Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica e Applicazioni
117° (1999), Vol. XXIII, fasc. 1, pagg. 117-123

V. E. LYANTSE (W. LANCE) (*)

Shadow of a Measure

ABSTRACT. — Riesz representation theorem implies conversion from a hyperfinite measure space $(T, 2^T, m)$ to a space $(\boldsymbol{T}, \boldsymbol{A}, \mu)$ with σ-additive measure μ in the framework of Nelson's Internal Set Theory [1-3].

Ombra di una misura

SUNTO. — Nel quadro della cosiddetta *Internal Set Theory* di Nelson, e valendosi del teorema di rappresentazione di Riesz-Markov, si trasforma uno spazio misurato iperfinito $(T, 2^T, m)$ in uno spazio misurato standard $(\boldsymbol{T}, \boldsymbol{A}, \mu)$, con μ misura σ-additiva.

An essential notion of Nonstandard Analysis (NSA) is those of the *shadow*. Let (X, d) be a standard metric space. A point $x \in X$ is said to be *nearstandard* (write $x \in {}^{nst}X$) iff there exists a standard $y \in X$ such that $d(x, y) \approx 0$ (¹). If such y exists, it is unique, is said to be the shadow of x and denoted by ${}^\circ x$. The importance of this notion is clear, for instance, from the following example. Let $(x_n)_{n \in \mathrm{N}}$ be a standard sequence in X. Then it is convergent iff for any $n \approx \infty$ (²) $x_n \in {}^{nst}X$ and the shadow ${}^\circ x_n$ is independent of $n \approx \infty$. In this case for all $n \approx \infty$ ${}^\circ x_n = \lim\limits_{k \to \infty} x_k$. Note that the map $x \mapsto {}^\circ x$ is noninjective: if $d(x_1, x_2) \approx 0$, then $x_1 \in {}^{nst}X$ implies $x_2 \in {}^{nst}X$ and ${}^\circ x_1 = {}^\circ x_2$.

A point $x \in X \setminus {}^{nst}X$ is said to be *remote*. The following *remoteness* theorem is known [4]. Let $(x_n)_{n \in \mathrm{N}}$ be such a sequence in X that $\forall\, p, q \in \mathrm{N}\ p \neq q \Rightarrow$ $\Rightarrow d(x_p, x_q) \gg 0$ (³). Then x_n is remote for some $n \approx \infty$. For instance, let $X = H$ be a

(*) Indirizzo dell'Autore: Department of Mathematics, L'viv State University, 290602 L'viv, Ukraine; E-mail: Wlanc@litech.lviv.ua

(**) Memoria presentata l'8 giugno 1999 da Mark Višik, uno dei XL.

(***) Research partially supported by a grant from the Fundamental Research State Foundation, Ukraine.

(¹) Read «is infinitesimal» for «≈ 0».

(²) Read «is infinite» for «$\approx \infty$».

(³) Read «is not infinitesimal» for «$\gg 0$».

standard Hilbert space and $(e_n)_{n\in N}$ a standard orthonormal basis of H. Then for each $n \approx \infty$ e_n is remote in spite of $\|e_n\| = 1$. (Note that each finite real number is nearstandard.)

It is worthwhile to extend the notion of nearstandardness as follows. Let X be a standard normed space and $l \in X^*$ (adjoint space).

1. Proposition: Suppose $\|l\| \ll \infty$ (4). Then l is *weakly nearstandard*, i.e. there exists $k \in {}^{\sigma}(X^*)$ (5), for which

$$\forall x \in {}^{\sigma}X \quad l(x) \approx k(x). \tag{1}$$

Such k is unique.

Proof: Let $l_0 \in ({}^{\sigma}\mathrm{C})^{({}^{\sigma}X)}$ (6) be the map defined by $\forall x \in {}^{\sigma}X \; l_0(x) = {}^{\circ}[l(x)]$. Since $\|l\| \ll \infty$, we have $\|l(x)\| \leq \|l\| \cdot \|x\| \ll \infty$ for standard x, therefore ${}^{\circ}[l(x)]$ is defined (as shadow of a finite complex number). Obviously, for standard $\alpha_1, \alpha_2 \in \mathrm{C}$, $x_1, x_2 \in X$ we have

$$l_0(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 l_0(x_1) + \alpha_2 l_0(x_2), \qquad |l_0(x_1)| \leq {}^{\circ}\|l\| \cdot \|x_1\|. \tag{2}$$

Recall that each map $f_0 \in ({}^{\sigma}F)^{({}^{\sigma}E)}$, where E, F are arbitrary standard sets, has a unique extension $f \in {}^{\sigma}(F^E)$, which is said to be the *standard extension* of f_0 ([1-3]). Define k as the standard extension of l_0. Transfer principle (7) and (2) imply that k is a linear continuous functional on X, that is $k \in X^*$. For standard x we have $k(x) = l_0(x) \approx l(x)$, i.e. (1) holds. Uniqueness of k is evident: a standard function which equals zero at standard points is identically zero. ■

2. Remark: The functional k described above is said to be the *shadow* of l (in the weak sense) and will be denoted by ${}^{\circ}l$. Emphasize that if l is nearstandard in the strong sense, i.e. $\|l - k\| \approx 0$ for some $k \in {}^{\sigma}(X^*)$, then it is weakly nearstandard and the last k is the same as k in (1), $k = {}^{\circ}l$.

3. Remark: Each $x \in X$ is in a natural way some element of X^{**}, with the same norm $\|x\|$. Therefore, if $\|x\| \ll \infty$, then x has the shadow ${}^{\bullet}x$ which belongs to X^{**}. If X is reflexive, then ${}^{\bullet}x$ can be regarded as some standard element ${}^{\circ}x$ of X, which is uniquely defined by $\forall\, l \in {}^{\sigma}(X^*)\; l(x) \approx l({}^{\circ}x)$. For instance, if $X = H$ is a standard Hilbert space, $x \in H$, and $\|x\| \ll \infty$, then the shadow ${}^{\circ}x$ of x is uniquely defined by conditions: ${}^{\circ}x \in {}^{\sigma}H$ and $\forall\, y \in {}^{\sigma}H\; (x|y) \approx ({}^{\circ}x|y)$.

Now consider an interesting special case. In what follows (T, d) denotes a stan-

(4) Read «is finite» or «is noninfinite» for «$\ll \infty$».
(5) Read «is a standard member of the set E» for «$\in {}^{\sigma}E$».
(6) F^E denotes the set of functions defined on the set E with values in the set F.
(7) Transfer principle is: $\forall^{\sigma} x\, p(x) \Rightarrow \forall x\, p(x)$, where property $p(x)$ can be expressed in language of usual mathematics. Linearity and boundness are such properties.

dard compact metric space, Λ the algebra of borelian subsets $\mathcal{E} \subsetneq \mathbf{T}$. By definition a *charge* on $\mathbf{T}$ is a σ-additive function defined on Λ with values in C. Each charge c generates a linear functional $l = l_c$ on the space $C(\mathbf{T})$ of continuous functions $x \in \mathrm{C}^{\mathbf{T}}$ (with norm $\|x\|_\infty := \max\limits_{t \in \mathbf{T}} |x(t)|$) by $l(x) := \int\limits_{\mathbf{T}} x \, dc$. Note that $\|l\| = \mathrm{var}_{\mathbf{T}} c$ (*).

4. Theorem: If $\mathrm{var}_{\mathbf{T}} c \ll \infty$, then c is *weakly nearstandard*, i.e. there exists a unique standard charge $°c$ such that

$$\forall \xi \in {}^{\sigma}C(\mathbf{T}) \int\limits_{\mathbf{T}} \xi \, d(°c) = {}^{\circ}\!\int\limits_{\mathbf{T}} \xi \, dc \,. \tag{3}$$

Proof: Define $\forall \xi \in C(\mathbf{T})$ $l(\xi) := \int\limits_{\mathbf{T}} \xi \, dc$. Then $l \in (C(\mathbf{T}))^*$ and $\|l\| = \mathrm{var}_{\mathbf{T}} c \ll \infty$. By proposition 1 l has the shadow $°l \in (C(\mathbf{T}))^*$. By Riesz-Markov representation theorem $°l$ is of the form $(°l)(\xi) = \int\limits_{\mathbf{T}} \xi \, d(°c)$ for some σ-additive charge $°c$ on Λ. Transfer principle (in the form $\exists x \; p(x) \Rightarrow \exists^{\sigma} x \; p(x)$) and uniqueness of $°c$ imply that $°c$ is standard. ■

5. Remark: Obviously, theorem 4 can be generalized to the case of locally compact $(\mathbf{T}, d)$. Theorem 4 prompts a method to transform a hyperfinite measure space to a standard measure space. The reader is invited to compare this construction with use of the Loeb measure [5-8]. Let us explain that a *hyperfinite measure space* is a triple $(T, 2^T, m)$ where T is a set such that $\mathrm{card}\, T \in \mathrm{N} \setminus {}^{\sigma}\mathrm{N}$ and m is an additive function with positive values on the algebra 2^T of all (internal) subsets $E \subsetneq T$. If m_t denotes the value $m\{t\}$ of m at one-point set $\{t\}$, then $\forall E \in 2^T$ $mE := \sum\limits_{t \in E} m_t$.

6. Theorem: Let a hyperfinite set T be a subset of $\mathbf{T}$ where $(\mathbf{T}, d)$ is a standard compact metric space. Any additive measure m defined on 2^T such that $mT \ll \infty$ generates on the algebra Λ of borelian subsets of $\mathbf{T}$ a standard σ-additive measure μ which is uniquely determined by

$$\forall \xi \in {}^{\sigma}C(\mathbf{T}) \int\limits_{\mathbf{T}} \xi(\tau) \, \mu(d\tau) = {}^{\circ}\!\sum_{t \in T} \xi(t) \, m_t \,. \tag{4}$$

Proof: The measure m on 2^T induces the measure $\widehat{m}$ on Λ by the formula: $\forall \mathcal{E} \in \Lambda$ $\widehat{m}\mathcal{E} = m(\mathcal{E} \cap T)$. This $\widehat{m}$ is trivially σ-additive. Indeed, let $\mathcal{E} \in \Lambda$ be a disjunctive union $\mathcal{E} = \bigcup\limits_{n \in \mathrm{N}} \mathcal{E}_n$, $\mathcal{E}_n \in \Lambda$. Only finite quantity of $\mathcal{E}_n \cap T$ is not empty. Therefore for some $k \in \mathrm{N}$ $\forall n > k$ $\widehat{m}\mathcal{E}_n = 0$. Since $\widehat{m}\mathbf{T} = mT \ll \infty$, by theorem 4 there exists a unique stan-

(*) $\mathrm{var}_{\mathcal{E}} c$ denotes the variation of c on the set $\mathcal{E}$.

dard charge μ defined on Λ such that (4) holds. Since for $\xi \geqslant 0$ the right side in (4) is $\geqslant 0$, this charge is a measure. ■

7. Example: Let $\boldsymbol{T} := [-1, +1] \subset \mathbb{R}$ and T be some hyperfinite subset of $\boldsymbol{T}$. For each $t \in T$ denote by $t + dt$ the member of T which follows immediately after t in the sense of $<$. Define $\forall E \in 2^T \; mE := \sum_{t \in E} dt$ where $dt := (t + dt) - t$. Then $(T, 2^T, m)$ is a hyperfinite measure space. Suppose that the first t equals -1, the last $t + dt$ equals $+1$, and $\forall t \in T \; dt \approx 0$. Then the standard measure space $(\boldsymbol{T}, \Lambda, \mu)$ generated by $(T, 2^T, m)$ according to theorem 6 is $([-1, +1], \Lambda, \mu)$ where μ is the standard Lebesgue measure. This is clear from (4).

8. Example: Let $\boldsymbol{T} = [-1, +1]$ and ε be some positive infinitesimal number. For any borelian $\mathcal{E} \subseteq \boldsymbol{T}$ put $m\mathcal{E} := (2\varepsilon)^{-1}\mu(\mathcal{E} \cap [-\varepsilon, +\varepsilon])$ where μ is the standard Lebesgue measure on $\boldsymbol{T}$. The shadow of μ (in the weak sense; see theorem 4) is the Dirac measure concentrated at 0. Indeed, $\forall \xi \in {}^{st}C[-1, +1]$

$$\int_{\boldsymbol{T}} \xi \, d({}^{*}\mu) = {}^{*}\!\left(\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \xi(\tau) \, d\tau \right) = \xi(0); \qquad \text{see (3)}. \qquad ■$$

9. Example: Let H be a standard (separable or not) Hilbert space and $A \in \mathcal{B}(H)$ a standard selfadjoint operator. Denote by H_0 a subspace of H such that ${}^{st}H \subset H_0$ and $\dim H_0 \in \mathbb{N}$. (The existence of such H_0 follows from the idealization principle of IST [1, 4, 9].) Let P be the orthoprojector $H \to H_0$. Denote by A' the restriction of PAP to H_0. Since A' is an operator in a finite-dimensional space, its spectrum $\sigma(A')$ consists of eigenvalues only. For each eigenvalue $\lambda \in \sigma(A')$ denote by P'_λ the corresponding eigenprojector of A'. Since A' is selfadjoint, P'_λ is an orthoprojector in H_0. Now for any $x \in {}^{st}H$ there arises a hyperfinite measure space $(T, 2^T, m)$, where $T := \sigma(A')$ and

$$\forall E \in 2^T \; mE = m_x E = \sum_{\lambda \in \sigma(A)} (P'_\lambda x \mid x). \tag{5}$$

Suppose that $\|x\| \ll \infty$. Then $\|m\| = mT = \|x\|^2 \ll \infty$. According to theorem 6 $(T, 2^T, m)$ generates a standard measure space $(\boldsymbol{T}, \Lambda, \mu)$ with a standard σ-additive measure $\mu (= \mu_x)$ such that $\boldsymbol{T} = [-\|A\|, +\|A\|]$ and Λ is the family of borelian sets $\mathcal{E} \subseteq \boldsymbol{T}$. Just the standard μ is uniquely determined by

$$\forall \xi \in {}^{st}C(\boldsymbol{T}) \int_{\boldsymbol{T}} \xi \, d\mu = {}^{*}\!\sum_{\lambda \in \sigma(A')} \xi(\lambda)(P'_\lambda x \mid x).$$

We claim that

$$\forall x \in H \quad \forall \delta \in \Lambda \quad \mu\delta = (P(\delta)\, x | x)\,, \tag{6}$$

where $\{P(\delta)\}_{\delta \in \Lambda}$ is the spectral family of A (compare with [7]).

Proof: It is easy to see that for any *standard* polynomial $p(\lambda)$ and any $x \in {}^{\sigma}H$ we have $p(A')\, x = p(A) x$; note that $\forall x \in {}^{\sigma}H \;\; Px = x$. Thus for $x \in {}^{\sigma}H \;\; (p(A)\, x | x) = (p(A')\, x | x) = \sum_{\lambda \in T} p(\lambda)(P'_{\lambda} x | x) \approx \int_T p\, d\mu$. Since the first and the last members of this chain are standard, they are equal. By transfer principle, the equality $(p(A)\, x | x) = \int_T p\, d\mu$, holds for all (optionally standard) p and x. This proves our assertion. ■

Now we want to give some complementary information about the measure μ defined by (4). The notation below are as in theorem 6.

Let Q be an embedding $2^T \to \Lambda$ such that $\forall t, s \in T \;\; t \neq s \Rightarrow Qt \cap Qs = \emptyset$ where $Qt := Q\{t\}$, $\forall E \in 2^T \;\; QE := \bigcup_{t \in E} Qt$, and $QT = T$ (or more generally T coincides with the standardization ${}^{S}(QT)$ of the set QT; see [10] where (T, Λ, Q) is named the *standard filling* of the (hyper)finite set T). Then each σ-additive measure ν defined on Λ induces some additive measure n on 2^T which is given by

$$\forall E \in 2^T \qquad nE = \nu QE\,. \tag{7}$$

10. Theorem: Suppose that the measure ν is standard and

$$1) \;\; \forall t \in T \;\; \operatorname{diam} Qt = 0\,, \qquad 2) \;\; \max \frac{m_t}{\nu_t} \ll \infty\,,$$

where $m_t := m\{t\}$, $n_t := \nu Qt$. Then the measure μ generated by m (see (4)) is absolutely continuous relative to the measure ν.

For proof we need the following

11. Lemma: Put for $\xi \in C(T)$, $t \in T$,

$$\mathcal{N}\xi(t) := \frac{1}{\nu Qt} \int_{Qt} \xi\, d\nu\,. \tag{8}$$

Then $\mathcal{N}$ is continuous as a map $L_2(T, \nu) \to L_2(T, n)$.

Proof: By Bunyakowski-Cauchy inequality $|\mathcal{N}\xi(t)|^2 \leqslant (1/n_t) \int_{Qt} |\xi|^2\, d\nu$. Therefore $\|\mathcal{N}\xi\|^2_{L_2(T,n)} = \sum_{t \in T} |\mathcal{N}\xi(t)|^2 n_t \leqslant \sum_{t \in T} \int_{Qt} |\xi|^2\, d\nu \leqslant \|\xi\|^2_{L_2(T,\nu)}$. ■

Proof of theorem 10: For $\xi \in {}^{\#}C(T)$ the condition $\operatorname{diam} Qt \approx 0$ implies $\mathcal{N}\xi(t) \approx \xi(t)$. Since $mT \ll \infty$, we have $\sum_{t\in T} \xi(t)\, m_t \approx \sum_{t\in T} \mathcal{N}\xi(t)\, m_t$. Once more by Bunyakowski-Cauchy inequality and by lemma 11,

$$\left|\sum_{t\in T}\xi(t)\, m_t\right|^2 \approx \left|\sum_{t\in T}\mathcal{N}\xi(t)\, m_t\right|^2 \leqslant mT \sum_{t\in T}\frac{m_t}{n_t}\,|\mathcal{N}\xi(t)|^2 n_t \leqslant$$

$$\leqslant \gamma^2 \|\mathcal{N}\xi\|^2_{L_2(T,\,\nu)} \leqslant \gamma^2 \|\xi\|^2_{L_2(T,\,\nu)},$$

where $\gamma^2 := mT\cdot\max(m_t/n_t) \ll \infty$. Since $\forall \xi \in {}^{\#}C(T)$ $\sum_{t\in T}\xi(t)\, m_t \approx \int_T \xi\, d\mu$, we see that

$$\left|\int_T \xi\, d\mu\right| \leqslant \gamma_1 \|\xi\|_{L_2(T,\,\nu)}$$

for some standard $\gamma_1 > 0$. By transfer this inequality holds for all (optionally standard) $\xi \in C(T)$. This means that the functional $\xi \mapsto \int_T \xi\, d\mu$ belongs to $L_2(T, \nu)^*$. By Riesz representation theorem (for Hilbert spaces)

$$\int_T \xi\, d\mu = \int_T \xi\,\bar{\eta}\, d\nu$$

for some $\eta \in L_2(T, \nu)$. Taking into account that $C(T)$ is dense in $L(T, \mu)$ and $L_2(T, \nu)$ we replace here ξ by characteristical function χ_δ of an arbitrary $\delta \in \Lambda$. We get

$$\forall \delta \in \Lambda \quad \mu\delta = \int_\delta \mu'_\nu\, d\nu,$$

where $\mu'_\nu := \eta$ is proved to be the Radon-Nikodym derivative of μ with respect to ν. ■

REFERENCES

[1] E. Nelson, *Internal Set Theory*, Bull. Amer. Math. Soc., 83, v. 6 (1977), 1165-1198.

[2] F. Diener - G. Reeb, *Analyse Non Standard*, 1989, Hermann, ESA, Paris.

[3] R. Lutz - M. Goze, *Nonstandard Analysis: a practical guide with applications*, Lect. Notes in Math., 881, 261p. Springer-Verlag, Berlin-New York, 1981.

[4] M. Davis, *Applied Nonstandard Analysis*, A Willey-Interscience Publication, New York-London-Sidney-Toronto, 1977.

[5] P. Loeb, *Conversion from nonstandard to standard measure spaces and applications in probability theory*, Trans. Amer. Math. Soc., 211 (1975), 114-122.

[6] N. Cutland, *Nonstandard measure theory and its applications*, Bull. London Math. Soc., 15 (1983), p. 529-589.

[7] T. Lindstrøm, *Invitation to Nonstandard Analysis*, in: N. Cutland, Nonstandard Analysis and its Applications, Cambridge University Press (1988), 1-105.

[8] S. Albeverio - J. Fenstad - R. Høeg-Krohn - T. Lindstrøm, *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press (1986).

[9] A. Bernstein, *The spectral theorem - a nonstandard approach*, Z. Math. Logic, Grundlagen Math., v. 18 (1972), 419-434.

[10] V. Lyantse (W. Lance), *Nearstandardness on finite set*, Dissert. Math., 369 (1997), 1-67, Warszawa, PWN.