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Some considerations concerning r-Gaussian random variables (**)

ABSTRACT. — A class of consistent estimators of the parameter r of a r-Gaussian random variable is given.

Un risultato riguardante le variabili aleatorie r-gaussiane

RIASSUNTO. — Si determina una classe di stimatori consistenti del parametro r relativo a variabili aleatorie r-gaussiane.

1. - INTRODUCTION

In some reliability problems it is important to have an asymptotically linear failure rate function. More precisely, it is often necessary to have lifetime distributions with a continuous density function f so that

$$\lim_{x \to +\infty} \frac{f(x)}{(1 - F(x))\,x} = \alpha\,, \tag{1.1}$$

where F is the associated distribution function and α is a positive real number. The class of the density functions which verify the condition (1.1) is not parameterizable. Nevertheless, it is easy to define a large parametric family $\mathcal{S}$ of survival functions in such a way that each element of $\mathcal{S}$ has a definitively continuous derivative which necessarily verifies (1.1). In this short paper, the family $\mathcal{S}$ is defined by means of the notion of the r-Gaussian random variable and some consistency results are obtained for suita-

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ble estimators of the «parameter» r, which characterizes the asymptotic behaviour of the element of $\mathcal{S}$.

2. - Notations

Let X be a real random variable and let S be the survival function $x \mapsto P\{X > x\}$. Given a real number r, X is said to have *r-Gaussian behaviour* if there exist a pair a, b of positive real numbers so that

$$S(x/a) \sim bx^{-r}\exp(-x^2/2),$$

namely

$$\lim_{x\to\infty} \frac{S(x/a)}{bx^{-r}\exp(-x^2/2)} = 1 .$$

In particular, if $a = 1$, X is said to be a *standard r*-Gaussian random variable. Note that a centered Gaussian random variable and a Weibull random variable with a shape parameter equal to 2, are a 1-Gaussian random variable and a 0-Gaussian random variable, with $b = 1$, respectively. Obviously the family of the centered Gaussian random variables is strictly contained in the family of the 1-Gaussian random variables. Moreover, the minimum of two independent random variables — standard r_1-Gaussian and r_2-Gaussian respectively — is a $(r_1 + r_2)$-Gaussian while the maximum is a standard $(r_1 \wedge r_2)$-Gaussian random variable. Finally, it must be noticed that, in many applications, b is given or, in the case of some classic distributions, b is equal to 1.

Now, let $\mathcal{S}$ be the class of survival functions with respect to particular r-Gaussian random variables. More precisely, the survival functions of $\mathcal{S}$ are definitively such that, for some $a > 0$,

$$S(x) \propto x^{-r}\exp(-x^2a^2/2).$$

It is easy to prove that each distribution function of $\mathcal{S}$ has a definitively continuous derivative f which satisfies the condition (1.1) with $\alpha = a^2$. Then, if the scale parameter a is known, it is important to determine the «parameter» r, in order to completely identify the asymptotic behaviour of S. In other words, we propose to study the estimators of r when standard r-Gaussian random variables are considered.

For this purpose, let $(X_n)_{n\geq 1}$ be a sequence of real independent standard r-Gaussian random variables and let

$$Y_n = X_1 \vee \ldots \vee X_n - \sqrt{2\log n}\,. \tag{2.1}$$

In the following section, we prove that the random variables Y_n, suitably normalized, are weakly but not strongly consistent estimators of r. Moreover, if $r = 0$, we show that $Y_n\sqrt{2\log n}$ converges stably to a Gumbel random variable.

3. - The main result

Suppose X_n are independent and identically distributed random variables. If X_1 is a standard r-Gaussian random variable, the following result may be proved.

(3.1) Theorem: *The random variable*

$$W_n = -\frac{2\sqrt{2\log n}}{\log(\log n)} Y_n$$

converges in probability to r and, almost surely, it turns out that

$$(3.2) \qquad r-2 = \liminf_n W_n, \qquad c = \limsup_n W_n,$$

where c is a constant so that $r \leqslant c \leqslant r+2$.

Proof: Fixed a real number x, let

$$d_n = -x\frac{\log(\log n)}{2\sqrt{2\log n}} + \sqrt{2\log n}\,.$$

We may observe

$$(3.3) \qquad \begin{aligned} P\{W_n > x\} &= P\{X_1 \vee \ldots \vee X_n < d_n\} \\ &\sim \exp(n\log(1 - P\{X_1 \geqslant d_n\})) \\ &\sim \exp(-nP\{X_1 \geqslant d_n\}) \\ &\sim \exp(-bd_n^{-r} e^{u_n/2}), \end{aligned}$$

where u_n denotes $x\log(\log n) - x^2(\log^2(\log n))/8\log n$. In other words, for $x \neq r$,

$$\lim_n P\{W_n > x\} = \lim_n \exp(-b2^{-r/2} e^{(x-r)\log\sqrt{\log n}}).$$

It is clear that the previous limit is 1 for $x < r$ and 0 for $x > r$. Thus W_n converges in distribution to r and consequently converges in probability to r. Now it must be proven that W_n does not converge almost surely. The random variables $\liminf_n W_n$, $\limsup_n W_n$ are measurable with respect to the symmetric σ-field and, thanks to the Hewitt-Savage Theorem, are degenerate. First, in order to prove the relation (3.2), it must be observed that, for $x > r$, the series

$$\sum_{n\geqslant 1} P\{W_n > x\} \quad \text{and} \quad \sum_{n\geqslant 1} \exp(-b2^{-r/2} e^{(x-r)\log\sqrt{\log n}})$$

have the same behaviour. Since the second series converges for $x > r+2$, the random variable $\limsup_n W_n$ is almost surely a constant less than or equal to $r+2$. Moreover, owing to Fatou's Lemma, $\limsup_n W_n$ is almost surely greater than or equal to r. Finally, it must still be proven that

$$\liminf_n W_n = r - 2 \quad \text{a.s.}$$

Note that

$$\{\liminf_n W_n < x\} = \{\limsup_n - W_n > -x\}$$

$$\subset \limsup_n \{X_1 \vee \ldots \vee X_n > d_n\} = \limsup_n \{X_n > d_n\}.$$

Similarly $\limsup_n \{X_n \geqslant d_n\} \subset \{\liminf_n W_n \leqslant x\}$. Thanks to the Borel-Cantelli Lemma, it suffices to show that the series

$$\sum_{n \geqslant 1} P\{X_n > d_n\}$$

converges for $x < r - 2$ and diverges for $x > r - 2$. This arises from the equivalence

$$P\{X_n > d_n\} \sim b 2^{-r/2} n^{-1} (\log n)^{(x-r)/2}.$$

The theorem is thus proved.

(3.4) REMARK: From the relation (3.3), which characterizes the asymptotic behaviour of $x \mapsto P\{W_n > x\}$, it follows that W_n also converges in any L^p-space to r. Analogous results hold for non-standard r-Gaussian random variables by considering $X_1 \vee \ldots \vee X_n - a\sqrt{2 \log n}$ instead of Y_n.

Moreover, if the random variables X_n are independent and their survival functions are all definitively bounded below (resp. bounded above) by the same survival function of a random variable having a standard r-Gaussian behaviour (resp. s-Gaussian), the sequence $(Y_n^- \log^\alpha n)$ (resp. $(Y_n^+ \log^\alpha n)$) converges almost surely to 0, for any $\alpha < 1/2$. In particular, these results are applied when X_n are *sub-Gaussian* random variables that is, if there exists a centered Gaussian random variable Z so that, for every real number c, $E[e^{cX_n}] \leqslant E[e^{cZ}]$ holds. As a matter of fact, in this case, the survival function of X_n is bounded above by the survival function of a 0-Gaussian random variable. For more detailed results and characterizations of sub-Gaussian random variables see Ostrovskii [2] and Talagrand [3].

When W_n converges in probability to 0, namely $r = 0$, it may be interesting to evaluate the infinitesimal order of W_n and, more precisely, to determine a suitable normalization of W_n, in such a way that the limit in distribution is not degenerate. For this purpose the following result is proved.

(3.5) THEOREM: *Suppose X_n are independent and identically distributed random* *variables. If X_1 is a standard 0-gaussian random variable, then $(Y_n \sqrt{2 \log n})$ converges stably to a Gumbel random variable.*

More precisely, for any real number x and for any non-negligible event H, it holds that

$$\lim_n P_H(Y_n \sqrt{2 \log n} \leq x) = \exp(-be^{-x}), \tag{3.6}$$

where b is equal to $\lim_{t \to \infty} P(X > t) \exp(t^2/2)$.

Proof: It suffices to prove the relation (3.6) for every real number x and for every event H equals to $\bigcap_{i \leq m} \{X_i \in A_i\}$, where A_i is a Borelian set of R. To this end, fixed x, let

$$v_n = x(2 \log n)^{-1/2} + \sqrt{2 \log n}\,.$$

Given H, from the 0-Gaussian behaviour of X_n, it turns out that

$$P_H\{Y_n \sqrt{2 \log n} \leq x\} = P_H\{Y_n < x(2 \log n)^{-1/2}\}$$
$$= P_H\{X_1 \vee \ldots \vee X_n < v_n\}$$

is equivalent to

$$\exp((n-m) \log(1 - be^{-v_n^2/2}))$$

and consequently to $\exp(-be^{-x})$. The theorem is thus complete.

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