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Stochastic Integral of Process measures in Banach spaces III. Square Integrable Martingale Measures

ABSTRACT. — In this paper we present the construction and the properties of the stochastic integral for a process measure X with values in a Banach space E, if X is summable. The approach is measure theoretic: we associate to X a measure I_X with values in L^p_E and apply the general integration theory with respect to a vector measure with finite semivariation. It has been proved previously that process measures with integrable variation or integrable semivariation are summable. The main result of the paper is that an orthogonal martingale measure M with values in a Hilbert space is summable and that the stochastic integral $H \cdot M$ is again an orthogonal martingale measure.

Integrazione stocastica rispetto a una misura-processo in uno spazio di Banach. Parte III: caso di una misura-martingala di quadrato integrabile

SUNTO. — Nella presente memoria si costruisce e si studia l'integrale stocastico rispetto a una misura-processo X, a valori in uno spazio di Banach E, nel caso in cui X sia sommabile. L'impostazione scelta è fondata sulla teoria della misura: essa consiste nell'associare a X una misura I_X a valori in L^p_E e nell'applicare la teoria generale dell'integrazione rispetto a una misura vettoriale con semivariazione finita. È stato già dimostrato, in precedenti lavori, che la sommabilità di X è assicurata quando la misura-processo X abbia variazione, o semivariazione, integrabile. Il principale risultato della presente memoria consiste nel dimostrare che, se M è una misura martingala ortogonale, a valori in uno spazio di Hilbert, allora M è sommabile, e l'integrale stocastico $H \cdot M$ è a sua volta una misura martingala ortogonale.

INTRODUCTION

This paper is a continuation of the papers [D.5] and [D.6], in which we studied the stochastic integral of process measures with values in Banach spaces and having

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integrable variation or integrable semivariation. In this paper we study the stochastic integral of square integrable martingale measures with values in Hilbert spaces.

The real valued orthogonal martingale measures have been introduced by J. Walsh [W]. He defined the stochastic integral using a method similar to Itô's isometry for the usual square integrable martingales. This method cannot be used for square integrable martingales in Banach spaces. At the same time, the classical stochastic integral for real valued square integrable martingales or for real valued orthogonal martingale measures is not a genuine integral, in the sense that it is not an integral with respect to a measure. It would be desirable, as in classical Measure Theory, to have a space of integrable functions, with a norm on it for which it is a Banach space, and an integral for integrable functions, which would be the stochastic integral. Also desirable would be to have Vitali and Lebesgue convergence theorems for the integrable functions. In order to fulfill this goal, we apply the general integration theory with respect to vector measures with finite semivariation, presented in [B-D.1] (Appendix I) and in [B-D.2]. This approach has been used in [B-D.1] and in [D.2] to define the stochastic integral for one parameter processes and in [D.3] and [D.4] for the stochastic integral of two parameter processes. The same approach has been used in [D.5] and [D.6] and is used in this paper again, to study the stochastic integral of process measures. We devote § 1 to the construction of the stochastic integral for those process measures which are summable. The framework for this construction consists of a probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions, a Lusin space L endowed with its Borel σ-algebra $\mathcal{L}$, $1 \leq p < \infty$ and E, F, G Banach spaces with $E \subset L(F, G)$. A p-process measure is a function $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ which, for fixed $B \in \mathcal{L}$ is a cadlag, adapted process and for fixed $t \geq 0$, is a σ-additive measure from $\mathcal{L}$ into L^p_E. We associate to X a finitely additive measure I_X with values in L^p_E. The process X is said to be *p-summable* if I_X can be extended to a σ-additive measure on the σ-algebra $\mathcal{P} \otimes \mathcal{L}$, where $\mathcal{P}$ is the predictable σ-algebra of $\mathbb{R}_+ \times \Omega$. In this case we can apply the general integration theory and define the integral $\int H dI_X$. The stochastic integral $H \cdot X$ is, essentially, the process measure defined by

$$(H \cdot X)_t(B) = \int_{[0,t] \times B} H dI_X.$$

In § 2 we present a series of properties of the stochastic integral. Among the main results of this paragraph we quote: the convergence theorems, which ensure that the space of integrable functions is complete (Corollary 2.9); the p-summability of the stochastic integral (Theorem 2.13); and Theorem 2.15 stating that if X is a martingale measure, then the stochastic integral $H \cdot X$ is also a martingale measure.

But a martingale measure is not necessarily summable, even if it is square integrable. An additional condition has been found by Walsh — the orthagonality — which guarantees its summability.

The main results of the paper are contained in § 3, devoted to orthogonal martin-

gale measures. We prove that an orthogonal martingale measure with values in a Hilbert space is 2-summable, therefore the stochastic integral $H \cdot X$ is defined for certain real valued processes (Theorem 3.9). Moreover, the stochastic integral of an orthogonal martingale measure is itself an orthogonal martingale measure (Theorem 3.13).

1. - Process measures and their stochastic integral.

1.1. *Notations.*

Throughout the paper we shall adopt the following notations. In general, we adopt the definitions and notations used in [D-M].

1) $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is a filtration satisfying the usual conditions. We assume $\mathcal{F}_\infty = \mathcal{F}$.

Any process X on $\Omega \times \mathbb{R}_+$ will be automatically considered to be extended with 0 for $t < 0$. The filtration is also extended with $\mathcal{F}_t = \mathcal{F}_0$ for $t < 0$.

$\mathcal{R}$ is the ring generated by the predictable rectangles of $\Omega \times \mathbb{R}_+$ of the form $\{0\} \times A$ with $A \in \mathcal{F}_0$ or of the form $(s, t] \times A$ with $A \in \mathcal{F}_s$. The σ-algebra generated by $\mathcal{R}$ is the σ-algebra $\mathcal{P}$ of predictable subsets of $\Omega \times \mathbb{R}_+$.

We denote by $\mathcal{R}_{-\infty}$ the ring of subsets of $\Omega \times \mathbb{R}$, generated by the predictable rectangles of the form $(s, t] \times A$ with $-\infty \leq s < t$ and $A \in \mathcal{F}_s$, and by $\mathcal{P}_{-\infty}$ the σ-algebra of predictable subsets of $\Omega \times \mathbb{R}$, generated by $\mathcal{R}_{-\infty}$. The ring $\mathcal{R}$ and the σ-algebra $\mathcal{P}$ are the traces on $\Omega \times \mathbb{R}_+$ of $\mathcal{R}_{-\infty}$ and $\mathcal{P}_{-\infty}$ respectively.

2) $(L, \mathcal{L})$ is a Lusin space, endowed with its Borel σ-algebra $\mathcal{L}$. Without loss of generality, we shall take $L = \mathbb{R}$ and $\mathcal{L} = \mathcal{B}(\mathbb{R})$. Moreover, we shall assume that any function defined on L vanishes on $(-\infty, 0]$, and that any measure on $\mathcal{L}$ has its support contained in $(0, \infty)$. The justification for this assumption is that the Lusin space L is homeomorphic to a Borel subset of the real line, which can be taken in $(0, \infty)$; then we extend any function on L with 0 outside L; and any measure μ on $\mathcal{L}$ is extended to $\mathcal{B}(\mathbb{R})$ by $\mu(A) = \mu(A \cap L)$, for $A \in \mathcal{B}(\mathbb{R})$.

$\mathcal{S}$ is the ring generated by the intervals $(x, y]$ in L, and $\mathcal{S}_{\mathbb{Q}}$ is the ring generated by the intervals $(x, y]$ in L with x, y rational. The σ-algebra generated by $\mathcal{S}$ and $\mathcal{S}_{\mathbb{Q}}$ is $\mathcal{L}$.

We denote by $\mathcal{R} \times \mathcal{S}$ the semiring of rectangular sets $A \times B$ with $A \in \mathcal{R}$ and $B \in \mathcal{S}$, and by $\mathcal{C} = r(\mathcal{R} \times \mathcal{S})$ the ring generated by $\mathcal{R} \times \mathcal{S}$.

We denote by $\mathcal{P} \times \mathcal{L}$ the semiring of rectangular sets $A \times B$ with $A \in \mathcal{P}$ and $B \in \mathcal{L}$ and by $\mathcal{P} \otimes \mathcal{L}$ the σ-algebra generated by $\mathcal{P} \times \mathcal{L}$.

3) E, F, G are Banach spaces with $E \subset L(F, G)$ isometrically. $1 \leq p < \infty$ and $L^p_E = L^p_E(P)$. We have $L^p_E \subset L(F, L^p_G)$, continuously.

4) Let $\mathcal{K}$ be a ring of subsets of L generating the σ-algebras $\mathcal{L}$. We can take, for example, $\mathcal{K} = \mathcal{L}$.

DEFINITION 1.1: *Let $X: \Omega \times \mathrm{R}_+ \times \mathcal{K} \to E$ be a function. Such a function is called a process set function on $\mathcal{K}$.*

a) *We say X is adapted (to the filtration $(\mathcal{F}_t)$) if for every $t \geq 0$ and $B \in \mathcal{K}$, the function $\omega \mapsto X_t(\omega, B)$ is $\mathcal{F}_t$-measurable.*

b) *X is right continuous (respectively left continuous, cadlag) if for every $\omega \in \Omega$ and $B \in \mathcal{L}$, the function $t \mapsto X_t(\omega, B)$ is right continuous (respectively left continuous, cadlag).*

c) *Let $1 \leq p < \infty$. X is called a p-process measure on $\mathcal{K}$, if it is cadlag and adapted, if for every $t \geq 0$ and $B \in \mathcal{L}$, the function $\omega \mapsto X_t(\omega, B)$ belongs to L^p_E, and if for every $t \geq 0$, the set function $B \mapsto X_t(\cdot, B)$ from $\mathcal{K}$ into L^p_E is σ-additive on $\mathcal{K}$.*

If X is a process measure on $\mathcal{L}$, we shall say, simply, that X is a p-process measure.

We note that if X is extended to $\Omega \times \mathrm{R} \times \mathcal{K}$ with $X_t = 0$ for $t < 0$, and if X has any one of the above properties on $\Omega \times \mathrm{R}_+ \times \mathcal{K}$, then its extension will have the same property on $\Omega \times \mathrm{R} \times \mathcal{K}$.

1.2. *The measure I_X and summable processes.*

Assume $X: \Omega \times \mathrm{R}_+ \times \mathcal{L} \to E$ is a p-process measure (on $\mathcal{L}$).

We define the finitely additive measure $I_X: \mathcal{R} \times \mathcal{L} \to L^p_E$ by

$$I_X(\{0\} \times A \times B) = 1_A X_0(B), \quad \text{for } A \in \mathcal{F}_0 \text{ and } B \in \mathcal{L},$$

and

$$I_X((s, t] \times A \times B) = 1_A (X_t(B) - X_s(B)), \quad \text{for } A \in \mathcal{F}_s \text{ and } B \in \mathcal{L}.$$

Then I_X is finitely additive on the semiring $\mathcal{R} \times \mathcal{L}$ and we can extend I_X by additivity to the ring $r(\mathcal{R} \times \mathcal{L})$. We further set

$$I_X(\{\infty\} \times A \times B) = 0, \quad \text{for } A \in \mathcal{F} \text{ and } B \in \mathcal{L}.$$

For any simple function

$$s = \sum_{1 \leq i \leq n} 1_{A_i} x_i, \quad \text{with } A_i \in r(\mathcal{R} \times \mathcal{L}) \text{ and } x_i \in F,$$

we define the integral $\int s dI_X$ in the usual way,

$$\int s dI_X = \sum_{1 \leq i \leq n} I_X(A_i)\, x_i \in L^p_G.$$

Since $L^p_E \subset L(F, L^p_G)$, we can define the semivariation of I_X relative to this embedding: For every set $C \in r(\mathcal{R} \times \mathcal{L})$, the semivariation $\hat{I}_X(C)$ of I_X on C, relative to the embed-

ding $L_E^p \subset L(F, L_G^p)$, (or simply, relative to the pair (F, G)) is defined by the following equality:

$$\tilde{I}_X(C) = \sup \left\| \sum_{i \in I} I_X(C_i) x_i \right\|_{L_E^p},$$

where the supremum is taken for all finite families $(C_i)_{i \in I}$ of disjoint subsets from $r(\mathcal{R} \times \mathcal{L})$ contained in C, and all finite families $(x_i)_{i \in I}$ of elements from F with $|x_i| \leq 1$.

If we want to emphasize the dependence of $\tilde{I}_X$ on the pair (F, G), we write $(\tilde{I}_X)_{F, L_E^p}$, or even $(\tilde{I}_X)_{F, G}$, if there is no danger of confusion.

The semivariation can be defined, equivalently, by the following equality:

$$\tilde{I}_X(C) = \sup \left\| \int s dI_X \right\|_{L_E^p},$$

where the supremum is taken of all $r(\mathcal{R} \times S)$-simple functions $s: \Omega \times \mathrm{R}_+ \times L \to F$ with $|s| \leq 1_C$.

Definition 1.2: *We say that an E-valued p-process measure X is p-summable relative to the pair (F, G), if I_X can be extended to a σ-additive measure $I_X: \mathcal{P} \otimes \mathcal{L} \to L_E^p$ with finite semivariation $(\tilde{I}_X)_{F, L_E^p}$.*

It is enough to assume that I_X has bounded semivariation $(\tilde{I}_X)_{F, L_E^p}$ on $r(\mathcal{R} \times S)$; then I_X has also finite semivariation on $\mathcal{P} \otimes \mathcal{L}$.

In [D.5], (Corollary 3.6) we proved that the process measures with integrable variation are 1-summable; in [D.6], (Theorem 3.1) it is proved that the process measures with p-integrable semivariation are p-summable, provided $c_0 \not\subset E$.

In this paper we shall prove that the *orthogonal* martingale measures with values in a *Hilbert* space E are 2-summable relative to the embedding $L_E^2 = L(\mathrm{R}, L_E^2)$ (Theorem 3.9); and that the *scalar* valued, *orthogonal* martingale measures are 2-summable relative to the embedding $L_E^2 \subset L(D, L_D^2)$, for any *Hilbert* space D.

Remarks: 1) Let $X: \Omega \times \mathrm{R}_+ \times \mathcal{L} \to E \subset L(F, G)$ be a p-process measure. For any set $B \in \mathcal{L}$ we can consider the process $X(B): \Omega \times \mathrm{R}_+ \to E \subset L(F, G)$ and define the additive measure $I_{X(B)}: \mathcal{R} \to L_E^p$ by

$$I_{X(B)}(\{0\} \times A) = 1_A X_0(B), \qquad \text{for } A \in \mathcal{F}_0,$$

and

$$I_{X(B)}((s, t] \times A) = 1_A (X_t(B) - X_s(B)), \qquad \text{for } A \in \mathcal{F}_s,$$

and then extended by additivity to $\mathcal{R}$. We have then

$$I_{X(B)}(C) = I_X(C \times B), \qquad \text{for } C \in \mathcal{R} \text{ and } B \in \mathcal{L}.$$

The definition of the semivariation $\bar{I}_{X(B)}(C)$ for $C \in \mathcal{R}$, relative to (F, L^p_E), is similar to that of $\bar{I}_X(C)$, replacing $r(\mathcal{R} \times \mathcal{L})$ with $\mathcal{R}$. We have then

$$\bar{I}_{X(B)}(C) \leq \bar{I}_X(C \times B), \quad \text{for } C \in \mathcal{R} \text{ and } B \in \mathcal{L}.$$

2) For usual, cadlag, adapted processes $X: \mathrm{R}_+ \times \Omega \to E$ we have the following criterion for the extension of I_X ([B-D.1], Theorem 2.5):

If $c_0 \not\subset E$, then I_X can be extended to a σ-additive measure $I: \mathcal{P} \to L^p_E$ iff I_X is bounded in L^p_E on $\mathcal{R}$.

For process measures we do not have a similar criterion of extension. However, we have the following result ([D.2], Corollary 1.3):

Assume there is a positive, bounded, σ-additive measure μ on $\mathcal{R} \times \mathcal{S}$ such that $I_X \ll \mu$. Then I_X and μ can be extended to σ-additive measures on $\mathcal{P} \otimes \mathcal{L}$, still denoted by I_X and μ, and we still have $I_X \ll \mu$ on $\mathcal{P} \otimes \mathcal{L}$.

We shall use this result to prove the 2-summability of orthogonal martingale measures (step c in the proof of Theorem 3.8).

3) If I_X has σ-additive extension to $\mathcal{P} \otimes \mathcal{L}$, then it has automatically finite semivariation relative to the embedding $L^p_E = L(\mathrm{R}, L^p_E)$, hence, in this case, X is p-summable relative to (R, E).

4) If X is p-summable relative to (F, G), we shall further extend I_X to a σ-additive measure with finite semivariation relative to (F, L^p_E), on the σ-algebra $\mathcal{P}[0, \infty] \otimes \mathcal{L}$ — where $\mathcal{P}[0, \infty]$ is the σ-algebra of predictable subsets of $[0, \infty] \times \Omega$ — by setting $I_X(C) = 0$ for $C \in \mathcal{P}[0, \infty] \otimes \mathcal{L} \setminus \mathcal{P} \otimes \mathcal{L}$.

For each set $B \in \mathcal{L}$, we shall also extend $I_{X(B)}$ to $\mathcal{P}[0, \infty]$ by setting $I_{X(B)}(C) = 0$, for $C \in \mathcal{P}[0, \infty] \setminus \mathcal{P}$. Then for every sets $C \in \mathcal{P}[0, \infty]$ and $B \in \mathcal{L}$ we have

$$I_{X(B)}(C) = I_X(C \times B)$$

and

$$\bar{I}_{X(B)}(C) \leq \bar{I}_X(C \times B).$$

We can also extend X itself to $\Omega \times [0, \infty] \times \mathcal{L}$:

THEOREM 1.3: *Let X be an E-valued p-summable process measure relative to (F, G). Then:*

(a) *For each $B \in \mathcal{L}$, there is a random variable $X_\infty(B) \in L^p_E$ such that $\lim_{t \to \infty} X_t(B) = X_\infty(B)$, in L^p_E.*

If the pointwise limit $X_{\infty-}(B)$ exists, then

$$X_{\infty-}(B) = X_\infty(B), \quad a.s.$$

We can, therefore, consider X defined on $\Omega \times [0, \infty] \times \mathcal{L}$.

(b) *For any set $B \in \mathcal{L}$ and any stopping time T we have $X_T(B) \in L^p_E$ and*

$$I_X([0, T] \times B) = X_T(B) = I_{X(B)}([0, T]);$$

(c) *For any $B \in \mathcal{L}$ and any predictable stopping time T we have $X_{T-}(B) \in L^p_E$ and*

$$I_X([0, T) \times B) = X_{T-}(B) = I_{X(B)}([0, T))$$

and

$$I_X([T] \times B) = \Delta X_T(B) = I_{X(B)}([T]).$$

(d) *Let $B \in \mathcal{L}$ and let $S \leq T$ be two stopping times. Then*

$$I_X((S, T] \times B) = X_T(B) - X_S(B) = I_{X(B)}((S, T]);$$

If S is predictable, then

$$I_X([S, T] \times B) = X_T(B) - X_{S-}(B) = I_{X(B)}([S, T]);$$

If T is predictable, then

$$I_X((S, T) \times B) = X_{T-}(B) - X_S(B) = I_X((S, T));$$

If both S and T are predictable, then

$$I_X([S, T) \times B) = X_{T-}(B) - X_{S-}(B) = I_X([S, T)).$$

The proof is similar to that of Theorem 2.2 in [B-D.1].

1.3. *The Integral* $\int H dI_X$.

Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ be a p-summable process measure. Consider the σ-additive measure $I_X: \mathcal{P} \otimes \mathcal{L} \to L^p_E \subset L(F, L^p_C)$, with finite semivariation $\hat{I}_X$ relative to (F, L^p_C). We can then apply the general integration theory, presented in ([B-D.1], Appendix I) and in [B-D.2], to define the integral with respect to I_X.

Let $Z \subset L^q_{C^*}$, $1/p + 1/q = 1$, be a norming space for L^p_C, that is, for every $f \in L^p_C$ we have

$$\|f\|_p = \sup\{|\langle f, z\rangle| : z \in Z, \|z\|_q \leq 1\} =$$

$$= \sup\left\{\left|\int \langle f(\omega), z(\omega)\rangle P(d\omega)\right| : z \in Z, \|z\|_q \leq 1\right\},$$

where $\langle f, z\rangle$ means the duality between L^p_G and $L^q_{G^*}$, while $\langle f(\omega), z(\omega)\rangle$ means the duality between G and G^*.

Denote $m = I_X$. For any function $z \in Z$, we consider the measure $m_z = (I_X)_z : \mathcal{P} \otimes \mathcal{L} \to F^*$, defined by

$$\langle y, m_z(A)\rangle = \langle m(A)\, y, z\rangle = \int \langle (I_X(A)(\omega)\, y, z(\omega)\rangle\, P(d\omega),$$

for $y \in F$ and $A \in \mathcal{P} \otimes \mathcal{L}$. Then the semivariation $\tilde{I}_X$ can be defined, equivalently, in terms of the norming space Z, by the equality

$$(\tilde{I}_X)_{F, L^p_G}(A) = \tilde{m}_{F, L^p_G}(A) = \sup\{\, |m_z|(A) : z \in Z,\ \|z\|_q \leq 1\},$$

where $|m_z|$ is the variation of m_z.

Let D be any Banach space. We extend the definition of $\tilde{I}_X$ for any predictable function $H : \Omega \times \mathbb{R}_+ \times L \to D$ (i.e. $\mathcal{P} \otimes \mathcal{L}$-measurable) by the equality

$$(\tilde{I}_X)_{F, L^p_G}(H) = \tilde{m}_{F, L^p_G}(H) = \sup \left| \int s\, dI_X \right|$$

where the supremum is taken for all $r(\mathcal{R} \times \mathcal{L})$-simple functions $s : \Omega \times \mathbb{R}_+ \times L \to F$ with $|s| \leq |H|$.

An equivalent definition is given in terms of the norming space Z:

$$\tilde{I}_{F, L^p_G}(H) = \sup \left\{ \int |H|\, d|(I_X)_z| : z \in Z,\ \|z\|_q \leq 1 \right\}.$$

When precision is necessary, we denote by $\mathcal{F}_D(I_{F,G})$ or $\mathcal{F}_D((I_X)_{F,G})$ or $\mathcal{F}_D((I_X)_{F, L^p_G})$ the set of all predictable functions $H : \Omega \times \mathbb{R}_+ \times L \to D$ with $\tilde{I}_{F, L^p_G}(H) < \infty$. Then $\mathcal{F}_D(I_{F,G})$ is a complete vector space for the seminorm $\tilde{I}_{F, L^p_G}$. We have the inclusion

$$\mathcal{F}_D((I_X)_{F,G}) \subset \bigcap_{z \in Z} L^1_F(|(I_X)_z|),$$

with equality, if Z is closed in $L^q_{G^*}$.

If $D = F$, to simplify the notation, we write $\mathcal{F}_{F,G}(I_X) = \mathcal{F}_F((I_X)_{F,G})$. In this case, we can define the integral $\int H dI_X \in Z^*$, for any $H \in \mathcal{F}_{F,G}(I_X)$, in the following way: Let $H \in \mathcal{F}_{F,G}(I_X)$. Then $H \in L^1_F(|(I_X)_z|)$ for any $z \in Z$, hence the integral $\int H d(I_X)_z$ is defined and is a scalar. We thus obtain a linear functional $z \mapsto \int H d(I_X)_z$ on Z. This linear functional is continuous:

$$\left| \int H d(I_X)_z \right| \leq \tilde{I}_X(H)\, |z|,$$

therefore, it is an element of Z^*, denoted $\int H dI_X$ and called the integral of H with re-

spect to I_X. We have therefore $\int H dI_X \in Z^*$,

$$\left\langle \int H dI_X, z \right\rangle = \int H d(I_X)_z, \quad \text{for } z \in Z,$$

and

$$\left| \int H dI_X \right| \leq \bar{I}_{F,G}(H).$$

The last inequality implies that the integral mapping $H \mapsto \int H dI_X$ from $\mathcal{F}_{F,G}(I_X)$ into Z^* is continuous.

If $H \in \mathcal{F}_{F,G}(I_X)$, then, for every $C \in \mathcal{P} \otimes \mathcal{L}$ we have $1_C H \in \mathcal{F}_{F,G}(I_X)$. We set

$$\int_C H dI_X = \int 1_C H dI_X.$$

If $H^n \to H$ in $\mathcal{F}_{F,G}(I_X)$, then $\int_C H^n dI_X \to \int_C H dI_X$ in Z^*, uniformly for $C \in \mathcal{P} \otimes \mathcal{L}$.

The property relating I_X and $I_{X(B)}$ in Remark 4, can be extended for functions.

THEOREM 1.4: *Let $H: \mathbb{R}_+ \times \Omega \times F$ be a predictable process and $B \in \mathcal{L}$ such that $H1_B \in \mathcal{F}_{F,G}(I_X)$. Then $H \in \mathcal{F}_{F,G}(I_{X(B)})$ and we have*

$$\int H1_B dI_X = \int H dI_{X(B)}$$

and

$$(\bar{I}_{X(B)})_{F,G}(H) \leq (\bar{I}_X)_{F,G}(H1_B).$$

PROOF: From Remark 4 we deduce that the property is true for H a predictable step process. Assume now H is predictable, $B \in \mathcal{L}$ and $H1_B \in \mathcal{F}_{F,G}(I_X)$. For any predictable simple process $s: \mathbb{R}_+ \times \Omega \to F$ with $|s| \leq |H|$ we have $|s1_B| \leq |H1_B|$ and

$$\left\| \int s dI_{X(B)} \right\|_{L^p_G} = \left\| \int s1_B dI_X \right\|_{L^p_G} \leq (\bar{I}_X)_{F,G}(H1_B).$$

Taking the supremum for all $|s| \leq |H|$ we get

$$(\bar{I}_{X(B)})_{F,G}(H) \leq (\bar{I}_X)_{F,G}(H1_B) < \infty,$$

hence $H \in \mathcal{F}_{F,G}(I_{X(B)})$.

To prove the equality in the statement, let (H^n) be a sequence of simple, predictable processes such that $H^n \to H$ pointwise and $|H^n| \leq |H|$ for each n. Then $H^n 1_B \to H1_B$ and $|H^n 1_B| \leq |H1_B|$. For each $z \in L^q_G$, we can apply the Lebesgue theo-

rem in the spaces $L_F^1((I_{X(B)})_t)$ and $L_F^1((I_X)_t)$ and deduce

$$\int H^n d(I_{X(B)})_t \to \int H d(I_{X(B)})_t$$

and

$$\int H^n 1_B d(I_X)_t \to \int H 1_B d(I_X)_t,$$

i.e.,

$$\left\langle \int H^n dI_{X(B)}, z \right\rangle \to \left\langle \int H dI_{X(B)}, z \right\rangle$$

and

$$\left\langle \int H^n 1_B dI_X, z \right\rangle \to \left\langle \int H 1_B dI_X, z \right\rangle.$$

Since for each n we have

$$\int H^n dI_{X(B)} = \int H^n 1_B dI_X,$$

we deduce that

$$\int H dI_{X(B)} = \int H 1_B dI_X.$$

1.4. *A convergence theorem.*

The following theorem ensures the convergence of the integrals, but not necessarily convergence in the topology of $\mathcal{F}_{F,G}(I_X)$. It is very useful in proving many other theorems.

THEOREM 1.5: *Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ be a p-summable process measure relative to (F, G).*

Let $(H^n)_{0 \leq n < \infty}$ be a sequence from $\mathcal{F}_{F,G}(I_X)$ such that $|H^n| \leq |H^0|$ for every n and assume $H^n \to H$ pointwise.

If $\int H^n dI_X \in L_G^p$ for $n \geq 1$ and if the sequence $\left(\int H^n dI_X\right)$ converges pointwise on Ω, weakly in G, then $\int H dI_X \in L_G^p$ and $\int H^n dI_X \to \int H dI_X$ in the $\sigma(L_G^p, L_{G^*}^q)$ topology of L_G^p, as well as pointwise, weakly in G.

If $\left(\int H^n dI_X\right)$ converges strongly in G, then $\int H^n dI_X \to \int H dI_X$ strongly in L_G^1.

The proof is similar to that of Theorem 3.1 in [B-D.1].

REMARK: Convergence theorems in the topology of $\mathcal{F}_{F,G}(I_X)$ will be stated in the next section (Theorems 2.8, 2.10, 2.11 and 2.12).

1.5. *The Stochastic Integral $H \cdot X$.*

Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E \subset L(F, G)$ be a p-summable process measure and let $H \in \mathcal{F}_{F,G}(I_X)$. For each $t \geq 0$ and $B \in \mathcal{L}$ consider the integral

$$\int_{[0,t] \times B} H dI_X \in Z^*.$$

We obtain a family $\left(\int_{[0,t] \times B} H dI_X \right)_{t \in \mathbb{R}_+, B \in \mathcal{L}}$ of elements in Z^*. We are interested in the functions H for which the integral $\int_{[0,t] \times B} H dI_X$ belongs to the subspace L^p_G of Z^*, for each $t \geq 0$ and $B \in \mathcal{L}$. In this case we denote by the same symbol the equivalence class $\int_{[0,t] \times B} H dI_X$ in L^p_G, as well as any representative of this class. If, in each equivalence class, we choose a representative, we obtain a process set function $\left(\int_{[0,t] \times B} H dI_X \right)_{t \geq 0, B \in \mathcal{L}}$ with values in G.

THEOREM 1.6: *Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E \subset L(F, G)$ be a p-summable process measure, relative to (F, G). Let $H \in \mathcal{F}_{F,G}(I_X)$ be such that $\int_{[0,t] \times B} H dI_X \in L^p_G$ for every $t \geq 0$ and $B \in \mathcal{L}$. Then*

$$\left[\left(\int_{[0,t] \times B} H dI_X \right)(\omega) \right]_{t \in \mathbb{R}_+, B \in \mathcal{L}}$$

is an adapted p-process measure with values in G.

See proof in [D.5], (Theorem 3.2).

REMARK: It does not follow that an arbitrary choice of representatives $\left(\int_{[0,t] \times B} H dI_X \right)(\omega)$, $t \geq 0$, $B \in \mathcal{L}$, is cadlag; or that there is a cadlag choice. This leads us to the following definition:

DEFINITION 1.7: *Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E \subset L(F, G)$ be a p-summable process measure. We denote by $L^p_{F,G}(X)$ the subspace of $\mathcal{F}_{F,G}(I_X)$ consisting of processes H satisfying the following two conditions:*

(α) $\int_{[0,t] \times B} H dI_X \in L^p_G$, *for every $t \geq 0$ and $B \in \mathcal{L}$;*

(β) *The process set function $\left(\int_{[0,t] \times B} H dI_X \right)_{t \geq 0, B \in \mathcal{L}}$ has a cadlag modification.*

The processes $H \in L^1_{F,G}(X)$ are said to be integrable with respect to X. Any cadlag modification of $\left(\int_{[0,t]\times B} HdI_X\right)_{t\geq 0,\, B\in\mathcal{L}}$ is called the stochastic integral of H with respect to X and is denoted $H\cdot X$:

$$(H\cdot X)_t(\omega, B) = \left(\int_{[0,t]\times B} HdI_X\right)(\omega), \quad a.s.$$

It follows that the stochastic integral $H\cdot X$ is defined up to an evanescent set, i.e. a subset of $N \times \mathrm{R}_+ \times L$ with N negligible.

From Theorem 1.6 and Definition 1.7 we deduce the following theorem.

THEOREM 1.8: *For every $H \in L^1_{F,G}(X)$, the stochastic integral $H\cdot X$: $\Omega \times \mathrm{R}_+ \times L \to G$ is a cadlag, adapted p-process measure.*

We shall see later on that $L^1_{F,G}(X)$ is complete for the seminorm $\tilde{I}_{F,G}$ (Theorem 2.9).

REMARK: For each set $B \in \mathcal{L}$ we can apply similar considerations for the process $X(B)$: $\Omega \times \mathrm{R}_+ \to E$. We define the measure $I_{X(B)}$, the semivariation $\tilde{I}_{X(B)}$ and the spaces $\mathcal{F}_{F,G}(I_{X(B)})$ and $L^1_{F,G}(X(B))$. (See [B-D.1]).

For $H \in L^1_{F,G}(X)$, the left limit of the stochastic integral $H\cdot X$ can also be expressed in terms of the integral $\int HdI_X$.

PROPOSITION 1.9: *If $H \in L^1_{F,G}(X)$, then for every $t \in [0, \infty]$ and $B \in \mathcal{L}$ we have $(H\cdot X)_{t-}(B) \in L^p_G$ and*

$$(H\cdot X)_{t-}(B) = \int_{[0,t)\times B} HdI_X, \quad a.s.$$

The mapping $t \mapsto (H\cdot X)_t(B)$ is cadlag in L^1_G.

PROOF: Let $t_n \nearrow t$ and $B \in \mathcal{L}$. Then $1_{[0,t_n]\times B}H \to 1_{[0,t)\times B}H$ pointwise and $|1_{[0,t_n]\times B}H| \leq |H|$. For each n we have

$$\int 1_{[0,t_n]\times B} HdI_X = (H\cdot X)_{t_n}(B) \in L^p_G$$

and $(H\cdot X)_{t_n}(B) \to (H\cdot X)_{t-}(B)$. By the convergence Theorem 1.5 we have $\int 1_{[0,t)\times B} HdI_X \in L^p_G$ and

$$\int 1_{[0,t_n]\times B} HdI_X \to \int 1_{[0,t)\times B} HdI_X,$$

pointwise and in L^1_G. Hence $(H\cdot X)_{t-}(B) = \int 1_{[0,t)\times B} HdI_X$.

2. - Properties of the Stochastic Integral

Throughout this paragraph we shall assume that $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E \subset L(F, G)$ is a p-summable process measure, and we shall prove additional properties of the integral $\int H dI_X$ and of the stochastic integral $H \cdot X$.

The reader interested in the stochastic integral of orthogonal martingale measures, may read this paragraph after § 3.

2.1. *The Stochastic Integral and Stopping Times.*

We prove first a proposition which shows that in the definition of the measure I_X, the intervals $(s, t]$ can be replaced with stochastic intervals $(S, T]$.

Proposition 2.1: *Let $S \leq T$ be stopping times, $B \in \mathcal{L}$ and $b: \Omega \to F$ an $\mathcal{F}_S$-measurable, bounded random variable. Then*

$$\int b 1_{(S, T]} 1_B dI_X = \int b 1_{(S, T]} dI_{X(B)} = b(X_T(B) - X_S(B)) .$$

If S is predictable and b is $\mathcal{F}_{S-}$-measurable, then

$$\int b 1_{[S, T]} 1_B dI_X = \int b 1_{[S, T]} dI_{X(B)} = b(X_T(B) - X_{S-}(B))$$

and

$$\int b 1_{\{S\}} 1_B dI_X = b \Delta X_S(B) .$$

Proof: Assume first that $b = 1_A y$ with $A \in \mathcal{F}_S$ and $y \in F$. Then

$$\int b 1_{(S, T]} 1_B dI_X = \int b 1_{(S, T]} dI_{X(B)} =$$

$$= I_{X(B)}((S_A, T_A]) y = (X_{T_A}(B) - X_{S_A}(B)) y = b(X_T(B) - X_S(B)) .$$

The equality remains valid for $\mathcal{F}_S$-simple functions b. Let now b be an F-valued, bounded, $\mathcal{F}_S$-measurable function, and let (b^n) be a sequence of $\mathcal{F}_S$-simple functions such that $b^n \to b$ pointwise and $|b^n| \leq |b|$ for every n. The hypothesis of the convergence Theorem 1.5 is satisfied for the sequence $H^n = b^n 1_{(S, T]} 1_B$. In fact

$$\int H^n dI_X = b^n (X_T(B) - X_S(B)) \to b(X_T(B) - X_S(B)) ,$$

pointwise on Ω, strongly in G. Then the limit $H = b 1_{(S, T]} 1_B$ belongs to L^p_G and $\int H^n dI_X \to \int H dI_X$, pointwise on Ω, weakly in G, that is,

$$\int b^n 1_{(S, T]} 1_B dI_X \to \int b 1_{(S, T]} 1_B dI_X ,$$

weakly in G. On the other hand,

$$\int b^n 1_{(S,T]} 1_B dI_X \to b(X_T(B) - X_S(B)),$$

pointwise on Ω, strongly in G. It follows that the two limits are equal:

$$\int b 1_{(S,T]} 1_B dI_X = b(X_T(B) - X_T(B)).$$

Assume now that S is predictable and b is $\mathcal{F}_{S-}$-measurable. If $b = 1_A y$ with $A \in \mathcal{F}_{S-}$ and $y \in F$, then S_A is a predictable stopping time and

$$\int b 1_{[S]} 1_B dI_X = \int 1_A y 1_{[S]} dI_X = \int 1_{[S_A]} y 1_B dI_X =$$

$$= \int_{[S_A]} y dI_{X(B)} = \Delta X_{S_A}(B)\, y = \Delta X_S(B)\, 1_A y = b \Delta X_S(B);$$

thus

$$\int b 1_{(S,T]} dI_X = \int b 1_{[S]} 1_B dI_X + \int b 1_{(S,T]} 1_B dI_X =$$

$$= b \Delta_S X(B) + b(X_T(B) - X_S(B)) = b(X_T(B) - X_{S-}(B)).$$

The two equalities above remain valid for $\mathcal{F}_{S-}$-simple functions b, and as above, using the convergence Theorem 1.5, we deduce that they are valid for bounded, $\mathcal{F}_{S-}$-measurable functions b.

Corollary 2.2: *Let*

$$H = H_0 1_{\{0\}} 1_{B_0} + \sum_{1 \leq i \leq n} H_i 1_{(T_i, T_{i+1}]} 1_{B_i}$$

be an elementary process, with H_i bounded, $\mathcal{F}_{T_i}$-measurable random variables and H_0 a bounded, $\mathcal{F}_0$-measurable random variable. Then H is integrable with respect to I_X and the integral $\int H dI_X$ can be computed pathwise:

$$\int H dI_X = H_0 X_0(B_0) + \sum_{1 \leq i \leq n} H_i (X_{T_{i+1}}(B_i) - X_{T_i}(B_i)).$$

Proposition 2.1 will be extended now with 1_B replaced by a predictable function H.

Theorem 2.3: 1) *Let $S \leq T$ be stopping times and assume that either*

(a) *$b: \Omega \to \mathbb{R}$ is bounded, $\mathcal{F}_S$-measurable and*

$$H \in \mathcal{F}_{F,G}(I_X), \quad \text{with } \int 1_{(S,T]} H dI_X \in L^p_G$$

or

(b) $h: \Omega \to F$ *is bounded,* $\mathcal{F}_T$*-measurable and*

$$H \in \mathcal{F}_{\mathcal{R},E}(I_X) \cap \mathcal{F}_{\mathcal{R}}((I_X)_{F,G}) \quad \textit{with} \int 1_{(S,T]} H dI_X \in L^p_E.$$

Then

$$\int h 1_{(S,T]} H dI_X = h \int 1_{(S,T]} H dI_X.$$

2) *Let S be a predictable stopping time and T a stopping time with $S \leq T$ and assume that either*

(a') $h: \Omega \to \mathbb{R}$ *is bounded,* $\mathcal{F}_{S-}$*-measurable and*

$$H \in \mathcal{F}_{T,G}(I_X) \quad \textit{with} \int 1_{(S,T]} H dI_X \in L^p_G,$$

or

(b') $h: \Omega \to F$ *is bounded,* $\mathcal{F}_{S-}$*-measurable and*

$$H \in \mathcal{F}_{\mathcal{R},E}(I_X) \cap \mathcal{F}_{\mathcal{R}}((I_X)_{F,G}) \quad \textit{with} \int 1_{(S,T]} H dI_X \in L^p_G.$$

Then

$$\int h 1_{(S,T]} H dI_X = h \int 1_{(S,T]} H dI_X.$$

Proof: Assume first hypothesis (a). Let $H = 1_{(s,t] \times A \times B} y$ with $A \in \mathcal{F}_s$, $B \in \mathcal{L}$ and $y \in F$. By Proposition 2.1 we have

$$\int h 1_{(S,T]} H dI_X = \int h 1_A y 1_{(S \vee s, T \wedge t]} 1_B dI_X =$$

$$= h 1_A y (X_{T \wedge t}(B) - X_{S \vee s}(B)) = h \int 1_{(S,T]} H dI_X \in L^p_G.$$

It follows that for $H = 1_{C \times B} y$ with $C \in \mathcal{R}$ and $B \in \mathcal{L}$ we have

$$\int h 1_{(S,T]} 1_{C \times B} y dI_X = h \int 1_{(S,T]} 1_{C \times B} y dI_X \in L^p_G.$$

For any $z \in L^q_{G^*}$, $1/p + 1/q = 1$, we have then

$$\int h 1_{(S,T]} 1_{C \times B} y d(I_X)_z = \int 1_{(S,T]} 1_{C \times B} y d(I_X)_{hz}.$$

The class of all sets $D \in \mathcal{P} \otimes \mathcal{L}$ satisfying the equality

$$\int h 1_{(S,T]} 1_D y d(I_X)_z = \int 1_{(S,T]} 1_D y d(I_X)_{hz},$$

for all $z \in L^q_{G^*}$ is a monotone class containing $\mathcal{R} \times \mathcal{L}$, hence the above equality holds

for all $D \in \mathcal{P} \otimes \mathcal{L}$ and $z \in L^p_{G^*}$. Then, for each F-valued, $\mathcal{P} \otimes \mathcal{L}$-simple function H and for $z \in L^p_{G^*}$ we have

$$\int b 1_{(S,T]} H d(I_X)_z = \int 1_{(S,T]} H d(I_X)_{bz}.$$

If $H \in \mathcal{F}_{F,G}(X)$, taking a sequence (H^n) of F-valued, $\mathcal{P} \otimes \mathcal{L}$-simple functions with $H^n \to H$ pointwise and $|H^n| \leq |H|$ and applying Lebesgue's dominated convergence theorem we deduce that the above equality holds for H.

To prove assertion 1), assume that $\int 1_{(S,T]} H dI_X \in L^p_G$. Then $b \int 1_{(S,T]} H dI_X \in L^p_G$ and

$$\left\langle b \int 1_{(S,T]} H dI_X, z \right\rangle = \left\langle \int 1_{(S,T]} H dI_X, bz \right\rangle =$$

$$= \int 1_{(S,T]} H d(I_X)_{bz} = \int b 1_{(S,T]} H d(I_X)_z = \left\langle \int b 1_{(S,T]} H dI_X, z \right\rangle.$$

Since $L^p_{G^*}$ is norming for both L^p_G and $(L^p_{G^*})^*$, we deduce that

$$\int b 1_{(S,T]} H dI_X = b \int 1_{(S,T]} H dI_X \in L^p_G,$$

and this proves assertion 1) under hypothesis (a).

Assume now hypothesis (b) and let $H: \Omega \times \mathbb{R}_+ \times L \to \mathbb{R}$ be $\mathcal{P} \otimes \mathcal{L}$-measurable with $\hat{I}_{F,G}(H) < \infty$ and $\hat{I}_{\mathbb{R},\mathbb{E}}(H) < \infty$. Assume also that $\int 1_{(S,T]} H dI_X \in L^p_G$. Consider first the case $b = b' y$ with $y \in F$ and b' real valued, bounded $\mathcal{F}_S$-measurable. Then, by Theorem AI.14 in [B-D.1] we have $\int 1_{(S,T]} H y dI_X = y \int 1_{(S,T]} H I_X$.

By the first part of the proof we have

$$\int b 1_{(S,T]} H dI_X = b' \int 1_{(S,T]} H y \, dI_X = b \int 1_{(S,T]} H dI_X.$$

This equality holds for any $\mathcal{F}_S$-simple function b. If now b is $\mathcal{F}_S$-measurable we take a sequence (b_n) of $\mathcal{F}_S$-simple functions with $b_n \to b$ pointwise and $|b_n| \leq |b|$ and use the convergence Theorem 1.5 for the sequence $H^n = b_n 1_{(S,T]} H$ to deduce assertion 1) under hypothesis (b). Assertion 2) is proved exactly the same way, replacing $(S, T]$ with $[S, T]$.

THEOREM 2.4: *Let $H \in L^1_{F,G}(X)$ and T a stopping time. Then $1_{[0,T]} H \in L^1_{F,G}(X)$ and*

$$(1_{[0,T]} H) \cdot X = (H \cdot X)^T.$$

If T is predictable, then $1_{[0,T]}H \in L^1_{F,G}(X)$ *and*

$$(1_{[0,T]}H)\cdot X = (H\cdot X)^T .$$

PROOF: We shall prove that $1_{(T,\infty]}H \in L^1_{F,G}$.

For this purpose, we assume first that T is a simple stopping time, of the form

$$T = \sum_{1 \leq i \leq n} 1_{A_i} t_i$$

with $0 \leq t_1 \leq \ldots \leq t_n \leq +\infty$ and $A_i \in \mathcal{F}_{t_i}$, mutually disjoint and with union $\bigcup_{1 \leq i \leq n} A_i = \Omega$. For each $\omega \in \Omega$ there is a unique i such that $\omega \in A_i$ and then $T(\omega) = t_i$. Then, for $B \in \mathcal{L}$ we have

$$(H\cdot X)_T(\omega, B) = (H\cdot X)_{t_i}(\omega, B) = \left(\int 1_{[0,t_i]\times B} H dI_X\right)(\omega),$$

hence

$$(H\cdot X)_T(B) = \sum_{1 \leq i \leq n} 1_{A_i} \int 1_{[0,t_i]\times B} H dI_X = \int_{[0,\infty]\times B} H dI_X - \sum_{1 \leq i \leq n} 1_{A_i} \int_{(t_i,\infty]\times B} H dI_X =$$

$$= \int_{[0,\infty]\times B} H dI_X - \sum_{1 \leq i \leq n} \int_{[t_i,\infty]\times B} 1_{A_i} H dI_X \in L^2_G,$$

according to Theorem 2.3. Then

$$(H\cdot X)_T(B) = \int_{[0,\infty]\times B} H dI_X - \int 1_{(T,\infty]} 1_B H dI_X =$$

$$= \int 1_{[0,T]} 1_B H dI_X .$$

Let now T be a general stopping time and let (T_n) be a decreasing sequence of simple stopping times with $T_n \downarrow T$. We have $1_B 1_{[0,T]} H \in \mathcal{F}_{F,G}(X)$ and

$$1_B 1_{[0,T_n]} H \to 1_B 1_{[0,T]} H, \quad \text{pointwise},$$

and also

$$\int 1_{[0,T_n]} 1_B H dI_X = (H\cdot X)_{T_n}(B) \to (H\cdot X)_T(B),$$

pointwise, by the right continuity of $H\cdot X$. We can apply the convergence Theorem 1.5

to the sequence $H^n = 1_{[0,T_n]} 1_B H$, and deduce that $\int 1_{[0,T]} 1_B H dI_X \in L^p_G$ and

$$\int 1_{[0,T_n]} 1_B H dI_X \to \int 1_{[0,T]} 1_B H dI_X,$$

pointwise and strongly in L^1_G. Since for each n we have

$$\int 1_{[0,T_n]} 1_B H dI_X = (H \cdot X)_{T_n}(B) \to (H \cdot X)_T(B)$$

we deduce that the two limits are equal a.s:

$$\int 1_{[0,T]} 1_B H dI_X = (H \cdot X)_T(B) \in L^p_G.$$

Replacing T with $T \wedge t$, we obtain

$$\int_{[0,t] \times B} 1_{[0,T]} H dI_X = (H \cdot X)_{T \wedge t}(B) \in L^p_G.$$

Since the mapping $t \mapsto (H \cdot X)_{T \wedge t}(B)$ is cadlag, we deduce that the mapping $t \mapsto \int_{[0,t] \times B} 1_{[0,T]} H dI_X$ is cadlag. It follows that $1_{[0,T]} H \in L^1_{F,G}(X)$ and

$$[(1_{[0,T]} H) \cdot X]_t(B) = (H \cdot X)_{T \wedge t}(B) = (H \cdot X)^T_t(B)$$

hence

$$(1_{[0,T]} H) \cdot X = (H \cdot X)^T.$$

Assume now T is predictable and let (T_n) be an increasing sequence of stopping times with $T_n \uparrow T$. Then

$$1_{[0,t] \times B} 1_{[0,T_n]} H \to 1_{[0,t] \times B} 1_{[0,T]} H, \quad \text{pointwise},$$

$$\int 1_{[0,t] \times B} 1_{[0,T_n]} H dI_X \in L^p_G$$

and

$$\int 1_{[0,t] \times B} 1_{[0,T_n]} H dI_X = (H \cdot X)^{T_n}_t(B) \to (H \cdot X)^{T^-}_t(B),$$

pointwise. Using the convergence Theorem 1.5, we deduce that

$$\int_{[0,t] \times B} 1_{[0,T]} H dI_X \in L^p_G$$

and

$$\int_{[0,t] \times B} 1_{[0,T_n]} H dI_X \to \int_{[0,t] \times B} 1_{[0,T]} H dI_X$$

pointwise and in L^1_G. The equality of the two limits gives

$$\int_{[0,t]\times B} 1_{[0,T]} H dI_X = (H\cdot X)^{T-}_t(B) \in L^p_G.$$

Since $t \mapsto (H\cdot X)^{T-}_t(B)$ is cadlag, it follows that $1_{[0,T]}H \in L^1_{F,G}(X)$ and

$$(1_{[0,T]}H)\cdot X = (H\cdot X)^{T-}.$$

The following theorem is an extension of Theorem 2.3.

THEOREM 2.5: 1) *Let $S \leq T$ be stopping times and assume that either*

(a) *$b: \Omega \to \mathbb{R}$ is bounded, $\mathcal{F}_S$-measurable and $H \in L^1_{F,G}(X)$, or*

(b) *$b: \Omega \to F$ is bounded, $\mathcal{F}_S$-measurable, $H \in L^1_{R,E}(X)$ and $(\bar{I}_X)_{F,G}(H) < \infty$.*

Then $1_{(S,T]}H$ and $b1_{(S,T]}H$ are integrable with respect to X and we have

$$(b1_{(S,T]}H)\cdot X = b[(1_{(S,T]}H)\cdot X].$$

2) *Let S be a predictable stopping time and T a stopping time with $S \leq T$ and assume that either*

(a') *$b: \Omega \to \mathbb{R}$ is bounded, $\mathcal{F}_{S-}$-measurable and $H \in L^1_{F,G}(X)$, or*

(b') *$b: \Omega \to F$ is bounded, $\mathcal{F}_{S-}$-measurable, $H \in L^1_{R,E}(X)$ and $(\bar{I}_X)_{F,G}(H) < \infty$.*

Then $1_{[S,T]}H$ and $b1_{[S,T]}H$ are integrable with respect to X and we have

$$(b1_{[S,T]}H)\cdot X = b[(1_{[S,T]}H)\cdot X].$$

PROOF: By Theorem 2.4, hypothesis (a) or (b), implies that the processes $1_{[0,T]}H$ and $1_{[0,S]}H$ are integrable with respect to X. From the equality

$$1_{(S,T]}H = 1_{[0,T]}H - 1_{[0,S]}H$$

it follows that $1_{(S,T]}H$ is integrable with respect to X, therefore $\int_{[0,t]\times B} 1_{(S,T]} H dI_X$ belongs to L^p_G under hypothesis (a), and to L^p_E under hypothesis (b).

By Theorem 2.3, we have then

$$\int_{[0,t]\times B} b1_{(S,T]} H dI_X = b \int_{[0,t]\times B} 1_{(S,T]} H dI_X$$

and the right hand side term belongs to L^p_G under hypothesis (a) and to L^p_E under hypothesis (b). Since the right hand side is equal to $b[(1_{(S,T]}H)\cdot X)_t(B)$, it is cadlag. It follows that $b1_{(S,T]}H$ is integrable with respect to X and

$$[(b1_{(S,T]}H)\cdot X]_t(B) = b[(1_{(S,T]}H)\cdot X]_t(B),$$

that is,

$$(b1_{]S,T]}H)\cdot X = b[(1_{]S,T]}H)\cdot X],$$

and assertion 1) is proved.

Assertion 2 is proved in the same way using those assertions in theorems 2.3 and 2.4 which assume that S is predictable.

The following theorems states the p-summability of X^T.

THEOREM 2.6: *Assume X is p-summable relative to (F, G) and let T be a stopping time. Then:*

(a) *X^T is p-summable relative to (F, G) and we have*

$$X^T = 1_{[0,T]}\cdot X$$

and

$$I_{X^T}(C) = I_X(([0, T] \times L) \cap C), \quad \text{for } C \in \mathcal{P} \otimes \mathcal{L}.$$

(b) *For every F-valued process H we have*

$$(\hat{I}_{X^T})_{F,L^p}(H) = (\hat{I}_X)_{F,L^p}(1_{[0,T]}H).$$

(c) *We have $H \in \mathcal{F}_{F,G}(X^T)$ iff $1_{[0,T]}H \in \mathcal{F}_{F,G}(X)$, and in this case we have*

$$\int H dI_{X^T} = \int 1_{[0,T]} H dI_X.$$

(d) *If $H \in L^1_{F,G}(X)$, then $1_{[0,T]}H \in L^1_{F,G}(X)$ and $H \in L^1_{F,G}(X^T)$. In this case we have*

$$(H\cdot X)^T = H\cdot X^T = (1_{[0,T]}H)\cdot X.$$

PROOF: For $C = (s, t] \times A \times B$ with $A \in \mathcal{F}_s$ and $B \in \mathcal{L}$ we have

$$\begin{aligned} I_{X^T}(C) &= 1_A(X_t^T(B) - X_s^T(B)) = 1_A(X_{T\wedge t}(B) - X_{T\wedge s}(B)) = \\ &= I_{X(B)}((A \times \mathbf{R}_+) \cap (T\wedge s, T\wedge t]) = \\ &= I_X[((A \times \mathbf{R}_+) \cap (T\wedge s, T\wedge t]) \times B] = \\ &= I_X(((0, T] \times L) \cap C), \end{aligned}$$

by Theorem 1.3(d). Then this equality remains valid for $C \in r(\mathcal{R} \times \mathcal{L})$. Since X is p-summable, the measure I_X is σ-additive on $\mathcal{P} \otimes \mathcal{L}$. From the above equality it follows that I_{X^T} can be extended to a σ-additive measure on $\mathcal{P} \otimes \mathcal{L}$ and we still have

$$I_{X^T}(C) = I_X(([0, T] \times L) \cap C), \quad \text{for } C \in \mathcal{P} \otimes \mathcal{L}.$$

The function $H \equiv 1$ is integrable with respect to X and $1 \cdot X = X$. By Theorem 2.4, $1_{[0,T]} = 1_{[0,T]} H$ is also integrable with respect to X and we have $1_{[0,T]} \cdot X = X^T$.

To prove that X^T is p-summable relative to (F, G), it remains to show that I_{X^T} has finite semivariation relative to $(F, L^q_{\mathcal{C}})$. This will follow from assertion (b) with $H \equiv 1$.

To prove assertion (b) we note that the above equality can be written

$$(I_{X^T})(C) = \int_C 1_{[0,T]} dI_X, \quad \text{for } C \in \mathcal{P} \otimes \mathcal{L}.$$

Then, for $z \in L^q_{\mathcal{C}^*}$ with $1/p + 1/q = 1$, and $C \in \mathcal{P} \otimes \mathcal{L}$ we have

$$(I_{X^T})_z(C) = \int_C 1_{[0,T]} d(I_X)_z,$$

therefore, for the variations of the measures we deduce

$$|(I_{X^T})_z|(C) = \int_C 1_{[0,T]} d|(I_X)_z|.$$

For any F-valued process H we have then

$$\int |H| d|(I_{X^T})_z| = \int 1_{[0,T]} |H| d|(I_X)_z|;$$

taking the supremum for $\|z\|_q \leq 1$ we get

$$(\bar{I}_{X^T})_{F, L^q_{\mathcal{C}}}(H) = (\bar{I}_X)_{F, L^q_{\mathcal{C}}}(1_{[0,T]} H),$$

and this proves (b).

Then $H \in \mathcal{F}_{F,G}(X^T)$ iff $1_{[0,T]} H \in \mathcal{F}_{F,G}(X)$ and using the equality proved before:

$$(I_{X^T})_z(C) = \int_C 1_{[0,T]} d(I_X)_z, \quad \text{for } C \in \mathcal{P} \otimes \mathcal{L},$$

we deduce, for $H \in \mathcal{F}_{F,G}(X^T)$,

$$\int H d(I_{X^T})_z = \int 1_{[0,T]} H d(I_X)_z,$$

for every $z \in L^q_{\mathcal{C}^*}$, therefore

$$\int H dI_{X^T} = \int 1_{[0,T]} H dI_X.$$

and this proves assertion (c). Replacing H with $1_{[0,t]\times B}H$ with $t \geq 0$ and $B \in \mathcal{L}$, we get

$$\int_{[0,t]\times B} H dI_{X^T} = \int_{[0,t]\times B} 1_{[0,T]} H dI_X.$$

From this equality, we deduce that $H \in L^p_{F,G}(X^T)$ iff $1_{[0,T]}H \in L^p_{F,G}(X)$ and then

$$H \cdot X^T = 1_{[0,T]}(H) \cdot X;$$

From Theorem 2.4, we also have

$$(1_{[0,T]}H) \cdot X = (H \cdot X)^T,$$

and this proves assertion (d) and the Theorem.

The following theorem is the analog of Theorem 2.6 for predictable stopping times.

THEOREM 3.7: *Assume X is p-summable relative of (F, G) and let T be a predictable stopping time. Then:*

(a) *X^{T-} is p-summable relative to (F, G) and we have*

$$X^{T-} = 1_{[0,T]} \cdot X$$

and

$$I_{X^{T-}}(C) = I_X(([0, T) \times L) \cap C), \quad \text{for } C \in \mathcal{P} \otimes \mathcal{L}.$$

(b) *For every F-valued process H we have*

$$(\hat{I}_{X^{T-}})_{F,L^p}(H) = (\hat{I}_X)_{F,L^p}(1_{[0,T]}H).$$

(c) *We have $H \in \mathcal{F}_{F,G}(X^{T-})$ iff $1_{[0,T]}H \in \mathcal{F}_{F,G}(X)$ and in this case we have*

$$\int H dI_{X^{T-}} = \int 1_{[0,T]} H dI_X.$$

(d) *If $H \in L^p_{F,G}(X)$, then $1_{[0,T]}H \in L^p_{F,G}(X)$ and $H \in L^p_{F,G}(X^{T-})$. In this case we have*

$$(H \cdot X)^{T-} = H \cdot X^{T-} = (1_{[0,T]}H) \cdot X.$$

The proof is the same as that of Theorem 2.6, using Theorems 1.3(d) and 2.4.

2.2. *Convergence theorems.*

Assume X is an E-valued, p-summable process measure relative to (F, G). Theorem 1.5 states a convergence property of the integrals $\int H^n dI_X$. In this

section we state convergence theorems for sequences (H^n) in the topology of $\mathcal{F}_{F,G}(I_X)$.

The following theorem will be used to prove that the space $L^1_{F,G}(X)$ is complete (Corollary 2.9).

THEOREM 2.8: *Let (H^n) be a sequence in $L^1_{F,G}(X)$ and assume that $H^n \to H$ in $\mathcal{F}_{F,G}(I_X)$. Then*

a) $H \in L^1_{F,G}(X)$;

b) *for every $t \geq 0$ and $B \in \mathcal{E}$, we have*

$$(H^n \cdot X)_t(B) \to (H \cdot X)_t(B), \quad \text{in } L^0_G$$

and there exists a subsequence (n_r) such that

$$(H^{n_r} \cdot X)_t(B) \to (H \cdot X)_t(B), \quad \text{a.s., as } r \to \infty,$$

uniformly on every compact time interval.

PROOF: Since (H^n) is a Cauchy sequence in $L^1_{F,G}(X)$, by taking a subsequence, if necessary, we can assume that $\hat{I}_{F,G}(H^n - H^{n+1}) \leq 1/4^n$ for each n. Let $t_0 > 0$. For each n denote $Z^n = H^n \cdot X$. For each $B \in \mathcal{E}$ define the stopping time

$$u_n = \inf\left\{t: |Z^n_t(B) - Z^{n+1}_t(B)| > \frac{1}{2^n}\right\} \wedge t_0.$$

Set $G_n = \{u_n < t_0\}$. For each stopping time v we have, by Theorem 2.4,

$$Z^n_v(B) = \int_{[0,v]} H^n \, dI_{X(B)} = \int_{[0,v]\times B} H^n \, dI_X;$$

hence

$$E(|Z^n_v(B) - Z^{n+1}_v(B)|) = E\left(\left|\int_{[0,v]\times B} (H^n - H^{n+1})\, dI_X\right|\right) =$$

$$= \left\|\int_{[0,v]\times B} (H^n - H^{n+1})\, dI_X\right\|_{L^1_G} \leq$$

$$\leq \left\|\int_{[0,v]\times B} (H^n - H^{n+1})\, dI_X\right\|_{L^0_G} \leq \hat{I}_{F,G}(H^n - H^{n+1}) \leq \frac{1}{4^n}.$$

In particular, for $v = u_n$ we have

$$E(|Z^n_{u_n}(B) - Z^{n+1}_{u_n}(B)|) \leq \frac{1}{4^n}.$$

On the other hand

$$1_{G_n} = 1_{\{u_n < t_0\}} \leq 1_{\{|Z^n_{u_n}(B) - Z^{n+1}_{u_n}(B)| \geq 1/2^n\}} \leq$$

$$\leq 2^n \left| Z^n_{u_n}(B) - Z^{n+1}_{u_n}(B) \right|,$$

therefore

$$P(G_n) \leq 2^n E(|Z^n_{u_n}(B) - Z^{n+1}_{u_n}(B)|) \leq \frac{1}{2^n}.$$

Let $G_0 = \limsup G_n$. Then $P(G_0) = 0$. For $\omega \notin G_0$, there is a k such that if $n \geq k$, then $\omega \notin G_n$, hence $u_n(\omega) = t_0$. Thus

$$\sup_{t < t_0} |Z^n_t(\omega, B) - Z^{n+1}_t(\omega, B)| \leq \frac{1}{2^n}.$$

It follows that for $\omega \notin G_0$, the sequence $(Z^n_t(\omega, B))$ is Cauchy in G, uniformly for $t < t_0$. The process $Z^{(0)}_t(\omega, B) := \lim Z^n_t(\omega, B)$ defined for $t < t_0$ and $\omega \notin G_0$, with values in G, is cadlag, adapted, and $|Z^n_t(\omega, B) - Z^{(0)}_t(\omega, B)|_G \leq 1/2^{n-1}$, hence $\|Z^n_t(B) - Z^{(0)}_t(B)\|_{L^p_G} \leq 1/2^{n-1}$. It follows that for $t < t_0$, we have $Z^{(0)}_t(B) \in L^p_G$ and $Z^n_t(B) \to Z^{(0)}_t(B)$ in L^p_G. On the other hand, $1_{[0,t] \times B} H^n \to 1_{[0,t] \times B} H$ in $\mathcal{F}_{F,G}(X)$, hence $Z^n_t(B) \to \int_{[0,t] \times B} H dI_X$ in $(L^q_{G^*})^*$, $1/p + 1/q = 1$. It follows that

$$\int_{[0,t] \times B} H dI_X = Z^{(0)}_t(B) \in L^p_G.$$

Taking $t_0 = k \in \mathbb{N}$, we obtain, as above, a process $Z^{(k)}_t(B)$ defined for $t < k$. We have then

$$Z^{(k)}_t(B) = Z^{(k+1)}_t(B), \quad a.s., \quad \text{for } t < k,$$

therefore, the limit $Z_t(B) = \lim_k Z^{(k)}_t(B)$ exists a.s., is cadlag, adapted and we have

$$Z_t(B) = Z^{(k)}_t(B), \quad a.s., \quad \text{for } t < k,$$

and

$$Z_t(B) = \int_{[0,t] \times B} H dI_X.$$

Since $Z_t(B) \in L^p_G$ and Z is cadlag, we deduce that $H \in L^1_{F,G}(X)$ and $(H \cdot X)_t = Z_t$ for each t.

COROLLARY 2.9: *The space $L^1_{F,G}(X)$ is complete.*

The following theorem shows that uniform convergence implies convergence in $\mathcal{F}_{F,G}(X)$.

THEOREM 2.10: *Let (H^n) be a sequence from $\mathcal{F}_{F,G}(X)$, converging uniformly, pointwise to a process H. Then*

$$H \in \mathcal{F}_{F,G}(X) \quad \text{and} \quad H^n \to H \text{ in } \mathcal{F}_{F,G}(X).$$

If $H^n \in L^1_{F,G}(X)$ for each n, then:

(a) *$H \in L^1_{F,G}(X)$ and $H^n \to H$ in $L^1_{F,G}(X)$;*

(b) *For every $t \geq 0$ and $B \in \mathcal{L}$, we have*

$$(H^n \cdot X)_t(B) \to (H \cdot X)_t(B), \quad \text{in } L^p_G;$$

(c) *There is a subsequence (n_r) such that*

$$(H^{n_r} \cdot X)_t(B) \to (H \cdot X)_t(B), \quad \text{a.s.}, \quad \text{as } r \to \infty,$$

uniformly on each compact time interval.

Assertions (a) and (c) follow from Theorem 2.8, and assertion (b) follows from the continuity in L^p_G of the integral.

For the Vitali and Lebesgue theorems, pointwise convergence does not ensure convergence in $L^1_{F,G}(X)$, unless $I_{F,G}$ is uniformly σ-additive. These theorems follow from Theorems 2.8 and 2.10, and from the general Vitali and Lebesgue convergence theorems in [B-D.1], Theorems AI.9 and AI.10.

THEOREM 2.11: (*Vitali*). *Let (H^n) be a sequence from $\mathcal{F}_{F,G}(X)$ and let $H: \mathbb{R}_+ \times \Omega \times L \to F$ be a predictable process. Assume that*

(1) $\bar{I}_{F,G}(H^n 1_C) \to 0$ *as* $\bar{I}_{F,G}(C) \to 0$, *uniformly in n, and either one of the conditions (2) or (2′) below:*

(2) *$H^n \to H$ in $\bar{I}_{F,G}$-measure;*

(2′) *$H^n \to H$ pointwise and $I_{F,G}$ is uniformly σ-additive (for example, if $F = \mathbb{R}$). Then*

(a) *$H \in \mathcal{F}_{F,G}(X)$ and $H^n \to H$ in $\mathcal{F}_{F,G}(X)$.*

Under hypotheses (1) *and* (2) *or* (2'), *assume that* $H^n \in L^1_{F,G}(X)$ *for each* n. *Then*

(b) $H \in L^1_{F,G}(X)$ *and* $H^n \to H$ *in* $L^1_{F,G}(X)$;

(c) *For every* $t \geq 0$ *and* $B \in \mathcal{L}$ *we have*

$$(H^n \cdot X)_t(B) \to (H \cdot X)_t(B), \quad \text{in } L^0_G;$$

(d) *There is a subsequence* (n_r) *such that*

$$(H^{n_r} \cdot X)_t(B) \to (H \cdot X)_t(B), \quad a.s., \quad \text{as } r \to \infty,$$

uniformly on compact time intervals.

THEOREM 2.12: (*Lebesgue*). *Let* (H^n) *be a sequence from* $\mathcal{F}_{F,G}(X)$ *and let* $H: \mathbb{R}_+ \times \times \Omega \times L \to F$ *be a predictable process. Assume that*

(1) *There is a process* $\phi \in \mathcal{F}_{\mathbb{R}}(\mathcal{B}, I_{F,G})$ = *closure in* $\mathcal{F}_{\mathbb{R}}(I_{F,G})$ *of bounded processes, such that* $|H^n| \leq \phi$ *for every* n, *and either one of the conditions* (2) *or* (2') *below:*

(2) $H^n \to H$ *in* $I_{F,G}$*-measure;*

(2') $H^n \to H$ *pointwise and* $I_{F,G}$ *is uniformly* σ*-additive* (*for example, if* $F = \mathbb{R}$). *Then*

(a) $H \in \mathcal{F}_{F,G}(\mathcal{B}, X)$ *and* $H^n \to H$ *in* $\mathcal{F}_{F,G}(X)$.
Assume, in addition, that $H^n \in L^1_{F,G}(X)$ *for each* n. *Then*

(b) $H \in L^1_{F,G}(X)$ *and* $H^n \to H$, *in* $L^1_{F,G}(X)$;

(c) *For every* $t \geq 0$ *and* $B \in \mathcal{L}$ *we have*

$$(H^n \cdot X)_t(B) \to (H \cdot X)_t(B), \quad \text{in } L^1_{F,G}(X);$$

(d) *There is a subsequence* (n_r) *such that*

$$(H^{n_r} \cdot X)_t(B) \to (H \cdot X)_t(B), \quad a.s., \quad \text{as } r \to \infty,$$

uniformly on compact time intervals.

In condition 2' of the Vitali and Lebesgue theorems, $I_{F,G}$ is the set of positive measures

$$\{ |(I_X)_z| : z \in L^0_{G^*}, \|z\|_\infty \leq 1 \}.$$

2.3. *Summability of the Stochastic Integral.*

Let X be an E-valued p-summable process measure relative to (F, G). If $H \in L^1_{F,G}(X)$, then the stochastic integral $H \cdot X$ is again a cadlag, adapted, p-process

measure with values in G. In this section we shall prove that under certain conditions, the process measure $H \cdot X$ is itself summable. This will be used to prove that the step functions are dense in $L^1_{\mathbb{R},L^2}$ in Theorem 3.11.

We consider first the case when H is scalar valued.

We notice that X is also p-summable relative to $(\mathbb{R}, E)$, but, in general, there is no relationship between the seminorms in $L^1_{F,G}(X)$ and $L^1_{\mathbb{R},E}(X)$.

THEOREM 2.13: *Let $H \in L^1_{\mathbb{R},E}(X) \cap \mathcal{F}_{\mathbb{R}}((I_X)_{F,G})$. Assume that $\int_C H dI_X \in L^p_E$ for every $C \in \mathcal{P} \otimes \mathcal{L}$. Then*

(a) *$H \cdot X$ is p-summable relative to (F, G) and we have*

$$I_{(H \cdot X)}(C) = \int_C H dI_X, \quad \text{for } C \in \mathcal{P} \otimes \mathcal{L}.$$

(b) *For any predictable function $K \geq 0$ we have*

$$(\tilde{I}_{H \cdot X})_{F,G}(K) = (\tilde{I}_X)_{F,G}(KH).$$

(c) *$K \in L^1_{F,G}(H \cdot X)$ iff $KH \in L^1_{F,G}(X)$ and in this case we have*

$$K \cdot (H \cdot X) = (KH) \cdot C.$$

(d) *Assume $(I_X)_{F,G}$ is uniformly σ-additive. Then $(I_{H \cdot X})_{F,G}$ is uniformly σ-additive iff $H \in \mathcal{F}_{\mathbb{R}}(\mathcal{B}, (I_X)_{F,G})$.*

PROOF: Since $\int_A H dI_X \in L^p_E$ for every $C \in \mathcal{P} \otimes \mathcal{L}$, the mapping $A \mapsto \int_A H dI_X$ from $\mathcal{P} \otimes \mathcal{L}$ into L^p_E is σ-additive (see [B-D.1], AI12(a) and [B-D.2], Proposition 17). We shall prove first that for every set $C = A \times B$ with $A \in \mathcal{R}$ and $B \in \mathcal{L}$ we have

(i) $(I_{H \cdot X})(C) = \int_C H dI_X$.

Let first $C = A \times \{0\} \times B$ with $A \in \mathcal{F}_0$ and $B \in \mathcal{L}$. Then by Theorem 2.5 we have

$$(I_{H \cdot X})(C) = 1_A((H \cdot X)(B)) =$$

$$= 1_A \int 1_{\{0\} \times B} H dI_X = \int 1_{A \times \{0\} \times B} H dI_X =$$

$$= \int_C H dI_X.$$

Let now $C = A \times (s, t] \times B$ with $A \in \mathcal{F}_s$ and $B \in \mathcal{L}$. Then, using again Theorem 2.5, we have

$$(I_{H\cdot X})(C) = 1_A((H\cdot X)_t(B) - (H\cdot X)_s(B)) = 1_A\left(\int_{[0,t]\times B} HdI_X - \int_{[0,s]\times B} HdI_X\right) =$$

$$= 1_A \int_{(s,t]\times B} HdI_X = \int_{A\times(s,t]\times B} HdI_X = \int_C HdI_X.$$

Then equality (i) remains valid for $C \in \mathcal{R} \times \mathcal{L}$. From this equality it follows that $I_{H\cdot X}$ can be extended to a σ-additive measure on $\mathcal{P} \otimes \mathcal{L}$, with values in L^p_E, and equality (i) remains valid for all the sets $C \in \mathcal{P} \otimes \mathcal{L}$.

We shall now prove

(ii) $(\tilde{I}_{H\cdot X})_{F,G}(C) = (\tilde{I}_X)_{F,G}(1_C H)$, for $C \in \mathcal{P} \otimes \mathcal{L}$.

Since $(\tilde{I}_X)_{F,G}(1_C H) \leq (\tilde{I}_X)_{F,G}(H) < \infty$, it will follow that $I_{H\cdot X}$ has finite semivariation relative to (F, G), therefore $H\cdot X$ is p-summable relative to (F, G).

To prove (ii), let $C \in \mathcal{P} \otimes \mathcal{L}$, $z \in L^q_{E^*}$, $1/p + 1/q = 1$ and $x \in F$. Consider the σ-additive measures $(I_{H\cdot X})_z : \mathcal{P} \otimes \mathcal{L} \to F^*$ and $(I_X)_z : \mathcal{P} \otimes \mathcal{L} \to F^*$. We have

$$\langle x, (I_{H\cdot X})_z(C)\rangle = \langle (I_{H\cdot X})(C)x, z\rangle =$$

$$= \left\langle \left(\int_C HdI_X\right)x, z\right\rangle = \left\langle \int_C xHdI_X, z\right\rangle = \int_C (x, Hd(I_X)_z) = \left\langle x, \int_C Hd(I_X)_z\right\rangle,$$

therefore

(iii) $(I_{H\cdot X})_z(C) = \int_C Hd(I_X)_z$, for $C \in \mathcal{P} \otimes \mathcal{L}$.

Since $(\tilde{I}_X)_{F,G}(H) < \infty$, it follows that $(I_X)_z$ has finite finite variation $|(I_X)_z|$; then $(I_{H\cdot X})_z$ has also finite variation $|(I_{H\cdot X})_z|$ and since H is scalar valued, we have ([D.1], Theorem 7, p. 278)

(iv) $|(I_{H\cdot X})_z|(C) = \int_C |H| d|(I_X)_z|$, for $C \in \mathcal{P} \otimes \mathcal{L}$.

Taking the supremum for $\|z\|_{q} \leq 1$, we get

$$(\tilde{I}_{H\cdot X})_{F,G}(C) = (\tilde{I}_X)(1_C H) < \infty, \qquad \text{for } C \in \mathcal{P} \otimes \mathcal{L}$$

and thus equality (ii) is proved, therefore $H\cdot X$ is p-summable relative to (F, G). This proves assertion (a).

To prove assertion (b) for $K \geq 0$ predictable, we use equality (iv) to deduce that

$$\int K d|(I_{H\cdot X})_z| = \int K|H| d|(I_X)_z|,$$

for every $z \in L^1_{G^*}$. Taking the supremum for $\|z\|_q \leq 1$, we get

$$(\tilde{I}_{H\cdot X})_{F,G}(K) = (\tilde{I}_X)_{F,G}(KH)$$

which is assertion (b).

To prove assertion (c), we notice first that from assertion (b) we have

$$K \in \mathcal{F}_{F,G}(I_{H\cdot X}) \quad \text{iff } KH \in \mathcal{F}_{F,G}(I_X).$$

From (iii) and (iv) it follows that for every $z \in L^q_{G^*}$ we have

$$K \in L^1_F((I_{H\cdot X})_z) \quad \text{iff } KH \in L^1_F((I_X)_z),$$

and in this case we have

$$\int K d(I_{H\cdot X})_z = \int KH d(I_X)_z,$$

therefore

$$\int K dI_{H\cdot X} = \int KH dI_X.$$

Replacing K with $1_{[0,t]\times B}K$, we deduce that

$$\int_{[0,t]\times B} K dI_{H\cdot X} = \int_{[0,t]\times B} KH dI_X.$$

It follows that the left hand side of this equality belongs to L^p_G and has a cadlag modification iff the right hand side has the same property. This means that

$$K \in L^1_{F,G}(H\cdot X) \quad \text{iff } KH \in L^1_{F,G}(X),$$

and in this case

$$K\cdot(H\cdot X) = (KH)\cdot X.$$

This proves assertion (c).

Finally, to prove assertion (d), we use assertion (b) with $K = 1_C$ and deduce that

$$(\tilde{I}_{H\cdot X})_{F,G}(C) = (\tilde{I}_X)_{F,G}(1_C H), \quad \text{for } C \in \mathcal{P}\otimes\mathcal{L}.$$

By ([B-D.1], Theorem AI.8 and [B-D.2], Theorem 13), we deduce that $(I_{H\cdot X})_{F,G}$

is uniformly σ-additive iff $H \in \mathcal{F}_{\mathbb{R}}(\mathcal{B}, (I_X)_{F,G})$, and this proves assertion (d) and the theorem.

We state now the analog of Theorem 2.13, with H vector valued.

THEOREM 2.14: *Let $H \in L^1_{F,G}(X)$ and assume that $\int_C H dI_X \in L^p_E$ for every $C \in \mathcal{P} \otimes \mathcal{L}$. Then*

(a) *$H \cdot X$ is p-summable relative to $(\mathbb{R}, G)$ and*

$$(I_{H \cdot X})(C) = \int_C H dI_X, \quad \text{for } C \in \mathcal{P} \otimes \mathcal{L}.$$

(b) *For any predictable function $K \geq 0$ we have*

$$(\bar{I}_{H \cdot X})_{\mathbb{R},G}(K) \leq (\bar{I}_X)_{F,G}(KH).$$

(c) *If K is a real valued predictable function and if $KH \in L^1_{F,G}(X)$, then $K \in L^1_{\mathbb{R},G}(H \cdot X)$ and we have*

$$K \cdot (H \cdot X) = (KH) \cdot X.$$

The proof is similar to that of Theorem 2.13, replacing equality (iv) in Theorem 2.13 with the inequality

$$|(I_{H \cdot X})_t|(C) \leq \int_C |H| d|(I_X)_t|$$

and equality (ii) with the inequality

$$(\bar{I}_{H \cdot X})_{\mathbb{R},G}(C) \leq (\bar{I}_X)_{F,G}(1_C H).$$

REMARK: $(I_{H \cdot X})_{\mathbb{R},G}$ is automatically uniformly σ-additive.

2.4. *Summable martingale measures.*

We say that a process measure $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ is a *martingale measure*, if for each $B \in \mathcal{L}$, the process $(X_t(B))_{t \geq 0}$ is a martingale. If, in addition, $X_t(B) \in L^p_E$ for each $t \geq 0$ and $B \in \mathcal{L}$, we say that X is a p-martingale measure.

A martingale measure X is not necessarily summable. But if it is, then the following theorem states that the stochastic integral $H \cdot X$ is again a martingale measure; and if L^p_E is reflexive, then $H \cdot X$ is itself p-summable (Corollary 2.16).

THEOREM 2.15: *Let $X: \Omega \times \mathbb{R}_+ \times \Omega \to E$ be a martingale measure, p-summable relative to (F, G) and let $H \in \mathcal{F}_{F,L^p_E}(X)$ such that $\int_{[0,t] \times B} H dI_X \in L^p_E$ for every $t \geq 0$*

and $B \in \mathcal{L}$. Then $H \in L^1_{\mathcal{F},\mathcal{L}}(X)$ and $H \cdot X$ is a uniformly integrable martingale measure, bounded in $L^1_{\mathcal{L}}$.

Proof: Let $t \geq 0$, $A \in \mathcal{F}_t$ and $B \in \mathcal{L}$ and prove that

$$E\left(1_A \int_{]0,\infty] \times B} H dI_X\right) = E\left(1_A \int_{]0,t] \times B} H dI_X\right),$$

that is

$$(*) \qquad E\left(1_A \int 1_{]t,\infty] \times B} H dI_X\right) = 0 .$$

If $H = 1_{\{0\} \times C \times B} x$ with $C \in \mathcal{F}_0$ and $x \in F$, then $(*)$ holds. Assume $H = 1_{]u,v] \times C \times B} x$ with $C \in \mathcal{F}_u$ and $x \in F$. If $v \leq t$, then $(*)$ holds. Assume $t < v$. Then

$$\int 1_{]t,\infty] \times B} H dI_X = 1_C x (X_v(B) - X_{t \vee u}(B)) ,$$

thus

$$1_A \int 1_{]t,\infty] \times B} H dI_X = 1_{A \cap C} x (X_v(B) - X_{t \vee u}(B)) .$$

Since $A \cap C \in \mathcal{F}_{t \vee u}$, taking expectations we obtain $(*)$. Thus $(*)$ holds for $r(\mathcal{R} \times \mathcal{L})$-simple functions H. Assume now H is $\mathcal{P} \otimes \mathcal{L}$-measurable. Let $y^* \in G^*$ and set $z = 1_A y^* \in L^\infty_{G^*}$. The $r(\mathcal{R} \times \mathcal{L})$-simple functions are dense in $L^1_F((I_X)_z)$. Let (H^n) be a sequence of $r(\mathcal{R} \times \mathcal{L})$-simple functions converging to H in $L^1_F((I_X)_z)$. Then

$$\int 1_{]t,\infty] \times B} H^n d(I_X)_z \to \int 1_{]t,\infty] \times B} H d(I_X)_z ,$$

that is,

$$\left\langle \int_{]t,\infty] \times B} H^n dI_X, z \right\rangle \to \left\langle \int_{]t,\infty] \times B} H dI_X, z \right\rangle .$$

Thus

$$E\left(\left\langle 1_A \int_{]t,\infty] \times B} H^n dI_X, y^* \right\rangle\right) \to E\left(\left\langle 1_A \int_{]t,\infty] \times B} H dI_X, y^* \right\rangle\right),$$

that is

$$\left\langle E\left(1_A \int_{]t,\infty] \times B} H^n dI_X\right), y^* \right\rangle \to \left\langle E\left(1_A \int_{]t,\infty] \times B} H dI_X\right), y^* \right\rangle .$$

Since H^n are $r(\mathcal{R} \times \mathcal{S})$-step functions, by the above, we have

$\left\langle E\left(1_A \int_{(t,\infty]\times B} H^n dI_X\right), y^*\right\rangle = 0$ for each n and each $y^* \in G^*$. It follows then that $E\left(1_A \int_{(t,\infty]\times B} H dI_X\right) = 0$. This means that $\left(\int_{[0,t]\times B} H dI_X\right)_{t\geq 0}$ is a uniformly integrable martingale, which has a cadlag modification ([B-D.3], Theorem 3). Since $B \mapsto \int_{[0,t]\times B} H dI_X$ is a σ-additive in $L^p_{\mathcal{E}}$, it follows that $H\cdot X$ is a martingale measure, belonging to $L^1_{F,L^p_{\mathcal{E}}}(X)$.

Corollary 2.16: *Let X be an E-valued martingale measure.*

a) *If L^p_E is reflexive and if X is p-summable relative to (F, G), then*

$$L^1_{F,L^p_{\mathcal{E}}}(X) = \mathcal{F}_{F,L^p_{\mathcal{E}}}(I_X)$$

and for every $H \in L^1_{F,L^p_{\mathcal{E}}}(X)$, the stochastic integral $H\cdot X$ is a martingale measure, p-summable relative to (R, G).

b) *If L^p_E is reflexive, if X is p-summable relative to (R, E), and if $H \in L^1_{\mathrm{R},L^p_{\mathcal{E}}}(X)$ satisfies also $(\tilde{I}_X)_{F,G}(H) < \infty$, then $H\cdot X$ is p-summable relative to (F, G).*

To prove the Corollary, we use Theorems 2.15 and 2.14 for assertion a) and Theorems 2.15 and 2.13 for assertion b).

3. - Orthogonal martingale measures in Hilbert spaces

In this paragraph we shall assume that E and G are Hilbert spaces. The inner product in E will be denoted $\langle x, y\rangle$, or $\langle x, y\rangle_E$, or simply xy, in order to distinguish it from the sharp bracket.

The main result of this paragraph, and of the paper is that an E-valued orthogonal martingale measure M is 2-summable relative to (R, E) (Theorem 3.9) and that the stochastic integral $H\cdot M$ is again an orthogonal martingale measure (Theorem 3.13). If M is a real valued orthogonal martingale measure, then it is 2-summable relative to the embedding $\mathrm{R} \subset L(D, D)$, for any Hilbert space D (Theorem 3.9).

3.1. *Definition of orthogonal martingale measures.*

Definition 3.1: *An E-valued orthogonal martingale measure (OMM) is a cadlag, square integrable martingale measure $M: \Omega \times \mathrm{R}_+ \times \mathcal{E} \to E$ satisfying the following conditions:*

α) *$M_0(B) = 0$, for every $B \in \mathcal{E}$;*

β) *For every $B, B' \in \mathcal{E}$ disjoint, the martingales $(M(B))_{t\geq 0}$ and $(M(B'))_{t\geq 0}$ are*

orthogonal, i.e., their sharp bracket vanishes:

$$\langle M(B), M(B')\rangle_t = 0, \quad \textit{for every } t \geq 0 .$$

Condition α) allows us to extend M with $M_t(B) = 0$ for $t < 0$ and $B \in \mathcal{L}$. If we set $\mathcal{F}_t = \mathcal{F}_0$ for $t < 0$, then the extended process set function $(M_t(B))_{t \in \mathbb{R},\, B \in \mathcal{L}}$ is still an orthogonal martingale measure with respect to the extended filtration $(\mathcal{F}_t)_{t \in \mathbb{R}}$.

Condition β) is equivalent to the condition that the process $(\langle M_t(B), M_t(B')\rangle)_{t \geq 0}$ is a martingale, for each $B, B' \in \mathcal{L}$ disjoint. In order to avoid confusion between the sharp bracket, we shall often denote the inner product of $M_t(B)$ and $M_t(B')$, simply, by $M_t(B)M_t(B')$.

3.2. *The process measure* $\langle M\rangle$.

Let M be an E-valued orthogonal martingale measure. For each set $B \in \mathcal{L}$. We shall denote by $\langle M(B)\rangle$ the sharp bracket of the martingale $M(B)$; $\langle M(B)\rangle$ is an increasing, right continuous, predictable process, with $\langle M(B)\rangle_0 = |M_0(B)|^2 = 0$ and $\langle M(B)\rangle_\infty$ integrable, such that

$$|M(B)|^2 - \langle M(B)\rangle$$

is a martingale. We shall denote by $\langle M\rangle$ the process set function defined by

$$\langle M\rangle_t(\omega, B) = \langle M(B)\rangle_t(\omega), \qquad \text{for } \omega \in \Omega,\ t \in \mathbb{R}_+ \text{ and } B \in \mathcal{L}.$$

Along with the extension of the martingale measure M with $M_t(\omega, B) = 0$ for $t < 0$, we extend also $\langle M\rangle$ with $\langle M\rangle_t(\omega, B) = 0$, for $t < 0$.

A first consequence of the fact that the martingale measure is orthogonal, is used in the following theorem to prove that the mapping $B \to \langle M(B)\rangle_t$ is finitely additive on $\mathcal{L}$, which allows us to prove that $\langle M\rangle$ is an 1-process measure.

THEOREM 3.2: *If M is an E-valued orthogonal martingale measure, then $\langle M\rangle$ is a positive, increasing, right continuous, predictable, 1-process measure.*

PROOF: The fact that $\langle M\rangle$ is positive, increasing, right continuous and predictable follows from the definition of the sharp bracket.

For each $t \leq +\infty$ and $B \in \mathcal{L}$, we have

$$E(\langle M\rangle_t(B)) = E(\langle M(B)\rangle_t) = E(|M_t(B)|^2) < \infty,$$

hence $\langle M\rangle_t(\cdot, B) \in L^1$.

For each $t \leq +\infty$, the mapping $B \mapsto \langle M\rangle_t(B)$ from $\mathcal{L}$ into L^1 is *additive*. In fact, if $B \cap B' = \phi$ in $\mathcal{L}$, then

$$M_t(B \cup B') = M_t(B) + M_t(B'), \qquad \text{in } L^2_E,$$

hence

$$M_t(B \cup B') = M_t(B) + M_t(B'), \quad a.s.$$

Then, a.s.,

$$\langle M(B \cup B')\rangle_t = \langle M(B) + M(B')\rangle_t =$$

$$= \langle M(B)\rangle_t + \langle M(B')\rangle_t + 2\langle M(B), M(B')\rangle_t = \langle M(B)\rangle_t + \langle M(B')\rangle_t,$$

since $\langle M(B), M(B')\rangle = 0$, by the definition of an orthogonal martingale measure. If now (B_n) is a decreasing sequence from $\mathcal{L}$ with $B_n \downarrow \phi$, then $M_t(B_n) \to 0$ in $L^2_{\mathcal{F}}$. It follows that

$$E(\langle M\rangle_t(B_n)) = E(\langle M(B_n)\rangle_t) = E(|M_t(B_n)|^2) \to 0,$$

hence $\langle M\rangle_t(B_n) \to 0$ in L^1. This proves that the mapping $B \to \langle M\rangle_t(B)$ from $\mathcal{L}$ into L^1 is σ-additive, that is, $\langle M\rangle$ is an 1-process measure.

Remark: We shall prove below that $\langle M\rangle$ is 1-summable, relative to (R, R). A consequence of Theorem 3.2 is that we can define $I_{\langle M\rangle}$ as a finitely additive measure from $r(\mathcal{R} \times \mathcal{L})$ into L^1. We are then able to consider the positive, finitely additive Doléans measure $\mu_{\langle M\rangle} = E(I_{\langle M\rangle})$.

We shall prove that both $I_{\langle M\rangle}$ and $\mu_{\langle M\rangle}$ can be extended to σ-additive measures on $\mathcal{P} \otimes \mathcal{L}$ (Theorem 3.8).

Another process set function associated to M and $\langle M\rangle$ is the submartingale set function $|M|^2$ defined by

$$|M|^2_t(\omega, B) = |M_t(\omega, B)|^2, \quad \text{for } \omega \in \Omega,\ t \leq +\infty \text{ and } B \in \mathcal{L}.$$

However, $|M|^2$ is not finitely additive on $\mathcal{L}$. In fact, if $B, B' \in \mathcal{L}$ are disjoint, we have

$$|M_t(B \cup B')|^2 = |M_t(B) + M_t(B')|^2 = |M_t(B)|^2 + |M_t(B')|^2 + 2M_t(B)\,M_t(B'),$$

and the inner product $M_t(B)M_t(B')$ is not necessarily 0.

We can define

$$I_{|M|^2}((s, t] \times A \times B) = 1_A(|M_t(B)|^2 - |M_s(B)|^2)$$

for $A \in \mathcal{F}_s$ and $B \in \mathcal{L}$, and then extend it to $\mathcal{R} \times \mathcal{L}$. But the mapping $B \mapsto I_{|M|^2}(A \times B)$ is not finitely additive on $\mathcal{L}$; hence $I_{|M|^2}$ cannot be extended to a finitely additive measure on $r(\mathcal{R} \times \mathcal{L})$.

However, the Doléans measure,

$$\mu_{|M|^2}((s, t] \times A \times B) = E(I_{|M|^2}((s, t] \times A \times B)),$$

defined for $A \in \mathcal{F}_s$ and $B \in \mathcal{L}$, can be extended to a positive, finitely additive measure

on $r(\mathcal{R} \times \mathcal{L})$, since, by the above equality we have

$$\mu_{|M|^2}((s, t] \times A \times B)) = E[1_A(|M_t(B)|^2 - |M_s(B)|^2)] =$$

$$= E[1_A(\langle M(B)\rangle_t - \langle M(B)\rangle_s)] = E[I_{\langle M\rangle}((s, t] \times A \times B)] = \mu_{\langle M\rangle}((s, t] \times A \times B),$$

and $\mu_{\langle M\rangle}$ is a finitely additive measure on $r(\mathcal{R} \times \mathcal{L})$.

We shall prove that $I_M \ll \mu_{\langle M\rangle}$ (Theorem 3.8). In order to prove that I_M can be extended to a σ-additive measure $I_M: \mathcal{P} \otimes \mathcal{L} \to L^2_E$, it is enough to prove that $\mu_{\langle M\rangle}$ is σ-additive on $r(\mathcal{R} \times \mathcal{L})$.

However, this cannot be done directly. We shall prove the existence of a positive, right continuous, increasing and incrementally increasing, two parameter process F associated to $\langle M\rangle$ and satisfying

$$F(\omega, t, x) = \langle M(-\infty, x]\rangle_t(\omega), \quad a.s.$$

(Theorem 3.3 infra). Then, there is a σ-additive, positive measure μ_F on $\mathcal{F} \otimes \mathcal{B}(\mathrm{R}) \otimes \mathcal{L}$ induced by F, and satisfying $\mu_F = \mu_{\langle M\rangle}$. (Theorem 3.4 infra). This ensures that $\mu_{\langle M\rangle}$ is σ-additive and that I_M can be extended to a σ-additive measure on $\mathcal{P} \otimes \mathcal{L}$.

3.3. *The two parameter process F and the measure μ_F.*

An important step in proving the 2-summability of an *OMM* is the construction of a two parameter process F associated to $\langle M\rangle$, in the following theorem. We consider M and $\langle M\rangle$ extended with 0 for $t < 0$.

THEOREM 3.3: *Let M be an E-valued orthogonal martingale measure. Then, there is an increasing, incrementally increasing, right continuous, two parameter process $F: \Omega \times \mathrm{R} \times L \to \mathrm{R}_+$ satisfying*

$$F(\omega, t, x) = \langle M(-\infty, x]\rangle_t(\omega), \quad a.s., \quad \textit{for } t \in \mathrm{R} \textit{ and } x \in L.$$

PROOF: By Theorem 3.2, if $B \cap B' = \phi$ in $\mathcal{L}$, we have, for each $t \in \mathrm{R}$,

$$\langle M(B \cup B')\rangle_t = \langle M(B)\rangle_t + \langle M(B')\rangle_t, \quad \text{in } L^1,$$

therefore

$$\langle M(B \cup B')\rangle_t(\omega) = \langle M(B)\rangle_t(\omega) + \langle M(B')\rangle_t(\omega), \quad a.s.$$

In particular, for $x \leq x'$ in L and $t \in \mathrm{R}$, taking $B = (-\infty, x]$ and $B' = (x, x']$, we obtain

$$\langle M(-\infty, x']\rangle_t(\omega) = \langle M(-\infty, x]\rangle_t(\omega) + \langle M(x, x']\rangle_t(\omega), \quad a.s.,$$

the negligible set $N(t, x, x')$ depending on t, x and x'. It follows that if $x \leq x'$ and

$t \in \mathrm{R}$ we have

$$\langle M(-\infty, x]\rangle_t(\omega) \leq \langle M(-\infty, x']\rangle_t(\omega), \quad \text{for } \omega \notin N(t, x, x').$$

Denote $N = \bigcup\{N(t, x, x'): t, x, x' \text{ rational}\}$. Then N is negligible. We now define the function $G: \Omega \times \mathrm{Q} \times \mathrm{Q} \to \mathrm{R}_+$ by

$$G(\omega, t, x) = \langle M(-\infty, x]\rangle_t(\omega), \quad \text{for } \omega \notin N,\ t, x \in \mathrm{Q},$$

and

$$G(\omega, t, x) = 0, \quad \text{for } \omega \in N.$$

Then

$$\langle M(x, x']\rangle_t(\omega) = G(\omega, t, x') - G(\omega, t, x),$$

for $\omega \notin N$, t, x, x' rational, $x \leq x'$.

The function G is increasing and incrementally increasing. In fact, if $\omega \notin N$, and $t \leq t'$, $x \leq x'$ are rational, then

$$G(\omega, t, x) = \langle M(-\infty, x]\rangle_t(\omega) \leq \langle M(-\infty, x']\rangle_t(\omega),$$

and for $\omega \in N$ we have $G(\omega, t, x) = G(\omega, t, x') = 0$; hence G is increasing. Also, for $\omega \notin N$ we have

$$\Delta_{(t,x),(t',x')} G(\omega) = G(\omega, t', x') + G(\omega, t, x) - G(\omega, t', x) - G(\omega, t, x') =$$

$$= \langle M(-\infty, x']\rangle_{t'}(\omega) - \langle M(-\infty, x]\rangle_{t'}(\omega) - (\langle M(-\infty, x']\rangle_t(\omega) - \langle M(-\infty, x]\rangle_t(\omega)) =$$

$$= \langle M(x, x']\rangle_{t'}(\omega) - \langle M(x, x']\rangle_t(\omega) \geq 0,$$

and for $\omega \in N$ we have $\Delta_{(t,x),(t',x')} G(\omega) = 0$; therefore G is incrementally increasing.

Since G is increasing, it has finite right limits at every point $(t, x) \in \mathrm{R} \times L$, not necessarily rationals. We now define the function $F: \Omega \times \mathrm{R} \times L \to \mathrm{R}_+$ by

$$F(\omega, t, x) = \lim_{\substack{t' \downarrow t \\ x' \downarrow x \\ t', x' \in \mathrm{Q}}} G(\omega, t', x').$$

It follows, first, that F is increasing and incrementally increasing. We shall prove now that F is right continuous. Let $\omega \in \Omega$, $(t, x) \in \mathrm{R} \times L$, and (t_n, x_n) a sequence in $\mathrm{R} \times L$ with $t_n \to t$ and $x_n \to x$, with $t_n \geq t$ and $x_n \geq x$. For each n let t_n' and x_n' be rational numbers such that $t_n < t_n' < t_n + 1/n$ and $x_n < x_n' < x_n + 1/n$. Then $t_n' \to t$, $t_n' > t$ and $x_n' \to x$, $x_n' > x$. If $\omega \notin N$, we have

$$F(\omega, t, x) = \lim G(\omega, t_n', x_n');$$

on the other hand

$$F(\omega, t, x) \leq F(\omega, t_n, x_n) \leq G(\omega, t_n', x_n'),$$

hence $\lim F(\omega, t, x_n) = F(\omega, t, x)$; therefore F is right continuous. If $\omega \in N$, F is also, evidently, right continuous.

Finally, we shall prove now that

$$F(\omega, t, x) = \langle M(-\infty, x]\rangle_t(\omega), \quad a.s., \quad \text{for } t \in \mathrm{R} \text{ and } x \in L.$$

Let $\omega \notin N$ and $(t, x) \in \mathrm{R} \times L$. Since the right limit of G at (t, x) exists and is equal to $F(\omega, t, x)$, the iterated limits of G at (t, x) also exist and are equal to $F(\omega, t, x)$:

$$F(\omega, t, x) = \lim_{\substack{t' \downarrow t \\ x' \downarrow x \\ t', x' \in \mathrm{Q}}} G(\omega, t', x') = \lim_{x' \downarrow x} \lim_{t' \downarrow t} G(\omega, t', x') =$$

$$= \lim_{x' \downarrow x} \lim_{t' \downarrow t} \langle M(-\infty, x']\rangle_{t'}(\omega) = \lim_{x' \downarrow x} \langle M(-\infty, x']\rangle_t(\omega),$$

since $\langle M\rangle$ is right continuous. Let $x_n' \downarrow x$ with x_n' rationals. By the above, we have

$$\langle M(-\infty, x_n']\rangle_t(\omega) \to F(\omega, t, x), \quad a.s.$$

On the other hand the mapping $B \to \langle M(B)\rangle$ is σ-additive in L^1 on $\mathcal{L}$, by Theorem 3.2. Then

$$\langle M(-\infty, x_n']\rangle_t \to \langle M(-\infty, x]\rangle_t, \quad \text{in } L^1.$$

It follows that the two limits of the sequence $(\langle M(-\infty, x_n']\rangle_t)_n$ are equal a.s.:

$$F(\omega, t, x) = \langle M(-\infty, x]\rangle_t(\omega), \quad a.s.$$

and the theorem is proved.

Remark: Theorem 3.3 is valid under more general conditions, for a right continuous p-process measure with values in a Banach space, with p-integrable variation (see [D.4], Theorem 2.4), or with p-integrable semivariation (see [D.5], Theorem 2.5).

We can now associate to F a σ-additive measure μ_F, which is equal to $\mu_{\langle M\rangle}$ on $r(\mathcal{R} \times \mathcal{S})$.

Theorem 3.4: *Let M be an E-valued orthogonal martingale measure and let $F: \Omega \times \times \mathrm{R} \times L$ be the function associated with $\langle M\rangle$ by Theorem 3.3:*

$$F(\omega, t, x) = \langle M(-\infty, x]\rangle_t(\omega), \quad a.s.$$

Then there is a positive, σ-additive measure μ_F defined on $\mathcal{F}\otimes\mathcal{B}(\mathrm{R})\otimes\mathcal{L}$ satisfying

$$\mu_F(C) = E\left(\int 1_C(\omega, t, x)\, F(\omega, dt, dx)\right), \quad \textit{for } C \in \mathcal{F}\otimes\mathcal{B}(\mathrm{R})\otimes\mathcal{L},$$

and

$$\mu_F(C) = \mu_{(M)}(C), \quad \textit{for } C \in r(\mathcal{R}\times\mathcal{S}).$$

Proof: For each $\omega\in\Omega$, the function $F(\omega)$: $\mathrm{R}\times L\to\mathrm{R}_+$ defined by $F(\omega)(t, x) = F(\omega, t, x)$, is right continuous and incrementally increasing. We associate with $F(\omega)$ a positive measure $m_{F(\omega)}$ defined first for rectangles $(t, t'] \times (x, x']$ by

$$m_{F(\omega)}((t, t'] \times (x, x']) = \Delta_{(t,x),(t',x')}F(\omega) =$$

$$= F(\omega, t', x') + F(\omega, t, x) - F(\omega, t', x) - F(\omega, t, x').$$

Since $F(\omega)$ is right continuous, $m_{F(\omega)}$ can be extended to a positive, σ-additive measure $m_{F(\omega)}$ on $\mathcal{B}(\mathrm{R})\otimes\mathcal{L}$.

Consider the space $L^1_k(m_{F(\omega)})$. For any function $f\in L^1_{k(m_{F(\omega)})}$ we can define the integral $\int f dm_{F(\omega)}$. This integral is also denoted $\int f dF(\omega)$ or $\int f(t, x)\, F(\omega, dt, dx)$ and is called the Stieltjes integral of f with respect to $F(\omega)$. In particular, for any set $C\in\mathcal{F}\otimes\mathcal{B}(\mathrm{R})\otimes\mathcal{L}$, the function $(t, x)\mapsto 1_C(\omega, t, x)$ is $\mathcal{B}(\mathrm{R})\otimes\mathcal{L}$-measurable, and $F(\omega)$-integrable, hence the integral $\int 1_C(\omega, t, x)\, F(\omega, dt, dx)$ is defined. We shall now prove that the function $\omega\mapsto\int 1_C(\omega, t, x)\, F(\omega, dt, dx)$ is P-integrable. This is true, first, for $C = A\times(t, t']\times(x, x']$ with $A\in\mathcal{F}$:

$$\int 1_C(\omega, t, x)\, F(\omega, dt, dx) = 1_A(\omega)\, \Delta_{(t,x),(t',x')}F(\omega)$$

and this function is P-integrable.

By a monotone class argument, we deduce that $\int 1_C(\omega, t, x)\, F(\omega, dt, dx)$ is P-integrable for every $C\in\mathcal{F}\otimes\mathcal{B}(\mathrm{R})\otimes\mathcal{L}$.

For each set $C\in\mathcal{F}\otimes\mathcal{B}(\mathrm{R})\otimes\mathcal{L}$ we set

$$\mu_F(C) = E\left(\int 1_C(\omega, t, x)\, F(\omega, dt, dx)\right).$$

Then μ_F is additive. It is also σ-additive. In fact, let (C_n) be a decreasing sequence from $\mathcal{F}\otimes\mathcal{B}(\mathrm{R})\otimes\mathcal{L}$ with $C_n\downarrow\phi$. For each $\omega\in\Omega$, the sequence $(C_n(\omega))$ of sections is decreasing to ϕ. Since $m_{F(\omega)}$ is σ-additive, we have $m_{F(\omega)}(C_n(\omega))\to 0$ for each $\omega\in\Omega$. Since the functions $\omega\mapsto m_{F(\omega)}(C_n(\omega)) = \int 1_{C_n}(\omega, t, x)\, F(\omega, dt, dx)$ are integrable

and tend decreasingly to 0, we deduce that

$$\lim \mu_F(C_n) = \lim E\left(\int 1_{C_n}(\omega, t, x)\, F(\omega, dt, dx)\right) = 0\,,$$

which shows that μ_F is σ-additive.

Let now $C = A \times (t, t'] \times (x, x']$ with $A \in \mathcal{F}_t$. Then

$$\mu_F(C) = E\left(\int 1_A(\omega) 1_{(t,t'] \times (x,x']} F(\omega, dt, dx)\right) = E\left(1_A(\omega) \int 1_{(t,t'] \times (x,x']}\, dm_{F(\omega)}\right) =$$

$$= E(1_A(\omega)\, m_{F(\omega)}((t, t'] \times (x, x'])) = E(1_A(\omega)\, \Delta_{(t,x),(t',x')} F(\omega)) =$$

$$= E(1_A(F(\cdot, t', x') + F(\cdot, t, x) - F(\cdot, t', x) - F(\cdot, t, x'))) =$$

$$= E(1_A(\langle M(x, x']\rangle_{t'} - \langle M(x, x']\rangle_t)) = E(I_{\langle M\rangle}(C)) = \mu_{\langle M\rangle}(C).$$

The equality $\mu_F(C) = \mu_{\langle M\rangle}(C)$ remains valid for $C \in r(\mathcal{R} \times \mathcal{L})$, and the theorem is proved.

3.4. *Orthogonality of I_M and additivity of $\|I_M\|_2^2$.*

The next step in proving the 2-summability of an *OMM* is to prove that the set function $C \to \|I_M(C)\|_{L_E^2}^2$ is finitely additive on $r(\mathcal{R} \times \mathcal{L})$ and that $\mu_{\langle M\rangle}(C) = \|I_M(C)\|_{L_E^2}^2$ on $r(\mathcal{R} \times \mathcal{L})$.

For this, we have to prove first an orthogonality property of I_M.

Proposition 3.5: *Let M be an E-valued (not necessarily orthogonal) martingale measure. Then*

(i) *I_M can be extended to a separately σ-additive bimeasure $I_M: \mathcal{P} \times \mathcal{L} \to L_E^2$, and to a finitely additive measure $I_M: r(\mathcal{P} \times \mathcal{L}) \to L_E^2$.*

(ii) *Let N be another E-valued (not necessarily orthogonal) martingale measure. Then for any disjoint sets $A, A' \in \mathcal{P}$, for any (not necessarily disjoint) sets $B, B' \in \mathcal{L}$ and for any $x, y \in F$ we have*

$$I_M(A \times B) \perp I_N(A' \times B'), \qquad \text{in } L_E^2$$

and

$$I_M(A \times B)\, x \perp I_N(A' \times B')\, y\,, \qquad \text{in } L_G^2.$$

Proof: We shall divide the proof into several steps. Let $A, A' \in \mathcal{P}$ disjoint, and $B, B' \in \mathcal{L}$.

a) Assume $A = \{0\} \times C$ with $C \in \mathcal{F}_0$ and $A' = \{0\} \times C'$ with $C' \in \mathcal{F}_0$. Then $C \cap C' = \phi$, therefore

$$\langle I_M(A \times B), I_N(A' \times B')\rangle_E = 1_C 1_{C'} \langle M_0(B), N_0(B')\rangle_E = 0 .$$

Taking expectations, we deduce that $I_M(A \times B) \perp I_N(A' \times B')$ in L^2_E.

b) Assume $A = \{0\} \times C$ with $C \in \mathcal{F}_0$ and $A' = (s, t] \times C'$ with $C' \in \mathcal{F}_s$. Then

$$E(\langle I_M(A \times B), I_N(A' \times B')\rangle_E \mid \mathcal{F}_s) = 1_C 1_{C'} E(\langle M_0(B), N_t(B') - N_s(B')\rangle_E \mid \mathcal{F}_s) =$$

$$= 1_C 1_{C'} \langle M_0(B), E(N_t(B') - N_s(B') \mid \mathcal{F}_s)\rangle_E = 0 .$$

Taking the expectation we deduce that

$$E(\langle I_M(A \times B), I_N(A' \times B')\rangle_E) = 0 ,$$

that is, $I_M(A \times B \perp I_N(A' \times B'))$ in L^2_E.

c) Assume now $A = (s, t] \times C$ with $C \in \mathcal{F}_s$ and $A' = (s', t'] \times C'$ with $C' \in \mathcal{F}_{s'}$. If C and C' are disjoint, then, we deduce as in step a) that $I_M(A \times B) \perp I_N(A' \times B')$ in L^2_E.

If $(s, t]$ and $(s', t']$ are disjoint, we can assume that $s' < t' < s < t$. Then

$$\langle I_M(A \times B), I_N(A' \times B')\rangle_{L^2_E} = E(\langle I_M(A \times B), I_N(A' \times B')\rangle_E) =$$

$$= E(1_C 1_{C'} \langle M_t(B) - M_s(B), N_{t'}(B') - N_{s'}(B')\rangle_E) =$$

$$= E(1_C 1_{C'} \langle M_t(B) - M_s(B), N_{t'}(B') - N_{s'}(B')\rangle_E) =$$

$$= E(1_C 1_{C'} \langle E(M_t(B) - M_s(B) \mid \mathcal{F}_s), N_{t'}(B') - N_{s'}(B')\rangle_E) = 0$$

since $(M_t(B))_{t \geq 0}$ is a martingale.

d) If $A, A' \in \mathcal{R}$, we can write A and A' as finite unions of disjoint predictable rectangular sets of the preceding kind. We then deduce that $I_M(A \times B) \perp I_N(A' \times \times B')$ in L^2_E.

e) For each $B \in \mathcal{E}$, the additive measure $I_M(A \times B)$ is bounded in L^2_E, for $A \in \mathcal{R}$. In fact, let $A \in \mathcal{R}$ and let $T > 0$ be such that $A \subset [0, T] \times \Omega$. Denote $A' = [0, T] \times \times \Omega \setminus A \in \mathcal{R}$. Since A and A' are disjoint, $I_M(A \times B)$ and $I_M(A' \times B)$ are orthogonal in L^2_E, therefore

$$\|I_M(A \times B)\|^2_{L^2_E} \leq \|I_M(A \times B)\|^2_{L^2_E} + \|I_M(A' \times B)\|^2_{L^2_E} =$$

$$= \|I_M((A \cup A') \times B)\|^2_{L^2_E} = \|M_T(B)\|^2_{L^2_E} \leq \sup_{t \geq 0} \|M_t(B)\|^2_{L^2_E} < \infty .$$

Since L^2_E is reflexive, we have $c_0 \not\subset L^2_E$. Since $M(B)$ is a square integrable martingale, and since the measure $A \mapsto I_M(B)(A) = I_M(A \times B)$ is bounded in L^2_E for $A \in \mathcal{R}$, by the extension theorem in [B-D.1] (Theorem 2.5), $I_{M(B)}$ can be extended to a σ-additive

measure $I_{M(B)}: \mathcal{P} \to L^2_E$. Thus we obtain a set function $I_M: \mathcal{P} \times \mathcal{L} \to L^2_E$ which is σ-additive on $\mathcal{P}$ for each fixed set $B \in \mathcal{L}$.

f) I_M is also σ-additive on $\mathcal{L}$ for each fixed set $A \in \mathcal{P}$. In fact, let $\mathcal{P}_1$ be the class of sets $A \in \mathcal{P}$ such that $B \mapsto I_M(A \times B)$ is σ-additive on $\mathcal{L}$. Then $\mathcal{P}_1$ contains the predictable rectangles: if $A = \{0\} \times C$ with $C \in \mathcal{F}_0$, then $I_M(A \times B) = 1_C M_0(B)$, and $B \mapsto M_0(B)$ is σ-additive in L^2_E; if $A = (s, t] \times C$ with $C \in \mathcal{F}_s$, then

$$I_M(A \times B) = 1_C(M_t(B) - M_s(B)) ,$$

hence $B \mapsto I_M(A \times B)$ is again σ-additive in L^2_E.

The class $\mathcal{P}_1$ is closed under finite disjoint unions, since $I_M(A \times B)$ is additive with respect to $A \in \mathcal{R}$. It follows that $\mathcal{P}_1$ contains the ring $\mathcal{R}$. On the other hand, $\mathcal{P}_1$ is a monotone class. In fact, let (A_n) be a monotone sequence from $\mathcal{P}_1$ and let $A = \lim A_n$. For each A_n, the set function $B \mapsto I_M(A_n, B)$ is σ-additive; for each $B \in \mathcal{L}$, we have $I_M(A_n \times B) \to I_M(A \times B)$ in L^2_E. By the Nikodym theorem ([D-S], IV.10.b), the set function $B \mapsto I_M(A \times B)$ is σ-additive on $\mathcal{L}$; hence $A \in \mathcal{P}_1$. It follows that $\mathcal{P}_1 = \mathcal{P}$.

In particular, I_M is separately finitely additive on $\mathcal{P} \times \mathcal{L}$, hence I_M can be extended to a finitely additive measure I_M on $r(\mathcal{P} \times \mathcal{L})$. This proves (i).

g) Consider now I_M and I_N extended as separately σ-additive bimeasures on $\mathcal{P} \times \mathcal{L}$. Let $A \in \mathcal{R}$ and $B, B' \in \mathcal{L}$. Denote by $\sum_A$ the class of sets $A' \in \mathcal{P}$ such that $I_M(A \times B) \perp I_N((A' \setminus A) \times B')$. Then $\sum_A$ is a monotone class containing $\mathcal{R}$, hence $\sum_A = \mathcal{P}$. It follows that for $A \in \mathcal{R}$ and $A' \in \mathcal{P}$ disjoint, we have $I_M(A \times B) \perp I_M(A' \times B')$ in L^2_E.

Let now $A' \in \mathcal{P}$ and denote by $\sum_{A'}'$ the class of sets $A \in \mathcal{P}$ such that $I_M(A \times \times B) \perp I_M((A' \setminus A) \times B')$. Again, $\sum_{A'}'$ is a monotone class containing $\mathcal{R}$, hence $\sum_{A'}' = \mathcal{P}$. It follows that for $A, A' \in \mathcal{P}$ disjoint we have $I_M(A \times B) \perp I_M(A' \times B')$ in L^2_E. This proves the first part of assertion (ii).

h) If $x, y \in F$, then $(M_t(B)\, x)_{t \geq 0,\, B \in \mathcal{L}}$ and $(N_t(B)\, y)_{t \geq 0,\, B \in \mathcal{L}}$ are square integrable martingale measures with values in G, and $I_{M_x}(A \times B) = I_M(A \times B)\, x$, and similarly, $I_{N_y}(A \times B) = I_N(A \times B y$. If $A \cap A' = \phi$, then, by the above,

$$I_M(A \times B)\, x \perp I_N(A' \times B)\, y , \qquad \text{in } L^2_G.$$

This completes the proof of the proposition.

Remark: If the rectangular sets $A \times B$ and $A' \times B'$ are disjoint, but $A \cap A' \neq \phi$, we do not necessarily have the orthogonality property of I_M in Proposition 3.5. However, if M is an *orthogonal* martingale measure and $N = M$, the orthogonality property of M is still valid.

THEOREM 3.6: *Let $M: \Omega \times \mathrm{R}_+ \times \mathcal{L} \to E$ be an orthogonal martingale measure. Then for any disjoint sets $C, C' \in r(\mathcal{P} \times \mathcal{L})$ and for any numbers $\alpha, \alpha' \in \mathrm{R}$ we have*

$$I_M(C)\,\alpha \perp I_M(C')\alpha', \quad \text{in } L^2_E.$$

PROOF: We shall devide the proof into several steps.

Assume first that $C = A \times B$ and $C' = A' \times B'$ are disjoint with $A, A' \in \mathcal{P}$ and $B, B' \in \mathcal{L}$. The case $A \cap A' = \phi$ is covered by Proposition 2.5. Therefore we shall assume $B \cap B' = \phi$.

a) Assume $A = A' = \{0\} \times D$ with $D \in \mathcal{F}_0$. Then

$$\langle I_M(C), I_M(C')\rangle_E = 1_D\langle M_0(B), M_0(B')\rangle = 0\,,$$

since M is an *OMM*, and $M_0(B) = 0$. Then, taking expectations, we deduce that $I_M(C) \perp I_M(C')$, in L^2_E.

b) Assume $A = A' = (s, t] \times D$ with $D \in \mathcal{F}_s$. Then

$$\langle I_M(A \times B), I_M(A \times B')\rangle_E = 1_D\langle M_t(B) - M_s(B), M_t(B') - M_s(B')\rangle_E =$$

$$= 1_D\langle M_t(B), M_t(B')\rangle_E - 1_D\langle M_t(B), M_s(B')\rangle_E - 1_D\langle M_s(B), M_t(B')\rangle_E +$$

$$+ 1_D\langle M_s(B), M_s(B')\rangle_E.$$

Since $B \cap B' = \phi$, the process $(\langle M_t(B), M_t(B')\rangle_E)_{t \geq 0}$ is a martingale (because M is an *OMM*), therefore

$$E(1_D\langle M_t(B), M_t(B')\rangle_E \,|\, \mathcal{F}_t) = 1_D\langle M_t(B), M_t(B)\rangle_E.$$

Also

$$E(1_D\langle M_t(B), M_s(B')\rangle_E \,|\, \mathcal{F}_t) = 1_D\langle M_t(B), M_s(B')\rangle_E,$$

$$E(1_D\langle M_s(B), M_t(B')\rangle_E \,|\, \mathcal{F}_t) = 1_D\langle M_s(B), M_t(B')\rangle_E$$

and

$$E(1_D\langle M_s(B), M_s(B')\rangle_E \,|\, \mathcal{F}_t) = 1_D\langle M_s(B), M_s(B)\rangle_E.$$

It follows that

$$E(\langle I_M(A \times B), I_M(A \times B')\rangle_E \,|\, \mathcal{F}_t) = 0\,,$$

consequently

$$E(\langle I_M(A \times B), I_M(A \times B')\rangle_E) = 0\,,$$

that is, $I_M(A \times B) \perp I_M(A \times B')$, in L^2_E.

c) From steps a) and b) it follows that for every set $A \in \mathcal{R}$ we have $I_M(A \times \times B) \perp I_M(A \times B')$, in L^2_E.

d) For $A \in \mathcal{P}$ we still have $I_M(A \times B) \perp I_M(A \times B')$ in L^2_E. In fact, denote by $\mathcal{M}$ the class of sets $A \in \mathcal{P}$ for which this is true. By step c), $\mathcal{M}$ contains $\mathcal{R}$. $\mathcal{M}$ is also closed under countable disjoint unions: let (A_n) the a sequence of disjoint sets from $\mathcal{M}$ with union A. Then

$$I_M(A \times B) = \sum_{1 \leq n < \infty} I_M(A_n \times B)$$

and

$$I_M(A \times B') = \sum_{1 \leq n < \infty} I_M(A_n \times B'),$$

in L^2_E. For each n we have $I_M(A_n \times B) \perp I_M(A_m \times B')$, in L^2_E. For $n \neq m$ we have $A_n \cap A_m = \phi$, hence, by Proposition 2.4 we have $I_M(A_n \times B) \perp I_M(A_m \times B')$, in L^2_E. It follows that

$$\langle I_M(A \times B), I_M(A \times B')\rangle_{L^2_E} = \sum_{n,m} \langle I_M(A_n \times B), I_M(A_m \times B')\rangle_{L^2_E} = 0,$$

therefore $A \in \mathcal{M}$. In particular

$$\Omega \times \mathrm{R}_+ = (\Omega \times \{0\}) \cup \left(\bigcup_{n=0}^{\infty} \Omega \times (n, n+1]\right) \in \mathcal{M}.$$

We deduce that $\mathcal{M}$ is a σ-algebra containing $\mathcal{R}$, hence $\mathcal{M} = \mathcal{P}$.

e) Let now $A, A' \in \mathcal{P}$ and $B, B' \in \mathcal{L}$ with $B \cap B' = \phi$. Then

$$\langle I_M(A \times B), I_M(A' \times B')\rangle_{L^2_E} =$$

$$= \langle I_M((A \setminus A') \times B) + I_M((A \cap A') \times B), I_M((A' \setminus A) \times B') +$$

$$+ I_M((A' \cap A) \times B')\rangle_{L^2_E} = 0.$$

We used the equalities $(A \setminus A') \cap (A' \setminus A) = \phi$, $(A \setminus A') \cap (A' \cap A) = \phi, (A \cap A') \cap (A' \setminus A) = \phi$, and Proposition 3.5.

f) Finally, if $C, C' \in r(\mathcal{P} \times \mathcal{L})$ are disjoint, then $C = \bigcup_{1 \leq i \leq \infty} A_i \times B_i$ with $A_i \times B_i$ mutually disjoint from $\mathcal{P} \times \mathcal{L}$ and $C' = \bigcup_{1 \leq j \leq \infty} A'_j \times B'_j$ mutually disjoint from $\mathcal{P} \times \mathcal{L}$. Then $I_M(A_i \times B_i) \perp I_M(A'_j \times B'_j)$ for each i and j, therefore

$$I_M(C) = \sum_i I_M(A_i \times B_i) \perp \sum_j I_M(A'_j \times B'_j) = I_M(C').$$

g) Finally, if $I_M(C) \perp I_M(C')$, in L^2_E and $\alpha, \alpha' \in \mathrm{R}$, then also $I_M(C)\alpha \perp \perp I_M(C')\alpha'$, in L^2_E.

REMARK: Theorem 3.6 will be extended in Theorem 3.10 for all sets $C \in \mathcal{P} \otimes \mathcal{L}$.

For real valued orthogonal martingale measures, we can replace the scalars α, α' in the orthogonality property by vectors x, x':

THEOREM 3.6': *Let M be a real valued orthogonal martingale measure and D a Hilbert space. Consider $\mathbb{R} \subset L(D, D)$. Then, for any disjoint sets C, $C' \in r(\mathcal{P} \times \mathcal{L})$ and for any elements $x, x' \in D$ we have*

$$I_M(C)x \perp I_M(C')x, \quad \text{in } L^2_D.$$

PROOF: $E(\langle I_M(C)x, I_M(C')x'\rangle_D) =$

$$= E(I_M(C)I_M(C')\langle x, x'\rangle_D) = E(I_M(C)I_M(C'))\langle x, x'\rangle = 0$$

since, by Theorem 2.5, $I_M(C) \perp I_M(C')$, in $L^2_{\mathbb{R}}$.

REMARK: In Theorem 3.10 we shall extend Theorem 3.6' to all sets $C \in \mathcal{P} \otimes \mathcal{L}$.

A first consequence of the orthogonality property of I_M is the additivity of the mapping $C \to \|I_M(C)\|^2_2$.

THEOREM 3.7: *Let $M: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ be an orthogonal martingale measure. Then the mapping $C \mapsto \|I_M(C)\|^2_{L^2_E}$ is finitely additive for C in $r(\mathcal{P} \times \mathcal{L})$.*

PROOF: Let $(C_i)_{1 \leq i \leq n}$ be a family of disjoint sets from $r(\mathcal{P} \times \mathcal{L})$ with union C. By Theorem 3.6, the family $(I_M(C_i))_{i \leq I}$ is orthogonal in L^2_E, therefore

$$\|I_M(C)\|^2_{L^2_E} = \left\| \sum_i I_M(C_i) \right\|^2_{L^2_E} = \sum_i \|I_M(C_i)\|^2_{L^2_E}.$$

3.5. *Summability of orthogonal martingale measures.*

We can now prove that if M is an E-valued orthogonal martingale measure, then I_M can be extended to a σ-additive measure on $\mathcal{P} \otimes \mathcal{L}$, hence M is 2-summable relative to $(\mathbb{R}, E)$. At the same time we prove that the sharp bracket $\langle M \rangle$ is a 1-summable process measure.

THEOREM 3.8: *Let M be an E-valued orthogonal martingale measure.*

1) *The measure I_M can be extended to a σ-additive measure $I_M: \mathcal{P} \otimes \mathcal{L} \to L^2_E$.*

2) *The measure $I_{\langle M \rangle}$ can be extended to σ-additive measure from $\mathcal{P} \otimes \mathcal{L}$ into L^1.*

3) *The positive measures $\mu_{\langle M \rangle}$ and $\|I_M(\cdot)\|^2_{L^2_E}$ can be extended to σ-additive measures*

on $\mathcal{P}\otimes\mathcal{L}$ *and we have*

$$\mu_{(M)}(C) = \|I_M(C)\|_{L^2_E} = \|I_{(M)}(C)\|_{L^1}, \qquad \text{for } C \in \mathcal{P}\otimes\mathcal{L}.$$

PROOF: The proof will be divided into several steps.

a) The σ-additivity of $\mu_{(M)}$ on $r(\mathcal{R}\times\mathcal{S})$ follows from the equality $\mu_{(M)}(C) = \mu_F(C)$ for $C \in r(\mathcal{R}\times\mathcal{S})$, in Theorem 3.4. Then $\mu_{(M)}$ can be extended to a σ-additive positive measure on the whole σ-algebra $\mathcal{P}\otimes\mathcal{L}$.

b) For every $C \in r(\mathcal{R}\times\mathcal{L})$ we have

$$\mu_{|M|^2}(C) = \mu_{(M)}(C) = \|I_M(C)\|^2_{L^2_E}.$$

In fact, let $C = A\times\{0\}\times B$ with $A \in \mathcal{F}_0$ and $B \in \mathcal{L}$. Then

$$I_M(C) = 1_A M_0(B) = 0\,, \qquad I_{|M|^2}(C) = 1_A |M_0(B)|^2 = 0$$

and

$$I_{(M)}(C) = 1_A \langle M(B)\rangle_0 = 0\,,$$

and the above equalities are satisfied.

Let now $C = A\times(t, t']\times B$ with $A \in \mathcal{F}_t$ and $B \in \mathcal{L}$. Then

$$\mu_{|M|^2}(C) = E(I_{|M|^2}(C)) = E[1_A(|M_{t'}(B)|^2 - |M_t(B)|^2)]$$

$$= E(|1_A(M_{t'}(B) - M_t(B))|^2) = E(|I_M(C)|^2) = \|I_M(C)\|^2_{L^2_E}$$

and

$$\mu_{(M)}(C) = E(I_{(M)}(C)) = E[1_A(\langle M(B)\rangle_{t'} - \langle M(B)\rangle_t)] = E[1_A(|M_{t'}(B)|^2 - |M_t(B)|^2)] =$$

$$= E[|1_A(M_{t'}(B) - M_t(B))|^2 = E(|I_M(C)|^2) = \|I_M(C)\|^2_{L^2_E}.$$

Since the two measures are finitely additive on $r(\mathcal{R}\times\mathcal{L})$ (see Theorem 3.7 for $\|I_M(\cdot)\|^2_{L^2_E}$) and are equal on $\mathcal{R}\times\mathcal{L}$, they are equal on $r(\mathcal{R}\times\mathcal{L})$.

c) Using step a) and the equality $\mu_{(M)}(C) = \|I_M(C)\|^2_{L^2_E}$ for $C \in r(\mathcal{R}\times\mathcal{L})$, in step b), it follows that $I_M \ll \mu$, therefore I_M can be extended to a σ-additive measure $I_M : \mathcal{P}\otimes\mathcal{L} \to L^2_E$, and this proves assertion 1.

d) For any disjoint sets $C, C' \in \mathcal{P} \otimes \mathcal{L}$ and any numbers $\alpha, \alpha' \in \mathbb{R}$ we have

$$I_M(C)\,\alpha \perp I_M(C')\,\alpha', \qquad \text{in } L^2_E.$$

This was already proved in Theorem 3.6 for $C, C' \in r(\mathcal{R} \times \mathcal{L})$. Let now $C \in r(\mathcal{R} \times \mathcal{L})$ and denote by $\sum_C$ the class of sets $C' \in \mathcal{P} \otimes \mathcal{L}$ such that $I_M(C) \perp I_M(C' \setminus C)$ in L^2_E. Then $\sum_C$ contains $r(\mathcal{P} \times \mathcal{L})$ and is a monotone class, hence $\sum_C = \mathcal{P} \otimes \mathcal{L}$. It follows that for $C \in r(\mathcal{R} \times \mathcal{L})$ and $C' \in \mathcal{P} \otimes \mathcal{L}$ with $C \cap C' = \phi$ we have $I_M(C) \perp I_M(C')$ in L^2_E.

Let now $C' \in \mathcal{P} \otimes \mathcal{L}$ and denote by $\sum_{C'}$ the class of sets $C \in \mathcal{P} \otimes \mathcal{L}$ such that $I_M(C) \perp I_M(C' \setminus C)$ in L^2_E. Again, $\sum_{C'}$ is a monotone class containing $r(\mathcal{R} \times \mathcal{L})$, therefore $\sum_{C'} = \mathcal{P} \otimes \mathcal{L}$. It follows that for $C, C' \in \mathcal{P} \otimes \mathcal{L}$ disjoint we have $I_M(C) \perp I_M(C')$, in L^2_E, and then also $I_M(C)\alpha \perp I_M(C')\alpha'$, in L^2_E.

e) The mapping $C \mapsto \|I_M(C)\|^2_{L^2_E}$ is σ-additive in L^2_E on $\mathcal{P} \otimes \mathcal{L}$.

In fact, using the proof of Theorem 3.7, we see that $\|I_M(\cdot)\|^2_{L^2_E}$ is finitely additive on $\mathcal{P} \otimes \mathcal{L}$. If $C_n \downarrow \phi$ in $\mathcal{P} \otimes \mathcal{L}$, then, by the σ-additivity of I_M we have $I_M(C_n) \to 0$ in L^2_E, therefore $\|I_M(C_n)\|^2_{L^2_E} \to 0$; hence $\|I_M(\cdot)\|^2_{L^2_E}$ is σ-additive on $\mathcal{P} \otimes \mathcal{L}$.

f) We have

$$\mu_{(M)}(C) = \|I_M(C)\|^2_{L^2_E}, \qquad \text{for } C \in \mathcal{P} \otimes \mathcal{L}.$$

In fact, the two measures are σ-additive on $\mathcal{P} \otimes \mathcal{L}$ and by step b), they coincide on $r(\mathcal{R} \times \mathcal{L})$, hence they are equal on $\mathcal{P} \otimes \mathcal{L}$.

g) $I_{(M)}$ can be extended to a σ-additive measure from $\mathcal{P} \otimes \mathcal{L}$ into L^1.

In fact, for $C \in r(\mathcal{R} \times \mathcal{L})$ we have

$$\mu_{(M)}(C) = E(I_{(M)}(C)) = \|I_{(M)}\|_{L^1}.$$

It follows that $I_{(M)} \ll \mu_{(M)}$, on $r(\mathcal{R} \times \mathcal{L})$. Since $\mu_{(M)}$ is σ-additive on $\mathcal{P} \otimes \mathcal{L}$, by step a), we deduce that $I_{(M)}$ can be extended to a σ-additive measure from $\mathcal{P} \otimes \mathcal{L}$ into L^1, and this proves assertion 2.

h) Since $I_{(M)}$ is σ-additive in L^1 on $\mathcal{P} \otimes \mathcal{L}$, it follows that $E(I_{(M)}(\cdot))$ is a σ-additive, positive measure on $\mathcal{P} \otimes \mathcal{L}$, that is, $\|I_{(M)}(\cdot)\|_{L^1}$ is a σ-additive, positive measure on $\mathcal{P} \otimes \mathcal{L}$.

Since, by step g), we have $\mu_{(M)} = \|I_{(M)}(\cdot)\|_{L^1}$ on $r(\mathcal{R} \times \mathcal{L})$ and since these measures are σ-additive on $\mathcal{P} \otimes \mathcal{L}$, they are equal on $\mathcal{P} \otimes \mathcal{L}$. This equality together with the equality proved in step f) completes the proof of assertion 3).

THEOREM 3.9: 1) *Let M be an E-valued orthogonal martingale measure. Then*

a) *M is 2-summable relative to $(\mathbb{R}, E)$ and*

$$(\tilde{I}_M)_{\mathbb{R}, L^2_E}(C) = \|I_M(C)\|_{L^2_E}, \quad \text{for } C \in \mathcal{P} \otimes \mathcal{L}.$$

b) *$I_{\langle M \rangle}$ is a 1-summable, positive process measure and we have*

$$(\tilde{I}_{\langle M \rangle})_{\mathbb{R}, L^1_E}(C) = \|I_{\langle M \rangle}(C)\|_{L^1_E}, \quad \text{for } C \in \mathcal{P} \otimes \mathcal{L}.$$

2) *If M is a real valued, orthogonal martingale measure and D is a Hilbert space and if we consider the embedding $\mathbb{R} \subset L(D, D)$, then M is 2-summable relative to (D, L^2_D) and we have*

$$(\tilde{I}_M)_{D, L^2_D}(C) = \|I_M(C)\|_{L^2_E} = (\tilde{I}_M)_{\mathbb{R}, L^2_E}, \quad \text{for } C \in \mathcal{P} \otimes \mathcal{L}.$$

PROOF: By Theorem 3.8, I_M can be extended to a σ-additive measure $I_M: \mathcal{P} \otimes \mathcal{L} \to \to L^2_E$, therefore I_M automatically has finite semivariation $(\tilde{I}_M)_{\mathbb{R}, L^2_E}$; consequently M is 2-summable relative to $(\mathbb{R}, E)$. Similarly, by Theorem 3.8, $I_{\langle M \rangle}$ can be extended to a σ-additive measure from $\mathcal{P} \otimes \mathcal{L}$ into L^1, hence, it has finite semivariation relative to $(\mathbb{R}, L^1_E)$, consequently $\langle M \rangle$ is 1-summable relative to $(\mathbb{R}, \mathbb{R})$.

It remains to prove the equalities in assertions 1a) and 1b). Let $C \in \mathcal{P} \otimes \mathcal{L}$, let $(C_i)_{1 \leq i \leq n}$ be a family of disjoint sets from $\mathcal{P} \otimes \mathcal{L}$ with union C, and $(\alpha_i)_{1 \leq i \leq n}$ a family of numbers with $|\alpha_i| \leq 1$. Then, by step d) in the proof of Theorem 3.8, the family $(I_M(C_i)\alpha_i)$ is orthogonal in L^2_E. Then

$$\|\sum I_M(C_i)\,\alpha_i\|^2_{L^2_E} = \sum \|I_M(C_i)\alpha_i\|^2_{L^2_E} \leq \sum \|I_M(C_i)\|^2_{L^2_E} = \|\sum I_M(C_i)\|^2_{L^2_E} = \|I_M(C)\|^2_{L^2_E}.$$

It follows that

$$\|\sum I_M(C_i)\,\alpha_i\|_{L^2_E} \leq \|I_M(C)\|_{L^2_E} \leq (\tilde{I}_M)_{\mathbb{R}, L^2_E}(C).$$

Taking the supremum in the left hand side term, we obtain the equality

$$(\tilde{I}_M)_{\mathbb{R}, L^2_E}(C) = \|I_M(C)\|_{L^2_E}$$

and this proves assertion 1a).

To prove the equality in assertion 1b, let $C \in \mathcal{P} \otimes \mathcal{L}$, $(C_i)_{1 \leq i \leq n}$ a family of disjoint sets from $\mathcal{P} \otimes \mathcal{L}$ and $(\alpha_i)_{1 \leq i \leq n}$ a family of real numbers with $|\alpha_i| \leq 1$. Then

$$\|\sum I_{\langle M \rangle}(C_i)\,\alpha_i\|_{L^1_E} \leq \sum \|I_{\langle M \rangle}(C_i)\,\alpha_i\|_{L^1_E} \leq$$

$$\leq \sum \|I_{\langle M \rangle}(C_i)\|_{L^1_E} = \|\sum I_{\langle M \rangle}(C_i)\|_{L^1_E} = \|I_{\langle M \rangle}(C)\|_{L^1_E} \leq (\tilde{I}_{\langle M \rangle})_{\mathbb{R}, L^1_E}(C).$$

Taking the supremum in the left hand side term, we get

$$(\tilde{I}_{(M)})_{\mathbb{R},\, L^2_{\mathbb{R}}}(C) = \|I_{(M)}(C)\|_{L^2_{\mathbb{R}}}.$$

Thus assertion 1b is also proved.

Assume now that M is a real valued orthogonal martingale and D a Hilbert space and prove assertion 2. We only have to prove the equality

$$(\tilde{I}_M)_{D,\, L^2_D}(C) = \|I_M(C)\|_{L^2_{\mathbb{R}}}, \qquad \text{for } C \in \mathcal{P} \otimes \mathcal{L}.$$

For this purpose, we prove first that for $C, C' \in \mathcal{P} \otimes \mathcal{L}$ disjoint and $x, x' \in D$, we have

$$I_M(C)\, x \perp I_M(C')\, x', \qquad \text{in } L^2_D.$$

This was already proved in Theorem 3.6′ for $C, C' \in r(\mathcal{P} \times \mathcal{L})$. The extension of this property for sets $C, C' \in \mathcal{P} \otimes \mathcal{L}$ is done exactly as in step d) of Theorem 3.8.

Let now $C \in \mathcal{P} \otimes \mathcal{L}$, $(C_i)_{1 \leq i \leq n}$ a family of disjoint sets from $\mathcal{P} \otimes \mathcal{L}$ with union C and $(x_i)_{1 \leq i \leq n}$ a family of elements from D with $|x_i| \leq 1$. Then the family $(I_M(C_i))_i$ is orthogonal in $L^2_{\mathbb{R}}$ and the family $(I_M(C_i)\, x_i)_i$ is orthogonal in L^2_D. We have

$$\|\sum I_M(C_i)\, x_i\|^2_{L^2_D} = \sum \|I_M(C_i)\, x_i\|^2_{L^2_D} \leq \sum \|I_M(C_i)\|^2_{L^2_{\mathbb{R}}} = \|\sum I_M(C_i)\|^2_{L^2_{\mathbb{R}}} = \|I_M(C)\|^2_{L^2_{\mathbb{R}}},$$

therefore

$$\|\sum I_M(C_i)\, x_i\|_{L^2_D} \leq \|I_M(C)\|_{L^2_{\mathbb{R}}}.$$

Taking the supremum in the left hand side term, we get

$$(\tilde{I}_M)_{D,\, L^2_D}(C) \leq \|I_M(C)\|_{L^2_{\mathbb{R}}}.$$

To prove the converse inequality, let $x \in D$ with $|x| = 1$. Then

$$\|I_M(C)\|_{L^2_{\mathbb{R}}} = \|I_M(C)\, x\|_{L^2_D} \leq (\tilde{I}_M)_{D,\, L^2_D}(C),$$

and the equality

$$(\tilde{I}_M)_{D,\, L^2_D}(C) = \|I_M(C)\|_{L^2_{\mathbb{R}}}$$

follows. The second equality

$$\|I_M(C)\|_{L^2_{\mathbb{R}}} = (\tilde{I}_M)_{\mathbb{R},\, L^2_{\mathbb{R}}}(C)$$

follows from assertion 1a) with $E = \mathbb{R}$. This proves assertion 2 completely.

From the proofs of Theorems 3.8 (step d) and 3.9 we obtain the following extension of the orthogonality properties stated in theorems 3.6 and 3.6′ for sets in $r(\mathcal{P} \times \mathcal{L})$.

THEOREM 3.10: 1) *Let M be an E-valued, orthogonal martingale measure. Then for any disjoint sets $C, C' \in \mathcal{P} \otimes \mathcal{L}$ and any numbers α, α' we have*

$$I_M(C)\alpha \perp I_M(C')\alpha', \quad \text{in } L^2_E.$$

2) *Let M be a real valued, orthogonal martingale measure and D a Hilbert space. Consider the embedding $\mathbb{R} \subset L(D, D)$. Then, for any disjoint sets $C, C' \in \mathcal{P} \otimes \mathcal{L}$ and any elements $x, x' \in D$ we have*

$$I_M(C)\, x \perp I_M(C')x', \quad \text{in } L^2_D.$$

3.6. *Approximation of integrable functions by step functions.*

In general, if X is a p-summable process measure, the step functions are not necessarily dense in the space $L^1_{F,G}(X)$. But if M is an orthogonal martingale measure, this property is true in the space $L^1_{\mathbb{R},L^2_E}(M)$.

THEOREM 3.11: *If M is an E-valued, orthogonal martingale measure, then*

$$L^1_{\mathbb{R},L^2_E}(M) = \mathcal{F}_{\mathbb{R},L^2_E}(I_M)$$

and the real valued, $\mathcal{R} \times \mathcal{L}$-simple functions are dense in $\mathcal{F}_{\mathbb{R},L^2_E}(I_M)$.

PROOF: Let $H \in \mathcal{F}_{\mathbb{R},L^2_E}(I_M)$ and prove that H can be approximated in $\mathcal{F}_{\mathbb{R},L^2_E}(I_M)$ by bounded functions.

We have $L^1_{\mathbb{R},L^2_E}(M) = \mathcal{F}_{\mathbb{R},L^2_E}(I_M)$, by Corollary 2.16. By Theorem 2.15, $H \cdot M$ is a square integrable martingale measure.

By Theorem 2.13, $H \cdot M$ is p-summable relative to $(\mathbb{R}, E)$ and we have

$$(\tilde{I}_{H \cdot M})_{\mathbb{R},L^2_E}(C) = (\tilde{I}_M)_{\mathbb{R},L^2_E}(1_C H).$$

Since $H \cdot M$ is p-summable relative to $(\mathbb{R}, L^2_E)$ the set of measures $(I_{H \cdot M})_{\mathbb{R},L^2_E}$ is uniformly σ-additive; hence if $C_n \in \mathcal{P} \otimes \mathcal{L}$ and $C_n \downarrow \phi$, then $(\tilde{I}_{H \cdot M})_{\mathbb{R},L^2_E}(C_n) \to 0$, therefore $(\tilde{I}_M)_{\mathbb{R},L^2_E}(1_{C_n} H) \to 0$. Then, by ([B-D.1], theorem AI.8(b)), we deduce that $H \in \mathcal{F}_{\mathbb{R},L^2_E}(\mathcal{B}, I_M)$. It follows then that

$$\mathcal{F}_{\mathbb{R},L^2_E}(I_M) = \mathcal{F}_{\mathbb{R},L^2_E}(\mathcal{B}, I_M).$$

Since, by ([B-D.1], theorem AI.11), the $\mathcal{R} \times \mathcal{L}$-simple functons are dense in $\mathcal{F}_{\mathbb{R},L^2_E}(\mathcal{B}, I_M)$, it follows that the $\mathcal{R} \times \mathcal{L}$-simple functions are dense in $\mathcal{F}_{\mathbb{R},L^2_E}(I_M)$, and this proves the theorem.

REMARK: If M is a real valued orthogonal martingale measure and if we consider $\mathbb{R} \subset L(D, D)$, for a Hilbert space D, then the simple, real valued, $\mathcal{R} \times \mathcal{L}$-simple functions are dense in $\mathcal{F}_{\mathbb{R},L^2_E}(I_M)$; but we do not know whether the D-valued $\mathcal{R} \times \mathcal{L}$-simple funcitons are dense in $\mathcal{F}_{D,L^2_D}(I_M)$.

As a consequence of Theorem 3.11 about the density of step functions in $L^1_{\mathbb{R},L^2_E}(M)$, we prove the following theorem, which is an extension of Theorem 3.10, replacing sets $C \in \mathcal{P} \otimes \mathcal{L}$ with predictable functions H.

THEOREM 3.12: *Let M be an E-valued orthogonal martingale measure. Then*

$$\mathcal{F}_{\mathbb{R},L^2_E}(I_M) = L^1_{\mathbb{R},L^2_E}(M) = L^2_{\mathbb{R}}(\mu_{(M)})$$

and for $H \in \mathcal{F}_{\mathbb{R},L^2_E}(I_M)$ we have

$$(\hat{I}_M)_{\mathbb{R},L^2_E}(H) = \left\| \int H dI_M \right\|_{L^2_E} = \|H\|_{L^2_{\mathbb{R}}(\mu_{(M)})}.$$

PROOF: The equality $\mathcal{F}_{\mathbb{R},L^2_E}(I_M) = L^1_{\mathbb{R},L^2_E}(M)$ follows from Corollary 2.16. We shall prove first the equality

$$\left\| \int K dI_M \right\|_{L^2_E} = \|K\|_{L^2_{\mathbb{R}}(\mu_{(M)})}$$

for $\mathcal{P} \otimes \mathcal{L}$-step functions $K = \sum 1_{C_i} \alpha_i$ with C_i mutually disjoint and $\alpha_i \in \mathbb{R}$. By Theorem 3.10, the family $(I_M(C_i))_i$ is orthogonal in L^2_E, hence

$$\left\| \int K dI_M \right\|^2_{L^2_E} = \|\sum I_M(C_i)\, \alpha_i\|^2_{L^2_E} = \sum \|I_M(C_i)\, \alpha_i\|^2_{L^2_E} = \sum \|I_M(C_i)\|^2_{L^2_E} |\alpha_i|^2 =$$

$$= \sum \mu_{(M)}(C_i)\, |\alpha_i|^2 = \int |K|^2 d\mu_{(M)}.$$

It follows that

$$\left\| \int K dI_M \right\|_{L^2_E} = \|K\|_{L^2_{\mathbb{R}}(\mu_{(M)})}.$$

Let now $H \in L^2_{\mathbb{R}}(\mu_{(M)})$. Taking the supremum in the above equality for all real valued, $\mathcal{P} \otimes \mathcal{L}$-simple functions K with $|K| \leq |H|$, we get

$$(\hat{I}_M)_{\mathbb{R},L^2_E}(H) = \|H\|_{L^2_{\mathbb{R}}(\mu_{(M)})} < \infty.$$

It follows that $H \in \mathcal{F}_{\mathbb{R},L^2_E}(I_M)$; therefore $L^2_{\mathbb{R}}(\mu_{(M)}) \subset \mathcal{F}_{\mathbb{R},L^2_E}(I_M)$. Since, by Theorem 3.11, the simple functions are dense in $\mathcal{F}_{\mathbb{R},L^2_E}(I_M)$, and since the simple functions are also dense in $L^2_{\mathbb{R}}(\mu_{(M)})$, the above isometry implies that $L^2_{\mathbb{R}}(\mu_{(M)}) = \mathcal{F}_{\mathbb{R},L^2_E}(I_M)$.

It remains to prove the equality

$$\left\| \int H dI_M \right\|_{L^2_E} = \|H\|_{L^2_{\mathbb{R}}(\mu_{(M)})}, \qquad \text{for } H \in L^2_{\mathbb{R}}(\mu_{(M)}).$$

Let $H \in L^2_{\mathbb{R}}(\mu_{(M)})$ and let (K^n) be a sequence of $\mathcal{P} \otimes \mathcal{L}$-simple functions converging to H in $L^2_{\mathbb{R}}(\mu_{(M)})$. Then $K^n \to H$ in $\mathcal{F}_{\mathbb{R},L^2_E}(I_M)$. Since the integral with respect to I_M is continuous, we have $\int K^n dI_M \to \int H dI_M$ in L^2_E, hence $\|\int K^n dI_M\|_{L^2_E} \to \|\int H dI_M\|_{L^2_E}$. We also

have $\|K^n\|_{L^2_R(\mu_{(M)})} \to \|H\|_{L^2_R(\mu_{(M)})}$. Since for each n we have

$$\left\| \int K^n \, dI_M \right\|_{L^2_E} = \|K^n\|_{L^2_R(\mu_{(M)})},$$

it follows that

$$\left\| \int H dI_M \right\|_{L^2_E} = \|H\|_{L^2_R(\mu_{(M)})},$$

and the theorem is proved.

REMARK: The equality $\| \int H dI_M \|_{L^2_E} = \|H\|_{L^2_R(\mu_{(M)})}$ proves the classical Itô isometry between the spaces $L^2_R(P)$ and $L^2_R(\mu_{(M)})$, which was used to define the stochastic integral $H \cdot M$ with respect to a real valued, square integrable martingale.

Our approach in the definition of the stochastic integral proves the isometry of the 3 spaces, $L^1_{R,\,L^2_E}$, $L^2_R(P)$ and $L^2_R(\mu_{(M)})$, in case M is an E-valued, orthogonal martingale measure; in particular, in the case when M is a real-valued orthogonal martingale measure.

Suppose now that M is a real valued orthogonal martingale measure and D is a Hilbert space and consider the embedding $R \subset L(D, D)$. Then we know that the step functions are dense in $L^1_{D,\,L^2_D}(\mathcal{B}, M)$ = closure of bounded functions, but not necessarily in $L^1_{D,\,L^2_D}(M)$. In this case we are only able to prove the isometry of $L^1_{D,\,L^2_D}(\mathcal{B}, M)$ with $L^2_D(P)$ and $L^2_D(\mu_{(M)})$.

THEOREM 3.12': *Let M be a real-valued, orthogonal martingale measure and D a Hilbert space. Consider the embedding $R \subset L(D, D)$, then*

$$\mathcal{F}_{D,\,L^2_D}(\mathcal{B}, M) = L^2_{D,\,L^2_D}(\mathcal{B}, M) = L^2_D(\mu_{(M)})$$

and for $H \in \mathcal{F}_{D,\,L^2_D}(\mathcal{B}, M)$ we have

$$(\hat{I}_M)_{D,\,L^2_D}(H) = \left\| \int H dI_M \right\|_{L^2_D} = \|H\|_{L^2_D(\mu_{(M)})}.$$

The proof is the same as that of Theorem 3.12, replacing the numbers α_i by elements $x_i \in D$ and using Theorem 3.10.

3.7. *Orthogonality of the stochastic integral.*

We are now able to prove that the stochastic integral $H \cdot M$ with respect to an orthogonal martingale measure M, is again an orthogonal martingale measure.

THEOREM 3.13: *Let M be an E-valued, orthogonal martingale measure and $H \in L^1_{R,\,L^2_E}(M)$. Then $H \cdot M$ is an orthogonal martingale measure.*

PROOF: From Theorem 2.15 it follows that $H \cdot M$ is a cadlag, square integrable martingale. Since, by the definition of *OMM* we have $M_0 = 0$, it follows that $(H \cdot M)_0 = 0$. It remains to prove that for any disjoint sets $B, C \in \mathcal{L}$, the martingales $(H \cdot M)(B)$ and $(H \cdot M)(C)$ are orthogonal i.e., their sharp bracket $\langle (H \cdot M)(B), (H \cdot M)(C) \rangle = 0$, or equivalently, their inner product in E, denoted $(H \cdot M)(B)(H \cdot M)(C)$ is a martingale.

We shall prove this first for a simple function H, of the form

$$H = 1_{\{0\} \times A_{0-} \times B_{0-}} r_{0-} + \sum_{i=0}^{n-1} 1_{(t_i, t_{i+1}] \times A_i \times B_i} r_i$$

with $A_{0-} \in \mathcal{F}_0$, $A_i \in \mathcal{F}_i$ for $0 \leq i < n$ and $B_{0-} \in \mathcal{L}$, $B_i \in \mathcal{L}$ for $0 \leq i < n$, and $r_{0-}, r_0, \ldots, r_n$ real numbers. Then

$$(H \cdot X)_t(B) = 1_{A_{0-}} X_0(B_{0-} \cap B) r_{0-} + \sum_{i=0}^{n-1} 1_{A_i} (X_{t_{i+1} \wedge t}(B_i \cap B) - X_{t_i \wedge t}(B_i \cap B)) r_i$$

and $X_0(B_{0-} \cap B) = 0$; therefore

$$(H \cdot X)(B) = \sum_{i=0}^{n-1} 1_{A_i} (X_{t_{i+1} \wedge t}(B_i \cap B) - X_{t_i \wedge t}(B_i \cap B)) r_i.$$

Similarly

$$(H \cdot X)(C) = \sum_{i=0}^{n-1} 1_{A_i} (X_{t_{i+1} \wedge t}(B_i \cap C) - X_{t_i \wedge t}(B_i \cap C)) r_i.$$

We shall denote

$$I_t = (H \cdot M)_t(B)(H \cdot M)_t(C)$$

and for each i and j,

$$I_t^{ij} = 1_{A_i \cap A_j} r_i r_j (M_{t_{i+1} \wedge t}(B_i \cap B) - M_{t_i \wedge t}(B_i \cap B))(M_{t_{j+1} \wedge t}(B_j \cap C) - M_{t_j \wedge t}(B_j \cap C)).$$

Then $I_t = \sum_{i,j} I_t^{ij}$. We remark that if $t \leq t_i$ or $t \leq t_j$, then $I_t^{ij} = 0$.

To prove that I is a martingale, we have to show that for $s < t$ we have $E(I_t^{ij} \mid \mathcal{F}_s) = I_s^{ij}$, for each i and j.

Let $s < t$ and $0 \leq i, j < n$. To make a choice, assume $i \leq j$. Then there are several possibilities.

a) $t_{j+1} \leq s$.

Then $t_{j+1} \wedge t = t_{j+1} \wedge s$, $t_j \wedge t = t_j \wedge s$, $t_{i+1} \wedge t = t_{i+1} \wedge s$ and $t_i \wedge t = t_i \wedge s$; therefore $I_t^{ij} = I_s^{ij}$, hence $E(I_t^{ij} \mid \mathcal{F}_s) = I_s^{ij}$.

b) $t_j < s < t_{j+1}$ and $i < j$.

Then $t_j \wedge t = t_j \wedge s$, $t_{i+1} \wedge t = t_{i+1} \wedge s$ and $t_i \wedge t = t_i \wedge s$. Also, $A_i, A_j \in \mathcal{F}_s$. We have

$$E(M_{t_{i+1} \wedge t}(B_j \cap C) \mid \mathcal{F}_s) = M_s(B_j \cap C) = M_{t_{i+1} \wedge s}(B_j \cap C).$$

It follows that $E(I_t^{ij} \mid \mathcal{F}_s) = I_s^{ij}$.

c) $t_j < s < t_{j+1}$ and $i = j$.

Then $t_i \wedge t = t_i \wedge s$ and $t_j \wedge t = t_j \wedge s$. Also $A_i, A_j \in \mathcal{F}_s$. We can write

$$I_t^{ij} = I_s^{ij} = 1_{A_i} r_i^2 [M_{t_{i+1} \wedge t}(B_i \cap B)\, M_{t_{i+1} \wedge t}(B_i \cap C) +$$
$$+ M_{t_i \wedge t}(B_i \cap B)\, M_{t_i \wedge t}(B_i \cap C) - M_{t_{i+1} \wedge t}(B_i \cap B)\, M_{t_i \wedge t}(B_i \cap C) -$$
$$- M_{t_i \wedge t}(B_i \cap B) M_{t_{i+1} \wedge t}(B_i \cap C)].$$

Since M is an orthogonal martingale measure, and the sets $B_i \cap B$ and $B_i \cap C$ are disjoint, the product $M(B_i \cap B)\, M(B_i \cap C)$ is a martingale, therefore,

$$E(M_{t_{i+1} \wedge t}(B_i \cap B)\, M_{t_{i+1} \wedge t}(B_i \cap C) \mid \mathcal{F}_s) =$$
$$= M_s(B_i \cap B)\, M_s(B_i \cap C) = M_{t_{i+1} \wedge s}(B_i \cap B)\, M_{t_{i+1} \wedge s}(B_i \cap C).$$

The terms $M_{t_i \wedge s}(B_i \cap B)$ and $M_{t_i \wedge s}(B_i \cap C)$ are $\mathcal{F}_s$-measurable. We have also

$$E(M_{t_{i+1} \wedge t}(B_i \cap B) \mid \mathcal{F}_s) = M_s(B_i \cap B) = M_{t_{i+1} \wedge s}(B_i \cap B)$$

and

$$E(M_{t_{i+1} \wedge t}(B_i \cap C) \mid \mathcal{F}_s) = M_{t_{i+1} \wedge s}(B_i \cap C).$$

It follows that $E(I_t^{ij} \mid \mathcal{F}_s) = I_s^{ij}$.

d) $s < t_j \leq t$ and $i < j$.

Then, by the remark above, $I_t^{ij} = 0$. We have also

$$E(I_t^{ij} \mid \mathcal{F}_{t_j}) = 1_{A_i} 1_{A_j} r_i r_j (M_{t_{i+1} \wedge t}(B_i \cap B) - M_{t_i \wedge t}(B_i \cap B)) \times$$
$$\times E(M_{t_{j+1} \wedge t}(B_j \cap C) - M_{t_j \wedge t}(B_j \cap C) \mid \mathcal{F}_{t_j}) = 0,$$

therefore

$$E(I_t^{ij} \mid \mathcal{F}_s) = E(I_t^{ij} \mid \mathcal{F}_{s_j} \mid \mathcal{F}_s) = 0 = I_s^{ij}.$$

e) $s < t_j$ and $i = j$.

Then $I_s^{ij} = 0$ and we have

$$I_t^{ij} = I_s^{ij} = 1_{A_i} r_i^2 [M_{t_{i+1} \wedge t}(B_i \cap B)\, M_{t_{i+1} \wedge t}(B_i \cap C) +$$

$$+ M_{t_i \wedge t}(B_i \cap B)\, M_{t_i \wedge t}(B_i \cap C) - M_{t_{i+1} \wedge t}(B_i \cap B)\, M_{t_i \wedge t}(B_i \cap C) -$$

$$- M_{t_i \wedge t}(B_i \cap B)\, M_{t_{i+1} \wedge t}(B_i \cap C)].$$

Since $B_i \cap B$ and $B_i \cap C$ are disjoint, and M is orthogonal, the product $M(B_i \cap B)\, M(B_i \cap C)$ is a martingale, therefore

$$E(M_{t_{i+1} \wedge t}(B_i \cap B)\, M_{t_{i+1} \wedge t}(B_i \cap C) \mid \mathcal{F}_{t_i}) = M_{t_i \wedge t}(B_i \cap B)\, M_{t_i \wedge t}(B_i \cap C)$$

and similarly

$$E(M_{t_i \wedge t}(B_i \cap B)\, M_{t_i \wedge t}(B_i \cap C) \mid \mathcal{F}_{t_i}) = M_{t_i \wedge t}(B_i \cap B)\, M_{t_i \wedge t}(B_i \cap C).$$

For the other two terms in I^{ij} we have

$$E(M_{t_{i+1} \wedge t}(B_i \cap B)\, M_{t_i \wedge t}(B_i \cap C) \mid \mathcal{F}_{t_i}) = M_{t_i \wedge t}(B_i \cap B)\, M_{t_i \wedge t}(B_i \cap C)$$

and

$$E(M_{t_i \wedge t}(B_i \cap B)\, M_{t_{i+1} \wedge t}(B_i \cap C) \mid \mathcal{F}_{t_i}) = M_{t_i \wedge t}(B_i \cap B)\, M_{t_i \wedge t}(B_i \cap C)).$$

Since $A_i \in \mathcal{F}_{t_i}$, we deduce that $E(I_t^{ij} \mid \mathcal{F}_{t_i}) = 0$, therefore $E(I_t^{ij} \mid \mathcal{F}_s) = E(I_t^{ij} \mid \mathcal{F}_{t_i} \mid \mathcal{F}_s) = 0 = I_s^{ij}$.

f) $s < t < t_j$.

Then $I_s^{ij} = 0$ and $A_i, A_j \in \mathcal{F}_{t_j}$. We have

$$E(M_{t_{i+1} \wedge t}(B_j \cap B) - M_{t_i \wedge t}(B_j \cap C) \mid \mathcal{F}_{t_j}) = 0,$$

hence $E(I_t^{ij} \mid \mathcal{F}_{t_j}) = 0$; therefore

$$E(I_t^{ij} \mid \mathcal{F}_s) = E(I_t^{ij} \mid \mathcal{F}_{t_j} \mid \mathcal{F}_s) = 0 = I_s^{ij}.$$

From the above we deduce that $E(I_t \mid \mathcal{F}_s) = I_s$, that is, I is a martingale. The conclusion is that if H is $\mathcal{P} \otimes \mathcal{L}$-simple, then $H \cdot M$ is an orthogonal martingale measure.

Let now $H \in L^1_{\mathbb{R}, L^2_\xi}(M)$. By Theorem 3.11 the $\mathcal{R} \times \mathcal{L}$-simple processes are dense in $L^1_{\mathbb{R}, L^2_\xi}(M)$, hence there is a sequence (H^n) of $\mathcal{R} \times \mathcal{L}$-simple functions such that $H^n \to H$ in $L^1_{\mathbb{R}, L^2_\xi}(M)$. Then for every $t \geq 0$ and $B \in \mathcal{L}$, we have

$$1_{[0,t] \times B} H^n \to 1_{[0,t] \times B} H, \quad \text{in } \mathcal{T}_{\mathbb{R}, L^2_\xi}(I_M).$$

Since the integral $\int H dI_M$ is continuous, we deduce that

$$\int_{[0,t]\times B} H^n dI_M \to \int_{[0,t]\times B} H dI_M, \quad \text{in } L^2_E,$$

that is

$$(H^n \cdot M)_t(B) \to (H \cdot M)_t(B), \quad \text{in } L^2_E.$$

Then, for $B, C \in \mathcal{L}$ we have

$$(H^n \cdot M)_t(B)(H^n \cdot M)_t(C) \to (H \cdot M)_t(B)(H \cdot M)_t(C), \quad \text{in } L^1_R.$$

Since the conditional expectations are continuous in L^1_E, for $s < t$ we get

$$E((H^n \cdot M)_t(B)(H^n \cdot M)(C) \mid \mathcal{F}_s) \to E((H \cdot M)_t(B)(H \cdot M)_t(C) \mid \mathcal{F}_s)$$

in L^1_R. If $B \cap C = \phi$, we have

$$E((H^n \cdot M)_t(B)(H^n \cdot M)_t(C) \mid \mathcal{F}_s) = (H^n \cdot M)_s(B)(H^n \cdot M)_s(C),$$

since $(H^n \cdot M)(B)(H^n \cdot M)(C)$ is a martingale for each n. It follows that

$$E((H \cdot M)_t(B)(H \cdot M)_t(C) \mid \mathcal{F}_s) = (H \cdot M)_s(B)(H \cdot M)_s(C),$$

therefore $(H \cdot M)(B)(H \cdot M)(C)$ is a martingale, and this completes the proof of the theorem.

For real valued martingales, the property in Theorem 3.13 is valid only for functions H in $L^1_{D,L^2_D}(\mathcal{B}, M)$:

THEOREM 3.13': *Let M be a real-valued, orthogonal martingaale measure and D a Hilbert space such that $\mathbb{R} \subset L(D, D)$. If $H \in L^1_{D,L^2_D}(\mathcal{B}, M)$ = closure of $\mathcal{P} \otimes \mathcal{L}$-step functions in $L^1_{D,L^2_D}(M)$, then $H \cdot M$ is an orthogonal martingale measure.*

The proof is the same as that of Theorem 3.13, replacing the numbers r_i by elements $x_i \in D$.

REMARK: We do not know whether $H \cdot M$ is an orthogonal martingale measure for $H \in L^1_{D,L^2_D}(M)$ since we do not know whether the $\mathcal{P} \otimes \mathcal{L}$-step functions are dense in $L^1_{D,L^2_D}(M)$.

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