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Weak Kato Measures and Schrödinger Problems for a Dirichlet Form (**)

ABSTRACT. — We give a definition of weak Kato Measures relative to a Dirichlet form of diffusion type and we prove the Harnack inequality, the local Hölder continuity an estimate on the decay of the energy and a Wiener type criterion on the boundary for the harmonics relative to a Schrödinger problem for the Dirichlet form with a weak Kato measure as potential.

Misure di Kato deboli e problemi di Schrödinger per una forma di Dirichlet

SUNTO. — Si definisce la classe delle misure di Kato deboli relative a una forma di Dirichlet tipo diffusione e si provano la diseguaglianza di Harnack, la hölderianità locale, una stima di decrescita della energia e un criterio tipo Wiener al bordo per le armoniche relative a un problema tipo Schrödinger per una forma di Dirichlet con una misura di Kato debole come potenziale.

1. - INTRODUCTION

The Kato space relative to the Laplace operator has been introduced in [23] and has been used in [2] as a suitable space for the potential in a Schrödinger equation (for the Laplace operator) assuring in particular the continuity of solutions and the Harnack inequality for positive local solutions; the proofs in [2] are of probabilistic type.

A generalization of Kato spaces to the case of uniformly elliptic operators has been given in [11] in relation with the study of the associated Schrödinger equation; the principal goal of [11] is an extension to the case under consideration of the results in [2]; the proofs in [11] are analitycal. Extensions to degenerate elliptic operators constructed with vector fields satisfying an Hörmander condition are given in [12] and in

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[24] for the weighted case. We have also to recall the recent paper of W. Hansen [20] (see also [15]), where the Harnack inequality has been proved in the case that the potential in the Schrödinger equation is only potentially bounded; in the above paper the operators in consideration have to satisfy a technical assumption, which holds for operators on polynomial Lie groups.

The properties of Kato space relative to an uniformly elliptic operator have been further studied in [14]. In the same paper the continuity of the local solutions of an uniformly elliptic problem with a source term in the Kato space is also proved.

In [9] the notion of Kato spaces has been introduced for Dirichlet forms of diffusion type and the properties of these spaces have been investigated. In [9] the continuity of local weak solutions of the equation $Lu = \sigma$ has been studied, where L is the operator associated with the form and σ belongs to the associated Kato space, moreover in [27] an Harnack inequality has been also proved for local positive weak solutions of the equation $Lu + u\mu = 0$, where the potential μ is a measure in the Kato space associated with the form.

In the present paper we introduce the notion of weak Kato spaces; the new notion coincides with the usual Kato spaces in the case of uniformly elliptic operators and of operators on Lie groups, but is weaker than the usual one in the case of weighted uniformly elliptic operators, of weighted degenerate elliptic operators constructed with vector fields satisfying an Hörmander condition and of general Dirichlet forms of diffusion type. We consider weak Kato spaces of measures related to Dirichlet forms of diffusion type and we investigate the properties of the space; moreover we prove an Harnack inequality for local positive weak solutions of the equation $Lu + u\mu = 0$, where L is the operator associated with the form and μ is a measure in the weak Kato space, (this result is weaker than the result in [20] in the cases where the technical assumption assumed in [20] holds) and we prove that a weak solution of $Lu + u\mu = \sigma$ is locally continuous provided μ and σ are in weak Kato space associated with the form; moreover the energy measure of the form is locally in weak Kato space. The boundary behavior of a weak solution of the previous problem is studied under the assumption that the points of the boundary in consideration are regular points.

We now specify the class of Dirichlet forms we will consider.

Let X be a locally compact Hausdorff space and m a positive measure on X with $\operatorname{supp}(m) = X$; we assume that we are given a *strongly local* (*of diffusion type*), *regular Dirichlet form* in the Hilbert space $L^2(X, m)$, in the sense of M. Fukushima, [19], whose domain is denoted by $D[a]$. Such a form a admits the following integral representation $a(u, v) = \int_X \alpha(u, v)(dx)$ for every $u, v \in D[a]$ where $\alpha(u, v)(dx)$ is a signed Radon measure on X, uniquely associated with the functions u, v. Moreover for any open subset Ω of X the restriction of $\alpha(u, v)$ to Ω depends only on the restrictions of u and v to Ω. By $D_0[a, \Omega]$ we denote the closure of $C_0(\Omega) \cap D[a]$ in $D[a]$. By $D_{\mathrm{loc}}[a, \Omega]$ we denote the space of all m-measurable functions u, v in X, that coincide m-a.e. on every compact subset of Ω with some function of $D[a]$. The measure $\alpha(u, v)$

is defined unambigously in Ω for all $u, v \in D_{\mathrm{loc}}[a, \Omega]$. We refer to [6], [7], [19] for the properties of $\alpha(u, v)$ with respect to Leibnitz, chain and troncature rules. We assume that the form a has a *separating core* denoted by $\mathcal{C}$, [6], [7] and we define a *distance* associated with the form by

$$d(x, y) = \sup\{\phi(x) - \phi(y)\,;\ \forall \phi \in \mathcal{C} \ \text{ with } \ \alpha(\phi, \phi) \leq m\}\,.$$

We denote $B(x, r) = \{y\,;\, d(x, y) < r\}$, $B(r)$ will be balls $B(x, r)$ with a fixed center x. We consider a fixed relatively compact open set X_0 assume:

(H_1) The distance d defines a topology on X equivalent to the initial one; moreover the following property holds: there exists constants $0 < R_0 < +\infty$, $\nu > 0$ and $c_0 > 0$, such that

$$0 < c_0\left(\frac{r}{R}\right)^{\nu} m(B(x, R)) \leq m(B(x, r))$$

for every $x \in X_0$ and every $0 < r < R_0$. We say that ν is an estimate of *the intrinsic dimension* in X_0.

We observe that (H_1) is verified if *a duplication property* holds for the balls $B(x, r)$ $(x \in X_0,\ 0 < r < R_0)$ that is

$$m(B(x, 2r)) \leq c_0^{\star}\, m(B(x, r))\,,$$

where $c_0^{\star}$ is a positive constant independent of x, r. In this case we have $\nu \geq lg_2\, c_0^{\star}$. Moreover we observe that, if (H_1) holds, we have $\inf\limits_{x \in X_0} m(B(x, R_0/2)) > 0$.

(H_2) For every ball $B(x, r)$ with $x \in X_0$, $0 < r < R_0$ and every $f \in D_{\mathrm{loc}}[a, B(x, kr)]$ the following *scaled Poincaré inequality* holds

$$\int\limits_{B(x, r)} |f - f_{x, r}|^2\, m(dx) \leq c_1 r^2 \int\limits_{B(x, kr)} \alpha(u, u)(dx)$$

where $f_{x, r}$ is the average of f on $B(x, r)$, c_1 are constants independent of x, r and $k \geq 1$.

We observe that from the assumptions (H_1) and (H_2) it follows that the Sobolev inequalities relative to ν hold, [8].

We recall, [7], that for a Dirichlet form, that satifies the above assumptions and $B(x, t) \subseteq B(x, 2t) \subseteq X_0$, $s > t$, there exists a cut-off function ϕ of $B(x, s)$ in $B(x, t)$ such that,

$$\alpha(\phi, \phi)(dx) \leq \frac{C_1}{(t - s)^2}\, m(dx)\,.$$

Associated with the form a we have a capacity defined by

$$\operatorname{cap}(E,\Omega)=\sup\left\{\int_\Omega \alpha(u,u)(dx);\ u\in D_0[a,\Omega]\cap C_0(\Omega),\, u\geq 1 \text{ on a neighborhood of } E\right\}$$

where $\overline{E}\subsetneq\Omega\subsetneq X_0$. Functions in $D_{\mathrm{loc}}[a,X_0]$ may be defined in Ω up to set of capacity zero, so the supremum (or the oscillation) of the function means the supremum (or the oscillation) up to sets of capacity zero.

Moreover a Green function $G_\Omega(x,y)=G_\Omega(y,x)$ relative to a relatively compact open set $\Omega\subset\subset X_0$ can be defined and that, in the case $\Omega=B(x_0,R)$, $B(x_0,2kR)\subset\subset X_0$ and $R\leq R_0$, the following estimate holds

$$G_{B(x_0,R)}(x_0,y)\approx\int_{d(x_0,y)}^{R}\frac{s^2}{m(B(x,s))}\frac{ds}{s} \tag{1.1}$$

for every y such that $d(x_0,y)\leq R/2$.

Finally we observe that our assumptions hold for Dirichlet forms associated to wide classes of

(a) weighted uniformly elliptic operators

(b) weighted degenerate elliptic operators generated by vector fields satisfying an Hörmander condition

(c) subelliptic operators

see [6], [7], [24] for more details.

2. - The spaces $K_w(\Omega)$ and $K_w^{loc}(\Omega)$.

Let Ω be an open set in X with *diam* $(\Omega)=\overline{R}/2$; then $\Omega\subsetneq B_{\overline{R}/2}$, where $B_{\overline{R}/2}$ is a suitable ball with radius $\overline{R}/2$. We assume that $B_{4k\overline{R}}\subset\subset X_0$, where $B_{4k\overline{R}}$ is a ball with the same center as $B_{\overline{R}/2}$ and radius $4k\overline{R}$. By $|\mu|$ we denote the total variation of the Radon measure μ on Ω.

Definition 2.1: *We denote by $K_w(\Omega)$ the set of all Radon measures μ on Ω such that*

$$\lim_{r\to 0}\eta_\mu(r)=0$$

where

$$\eta_\mu(r)=\sup_{x\in\overline{\Omega}}\int_{\Omega\cap B(x,r)}\left(\int_{d(x,y)}^{40r}\frac{s^2}{m(B(x,s))}\frac{ds}{s}\right)|\mu|(dy).$$

We say that $\eta_\mu(r)$ is the K_w-modulus of the measure μ.

By $K_w^{\mathrm{loc}}(\Omega)$ *we denote the set of all Radon measures* μ *on* Ω *such that* $\mu \in K(\Omega')$ *for every open set* $\Omega' \subset\subset \Omega$.

It is easy to see that $K_w(\Omega)$ and $K_w^{\mathrm{loc}}(\Omega)$ are vector spaces.

PROPOSITION 2.2: *If* $\mu \in K(\Omega)$; *then* $|\mu|(\Omega) < +\infty$.

PROOF: By the definition, there exists $r_0 \leqslant \overline{R}/2$, such that

$$C|\mu|(\Omega \cap B(x, r_0)) \frac{r_0^2}{m(B(x, r_0))} \leqslant \int_{\Omega \cap B(x, r_0)} \left(\int_{d(x, y)}^{40 r_0} \frac{s^2}{m(B(x, s))} \frac{ds}{s} \right) |\mu|(dy) \leqslant 1$$

for every $x \in \Omega$, where C is an absolute constant. There exists $x_1, x_2, \ldots, x_n \in \Omega$ such that

$$\overline{\Omega} \subseteq \bigcup_{i=1}^{n} B(x_i, r_0)$$

hence

$$|\mu|(\Omega) \leqslant \sum_{i=1}^{n} |\mu|(\Omega \cap B(x_i, r_0)) \leqslant$$

$$\leqslant C^{-1} n \frac{\sup_{x=1,\ldots,n} m(B(x_i, r_0))}{r_0^2} \leqslant$$

$$\leqslant C^{-1} n \frac{m(B_{2\overline{R}})}{r_0^2} .$$

PROPOSITION 2.3: *Let* μ *be a measure in* $K_w(\Omega)$ *then*

$$\sup_{x \in \overline{\Omega}} \int_{\Omega} \left(\int_{d(x, y)}^{\overline{R}} \frac{s^2}{m(B(x, s))} \frac{ds}{s} \right) |\mu|(dy) < +\infty .$$

PROOF: Let μ be in $K_w(\Omega)$. Moreover there exists $0 < r_0 \leqslant \overline{R}/40$ such that

$$\int_{\Omega \cap B(x, r_0)} \left(\int_{d(x, y)}^{40 r_0} \frac{s^2}{m(B(x, s))} \frac{ds}{s} \right) |\mu|(dy) < 1$$

for every x in $\overline{\Omega}$. Then

$$\int\limits_{\Omega \cap B(x,\, r_0)} \left(\int\limits_{d(x,\, y)}^{\overline{R}} \frac{s^2}{m(B(x,\, s))} \frac{ds}{s} \right) |\mu|(dy) \leqslant$$

$$\leqslant \int\limits_{\Omega \cap B(x,\, r_0)} \left(\int\limits_{d(x,\, y)}^{40 r_0} \frac{s^2}{m(B(x,\, s)} \frac{ds}{s} \right) |\mu|(dy) +$$

$$+ \int\limits_{\Omega \cap B(x,\, r_0)} \left(\int\limits_{40 r_0}^{\overline{R}} \frac{s^2}{m(B(x,\, s))} \frac{ds}{s} \right) |\mu|(dy) \leqslant$$

$$\leqslant 1 + C_1(r_0)\, |\mu|(\Omega)\,.$$

Since

$$\int\limits_{\Omega - B(x,\, r_0)} \left(\int\limits_{d(x,\, y)}^{\overline{R}} \frac{s^2}{m(B(x,\, s))} \frac{ds}{s} \right) |\mu|(dy) \leqslant C_2(r_0)\, |\mu|(\Omega)$$

where $C_2(r_0)$ is a constant depending only on r_0. We have

$$\int\limits_{\Omega} \left(\int\limits_{d(x,\, y)}^{\overline{R}} \frac{s^2}{m(B(x,\, s))} \frac{ds}{s} \right) |\mu|(dy) \leqslant 1 + (C_1(r_0) + C_2(r_0))\, |\mu|(\Omega)$$

for every $x \in \overline{\Omega}$; this concludes the proof because from Proposition 2.2.

An easy consequence of Proposition 2.3. is that a measure in $K_w(\Omega)$ belongs to the space $D_0'[a,\, \Omega]$ ($D_0'[a,\, \Omega]$ denotes the dual space of $D_0[a,\, \Omega]$, see [9] for the a characterization of the space $D_0'[a,\, \Omega]$).

DEFINITION 2.4: *Let* $\mu \in K_w(\Omega)$, *we define*

$$\|\mu\|_{K_w(\Omega)} = \sup_{x \in \overline{\Omega}} \int\limits_{\Omega} \left(\int\limits_{d(x,\, y)}^{\overline{R}} \frac{s^2}{m(B(x,\, s))} \frac{ds}{s} \right) |\mu|(dy)\,.$$

It is easy to verify that $\|.\|_{K_w(\Omega)}$ is a norm on $K_w(\Omega)$.

REMARK 2.1: We observe that $m(B(x,\, s))$ is an increasing function of s then for almost every s we have $m(\partial B(x,\, s)) = 0$. So we have that $m(B(x,\, s))$ is continuous at a point x for almost every s; as consequence, for a fixed $0 < R < \overline{R}$, $\left(\int\limits_{d(x,\, y)}^{R} \frac{s^2}{m(B(x,\, s))} \frac{ds}{s} \right)$ is continuous with respect to $x \in \Omega$ for $y \in \overline{\Omega} - \{x\}$ uniformly for y in $\Omega - B(x, \sigma)$, $\sigma > 0$. Then we have that

$$\int_{\Omega} \left(\int_{d(x,y)}^{R} \frac{s^2}{m(B(x,s))} \frac{ds}{s} \right) |\mu|(dy)$$

is continuous with respect to x in $\partial\Omega$ (we observe that for $x \in \partial\Omega$ we have $\lim_{\sigma \to 0} |\mu|(\Omega \cap B(x, \sigma)) = 0$). As a consequence, we have that in Definition 2.1, in Proposition 2.2 and in Definition 2.3 we can take equivalently the supremum on Ω.

Consider now $\left(\int_{d(x,y)}^{\overline{R}} \frac{s^2}{m(B(x,s))} \frac{ds}{s} \right)$; the integral is continuous in y on $\Omega - B(x, \sigma)$ for $\sigma > 0$ and, if $\left(\int_{0}^{\overline{R}} \frac{s^2}{m(B(x,s))} \frac{ds}{s} \right) < +\infty$, the integral is continuous for $y \in \overline{\Omega}$.

Remark 2.2: Using the assumption (H_1) we easily obtain

$$|\mu|(\Omega) \leq C \|\mu\|_{K_w(\Omega)} \frac{m(B_{\overline{R}})}{\overline{R}^2}$$

Theorem 2.4: *The space $K_w(\Omega)$ with the norm $\|.\|_{K_w(\Omega)}$ is a Banach space.*

Proof: Let (μ_h) be a Cauchy sequence in $K_w(\Omega)$. By the Remark 2.2 we have that $|\mu_h|(\Omega)$ is bounded and

$$\lim_{h,k \to 0} |\mu_h - \mu_k|(\Omega) = 0 .$$

By the completeness of the space of all bounded Radon measures on Ω, there exists a Radon measure μ on Ω such that $|\mu_h - \mu|(\Omega) \to 0$ as $h \to 0$.

Since (μ_h) is a Cauchy sequence in $K_w(\Omega)$, for every $\varepsilon > 0$ there is h_ε such that

$$\sup_{x \in \overline{\Omega}} \int_{\Omega} \left(\int_{d(x,y)}^{\overline{R}} \frac{s^2}{m(B(x,s))} \frac{ds}{s} \right) |\mu_h - \mu_k|(dy) \leq \varepsilon$$

for every $h, k \geq h_\varepsilon$.

We recall that, since $|\mu| \in D_0'[a, \Omega]$ if $|\mu|(\{x\}) \neq 0$ then x is a point of positive capacity and $\int_{0}^{\overline{R}} \frac{s^2}{m(B(x,s))} \frac{ds}{s}$ is finite.

Taking into account Remark 2.1 and passing to the limit as $k \to +\infty$ we obtain, for every $x \in \overline{\Omega}$

$$\int\limits_{\Omega - B(x,\sigma)} \left(\int\limits_{d(x,y)}^{\overline{R}} \frac{s^2}{m(B(x,s))} \frac{ds}{s} \right) |\mu_h - \mu|(dy) \leqslant \varepsilon . \tag{2.1}$$

if $|\mu|(\{x\}) = 0$ (where $\sigma > 0$) and

$$\int\limits_{\Omega} \left(\int\limits_{d(x,y)}^{\overline{R}} \frac{s^2}{m(B(x,s))} \frac{ds}{s} \right) |\mu_h - \mu|(dy) \leqslant \varepsilon . \tag{2.1'}$$

if $|\mu|(\{x\}) \neq 0$. Then, for $r \leqslant \overline{R}/40$

$$\int\limits_{\Omega - B(x,\sigma)} \left(\int\limits_{d(x,y)}^{40r} \frac{s^2}{m(B(x,s))} \frac{ds}{s} \right) |\mu|(dy) \leqslant \tag{2.2}$$

$$\leqslant \int\limits_{\Omega} \left(\int\limits_{d(x,y)}^{40r} \frac{s^2}{m(B(x,s))} \frac{ds}{s} \right) |\mu_{h_\varepsilon}| + \varepsilon$$

if $|\mu|(\{x\}) = 0$ and

$$\int\limits_{\Omega} \left(\int\limits_{d(x,y)}^{40r} \frac{s^2}{m(B(x,s))} \frac{ds}{s} \right) |\mu|(dy) \leqslant \tag{2.2'}$$

$$\leqslant \int\limits_{\Omega} \left(\int\limits_{d(x,y)}^{40r} \frac{s^2}{m(B(x,s))} \frac{ds}{s} \right) |\mu_{h_\varepsilon}| + \varepsilon$$

if $|\mu|(\{x\}) \neq 0$. From (2.2) and (2.2′) we obtain

$$\int\limits_{\Omega} \left(\int\limits_{d(x,y)}^{40r} \frac{s^2}{m(B(x,s))} \frac{ds}{s} \right) |\mu|(dy) \leqslant \tag{2.3}$$

$$\leqslant \int\limits_{\Omega} \left(\int\limits_{d(x,y)}^{40r} \frac{s^2}{m(B(x,s))} \frac{ds}{s} \right) |\mu_{h_\varepsilon}| + \varepsilon .$$

By the same methods we have that (2.3) holds for Ω replaced by $\Omega \cap B(x, r)$; then we obtain that $\mu \in K_w(\Omega)$.

From (2.1) (2.1′) we obtain (passing to the limit in (2.1) as $\sigma \to 0$) that μ_h converges to μ in $K_w(\Omega)$.

REMARK 2.3: The main difference between the spaces $K(\Omega)$ defined in [9] and $K_w(\Omega)$ is that the measures in $K(\Omega)$ do not charge the points, but the measures in $K_w(\Omega)$ may charge points of positive capacity. We recall that in the general case of Dirichlet forms (and in particular in the case of weighted problems) we can consider measures in $D_0'[a, \Omega]$ ($D_0'[a, \Omega]$ denotes the dual space of $D_0[a, \Omega]$) which are supported by points of positive capacity and which belong to the space $K_w(\Omega)$ but do not belong to the space $K(\Omega)$.

3. - HARNACK INEQUALITY

Let a be a Dirichlet form satisfying the assumptions given in section 1. and Let μ be a measure in $K_w(\Omega)$. The main goal of this section is the proof of the following Harnack inequality:

THEOREM 3.1: *Let u be a positive local solution of the Schrödinger type problem relative to the measure μ and to the Dirichlet form a, i.e.*

$$\int_\Omega \alpha(u, v)(dx) + \int_\Omega uvm(dx) = 0$$

$$\forall v \in D_0[a, \Omega], \ \operatorname{supp}(v) \subseteq \Omega ; \qquad u \in D_{\mathrm{loc}}[a, \Omega], \ u > 0 .$$

Then there exists $\overline{R}_1 > 0$ *such that for* $r \leq \min\left(\overline{R}_1, \frac{\overline{R}}{4}\right)$, $B(x, 4r) \subseteq \Omega$ *we have*

$$\sup_{B(x, r)} u \leq C \inf_{B(x, r)} u$$

where C is a structural constant.

We begin by proving the following embedding result:

PROPOSITION 3.2: *Let* μ *be in* $K_w(\Omega)$ *and* $u \in D_{\mathrm{loc}}[a, B(x, t)]$, $\overline{B(x, 2t)} \subseteq X_0$ *and* $\varepsilon > 0$, *there exists* $\overline{R}(\varepsilon) > 0$ *such that*

$$\int_{B(x, t)} |u|^2 \phi^2 |\mu|(dx) \leq$$

$$\leq \varepsilon \int_{B(x, t)} \phi^2 \alpha(u, u)(dx) + \frac{C(\varepsilon)}{(t-s)^2} \int_{B(x, t) - B(x, s)} |u|^2 m(dx)$$

where $0 < s < t \leq \overline{R}(\varepsilon)$, ϕ *is the cut-off function of* $B(x, s)$ *in* $B(x, t)$ *defined in section 1 and* $\overline{R}(\varepsilon)$, $C(\varepsilon)$ *are constants depending only on* ε.

PROOF: We use the same methods as in [3].

We can assume, without loss of generality $\mu \geqslant 0$, $u \in L^\infty(\Omega)$.

Let w be the weak solution of the problem

$$a(w, v) = \int_{B(x, 2t)} v\mu(dx)$$

$$w \in D_0[a, B(x, 2t)], \quad \forall v \in D_0[a, B(x, 2t)];$$

We have

$$\int_{B(x, s)} |u|^2 \mu(dx) \leqslant \int_{B(x, t)} |u|^2 \phi^2 \mu(dx) = \tag{3.1}$$

$$= \int_{B(x, t)} \alpha(w, |u|^2 \phi^2)(dx) \leqslant$$

$$\leqslant 2 \int_{B(x, t)} u\phi^2 \alpha(w, u)(dx) + 2 \int_{B(x, t)} u^2 \phi\alpha(w, \phi)(dx) \leqslant$$

$$\leqslant \frac{\varepsilon}{2} \int_{B(x, t)} \phi^2 \alpha(u, u)(dx) +$$

$$+ \frac{C_2}{\varepsilon}\left[\int_{B(x, t)} u^2 \alpha(\phi, \phi)(dx) + \int_{B(x, t)} u^2 \phi^2 \alpha(w, w)\, dx\right] \leqslant$$

$$\leqslant \frac{\varepsilon}{2} \int_{B(x, t)} \phi^2 \alpha(u, u)(dx) +$$

$$+ \frac{C_3}{\varepsilon}\left[\frac{1}{(t-s)^2} \int_{B(x, t)} u^2 m(dx) + \int_{B(x, t)} u^2 \phi^2 \alpha(w, w)(dx)\right].$$

We now estimate the term

$$\int_{B(x, t)} u^2 \phi^2 \alpha(w, w)(dx).$$

We have

$$\int_{B(x, t)} u^2 \phi^2 \alpha(w, w)(dx) =$$

$$= \int_{B(x, t)} \alpha(w, wu^2 \phi^2)(dx) - 2 \int_{B(x, t)} wu\phi\alpha(w, u\phi)(dx) \leqslant$$

$$\leqslant \int_{B(x,\,t)} wu^2\phi^2\mu(dx) - 2\int_{B(x,\,t)} wu^2\phi\alpha(w,\,\phi)(dx) -$$

$$-2\int_{B(x,\,t)} wu\phi^2\alpha(w,\,u)(dx)\,.$$

We recall that $C_5\eta_\mu(t) \geqslant \sup\limits_{B(x,\,8t)} w$. Then we choose $\overline{R}(\varepsilon)$ such that for $t \leqslant \overline{R}(\varepsilon)$ we have $\eta_\mu(t) \leqslant \inf \dfrac{\varepsilon}{8C_3C_4C_5^2}$ (we assume $C_3,\,C_4,\,C_5 \geqslant 1,\ \varepsilon < 1$), we obtain

$$\int_{B(x,\,t)} u^2\phi^2\alpha(w,\,w)(dx) \leqslant \frac{\varepsilon}{4C_3}\int_{B(x,\,t)} u^2\phi^2\mu(dx) +$$

$$+\frac{\varepsilon^2}{4C_3}\int_{B(x,\,t)}\phi^2\alpha(u,\,u)(dx) + \frac{C_6}{(t-s)^2}\int_{B(x,\,t)-B(x,\,s)} u^2m(dx)$$

using in (3.1) the above estimate we obtain the result.

By an easy modification of Moser iteration method and by the Lemma 5.2 in [7] (taking into account the Sobolev inequalities proved by (H_2), [8]), we prove that

Proposition 3.3: *For a positive subsolution of the problem* (2.1) *the following estimate holds*

$$\sup_{B(x,\,r)} u \leqslant C_1\left(\frac{1}{m(B(x,\,2r))}\int_{B(x,\,2r)} u^q\,m(dx)\right)^{1/q} \tag{3.2}$$

where $q > 0$, $B(x,\,8kr) \subseteq \Omega$, $r \leqslant \overline{R}_2 = \overline{R}(1/2)$ *and* C_1 *depends on* q.

We are now ready for the proof of Theorem 3.1:

Proof of Theorem 3.1: Consider now the function $\log u$, where u is a positive supersolution in Ω, i.e.

$$\int_\Omega \alpha(u,\,v)(dx) + \int_\Omega uvm(dx) \geqslant 0$$

$$\forall v \in D_0[a,\,\Omega],\ \operatorname{supp}(v) \subseteq \Omega,\ v \geqslant 0\,; \qquad u \in D_{\mathrm{loc}}[a,\,\Omega],\ \ u > 0\,.$$

By the same methods in [7] we prove that for every positive local solution of (2.1) the following estimate holds

$$\int_{B(x,\,kr)} \alpha(\log u,\,\log u)(dx) \leqslant C_2(1 + (|\mu|(B(x,\,2r))\,\frac{r^2}{m(B(x,\,r))}\,)\,\frac{m(B(x,\,r))}{r^2} \tag{3.3}$$

where $B(x,\,2kr) \subseteq \Omega$ and C_2 is a structural constant.

We observe that the term $\mu(B(x, 2r)) \frac{r^2}{m(B(x, r))}$ is bounded (we use the assumption $\mu \in K_w(\Omega)$); then, using the Poincaré inequality we have

$$\frac{1}{m(B(x, r))} \int_{B(x, r)} |\log u - (\log u)_{x, r}|^2 m(dx) \leq C_3. \tag{3.4}$$

As in [7] using (3.4) we prove that there exists $\gamma > 0$ such that

$$\left(\frac{1}{m(B(x, r))} \int_{B(x, r)} u^{\gamma} m(dx) \right) \left(\frac{1}{m(B(x, r))} \int_{B(x, r)} u^{-\gamma} m(dx) \right) \leq C_4 \tag{3.5}$$

where $B(x, 2r) \subseteq \Omega$, $r \leq \overline{R}$, and C_4 depends only on C_3.

Let now u be a solution in Ω. The function $\tilde{u} = u + \sigma$, $\sigma > 0$ is again a local solution, then $1/\tilde{u}$ is a positive subsolution; then by (3.2), (3.5)

$$\inf_{B(x, r)} \tilde{u} \geq C_1 \left(\frac{1}{m(B(x, 2r))} \int_{B(x, 2r)} \tilde{u}^{-\gamma} m(dx) \right)^{-1/\gamma} \geq$$

$$\geq C_5 \left(\frac{1}{m(B(x, 2r))} \int_{B(x, 2r)} \tilde{u}^{\gamma} m(dx) \right)^{1/\gamma} \geq C_6 \sup_{B(x, r)} \tilde{u}$$

where C_6 depends on C_1 and C_4, but is independent of σ. Letting $\sigma \to 0$ we obtain the result.

Taking into account Proposition 3.2. we can prove as in [7] the existence of a Green function for Dirichlet problem the Schrödinger type operator associated with the form a and the potential μ, moreover as in [7] from the Harnack inequality we have estimates on the Green function:

THEOREM 3.4: *Let $B(x, 4R) \subset\subset X_0$ and let $G_{B(x, R)}(x, y)$ be the Green function for Dirichlet problem for the Schrödinger type operator associated with the form a and the potential μ ($\mu \in K_w^{\mathrm{loc}}(X_0)$), relative to $B(x, R)$ with singularity at x. Then $G_{B(x, R)}(x, y)$ is continuous in y in $B(x, R) - \{x\}$ and*

$$G_{B(x, R)}(x, y) \approx \int_{d(x, y)}^{R} \frac{s^2}{m(B(x, s))} \frac{ds}{s} \tag{3.5}$$

for every y such that $d(x, y) \leq R/2$.

From Proposition 3.3 and Theorem 3.4 we obtain:

PROPOSITION 3.5: *Let u be a subsolution in $B(x, 4r)$ for Dirichlet problem for the Schrödinger type operator associated with the form a and the potential μ and the source* *term* σ $(\mu, \sigma \in K_w^{\mathrm{loc}}(X_0))$, *where* $B(x, 16r) \subseteq\subseteq X_0$ *and* $r \leqslant \overline{R}_2$, *i.e.*

$$\int_{B(x,4r)} \alpha(u, v)(dx) + \int_{B(x,4r)} uv\,\mu(dx) \leqslant \int_{B(x,4r)} v\,\sigma(dx)$$

$$\forall v \in D_0[a, B(x, 4r)],\ \mathrm{supp}\,(v) \subseteq B(x, 4r)\ u \in D_{\mathrm{loc}}[a, B(x, 4r)]\,.$$

Then

$$\sup_{B(x,r)} u \leqslant C\left[\left(\frac{1}{m(B(x, 2r))}\int_{B(x,2r)} (u^+)^2\, m(dx)\right)^{1/2} + \eta_\sigma(2r)\right]. \tag{3.6}$$

PROOF: Let $\tilde{u}$ be the solution of the problem

$$\int_{B(x,2r)} \alpha(\tilde{u}, v)(dx) + \int_{B(x,2r)} \tilde{u} v\,\mu(dx) = 0$$

$$\forall v \in D_0[a, B(x, 2r)]\,, \qquad u - \tilde{u} \in D_0[a, B(x, 2r)]\,.$$

Then $w = u - \tilde{u}$ is such that

$$\int_{B(x,2r)} \alpha(w, v)(dx) + \int_{B(x,2r)} wv\,\mu(dx) = \int_{B(x,2r)} w\sigma(dx)$$

$$\forall v \in D_0[B(x, 2r), a]\,, \qquad u - \tilde{u} \in D_0[B(x, 2r), a]\,.$$

From the estimates on the Green function in Theorem 3.4. we obtain

$$\sup_{B(x,2r)} |w| \leqslant C_1\,\eta_\sigma(2r)\,. \tag{3.6}$$

Moreover from Proposition 3.3. we obtain

$$\sup_{B(x,r)} \tilde{u} \leqslant C_2\left(\frac{1}{m(B(x, 2r))}\int_{B(x,2r)} \tilde{u}^2\, m(dx)\right)^{1/2}. \tag{3.7}$$

From (3.6) and (3.7) the result easily follows.

4. - CONTINUITY AND ENERGY DECAY

In the following we denote

$$\delta(\varrho, x)) = \frac{\mathrm{cap}\,(B(x, \varrho) \cap \Omega, B(x, 2\varrho))}{\mathrm{cap}\,(B(x, \varrho), B(x, 2\varrho))}$$

where cap denotes the capacity associated with the form a. We begin this section by a result concerning the continuity and the decay of the energy for local solutions:

THEOREM 4.1: *Let u be a local solution in Ω for the Dirichlet problem for the Schrödinger type operator associated with the form a and the potential μ and the source term σ ($\mu, \sigma \in K_w^{\mathrm{loc}}(\Omega)$), i.e.*

$$\int_\Omega \alpha(u, v)(dx) + \int_\Omega uv\,\mu(dx) \leqslant \int_\Omega v\,\sigma(dx)$$

$$\forall v \in D_0[a, \Omega],\ supp\,(v) \subseteq \Omega\,; \qquad u \in D_{\mathrm{loc}}[a, \Omega]$$

Then we have

$$\int_{B(x,\, r)} G^x_{B(x,\, q_0^{-1} r)}\,\alpha(u, u)(dx) \leqslant$$

$$\leqslant C\left(\frac{r}{R}\right)^\beta \int_{B(x,\, R)} G^x_{B(x,\, q_0^{-1} R)}\,\alpha(u, u)(dx) +$$

$$+ C\left[\left(\overline{R}^{-\nu}\int_\Omega u^2\, m(dx) + \eta_\sigma^2(2q_0^{-1}R)\right)\eta_\mu^2(2q_0^{-1}R) + \eta_\sigma^2(2q_0^{-1}R)\right]$$

and

$$\operatorname*{osc}_{B(x,r)} u^2 \leqslant C\left[\left(\frac{r}{R}\right)^\beta \operatorname*{osc}_{B(x,R)} u^2 + \left(\overline{R}_0^{-\nu}\int_\Omega u^2 m(dx) + \eta_\sigma^2(2q_0^{-1}R)\right)\eta_\mu^2(2q_0^{-1}R) + \eta_\sigma^2(2q_0^{-1}R)\right]$$

where $B(x, 16q_0^{-1}R) \subseteq B(x, 4\,\overline{R}_0) \subset\subset \Omega$, q_0 *as in* [7], *and* $r \leqslant q_0 R$, $\overline{R}_0 = \min\left(\overline{R}_1, \overline{R}_2, \frac{1}{4} d(x, \partial\Omega)\right)$.

PROOF: Let $B(x, r)$ be such that $B(x, r) \subseteq B(x, q_0 R)$. Consider the problem

$$\int_{B(x,\, q_0^{-1}R)} \alpha(\tilde{u}, v)(dx) = 0 \tag{4.1}$$

$$\forall v \in D_0[a, B(x, q_0^{-1}R)], \qquad u - \tilde{u} \in D_0[a, B(x, q_0^{-1}R)]$$

The problem (4.1) has a unique solution and we have, [7],

$$\int_{B(x,\, r)} G^x_{B(x,\, q_0^{-1} r)}\,\alpha(\tilde{u}, \tilde{u})(dx) \leqslant C\left(\frac{r}{R}\right)^\beta \int_{B(x,\, R)} G^x_{B(x,\, q_0^{-1} R)}\,\alpha(\tilde{u}, \tilde{u})(dx)$$

and

$$\operatorname*{osc}_{B(x,\,r)} \tilde{u}^2 \leqslant C\left(\frac{r}{R}\right)^{\beta} \operatorname*{osc}_{B(x,\,R)} \tilde{u}^2 .$$

Let now be $w = \tilde{u} - u$ we have that w is the solution of the problem

$$(4.2) \qquad \int_{B(x,\, q_0^{-1}R)} \alpha(w,\, v)(dx) = \int_{B(x,\, q_0^{-1}R)} uv\mu(dx) - \int_{B(x,\, q_0^{-1}R)} v\sigma(dx)$$

$$\forall v \in D_0[a,\, B(x,\, q_0^{-1}R)]\,, \qquad w \in D_0[a,\, B(x,\, q_0^{-1}R)]\,.$$

We have

$$w(y) = \int_{B(x,\, q_0^{-1}R)} G^y_{B(x,\, q_0^{-1}R)}(u\mu(dx) + \sigma(dx)) \leqslant$$

$$\leqslant \left(\sup_{B(x,\, q_0^{-1}R)} |u|\right) \int_{B(y,\, 2q_0^{-1}R)} G^y_{B(y,\, 8q_0^{-1}R)}\mu(dx) + \eta_\sigma(2q_0^{-1}R) \leqslant$$

$$\leqslant C\left(\left(\sup_{B(x,\, q_0^{-1}R)} |u|\right)\eta_\mu(2q_0^{-1}R) + \eta_\sigma(2q_0^{-1}R)\right)$$

and

$$\int_{B(x,\, q_0^{-1}R)} G^x_{B(x,\, q_0^{-1}R)}\,\alpha(w,\, w)(dx) \leqslant C\left[\left(\sup_{B(x,\, q_0^{-1}R)} |u|^2\right)\eta^2_\mu(2q_0^{-1}R) + \eta^2_\sigma(2q_0^{-1}R)\right].$$

Then, taking into account the local L^∞-estimate Proposition 3.5., we obtain

$$\int_{B(x,\, r)} G^x_{B(x,\, q_0^{-1}r)}\,\alpha(u,\, u)(dx) \leqslant$$

$$\leqslant C\left(\frac{r}{R}\right)^{\beta} \int_{B(x,\, R)} G^x_{B(x,\, q_0^{-1}R)}\,\alpha(u,\, u)(dx) +$$

$$+ C\left[\left(\overline{R}_0^{-\nu}\int_\Omega u^2\, m(dx) + \eta^2_\sigma(2q_0^{-1}R)\right)\eta^2_\mu(2q_0^{-1}R) + \eta^2_\sigma(2q_0^{-1}R)\right]$$

and

$$\operatorname*{osc}_{B(x,r)} u^2 \leqslant C\left[\left(\frac{r}{R}\right)^{\beta} \operatorname*{osc}_{B(x,R)} u^2 + \left(\overline{R}_0^{-\nu}\int_{\Omega} u^2 m(dx) + \eta_\sigma^2(2q_0^{-1}R)\right)\eta_\mu^2(2q_0^{-1}R) + \eta_\sigma^2(2q_0^{-1}R)\right].$$

From Theorem 4.1 we easily obtain:

COROLLARY 4.2: *Let u be a local solution in Ω for the Dirichlet problem for the Schrödinger type operator associated with the form a and the potential μ and the source term σ (μ, $\sigma \in K_w^{\mathrm{loc}}(\Omega)$); then u is locally continuous in Ω.*

Now we consider the boundary behavior:

THEOREM 4.3: *Let u be a local solution in Ω for the Dirichlet problem for the Schrödinger type operator associated with the form a, the potential μ, the source term σ (μ, $\sigma \in K_w^{\mathrm{loc}}(\Omega)$) and the boundary data g ($g \in D_{\mathrm{loc}}[X_0, a] \cap L^\infty(X_0, m)$) i.e.*

$$\int_\Omega \alpha(u, v)(dx) + \int_\Omega uv\,\mu(dx) = \int_\Omega v\,\sigma(dx)$$

$$\forall v \in D_0[a, \Omega], \qquad u - g \in D_0[a, \Omega].$$

Then we have

$$\int_{B(x,\,r)} G^x_{B(x,\,q_0^{-1}r)}\,\alpha((u-k)^\pm, (u-k)^\pm)(dx) \leqslant$$

$$\leqslant C\exp\left(-\beta\int_r^{q_0^{-1}R} \delta(\varrho, x)\,\frac{d\varrho}{\varrho}\right)\int_{B(x,\,R)} G^x_{B(x,\,q_0^{-1}R)}\,\alpha((u-k)^\pm, (u-k)^\pm)(dx) +$$

$$+ C\left[\left(\overline{R}_0^{-\nu}\int_\Omega u^2\, m(dx) + \sup_{\partial\Omega} g^2 + \eta_\sigma^2(2q_0^{-1}R)\right)\eta_\mu^2(2q_0^{-1}R) + \eta_\sigma^2(2q_0^{-1}R)\right]$$

where $k \geqslant \sup\limits_{B(x,\,2q_0^{-1}R)\cap\partial\Omega} g$ *and*

$$\operatorname*{osc}_{B(x,\,r)} u^2 \leqslant C\exp\left(-\beta\int_r^{q_0^{-1}R}\delta(\varrho, x)\,\frac{d\varrho}{\varrho}\right)\operatorname*{osc}_{B(x,\,R)} u^2 +$$

$$+ C\left[\left([(\overline{R}_0^{-\nu}\int_\Omega u^2 m(dx) + \sup_{\partial\Omega} g^2 + \eta_\sigma^2(2q_0^{-1}R)\right)\eta_\mu^2(2q_0^{-1}R) + \eta_\sigma^2(2q_0^{-1}R) + \operatorname*{osc}_{B(x,2q_0^{-1}R)\cap\partial\Omega} g^2\right]$$

where β is a positive structural constant, $x \in \partial\Omega$, $B(x, 16q_0^{-1}R) \subseteq B(x, 4\overline{R}) \subset\subset X_0$, q_0 as in [4] [7], *and* $r \leqslant q_0 R$, $\overline{R} = \min\left(\overline{R}_1, \overline{R}_2, \frac{1}{4}d(x, \partial\Omega)\right)$.

PROOF: The same methods used in the proof of Theorem 4.1 allow us to prove the Theorem 4.3.

Consider the problem

$$\int_{B(x,\, q_0^{-1}R)\cap\Omega} \alpha(\tilde{u},\, v)(dx) = 0 \tag{4.3}$$

$$\forall v \in D_0[a,\, B(x,\, q_0^{-1}R)\cap\Omega], \qquad u-\tilde{u}\in D_0[a,\, B(x,\, q_0^{-1}R)\cap\Omega].$$

The problem (4.2) has a unique solution and we have, [4] [5] (see also [7]),

$$\int_{B(x,\, r)\cap\Omega} G^x_{B(x,\, q_0^{-1}r)\cap\Omega}\,\alpha((\tilde{u}-k)^+,(\tilde{u}-k)^+)(dx) \leqslant$$

$$\leqslant C\exp\left(-\beta\int_r^{q_0^{-1}R}\delta(\varrho,\, x)\,\frac{d\varrho}{\varrho}\right)\int_{B(x,\, R)\cap\Omega} G^x_{B(x,\, q_0^{-1}R)}\,\alpha((\tilde{u}-k)^+,(\tilde{u}-k)^+)(dx)$$

for $k \geqslant \sup\limits_{B(x,\, 2q_0^{-1}R)\cap\partial\Omega} g$,

$$\int_{B(x,\, r)\cap\Omega} G^x_{B(x,\, q_0^{-1}r)\cap\Omega}\,\alpha((\tilde{u}-k)^-,(\tilde{u}-k)^-)(dx) \leqslant$$

$$\leqslant C\exp\left(-\beta\int_r^{q_0^{-1}R}\delta(\varrho,\, x)\,\frac{d\varrho}{\varrho}\right)\int_{B(x,\, R)\cap\Omega} G^x_{B(x,\, q_0^{-1}R)}\,\alpha((\tilde{u}-k)^-,(\tilde{u}-k)^-)(dx)$$

for $k \leqslant \inf\limits_{B(x,\, 2q_0^{-1}R)\cap\partial\Omega} g$ and

$$\operatorname*{osc}_{B(x,\, r)\cap\Omega}\tilde{u}^2 \leqslant C\exp\left(-\beta\int_r^{q_0^{-1}R}\delta(\varrho,\, x)\,\frac{d\varrho}{\varrho}\right)\operatorname*{osc}_{B(x,\, R)\cap\Omega}\tilde{u}^2 + \sup_{B(x,\, R)\cap\Omega} g^2.$$

Let now be $w = \tilde{u}-u$ we have that w is the solution of the problem

$$\int_{B(x,\, q_0^{-1}R)\cap\Omega}\alpha(w,\, v)(dx) = \int_{B(x,\, q_0^{-1}R)\cap\Omega)} uv\mu(dx) - \int_{B(x,\, q_0^{-1}R)\cap\Omega)} v\sigma(dx) \tag{4.4}$$

$$\forall v\in D_0[a,\, B(x,\, q_0^{-1}R)\cap\Omega], \qquad w\in D_0[a,\, B(x,\, q_0^{-1}R)\cap\Omega].$$

We have

$$|w(y)| = \left| \int_{B(x,\, q_0^{-1}R)\cap\Omega} G^y_{B(x,\, q_0^{-1}R)\cap\Omega}(u\mu(dx) + \sigma(dx) \right| \leqslant$$

$$\leqslant C\left(\left(\sup_{B(x,\, q_0^{-1}R)\cap\Omega} |u|\right)\eta_\mu(2q_0^{-1}R) + \eta_\sigma(2q_0^{-1}R)\right)$$

and

$$\int_{B(x,q_0^{-1}R)\cap\Omega} G^x_{B(x,q_0^{-1}R)}\alpha(w,w)(dx) \leqslant C\left[\left(\sup_{B(x,q_0^{-1}R)\cap\Omega} |u|^2\right)\eta^2_\mu(2q_0^{-1}R) + \eta^2_\sigma(2q_0^{-1}R)\right].$$

Since $(u-k)^\pm$ are subsolutions in $B(x,\, 2q_0^{-1}R)$, taking into account the local L^∞-estimate in Proposition 3.5., we obtain

$$\int_{B(x,\, r)\cap\Omega} G^x_{B(x,\, q_0^{-1}r)\cap\Omega}\alpha((u-k)^+,(u-k)^+)(dx) \leqslant$$

$$\leqslant C\exp\left(-\beta\int_r^{q_0^{-1}R}\delta(\varrho,\, x)\,\frac{d\varrho}{\varrho}\right)\int_{B(x,\, R)\cap\Omega} G^x_{B(x,\, q_0^{-1}R)}\alpha((u-k)^+,(u-k)^+)(dx) +$$

$$+ C\left[\left(\overline{R}_0^{-\nu}\int_\Omega u^2\, m(dx) + \sup_{\partial\Omega} g^2 + \eta^2_\sigma(2q_0^{-1}R)\right)\eta^2_\mu(2q_0^{-1}R) + \eta^2_\sigma(2q_0^{-1}R)\right]$$

for $k \geqslant \sup_{B(2q_0^{-1}R)\cap\partial\Omega} g$,

$$\int_{B(x,\, r)\cap\Omega} G^x_{B(x,\, q_0^{-1}r)\cap\Omega}\alpha((u-k)^-,(u-k)^-)(dx) \leqslant$$

$$\leqslant C\exp\left(-\beta\int_r^{q_0^{-1}R}\delta(\varrho,\, x)\,\frac{d\varrho}{\varrho}\right)\int_{B(x,\, R)\cap\Omega} G^x_{B(x,\, q_0^{-1}R)}\alpha((u-k)^-,(u-k)^-)(dx) +$$

$$+ C\left[\left(\overline{R}_0^{-\nu}\int_\Omega u^2\, m(dx) + \sup_{\partial\Omega} g^2 + \eta^2_\sigma(2q_0^{-1}R)\right)\eta^2_\mu(2q_0^{-1}R) + \eta^2_\sigma(2q_0^{-1}R)\right]$$

for $k \leq \inf_{B(x,R) \cap \partial\Omega} g^2$ and

$$\operatorname*{osc}_{B(x,r) \cap \Omega} u^2 \leq C \exp\left(-\beta \int_r^{q_0^{-1}R} \delta(\varrho, x) \frac{d\varrho}{\varrho} \right) \operatorname*{osc}_{B(x,R) \cap \Omega} u^2 +$$

$$+ C \left[\left(\overline{R}_0^{-\nu} \int_\Omega u^2 m(dx) + \sup_{\partial\Omega} g^2 + \eta_\sigma^2 (2q_0^1 R) \right) \eta_\mu^2 (2q_0^{-1}R) + \eta_\sigma^2 (2q_0^{-1}R) + \operatorname*{osc}_{B(x, 2q_0^{-1}R) \cap \partial\Omega} g^2 \right].$$

From Theorem 4.3 we easily obtain:

COROLLARY 4.4: *Let*

$$\int_0^1 \delta(\varrho, x) \frac{d\varrho}{\varrho} = +\infty$$

then the function u considered in Theorem 4.3 *is continuous at x.*

An easy perturbation argument allow us to prove also the necessary part of the Wiener Criterion, then:

THEOREM 4.5: *A point $x \in \partial\Omega$ is regular for the Schrödinger type problem relative to a and to the potential $\mu \in K_w(\Omega)$ iff*

$$\int_0^1 \delta(\varrho, x) \frac{d\varrho}{\varrho} = +\infty.$$

Then the regular points of $\partial\Omega$ for the Schrödinger problem are independent of the potential $\mu \in K_w(\Omega)$ and are the same as the ones for the Dirichlet form a.

PROOF: The Corollary 4.4. gives that if

$$\int_0^1 \delta(\varrho, x) \frac{d\varrho}{\varrho, x} = +\infty$$

then x is a regular point (for the Schrödinger problem).

Assume that for a regular point (for the Schrödinger problem) x_0 we have

$$\int_0^1 \delta(\varrho, x_0) \frac{d\varrho}{\varrho} < +\infty.$$

As observed in [4], using the same methods as in [5] [14], we can prove that there exists $\bar{r}$ such that for $r \leqslant \bar{r}$

$$v_r(x) \leqslant \frac{1}{4}$$

q.e. in $B(x_0, s) \cap \Omega^c$ for s suitable, where $v_r(x)$ is the potential with respect to the form a of $\Omega_r = \Omega^c \cap B(x_0, r)$ in $B(x_0, 2r)$. Consider now the solution w_r of the problem

$$a(w, v) + \int_\Omega wv\mu(dx) = 0$$

$$\forall v \in D_0[B(x_0, 2r)], \qquad \phi(v-1) \in D_0[B(x_0, 2r) - \Omega_r]$$

$$w \in D_0[B(x_0, 2r)], \qquad \phi(w-1) \in D_0[B(x_0, 2r) - \Omega_r]$$

where ϕ is the cut-off between the balls $B(x_0, r)$ and $B(x_0, 2r)$.

Since x_0 is regular we have that we can choose s such that

$$w(x) \geqslant \frac{3}{4}$$

q.e. in $B(x_0, s) \cap \Omega^c$. Taking into account that $0 \leqslant w \leqslant 1$, an easy comparison argument gives that, for s suitable, we have

$$w(x) \leqslant \frac{1}{2}$$

q.e. in $B(x_0, s) \cap (B(x_0, 2r) - \Omega_r)$, then we have a contradiction and the result is proved.

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