neq \emptyset$, then ϱ is related to the distance function to S_ϱ.

Let L be the uniformly elliptic differential operator

$$Lu = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i\, u_{x_i} + a\, u \tag{1}$$

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with coefficients $a_{ij} = a_{ji} \in L^{\infty}(\Omega)$, $i, j = 1, \ldots, n$. We study the following Dirichlet problem

$$(2) \qquad u \in W_s^2(\Omega) \cap \overset{\circ}{W}{}_{s-1}^1(\Omega), \qquad Lu + \lambda\beta u = f, \qquad f \in L_s^2(\Omega),$$

where β is a positive function, λ, $s \in R$, $W_s^2(\Omega)$, $\overset{\circ}{W}{}_{s-1}^1(\Omega)$ and $L_s^2(\Omega)$ are some weighted Sobolev spaces (see Section 1 for definitions) and the weight functions are suitable powers of ϱ.

In a recent paper (see [9]) problem (2) has been studied under the following hypotheses on coefficients a_{ij} of L:

$$(3) \qquad (a_{ij})_{x_k} \in L^q_{loc}(\overline{\Omega} \setminus S_\varrho), \qquad \sup_{x \in \Omega} \|(a_{ij})_{x_k}\|_{L^q(\Omega \cap B(x, \varrho(x)))} < +\infty, \qquad i, j, k = 1, \ldots, n,$$

where $q > 2$ if $n = 2$, $q = n$ if $n \geqslant 3$ and $B(x, \varrho(x))$ is the open ball centered at x of radius $\varrho(x)$. In such a paper similar hypotheses are made on a_i, a and β. If $\partial\Omega$ is singular, further conditions on ϱ, a_{ij}, a_i, a, β and λ are given in order to problem (2) be uniquely solvable.

We observe that if, in particular, ϱ is positive constant and Ω is a bounded open subset of R^n, $n \geqslant 3$, condition (3) becomes

$$(a_{ij})_{x_k} \in L^n(\Omega), \qquad i, j, k = 1, \ldots, n,$$

that is the classical hypothesis of C. Miranda (see [16]).

This means that the result in [9] extends that one contained in [16] to the case Ω unbounded open set with singular boundary and $(a_{ij})_{x_k}$, a_i and a singular functions near to S_ϱ.

Further generalizations of Miranda's result can be found in literature. For example in [2], [10], [11], [12] the coefficients a_{ij} of the operator L belong to wider functional spaces and in [19] Ω is an unbounded open set.

In this paper we study problem (2) with coefficients a_{ij} which satisfy a condition more general than (3).

In fact we suppose (see conditon α) in Section 3) that a_{ij} do not necessarily satisfy the last requirement in (3) and can be split in the following way

$$(4) \qquad a_{ij} = \alpha_{ij} + \gamma_{ij}, \qquad i, j = 1, \ldots, n,$$

where α_{ij} are bounded simmetric functions which verify an uniformly elliptic condition and hypothesis (3), γ_{ij} are sufficiently small near to S_ϱ and at infinity.

In Section 6 we give an example of a function with a behaviour similar to γ_{ij}, but which does not satisfy the last requirement in (3).

We emphasize that in these weaker hypotheses on a_{ij} it is not possible to use, as in [9], some results of variational type contained in [5].

In this paper we are able to get a priori bounds for solutions of problem (2) (see sections 3 and 4) as follows. On the part of Ω close to S_ϱ or to infinity, we suitably

adapt to α_{ij} in (4) some results contained in [9] and make use of the smallness hypotheses on γ_{ij}. On the rest of Ω we use some imbedding and compactness theorems contained in [8].

From a priori bounds we deduce some existence and uniqueness results in Section 5.

1. - Notations and function classes

Let E be a Lebesgue measurable subset of R^n and $\Sigma(E)$ the σ-algebra of Lebesgue measurable subsets of E.

For any $A \in \Sigma(E)$, $|A|$ is the Lebesgue measure of A, $\mathcal{D}(A)$ is the class of restrictions to A of functions $\zeta \in C_o^\infty(R^n)$ such that supp $\zeta \cap \overline{A} \subset A$, $L^p_{loc}(A)$ is the class of functions $f : A \to C$ such that $\zeta f \in L^p(A)$ for any $\zeta \in \mathcal{D}(A)$. We set

$$|f|_{p,A} = \|f\|_{L^p(A)}, \qquad 1 \leq p \leq +\infty .$$

We put

$$B(x, r) = \{y \in R^n : |y - x| < r\}, \qquad B_r = B(0, r) \ \forall x \in R^n, \ \forall r \in R_+ .$$

Let Ω be an open subset of R^n. We set

$$\Omega(x, r) = \Omega \cap B(x, r) \qquad \forall x \in \Omega, \ \forall r \in R_+ .$$

We call $\mathcal{A}(\Omega)$ the class of functions $\varrho : \Omega \to R_+$ satisfying

$$\sup_{\substack{x, y \in \Omega \\ |x-y| < \varrho(y)}} \left| \log \frac{\varrho(x)}{\varrho(y)} \right| < +\infty .$$

It is easy to see that $\varrho \in \mathcal{A}(\Omega)$ if and only if $\varrho : \Omega \to R_+$ and there exists a constant $\gamma \in R_+$ such that

$$\gamma^{-1} \varrho(y) \leq \varrho(x) \leq \gamma \varrho(y) \qquad \forall x \in \Omega, \ \forall y \in \Omega(x, \varrho(x)) .$$

Some examples of functions $\varrho \in \mathcal{A}(\Omega)$ are given in [22] where it is also observed that $\mathcal{A}(\Omega)$ contains the class of positive Lipschitz functions with Lipschitz constant less than 1.

For any $\varrho \in \mathcal{A}(\Omega)$ we set

$$S_\varrho = \{y \in \partial\Omega : \lim_{x \to y} \varrho(x) = 0\} .$$

As shown in [8], S_ϱ is a closed subset in $\partial\Omega$. Moreover if $S_\varrho \neq \emptyset$ it results (see [22])

$$\varrho(x) \leq \operatorname{dist}(x, S_\varrho) \qquad \forall x \in \Omega . \tag{1.1}$$

It is well-known (see, *e.g.*, Theor. 2, Chap. VI in [18] and Lemma 3.6.1 in [24]) that

there exist $\alpha \in C^{\infty}(\Omega) \cap C^{0,1}(\overline{\Omega})$, $c_1, c_2 \in R_+$ such that

$$c_1 \operatorname{dist}(x, S_\varrho) \leqslant \alpha(x) \leqslant c_2 \operatorname{dist}(x, S_\varrho) \qquad \forall x \in \Omega .$$

We put

$$\Omega_k = \{x \in \Omega : |x| < k,\ \alpha(x) > 1/k\} \qquad \forall k \in N .$$

If $f \in \mathcal{D}(\overline{R}_+)$ is a fixed function such that

$$0 \leqslant f \leqslant 1, \qquad f(t) = 1 \text{ if } t \leqslant 1/2, \qquad f(t) = 0 \text{ if } t \geqslant 1 ,$$

we define the functions

$$\psi_k : x \in \overline{\Omega} \to (1 - f(k\alpha(x)))\, f(|x|/2k) \qquad \forall k \in N .$$

We remark that, for any $k \in N$, ψ_k belongs to $\mathcal{D}(\overline{\Omega} \setminus S_\varrho)$ and the following conditions hold

$$0 \leqslant \psi_k \leqslant 1 , \qquad \psi_{k|\overline{\Omega}_k} = 1, \qquad \operatorname{supp} \psi_k \subset \overline{\Omega}_{2k}.$$

Let $\mathcal{A}_o(\Omega)$ be the class of measurable functions $\varrho \in \mathcal{A}(\Omega)$. If $\varrho \in \mathcal{A}_o(\Omega)$, then (see [8])

$$\varrho \in L^{\infty}_{loc}(\overline{\Omega}), \qquad \varrho^{-1} \in L^{\infty}_{loc}(\overline{\Omega} \setminus S_\varrho) . \tag{1.2}$$

Further examples and properties of functions of $\mathcal{A}(\Omega)$ can be found in [22], [20], [4], [8], [7].

If $r \in N$, $1 \leqslant p \leqslant +\infty$, $s \in R$ and $\varrho \in \mathcal{A}_o(\Omega)$, we denote by $W^{r,p}_s(\Omega)$ the space of distributions u on Ω such that $\varrho^{s+|\alpha|-r} \partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leqslant r$ endowed with the norm

$$\|u\|_{W^{r,p}_s(\Omega)} = \sum_{|\alpha| \leqslant r} |\varrho^{s+|\alpha|-r} \partial^\alpha u|_{p,\Omega} . \tag{1.3}$$

Moreover we denote by $\overset{\circ}{W}{}^{r,p}_s(\Omega)$ the closure of $C^{\infty}_o(\Omega)$ in $W^{r,p}_s(\Omega)$. We put

$$W^{0,p}_s(\Omega) = L^p_s(\Omega), \qquad W^{r,2}_s(\Omega) = W^r_s(\Omega), \qquad \overset{\circ}{W}{}^{r,2}_s(\Omega) = \overset{\circ}{W}{}^r_s(\Omega) .$$

For some properties of weighted Sobolev spaces, where the weight functions are powers of a function $\varrho \in \mathcal{A}(\Omega)$, see, *e.g.*, [3], [14], [17], [15], [21], [4], [8], [7].

If $1 \leqslant p < +\infty$, $s \in R$ and $\varrho \in \mathcal{A}_o(\Omega)$, we set

$$\Omega(x) = \Omega(x, \varrho(x)) \qquad \forall x \in \Omega , \tag{1.4}$$

and consider the spaces $K^p_s(\Omega)$, $\widetilde{K}^p_s(\Omega)$, $\overset{\circ}{K}{}^p_s(\Omega)$ defined in [4] in correspondence of the family of open sets defined by (1.4). Let us recall that:

$K_s^p(\Omega)$ is the space of functions $g \in L^p_{loc}(\overline{\Omega} \setminus S_\varrho)$ such that

$$\|g\|_{K_s^p(\Omega)} = \sup_{x \in \Omega} (\varrho^{s-n/p}(x)\, |g|_{p,\,\Omega(x)}) < +\infty\,, \tag{1.5}$$

endowed with the norm defined by (1.5),

$\widetilde{K}_s^p(\Omega)$ is the closure of $L_s^\infty(\Omega)$ in $K_s^p(\Omega)$,

$\overset{\circ}{K}{}_s^p(\Omega)$ is the closure of $C_o^\infty(\Omega)$ in $K_s^p(\Omega)$.

For some properties of the spaces $K_s^p(\Omega)$, $\widetilde{K}_s^p(\Omega)$ and $\overset{\circ}{K}{}_s^p(\Omega)$ we refer to [4], [8], [7].

Remark 1.1: Let us fix $\varrho \in \mathcal{A}_o(\Omega)$, $1 \leqslant p < +\infty$, $s \in R$. We observe that if $g \in L^p_{loc}(\overline{\Omega} \setminus S_\varrho)$, then, for any $\zeta \in \mathcal{D}(\overline{\Omega} \setminus S_\varrho)$, we have $\zeta g \in \overset{\circ}{K}{}_s^p(\Omega)$.

Now, if we fix $\zeta \in \mathcal{D}(\overline{\Omega} \setminus S_\varrho)$, then $\zeta g \in L^p(\Omega)$ and so there exists a sequence of functions $(g_n)_{n \in N}$, $g_n \in C_o^\infty(\Omega)$, such that

$$g_n \to \zeta g \quad \text{in } L^p(\Omega)\,.$$

For every fixed $\psi \in \mathcal{D}(\overline{\Omega} \setminus S_\varrho)$ with $\psi_{|\operatorname{supp}\zeta} = 1$, evidently we have

$$\psi g_n \to \zeta g \quad \text{in } L^p(\Omega)\,. \tag{1.6}$$

From (1.2) and (1.6) we obtain that $\zeta g \in K_s^p(\Omega)$ and

$$\psi g_n \to \zeta g \quad \text{in } K_s^p(\Omega)\,,$$

so $\zeta g \in \overset{\circ}{K}{}_s^p(\Omega)$. ■

2. - A preliminar lemma

Let us suppose $n \geqslant 2$ and fix $\varrho \in \mathcal{A}(\Omega) \cap L^\infty(\Omega)$ such that $S = S_\varrho \neq \emptyset$.

Set

$$B_+ = \{x \in B_1 \colon x_n > 0\}\,, \qquad B_o = \{x \in B_1 \colon x_n = 0\}\,,$$

we suppose that there exists an open subset Ω^* of R^n such that

$h_1)$ there are a $d \in R_+$, an open cover $\{U_i\}_{i \in I}$ of $\partial\Omega^*$ and, for any $i \in I$, a C^2-diffeomorphism $\psi_i \colon \overline{U}_i \to \overline{B}_1$ such that

$$\psi_i(U_i \cap \Omega^*) = B_+\,, \qquad \psi_i(U_i \cap \partial\Omega^*) = B_o; \tag{2.1}$$

(2.2) the components of ψ_i and ψ_i^{-1} and of their first and second derivatives are bounded by a constant independent of i;

(2.3) for any $x \in \Omega^*_d$ there exists an $i \in I$ such that $B(x, d) \subset U_i$ and, for any $x \in \Omega^* \setminus \Omega^*_d$, $B(x, d) \subset \Omega^*$, where $\Omega^*_d = \{x \in \Omega^* \colon \operatorname{dist}(x, \partial\Omega^*) < d\}$;

(2.4) $$\Omega \subset \Omega^*, \qquad \partial\Omega \setminus S \subset \partial\Omega^*.$$

REMARK 2.1: It is easy to prove that (2.1), (2.2) and (2.3) hold when Ω^* has the uniform C^2-regularity property defined in Section 4.6 in [1]. ■

Let us consider in Ω the second order linear differential operator

(2.5) $$Lu = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + au$$

with the following conditions on the coefficients:

h_2) $$a_{ij} = a_{ji}, \qquad a_{ij} \in L^\infty(\Omega), \qquad i, j = 1, \dots, n,$$

$$\sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geqslant \nu |\xi|^2 \qquad \forall \xi \in R^n, \text{ a.e. in } \Omega,$$

where ν is a positive constant independent of x and ξ;

h_3) $$(a_{ij})_{x_k} \in L^q_{loc}(\overline{\Omega} \setminus S), \qquad i, j, k = 1, \dots n,$$

where

$$q > 2 \quad \text{if} \quad n = 2, \qquad q = n \quad \text{if} \quad n \geqslant 3;$$

h_4) $$a_i \in \widetilde{K}_1^q(\Omega), \qquad i = 1, \dots, n, \qquad a \in \widetilde{K}_2^t(\Omega),$$

where

$$t = 2 \quad \text{if} \quad 2 \leqslant n < 4, \qquad t > 2 \quad \text{if} \quad n = 4, \qquad t = \frac{n}{2} \quad \text{if} \quad n > 4.$$

In the sequel

$$u_x = \left(\sum_{i=1}^{n} u_{x_i}^2 \right)^{1/2}, \qquad u_{xx} = \left(\sum_{i,j=1}^{n} u_{x_i x_j}^2 \right)^{1/2}.$$

We consider a function $\beta : \Omega \to R_+$ such that the following hypothesis holds:

h_5) $$\beta \in \widetilde{K}_2^t(\Omega), \qquad \exists \delta \in \widetilde{K}_1^q(\Omega) \quad \text{such that } \beta_x \leqslant \beta\delta.$$

An example of function β which satisfies the hypothesis h_5) can be given in the following way. From Remark 3.1 in [8] and Theorem 3.2 in [22], there exist $\sigma \in \mathcal{A}(\Omega) \cap C^\infty(\Omega) \cap C^{0,1}(\overline{\Omega})$, $c_0, c_1, c_2 \in R_+$ such that

(2.6) $$c_1 \varrho(x) \leqslant \sigma(x) \leqslant c_2 \varrho(x) \qquad \forall x \in \Omega,$$

(2.7) $$\sigma_x \leqslant c_0 \qquad \forall x \in \Omega.$$

So, the function

$$\beta = \frac{1}{\sigma^2} \tag{2.8}$$

satisfies the hypothesis h_5).

Indeed, $\beta \in L_2^\infty(\Omega)$ and then $\beta \in \tilde{K}_2^t(\Omega)$. If we put $\delta = \frac{2\sigma_x}{\sigma}$, $\delta \in L_1^\infty(\Omega)$ and so $\delta \in \tilde{K}_1^q(\Omega)$. Furthermore we have $\beta_x = \beta\delta$.

Another example of function which satisfies the hypothesis h_5) is the function

$$\beta(x) = \frac{1}{(1 + |x|^2)^\tau}, \qquad x \in \Omega, \ \tau > 0. \tag{2.9}$$

Indeed, $\beta \in L_2^\infty(\Omega)$ and then $\beta \in \tilde{K}_2^t(\Omega)$. If we set $\delta = \frac{2\tau|x|}{1 + |x|^2}$, $\delta \in L_1^\infty(\Omega)$ and then $\delta \in \tilde{K}_1^q(\Omega)$. Moreover we have $\beta_x = \beta\delta$.

We also observe that, from Lemma 2.1 in [8], $\beta \in \overset{\circ}{K}{}_2^t(\Omega)$.

Remark 2.2: One can show that under hypotheses h_1)-h_5), from Remark 3.1 and Theorem 3.1 in [8] (see also [7]), it follows that for any $s, \lambda \in R$ the operator

$$u \in W_s^2(\Omega) \to Lu + \lambda\beta u \in L_s^2(\Omega)$$

is bounded. ■

We put

$$L_o u = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j},$$

$$\tilde{g} = \sum_{i,j=1}^{n} ((a_{ij})_x + |a_{ij}|\delta),$$

where δ is the function defined in h_5).

Let us fix a bounded open subset V of R^n such that

$$V \subset \Omega \quad \text{or} \quad V \cap \partial\Omega \neq \emptyset \quad \text{and} \quad V \subset U_i \setminus S \quad \text{for some } i \in I. \tag{2.10}$$

Lemma 2.1: *If the hypotheses h_1)-h_3) and h_5) hold, then, for any $\lambda \geqslant 0$ and for any function v satisfying*

$$v \in W^2(\Omega) \cap \overset{\circ}{W}{}^1(\Omega), \qquad \operatorname{supp} v \subset V,$$

we have the bound

$$|v_{xx}|_{2,\Omega} \leqslant c(|L_o v + \lambda\beta v|_{2,\Omega} + |\tilde{g} v_x|_{2,\Omega} + |v|_{2,\Omega}), \tag{2.11}$$

where $c \in R_+$ *is independent of* v *and* λ.

PROOF: Proceeding as in the proof of Lemma 5.1 in [7] (see also Section 7 in [6]), we obtain

$$|v_{xx}|^2_{2,\Omega} \leqslant c_1(|L_o v|^2_{2,\Omega} + |g v_x|^2_{2,\Omega} + |v|^2_{2,\Omega}), \tag{2.12}$$

where $g = \sum_{i,j=1}^{n} (a_{ij})_x$ and $c_1 \in R_+$ is independent of v and λ.

By means of known techniques (see, *e.g.*, [13], [6], [7]) we have

$$\int_\Omega (L_o v + \lambda\beta v)^2\, dx \geqslant \int_\Omega (L_o v)^2\, dx + \lambda^2 \int_\Omega \beta^2 v^2\, dx + \tag{2.13}$$

$$+ 2\lambda\nu \int_\Omega \beta v_x^2\, dx - 2\lambda \int_\Omega \beta\tilde{g}\, |v| v_x\, dx\,.$$

Moreover we have

$$\int_\Omega \beta\tilde{g}\, |v| v_x\, dx \leqslant \frac{\lambda}{2} \int_\Omega \beta^2 v^2\, dx + \frac{1}{2\lambda} \int_\Omega |\tilde{g} v_x|^2\, dx\,. \tag{2.14}$$

From (2.12), (2.13) and (2.14) we deduce the result. ■

3. - A PRIORI BOUNDS

Let us suppose that a_{ij} satisfy the following further condition:

α) there exist functions α_{ij} such that

$$\alpha_{ij} = \alpha_{ji} \in L^\infty(\Omega), \qquad (\alpha_{ij})_{x_k} \in \tilde{K}^q_1(\Omega), \qquad i, j, k = 1, \ldots, n\,,$$

$$\sum_{i,j=1}^{n} \alpha_{ij}\xi_i\xi_j \geqslant \nu_o |\xi|^2 \qquad \forall \xi \in R^n, \qquad \text{a.e. in } \Omega\,,$$

where ν_o is a positive constant independent of x and ξ and for any $\varepsilon \in R_+$ there exists $k_\varepsilon \in N$ such that:

$$\operatorname*{ess\,sup}_{\Omega\setminus\Omega_{k_\varepsilon}} \sum_{i,j=1}^{n} |\alpha_{ij} - a_{ij}| < \varepsilon\,,$$

where Ω_k, $k \in N$, are the sets defined in Section 1.

LEMMA 3.1: *If the conditions h_1)-h_5), α) hold and λ_1 is a real number, then there exists a constant $c \in R_+$ such that*

$$|v_{xx}|_{2,\Omega} \leq c(|Lv + \lambda\beta v|_{2,\Omega} + |\varrho^{-1} v_x|_{2,\Omega} + |\varrho^{-2} v|_{2,\Omega}), \tag{3.1}$$

for any $\lambda \in [\lambda_1, +\infty[$ and for any function v satisfying

$$v \in W^2(\Omega) \cap \overset{\circ}{W}{}^1(\Omega), \qquad \operatorname{supp} v \subset V.$$

PROOF: Let us suppose $\lambda \geq 0$ and consider the functions ψ_k, $k \in N$, introduced in Section 1.

Applying Lemma 2.1 in [9] to the function $(1-\psi_k)v$ in the case $L = -\sum_{i,j=1}^{n} \alpha_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$, we get

$$\begin{aligned} &|((1-\psi_k)\, v)_{xx}|_{2,\Omega} \leq \\ &\leq c_1 \Bigg(\Bigg| -\sum_{i,j=1}^{n} \alpha_{ij} ((1-\psi_k)\, v)_{x_i x_j} + \lambda\beta(1-\psi_k)\, v \Bigg|_{2,\Omega} + \\ &+ |\varrho^{-1}((1-\psi_k)\, v)_x|_{2,\Omega} + |\varrho^{-2}(1-\psi_k)\, v|_{2,\Omega} \Bigg), \end{aligned} \tag{3.2}$$

where $c_1 \in R_+$ is independent of λ, v and k.

Moreover we have

$$\begin{aligned} &\Bigg| -\sum_{i,j=1}^{n} \alpha_{ij} ((1-\psi_k)\, v)_{x_i x_j} + \lambda\beta(1-\psi_k)\, v \Bigg|_{2,\Omega} \leq \\ &\leq |L_o((1-\psi_k)\, v) + \lambda\beta(1-\psi_k)\, v|_{2,\Omega} + \\ &+ \Bigg| \sum_{i,j=1}^{n} (\alpha_{ij} - a_{ij}) ((1-\psi_k)\, v)_{x_i x_j} \Bigg|_{2,\Omega} \leq \\ &\leq c_2 \Bigg(|(1-\psi_k)(L_o v + \lambda\beta v)|_{2,\Omega} + |(1-\psi_k)_x v_x|_{2,\Omega} + \\ &+ |(1-\psi_k)_{xx} v|_{2,\Omega} + \Bigg| ((1-\psi_k)\, v)_{xx} \sum_{i,j=1}^{n} (\alpha_{ij} - a_{ij}) \Bigg|_{2,\Omega} \Bigg), \end{aligned} \tag{3.3}$$

where $c_2 \in R_+$ is independent of λ, v and k.

From (3.2), (3.3) and α) we easily deduce that there exists $k_o \in N$ such that

$$|((1-\psi_{k_o})\, v)_{xx}|_{2,\Omega} \leq c_3(|L_o v + \lambda\beta v|_{2,\Omega} + |\varrho^{-1} v_x|_{2,\Omega} + |\varrho^{-2} v|_{2,\Omega}), \tag{3.4}$$

where $c_3 \in R_+$ is independent of λ and v. From now on, we denote by c_j, $j = 4, 5, 6$, positive constants independent of λ and v.

Applying Lemma 2.1 to the function $\psi_{k_o} v$, we get

$$(3.5) \qquad |(\psi_{k_o} v)_{xx}|_{2,\Omega} \leq c_4 (|L_o(\psi_{k_o} v) + \lambda\beta\psi_{k_o} v|_{2,\Omega} + \\ + |\tilde{g}(\psi_{k_o} v)_x|_{2,\Omega} + |\psi_{k_o} v|_{2,\Omega}) .$$

Let $\chi \in \mathcal{D}(\overline{\Omega} \setminus S)$ be a function such that $\chi_{|\operatorname{supp}\psi_{k_o}} = 1$. Since the function $\chi\tilde{g}$ belongs to the space $\tilde{K}_1^q(\Omega)$ (see Remark 1.1), from the estimate (3.6) in [8] we deduce

$$(3.6) \qquad |\chi\tilde{g}(\psi_{k_o} v)_x|_{2,\Omega} \leq \varepsilon \|(\psi_{k_o} v)_x\|_{W_o^1(\Omega)} + c_1(\varepsilon)\|(\psi_{k_o} v)_x\|_{L^2_{-1}(\Omega)} \leq \\ \leq \varepsilon |(\psi_{k_o} v)_{xx}|_{2,\Omega} + c_2(\varepsilon)|\varrho^{-1}(\psi_{k_o} v)_x|_{2,\Omega}.$$

Then from (3.5) and (3.6) we have

$$(3.7) \qquad |(\psi_{k_o} v)_{xx}|_{2,\Omega} \leq c_5 (|L_o v + \lambda\beta v|_{2,\Omega} + |\varrho^{-1} v_x|_{2,\Omega} + |\varrho^{-2} v|_{2,\Omega}) .$$

From (3.4) and (3.7) we get

$$(3.8) \qquad |v_{xx}|_{2,\Omega} \leq c_6 (|L_o v + \lambda\beta v|_{2,\Omega} + |\varrho^{-1} v_x|_{2,\Omega} + |\varrho^{-2} v|_{2,\Omega}) .$$

If $\lambda_1 < 0$, we fix $\lambda \in [\lambda_1, 0[$. From (3.6) in [8] we get

$$(3.9) \qquad |\lambda\beta v|_{2,\Omega} \leq |\lambda_1| (\varepsilon |v_{xx}|_{2,\Omega} + c_3(\varepsilon)|\varrho^{-1} v_x|_{2,\Omega} + c_4(\varepsilon)|\varrho^{-2} v|_{2,\Omega}) .$$

From (3.8) for $\lambda = 0$ and (3.9) we deduce (3.1) with L_o instead of L.

Finally, again by (3.6) in [8] we can obtain the result. ■

We denote by $W^2_{loc}(\overline{\Omega} \setminus S)$ (respectively $\overset{\circ}{W}{}^1_{loc}(\overline{\Omega} \setminus S)$) the space of all functions $u : \Omega \to R$ such that $\zeta u \in W^2(\Omega)$ (respectively $\overset{\circ}{W}{}^1(\Omega)$) for any $\zeta \in \mathcal{D}(\overline{\Omega} \setminus S)$.

THEOREM 3.1: *In the same hypotheses of Lemma* 3.1, *for any* $u : \Omega \to R$ *such that*

$$(3.10) \qquad \begin{aligned} &u \in W^2_{loc}(\overline{\Omega} \setminus S) \cap \overset{\circ}{W}{}^1_{loc}(\overline{\Omega} \setminus S) \cap L^2_{s-2}(\Omega) , \\ &Lu + \lambda'\beta u \in L^2_s(\Omega), \qquad \textit{for some } s, \lambda' \in R , \end{aligned}$$

we have $u \in W^2_s(\Omega)$. *Moreover, for any* $\lambda_1 \in R$, *we have the bound*

$$(3.11) \qquad \|u\|_{W^2_s(\Omega)} \leq c(\|Lu + \lambda\beta u\|_{L^2_s(\Omega)} + \|u\|_{L^2_{s-2}(\Omega)}) \qquad \forall \lambda \geq \lambda_1 ,$$

where the constant $c \in R_+$ *is independent of* u *and* λ.

PROOF: The result follows from Lemma 3.1 applying arguments similar to those one used in [9] in order to get Theorem 2.1 from Lemma 2.1. ■

COROLLARY 3.1: *In the same hypotheses of Theorem* 3.1 *and if*

$$\exists \mu \in R_+ : \beta \geqslant \mu \varrho^{-2} \quad \textit{a.e. in } \Omega, \tag{3.12}$$

then for any $s \in R$ *there exist* $c, \lambda_o \in R_+$ *such that*

$$\|u\|_{W_s^2(\Omega)} \leqslant c \|Lu + \lambda \beta u\|_{L_s^2(\Omega)} \tag{3.13}$$

$$\forall u \in W_s^2(\Omega) \cap \overset{\circ}{W}{}_{s-1}^1(\Omega), \qquad \forall \lambda \geqslant \lambda_o.$$

PROOF: By means of known techniques (see, e.g., [23], [19]) and Theorem 3.1, we have

$$\begin{aligned} \lambda \|u\|_{L_{s-2}^2(\Omega)} &\leqslant \mu^{-1} \|\lambda \beta u\|_{L_s^2(\Omega)} \leqslant \mu^{-1} \|Lu + \lambda \beta u\|_{L_s^2(\Omega)} + \\ &+ \mu^{-1} \|Lu\|_{L_s^2(\Omega)} \leqslant \mu^{-1} \|Lu + \lambda \beta u\|_{L_s^2(\Omega)} + c_1 \|u\|_{W_s^2(\Omega)} \leqslant \\ &\leqslant c_2 \|Lu + \lambda \beta u\|_{L_s^2(\Omega)} + c_3 \|u\|_{L_{s-2}^2(\Omega)} \end{aligned} \tag{3.14}$$

$$\forall u \in W_s^2(\Omega) \cap \overset{\circ}{W}{}_{s-1}^1(\Omega), \qquad \forall \lambda \in R_+,$$

where $c_1, c_2, c_3 \in R_+$ are independent of u and λ.

From (3.11) and (3.14) we deduce the result. ■

4. - FURTHER A PRIORI BOUNDS

We assume that the following further hypotheses hold:

h_6) the function σ which satisfies (2.6) and (2.7) is such that

$$\sigma_x \in \overset{\circ}{K}{}_o^q(\Omega),$$

where q is the number defined in the hypothesis h_3);

h_7)
$$(\alpha_{ij})_{x_k}, a_i \in \overset{\circ}{K}{}_1^q(\Omega), \qquad i, j, k = 1, \ldots, n,$$

$$a = a' + a'', \qquad a' \in \overset{\circ}{K}{}_2^t(\Omega),$$

$$a'' \geqslant \mu_o \varrho^{-2} \quad \text{a.e. in } \Omega,$$

where t is the number defined in the hypothesis h_4) and $\mu_o \in R_+$ is independent of x.

An example of function $\varrho \in \mathcal{A}_o(\Omega)$ satisfying the condition h_6) can be found in [9].

THEOREM 4.1: *If the hypotheses h_1)-h_7) and α) hold, then for any $s \in R$ there exist $c \in R_+$ and an open set $\Omega_o \subset\subset \overline{\Omega} \setminus S$ such that*

$$\|u\|_{W_s^2(\Omega)} \leqslant c(\|Lu + \lambda\beta u\|_{L_s^2(\Omega)} + |u|_{2,\Omega_o}) \tag{4.1}$$

$$\forall u \in W_s^2(\Omega) \cap \overset{\circ}{W}{}_{s-1}^1(\Omega), \qquad \forall \lambda \geqslant 0 .$$

PROOF: Proceeding as in the proof of Lemma 3.1, let us consider the functions ψ_k, $k \in N$, defined in Section 1.

From Theorem 3.1 in [9] we have

$$\begin{aligned}
(4.2)\qquad & \|(1-\psi_k)\,u\|_{W_s^2(\Omega)} \leqslant \\
& \leqslant c_1 \left\| -\sum_{i,j=1}^{n} \alpha_{ij}((1-\psi_k)\,u)_{x_i x_j} + a''(1-\psi_k)\,u + \lambda\beta(1-\psi_k)\,u \right\|_{L_s^2(\Omega)} \leqslant \\
& \leqslant c_1 \left(\left\| L_o((1-\psi_k)\,u) + a''(1-\psi_k)\,u + \lambda\beta(1-\psi_k)\,u \right\|_{L_s^2(\Omega)} + \right. \\
& \left. + \left\| \sum_{i,j=1}^{n} (\alpha_{ij} - a_{ij})((1-\psi_k)\,u)_{x_i x_j} \right\|_{L_s^2(\Omega)} \right) \leqslant \\
& \leqslant c_2 \left(\|(1-\psi_k)(L_o u + a'' u + \lambda\beta u)\|_{L_s^2(\Omega)} + \|(1-\psi_k)_x u_x\|_{L_s^2(\Omega)} + \right. \\
& \left. + \|(1-\psi_k)_{xx} u\|_{L_s^2(\Omega)} + \left\| ((1-\psi_k)\,u)_{xx} \sum_{i,j=1}^{n} (\alpha_{ij} - a_{ij}) \right\|_{L_s^2(\Omega)} \right),
\end{aligned}$$

where $c_1, c_2 \in R_+$ are independent of λ, u and k.

From hypothesis α) and from (4.2) we deduce that there exists $k_o \in N$ such that

$$\begin{aligned}
(4.3)\qquad & \|(1-\psi_{k_o})\,u\|_{W_s^2(\Omega)} \leqslant c_3\,(\|L_o u + a'' u + \lambda\beta u\|_{L_s^2(\Omega)} + \\
& + \|(1-\psi_{k_o})_x u_x\|_{L_s^2(\Omega)} + \|(1-\psi_{k_o})_{xx} u\|_{L_s^2(\Omega)})\,,
\end{aligned}$$

where $c_3 \in R_+$ is independent of λ and u, as all the positive constants appearing in the rest of the proof.

On the other hand, from Theorem 3.1 we get

$$\begin{aligned}
(4.4)\qquad & \|\psi_{k_o} u\|_{W_s^2(\Omega)} \leqslant c_4\,(\|L_o(\psi_{k_o} u) + a''\psi_{k_o} u + \lambda\beta\psi_{k_o} u\|_{L_s^2(\Omega)} + \\
& + \|\psi_{k_o} u\|_{L_{s-2}^2(\Omega)}) \leqslant c_5\,(\|\psi_{k_o}(L_o u + a'' u + \lambda\beta u)\|_{L_s^2(\Omega)} + \\
& + \|(\psi_{k_o})_x u_x\|_{L_s^2(\Omega)} + \|(\psi_{k_o})_{xx} u\|_{L_s^2(\Omega)} + \|\psi_{k_o} u\|_{L_{s-2}^2(\Omega)})\,.
\end{aligned}$$

From (4.3) and (4.4) it follows

$$\|u\|_{W_s^2(\Omega)} \leq c_6 (\|L_o u + a'' u + \lambda\beta u\|_{L_s^2(\Omega)} + \|(\psi_{k_o})_x u_x\|_{L_s^2(\Omega)} + \tag{4.5}$$

$$+ \|(\psi_{k_o})_{xx} u\|_{L_s^2(\Omega)} + \|\psi_{k_o} u\|_{L_{s-2}^2(\Omega)}) .$$

We remark that, by (1.2), we have

$$\|(\psi_{k_o})_x u_x\|_{L_s^2(\Omega)} \leq c_7 \, |u_x|_{2,\, \mathrm{supp}\, \psi_{k_o}}, \tag{4.6}$$

$$\|(\psi_{k_o})_{xx} u\|_{L_s^2(\Omega)} + \|\psi_{k_o} u\|_{L_{s-2}^2(\Omega)} \leq c_8 \, |u|_{2,\, \mathrm{supp}\, \psi_{k_o}}. \tag{4.7}$$

Moreover, from (3.7) in [8], for any $\varepsilon \in R_+$ there exist $c(\varepsilon) \in R_+$ and an open set $\Omega_\varepsilon \subset\subset \Omega$ such that

$$\sum_{i=1}^{n} \|a_i u_{x_i}\|_{L_s^2(\Omega)} + \|a' u\|_{L_s^2(\Omega)} \leq \tag{4.8}$$

$$\leq \varepsilon \|u\|_{W_s^2(\Omega)} + c(\varepsilon)(|u_x|_{2,\, \Omega_\varepsilon} + |u|_{2,\, \Omega_\varepsilon}) .$$

From (4.5)-(4.8), taking in mind (1.2), we deduce the assertion. ■

Corollary 4.1: *If the hypotheses of Theorem* 4.1 *are satisfied and if*

$$\beta^{-1} \in L_{loc}^{\infty}(\overline{\Omega} \setminus S) , \tag{4.9}$$

then for any $s \in R$ *there exist* c, $\lambda_o \in R_+$ *such that*

$$\|u\|_{W_s^2(\Omega)} \leq c \|Lu + \lambda\beta u\|_{L_s^2(\Omega)} \tag{4.10}$$

$$\forall u \in W_s^2(\Omega) \cap \overset{\circ}{W}{}_{s-1}^1(\Omega) , \qquad \forall \lambda \geq \lambda_o .$$

Proof: From (4.9) and (1.2) it follows that

$$\lambda |u|_{2,\, \Omega_o} \leq c_1 \, |\varrho^s \lambda\beta u|_{2,\, \Omega_o} \leq c_1 \|\lambda\beta u\|_{L_s^2(\Omega)} \tag{4.11}$$

$$\forall u \in W_s^2(\Omega) \cap \overset{\circ}{W}{}_{s-1}^1(\Omega) , \qquad \forall \lambda \in R_+ ,$$

where Ω_o is the open set of Theorem 4.1 and $c_1 \in R_+$ is independent of u and λ.

Proceeding as in the proof of Corollary 3.1, using Theorem 4.1 instead of Theorem 3.1, we obtain the result. ■

5. - Existence theorems

Theorem 5.1: *If either the hypotheses* h_1)-h_6), α) *and* (3.12) *or the hypotheses* $h_1) - h_7$), α) *and* (4.9) *hold, then for any* $s \in R$ *there exists* $\lambda_o \in R_+$ *such that for any*

$\lambda \geqslant \lambda_o$ the problem

$$(5.1) \qquad u \in W_s^2(\Omega) \cap \overset{\circ}{W}{}_{s-1}^1(\Omega), \qquad Lu + \lambda\beta u = f, \qquad f \in L_s^2(\Omega)$$

is uniquely solvable.

PROOF: We denote by A the operator $-\Delta$ if $h_1)$-$h_6)$, $\alpha)$ and (3.12) hold, the operator $-\Delta + a''$ in the other case.

For any $\tau \in [0, 1]$ we set

$$L_\tau = (1-\tau)A + \tau L .$$

Using Corollary 3.1 when $h_1)$-$h_6)$, $\alpha)$ and (3.12) hold, Corollary 4.1 in the other case, we deduce that there exist $c, \lambda_o \in R_+$ such that

$$\|u\|_{W_s^2(\Omega)} \leqslant c\|L_\tau u + \lambda\beta u\|_{L_s^2(\Omega)}$$

$$\forall u \in W_s^2(\Omega) \cap \overset{\circ}{W}{}_{s-1}^1(\Omega), \qquad \forall \lambda \geqslant \lambda_o, \qquad \forall \tau \in [0, 1].$$

If we fix $\lambda \geqslant \lambda_o$, by Theorem 3.2 in [9], the problem

$$u \in W_s^2(\Omega) \cap \overset{\circ}{W}{}_{s-1}^1(\Omega), \qquad Au + \lambda\beta u = f, \qquad f \in L_s^2(\Omega)$$

is uniquely solvable.

Observing that

$$L_\tau + \lambda\beta = (1-\tau)(A + \lambda\beta) + \tau(L + \lambda\beta),$$

we obtain the result by means of the method of continuity. ■

THEOREM 5.2: *If the hypotheses $h_1)$-$h_4)$, $h_6)$, $h_7)$ and $\alpha)$ are satisfied, then for any $s \in R$ the problem*

$$(5.2) \qquad u \in W_s^2(\Omega) \cap \overset{\circ}{W}{}_{s-1}^1(\Omega), \qquad Lu = f, \qquad f \in L_s^2(\Omega)$$

is an index problem with index equal to zero.

PROOF: We consider the function β defined from (2.9) with $\tau = 1$.

Since β satisfies the hypotheses $h_5)$ and (4.9), by Theorem 5.1 the problem (5.1) is uniquely solvable for λ large enough.

On the other hand, since $\beta \in \overset{\circ}{K}{}_2^t(\Omega)$, from Lemma 2 in [4] and (3.8) in [8] the operator

$$u \in W_s^2(\Omega) \to \beta u \in L_s^2(\Omega)$$

is compact.

So, from well-known results, we deduce the assertion. ■

6. - Appendix

Let us give an example of function f with a behaviour similar to $\gamma_{ij} = a_{ij} - \alpha_{ij}$ (see Section 3) and such that $f_{x_k} \notin K_1^q(\Omega)$ for some $k \in \{1, \dots, n\}$.

Let us set $\Omega = R_+ \times]0, 2[$ and $\varrho(x) = \left(\frac{x_1}{2 + x_1}\right)^\tau$ with $\tau \in [1, 2[$.

We fix $\delta \in R_+$ and consider the function $f : \Omega \to R$ defined by

$$f(x) = \begin{cases} x_2[(4n^2 - 1)\, x_1 - 2n + 1] \arctan \dfrac{1}{n^\delta} & x \in \left[\dfrac{1}{2n+1}, \dfrac{1}{2n}\right[\times]0, 2[\\ x_2[2n - 1 - (2n-1)^2 x_1] \arctan \dfrac{1}{n^\delta} & x \in \left[\dfrac{1}{2n}, \dfrac{1}{2n-1}\right[\times]0, 2[\\ 0 & x \in [1, +\infty[\times]0, 2[. \end{cases}$$

Evidently $f \in L^\infty(\Omega)$ and $\lim\limits_{|x| \to +\infty} f(x) = 0$.

On the other hand we have $0 \leqslant f(x) < 2 \arctan \frac{1}{n^\delta}$ and then $\lim\limits_{x \to x_o} f(x) = 0$ $\forall x_o \in S = \{x \in \partial\Omega : x_1 = 0\}$.

Moreover we note that $f_{x_1}, f_{x_2} \in L^\infty_{loc}(\overline{\Omega} \setminus S)$ from which we deduce that $f_{x_1}, f_{x_2} \in L^q_{loc}(\overline{\Omega} \setminus S)$.

Now we fix a sequence $(x^n)_{n \in N}$, $x^n \in \Omega$, such that

$$x_1^n = \frac{1}{2n-1} \qquad x_2^n = 1 .$$

For n large enough we have

$$\left[\frac{1}{2n}, \frac{1}{2n-1}\right] \times \left[1, 1 + \frac{1}{2}\varrho(x^n)\right] \subset \Omega(x^n) = B(x^n, \varrho(x^n)) .$$

Then we get

$$[\varrho(x^n)]^{q-2} \int_{\Omega(x^n)} |f_{x_1}|^q \, dx \geqslant$$

$$\geqslant \frac{1}{2} [\varrho(x^n)]^{q-1} \int_{1/2n}^{1/(2n-1)} \left[(2n-1)^2 \arctan \frac{1}{n^\delta}\right]^q dx_1 =$$

$$= \frac{1}{2} \left(\frac{1}{4n-1}\right)^{\tau(q-1)} \frac{(2n-1)^{2q-1}}{2n} \left(\arctan \frac{1}{n^\delta}\right)^q .$$

Finally for $\delta < \frac{(2-\tau)(q-1)}{q}$ we obtain

$$\lim_{n \to +\infty} [\varrho(x^n)]^{q-2} \int_{\Omega(x^n)} |f_{x_1}|^q \, dx = +\infty ,$$

hence $f_{x_1} \notin K_1^q(\Omega)$.

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