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On the Banach-Stone Theorem, II (**)

To the memory of A.W.

SUMMARY. — Given the Banach spaces $C(M)$ and $C(N)$ of all complex-valued continuous functions on two compact Hausdorff spaces M and N, the classical Banach-Stone theorem characterizing the linear isometries of $C(M)$ *onto* $C(N)$ has been recently extended to linear isometries of $C(M)$ *into* $C(N)$. The present paper is devoted to a further extension of the Banach-Stone theorem to the case in which M and N are locally compact, paracompact spaces, $C(M)$ and $C(N)$ being endowed with the compact-open topology. Locally equicontinuous semigroups of linear isometries of $C(M)$ into $C(N)$ are also investigated.

Sul teorema di Banach-Stone, II

RIASSUNTO. — Dati due spazi compatti di Hausdorff M e N, e gli spazi di Banach $C(M)$ e $C(N)$ delle funzioni continue, a valori complessi, su M e su N, una Nota recente estende ad isometrie lineari di $C(M)$ in $C(N)$ un classico teorema di Banach-Stone caratterizzante le isometrie surgettive. Questo lavoro extende ulteriormente il teorema di Banach-Stone al caso in cui M e N siano spazi localmente compatti e paracompatti, investigando inoltre semigruppi localmente equicontinui di isometrie lineari di $C(M)$ in $C(N)$.

According to the Banach-Stone theorem, two compact Hausdorff spaces M and N are homeomorphic if there is an isometry A of the space $C(M)$ of all continuous functions on M onto the space $C(N)$, both spaces being endowed with the metric topology of uniform convergence. If such an isometry $A \in \mathcal{L}(C(M), C(N))$ exists, there are a homeomorphism $\psi: N \to M$ and a function $\alpha \in C(N)$, with $|\alpha(y)| = 1$ at all $y \in N$, such that

(1) $$(Af)(y) = \alpha(y) \cdot f(\psi(y)), \qquad \forall f \in C(M)$$

and for all $y \in N$.

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The question how to describe non-surjective linear isometries $A\colon C(M) \to C(N)$ was investigated by W. Holsztyński, [4], who proved that, if $A \in \mathcal{L}(C(M), C(N))$ is such an isometry, there exist a closed set $N_0 \subset N$, a surjective, continuous map $\psi\colon N_0 \to N$, a function $\alpha \in C(N)$, with $\|\alpha\| = 1$ and $|\alpha(y)| = 1$ at all $y \in N_0$, such that (1) holds *for all* $y \in N_0$.

Under which conditions is $N_0 = N$?

Let $B(M)$, $B(N)$ be the open unit balls of $C(M)$, $C(N)$, and let $\Gamma(\overline{B(M)})$, $\Gamma(\overline{B(N)})$ be the sets of all extreme points of their closures $\overline{B(M)}$, $\overline{B(N)}$. Let $A \in \mathcal{L}(C(M), C(N))$.

According to [14], if, and only if,

$$A\Gamma(\overline{B(M)}) \subset \Gamma(\overline{B(N)}),$$

there exist a continuous map $\psi\colon N \to M$ and a function $\alpha \in C(N)$, with $|\alpha(y)| = 1$ at all $y \in N$, such that (1) holds for all $y \in N$. This equation implies that A is injective if, and only if, ψ is surjective, if, and only if, A is an isometry.

A key point in the proof is the fact that $\Gamma(\overline{B(M)})$ and $\Gamma(\overline{B(N)})$ coincide with the sets $\Theta(M)$ and $\Theta(N)$ of all complex-valued continuous functions whose values have modulus one at all points of M and N respectively.

Thus, the linear isometries $A\colon C(M) \to C(N)$ are characterized by the fact that they are injective and satisfy the condition

$$A\Theta(M) \subset \Theta(N). \tag{2}$$

This shows, incidentally, that any linear map $A \in \mathcal{L}(C(M), C(N))$ satisfying (2) is an isometry for all equivalent norms on $C(M)$ and $C(N)$ for which

$$\Gamma(\overline{B(M)}) \supset \Theta(M), \qquad \Gamma(\overline{B(N)}) \subset \Theta(N).$$

But, more in general, the fact that $\Theta(M)$ and $\Theta(N)$ depend only on the topology of M and N offers the possibility of extending these results to a wider context than the one in which M and N are assumed to be compact.

In this article, the characterization (1) of all continuous, linear, injective maps $A\colon C(M) \to C(N)$ satisfying (2) will be carried out to the case in which M and N are locally compact Hausdorff spaces, and $C(M)$ and $C(N)$ are endowed with the locally convex topology of uniform convergence on compact sets.

Assuming furthermore that the locally compact Hausdorff spaces M and N are exhausted by increasing sequences of compacts sets - and, by consequence, $C(M)$ and $C(N)$ are Fréchet spaces - we will investigate a holomorphic function F mapping injectively an open, convex, balanced neighbourhood V of 0 in $C(M)$ into an open, convex neighbourhood W of 0 in $C(N)$. It will be shown that, if $\Theta(M)$ is contained in the set $\Gamma(\overline{V})$ of all complex extreme points of the closure $\overline{V}$ of V, if $F(0) = 0$ and if a suitable extension of F maps $\Theta(M)$ into $\Gamma(\overline{W})$, then F is the restriction to V of a continuous linear map $C(M) \to C(N)$.

Assuming $M = N$ and connected, we will consider a continuous semigroup

$T\colon \boldsymbol{R}_+ \to \mathcal{L}(C(M))$ all whose elements are injective and map $\Theta(M)$ into $\Theta(N)$. The semigroup T turns out to be locally equicontinuous and to define a continuous cocycle $\alpha\colon \boldsymbol{R}_+ \to \Theta(M)$ and a continuous semiflow $\phi\colon \boldsymbol{R}_+ \times M \to M$. It will be shown that the main results established in [15] in the case in which M is compact (and therefore T is strongly continuous) still hold in the more general context in which M is locally compact and paracompact. It will be shown also that, if T is not trivial, it cannot be the restriction to $\boldsymbol{R}_+$ of a holomorphic map of a neighbourhood of $\boldsymbol{R}_+$ into $\mathcal{L}(C(M))$.

1. Let M be a locally compact, Hausdorff space, and let $C(M)$ be the complex vector space of all complex-valued continuous functions on M, endowed with the locally convex topology of uniform convergence on all compact sets of M.

If K is a compact set in M, the function $p_K\colon M \to \boldsymbol{R}_+$ defined on $f \in C(M)$ by

$$p_K(f) = \sup\{|f(x)| : x \in K\},$$

is a continuous seminorm on $C(M)$. When K varies among all compact subsets of M, the family $\{p_K\}$ defines the topology of $C(M)$.

Let

$$B_K = \{f \in C(M) \colon p_K(f) < 1\}$$

be the open unit ball of p_K, and let $\Gamma_K(M)$ be the set of all (complex = real) extreme points of the closure $\overline{B_K(M)}$ of $B_K(M)$.

According to [14], if $u \in \Gamma_K(M)$, then $|u(x)| = 1$ at all $x \in K$. If $K \neq M$, $M \setminus K$ is open and non-empty. For any $v \in C(M) \setminus \{0\}$ with $\operatorname{Supp} v \subset M \setminus K$,

$$p_K(u + \zeta v) = 1\,, \qquad \forall \zeta \in \boldsymbol{C}\,.$$

Hence, $\Gamma_K(M) = \emptyset$, and, as a consequence, the following lemma holds.

LEMMA 1: *If M is not compact, for any non-empty compact set $K \subset M$ $\Gamma_K(M)$ is empty.*

Let

$$\Theta(M) = \{f \in C(M) \colon |f(x)| = 1,\ \forall x \in M\}.$$

For every continuous linear form λ on $C(M)$ there is regular complex Borel measure μ with compact support in M, such that

$$\langle f, \lambda\rangle = \int f(x)\, d\mu(x) := (f, \mu)$$

for all $f \in C(M)$, [5].

Let $\Delta = \{\zeta \in \boldsymbol{C}\colon |\zeta| < 1\}$. Proceeding as in [14], one proves

LEMMA 2: *If the compactly supported, regular, complex Borel measure μ on M is such that $|(u, \mu)| = 1$ for all $u \in \Theta(M)$, there are a point $x \in M$ and a constant $a \in \partial\Delta$ such that*

$$\mu = a\delta_x, \tag{3}$$

where δ_x is the measure with mass 1 concentrated at the point x; i.e.,

$$(f, \mu) = af(x), \qquad \forall f \in C(M).$$

Let N be a locally compact Hausdorff space and let $A \in \mathcal{L}(C(M), C(N))$. Let $C(M)'$ and $C(N)'$ be the strong duals of $C(M)$ and $C(N)$ and let $A' \in \mathcal{L}(C(N)', C(M)')$ be the adjoint of A.

LEMMA 3: *If, and only if,*

$$A\Theta(M) \subset \Theta(N), \tag{4}$$

for every $y \in N$ there exist $x \in M$ and $a \in \partial\Delta$ such that

$$A'\delta_y = a\delta_x. \tag{5}$$

PROOF: For every $y \in N$, the map $f \mapsto (Af)(y) = (Af, \delta_y)$ of $C(M)$ into C is a continuous linear form on $C(M)$. Hence [5], there is a compactly supported, regular, complex Borel measure μ on M such that

$$(f, A'\delta_y) = (f, \mu)$$

for all $f \in C(M)$. If (4) holds, then

$$|(u, \mu)| = |(Au)(y)| = 1, \qquad \forall u \in \Theta(M).$$

By Lemma 2, μ is expressed by (3) for some $x \in M$ and $a \in \partial\Delta$. Hence (5) holds.

Viceversa, if this latter equation is satisfied, for any $u \in \Theta(M)$

$$|(Au)(y)| = |(u, A'\delta_y)| = |a(u, \delta_x)| = |u(x)| = 1. \quad \blacksquare$$

As a consequence, the following theorem holds, extending Theorem 1 of [14].

THEOREM 1: *If, and only if,* (4) *holds, there exist a function $\alpha \in \Theta(N)$ and a continuous map $\psi: N \to M$ such that*

$$Af = \alpha \cdot (f \circ \psi), \qquad \forall f \in C(M). \tag{6}$$

If (4) *holds, α and ψ are unique.*

If $H \subset N$ is compact, then

$$p_H(Af) = \sup\{|(Af)(y)| : y \in H\} = \sup\{|f(\psi(y))| : y \in N\}$$
$$= \sup\{|f(x)| : x \in \psi(H)\} = p_{\psi(H)}(f)$$

for all $f \in C(M)$.

If $Af = 0$ for some $f \in C(M)$, then $f(\psi(N)) = \{0\}$. Thus, if $\psi(N)$ is dense in M, A is injective.

Viceversa, if $\overline{\psi(N)} \neq M$, there is some $f \in C(M)\backslash\{0\}$, with $\operatorname{Supp} f \subset M \backslash \overline{\psi(N)}$. Therefore

$$(Af)(y) = f(\psi(y)) = 0, \qquad \forall y \in N,$$

proving thereby

Lemma 4: *If* (4) *holds, A is injective if, and only if, $\psi(N)$ is dense in M.*

If $A \in \mathcal{L}(C(M), C(N))$ is bijective, and if

$$A\Theta(M) = \Theta(N), \tag{7}$$

there exist $\varpi \in \Theta(M)$ and a continuous map $\varphi: M \to N$, with $\overline{\varphi(M)} = N$, such that

$$A^{-1}g = \varpi \cdot (g \circ \varphi), \qquad \forall g \in C(N).$$

For all $f \in C(M)$,

$$f = A^{-1} \circ Af = \varpi \cdot (\alpha \circ \varphi) \cdot (f \circ \psi \circ \varphi).$$

Choosing $f = 1$, then

$$\varpi = \frac{1}{\alpha \circ \varphi},$$

and therefore

$$f = f \circ \psi \circ \varphi$$

for all $f \in C(M)$. That is equivalent to $\psi \circ \varphi =$ identity on M, showing that ψ is surjective and φ is injective. A similar argument applied to $A \circ A^{-1}$ shows that φ is also surjective, and in conclusion, that ψ and φ are homeomorphisms and $\varphi = \psi^{-1}$. Hence the following theorem holds.

Theorem 2: *If $A \in \mathcal{L}(C(M), C(N))$ is bijective, and if* (7) *is satisfied, then* (6) *holds and ψ is a homeomorphism of N onto M.*

2. Let the locally compact, Hausdorff space M be such that there exists a sequence $\{K_n\colon n=0, 1, \dots\}$ of compact sets $K_n \subset M$ for which $K_n \subset K_{n+1}$ and every compact subset K of M is contained in some K_n.

The function $d_M\colon C(M) \times C(M) \to \boldsymbol{R}_+$ defined on $f, g \in C(M)$ by

$$d_M(f, g) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \frac{p_{K_n}(f-g)}{1+p_{K_n}(f-g)} = d_M(0, f-g),$$

is a complete distance inducing on $C(M)$ the topology of uniform convergence on compact sets of M, with respect to which $C(M)$ is a Fréchet space. Since

$$\frac{p_{K_n}(f-g)}{1+p_{K_n}(f-g)} < 1,$$

then

$$d_M(f, g) < 2, \qquad \forall f, g \in C(M).$$

If $u \in \Theta(M)$,

$$d_M(0, u) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \frac{p_{K_n}(u)}{1+p_{K_n}(u)} = \sum_{n=0}^{+\infty} \frac{1}{2^n} \cdot \frac{1}{2} = 1$$

If $g \in C(M) \setminus \{0\}$, there is some $x_0 \in M$ at which $g(x_0) \neq 0$. Let $n_0 \geqslant 0$ be defined by the conditions: $x_0 \in K_{n_0}$ and $g_{|K_n} = 0$ whenever $n < n_0$. Let $\zeta \in \Delta \setminus \{0\}$ be such that $|u(x_0) + \zeta g(x_0)| > 1$. Then

$$p_{K_n}(u+\zeta g) \geqslant p_{K_{n_0}}(u+\zeta g) > 1$$

for all $n \geqslant n_0$, and

$$p_{K_n}(u+\zeta g) = p_{K_n}(u) = 1$$

whenever $n < n_0$.

Hence

$$d_M(0, u+\zeta g) = \sum_{n=0}^{n_0-1} \frac{1}{2^n} \frac{p_{K_n}(u)}{1+p_{K_n}(u)} + \sum_{n=n_0}^{+\infty} \frac{1}{2^n} \frac{p_{K_n}(u+\zeta g)}{1+p_{K_n}(u+\zeta g)} >$$

$$> \sum_{n=0}^{n_0-1} \frac{1}{2^n} \frac{p_{K_n}(u)}{1+p_{K_n}(u)} + \sum_{n=n_0}^{+\infty} \frac{1}{2^n} \frac{p_{K_n}(u)}{1+p_{K_n}(u)} = \sum_{n=0}^{+\infty} \frac{1}{2^n} \frac{p_{K_n}(u)}{1+p_{K_n}(u)} = d_M(0, u) = 1.$$

That shows that all $u \in \Theta(M)$ are (complex = real) extreme points of the closure $\overline{B(M)}$ of the open unit ball

$$B(M) = \{f \in C(M)\colon d_M(0, f) < 1\}.$$

If $|f(x)| \geqslant 1$ at all $x \in M$ and $|f(x_0)| > 1$ for some point $x_0 \in K_{n_0} \subset M$, then

$$\frac{p_{K_n}(f)}{1+p_{K_n}(f)} \geqslant \frac{p_{K_{n0}}(f)}{1+p_{K_{n0}}(f)} > \frac{1}{2}$$

for all $n \geqslant n_0$, and therefore $d_M(0, f) > 1$. Hence, if a function $f \in C(M)$ is such that $f \in \overline{B(M)}$ but $f \notin \Theta(M)$, there exists some point $x_0 \in M$ at which $|f(x_0)| < 1$. Let U be an open neighbourhood of x_0 in M and let $\varepsilon \in (0, 1)$ be such that $|f(x)| < 1 - \varepsilon$ at all $x \in U$. If $g \in C(M) \setminus \{0\}$ has compact support contained in U, and is such that $|g(x)| < \varepsilon$ for all $x \in M$, then

$$|f(x)| + |g(x)| \leqslant 1, \qquad \forall x \in M.$$

Hence $p_{K_n}(f + \zeta g) \leqslant 1$ for all $\zeta \in \Delta$ and $n = 0, 1, \ldots,$ and therefore

$$d_M(f + \zeta g) \leqslant 1, \qquad \forall \zeta \in \Delta.$$

In conclusion, the following lemma holds.

Lemma 5: *$\Theta(M)$ is the set of all complex extreme points of $\overline{B(M)}$.*

Lemma 6: *For any continuous seminorm q on $C(M)$ there are a compact set $K \subset M$ and a positive constant c such that*

$$q(f) \leqslant c p_K(f), \qquad \forall f \in C(M).$$

Proof: The lemma will be established by showing that K and c exist such that

$$p_K(f) \leqslant 1 \Rightarrow q(f) \leqslant c.$$

Suppose that, for every $n = 1, 2, \ldots$ there exists a function $f_n \in C(M)$ for which

$$p_{K_n}(f_n) \leqslant 1 \quad \text{and} \quad q(f_n) \geqslant n.$$

For $m > 0$,

$$p_{K_n}(f_{n+m}) \leqslant p_{K_{n+m}}(f_{n+m}) \leqslant 1$$

and

$$q(f_{n+m}) \geqslant n + m > n.$$

Hence f_n can be replaced by f_{n+m}, so that

$$f_{n+m} - f_n = 0 \quad \text{on } K_n.$$

Then,

$$d_M(f_n, f_{n+m}) = \sum_{\nu=0}^{n} \frac{1}{2^\nu} \frac{p_{K_\nu}(f_{n+m} - f_n)}{1 + p_{K_\nu}(f_{n+m} - f_n)} + \sum_{\nu=n+1}^{+\infty} \frac{1}{2^\nu} \frac{p_{K_\nu}(f_{n+m} - f_n)}{1 + p_{K_\nu}(f_{n+m} - f_n)} =$$

$$= \frac{1}{2^{n+1}} \sum_{\nu=n+1}^{+\infty} \frac{1}{2^{\nu-n-1}} \frac{p_{K_\nu}(f_{n+m} - f_n)}{1 + p_{K_\nu}(f_{n+m} - f_n)} \leqslant \frac{1}{2^{n+1}} 2 = \frac{1}{2^n},$$

showing that $\{f_n\}$ is a Cauchy sequence. Letting $f = \lim_{n \to +\infty} f_n$, then

$$\lim_{n \to +\infty} q(f_n) = q(f),$$

contraddicting the fact that $q(f_n) \geqslant n$. ■

Assume now that the locally compact, Hausdorff space N satisfies the same hypotheses stated for M at the beginning of this section. Let $\{L_n \colon n = 0, 1, \ldots\}$ be a sequence of compact sets $L_n \subset N$ such that $L_n \subset L_{n+1}$ and that every compact subset of N is contained in some L_n. Let d_N be the distance on $C(N)$ defined on $h, k \in C(N)$ by

$$d_N(h, k) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \frac{p_{L_n}(h-k)}{1 + p_{L_n}(h-k)} = d_N(0, h-k).$$

By Lemma 5, $\Theta(N)$ is the set of all complex extreme points of the closure $\overline{B(N)}$ of the open unit ball $B(N)$ of $C(N)$ for the distance d_N. Hence if $A \in \mathcal{L}(C(M), C(N))$ maps the set of all complex extreme points of $\overline{B(M)}$ into the set of all complex extreme points of $\overline{B(N)}$, the results of n.1 hold for A.

3. Under the same hypotheses on M and N introduced in n. 2, let $V \subset M$ and $W \subset N$ be an open, convex, balanced neighbourhood of 0 in $C(M)$ and an open convex neighbourhood of 0 in $C(N)$. Denoting by $\Gamma(\overline{V})$ and $\Gamma(\overline{W})$ the sets of all complex extreme points of $\overline{V}$ and of $\overline{W}$, the following proposition will now be established.

PROPOSITION 1: *Let $F \colon V \to C(N)$ be a holomorphic (i.e. Gateaux analytic and continuous* (1)*) map such that: $F(V) \subset \overline{W}$ and $F(0) = 0$.*

If $\Theta(M) \subset \overline{V}$ and, for any $u \in \Theta(M)$ there is some $\varrho \in \Delta \backslash \{0\}$ such that

$$\frac{1}{\varrho} F(\varrho u) \in \Gamma(\overline{W}),$$

then F is the restriction to V of a continuous linear map $C(M) \to C(N)$.

(1) For all notions concerning holomorphic functions on Fréchet spaces, see, *e.g.*, [10] or [9].

PROOF: There is sequence $\{a_n: n = 1, 2, \dots\}$ in $C(N)$, depending on u, for which

$$\frac{1}{\zeta} F(\zeta u) = a_1 + \zeta a_2 + \zeta^2 a_3 + \dots$$

for all $\zeta \in \Delta \setminus \{0\}$.

By the strong maximum principle for holomorphic functions with values in locally convex spaces, [13],

$$a_2 = a_3 = \dots = 0 , \tag{8}$$

i.e., $F(\zeta u) = \zeta a_1$, or also

$$F(\zeta u) = \zeta dF(0)\, u$$

for all $\zeta \in \Delta$ and all $u \in \Theta(M)$.

There exist a sequence $\{P_n: n = 1, 2, \dots\}$ of homogeneous polynomials $P_n: C(M) \to C(N)$ of degree n, such that

$$F(f) = \sum_{n=1}^{+\infty} P_n(f), \qquad \forall f \in V .$$

If $\lambda \in C(M)'$, the scalar-valued holomorphic function $f \mapsto \langle F(f), \lambda \rangle$ is expressed by

$$\langle F(f), \lambda \rangle = \sum_{n=1}^{+\infty} \langle P_n(f), \lambda \rangle ,$$

and the polynomials $f \mapsto \langle P_n(f), \lambda \rangle$ are continuous.

Let $f \in V$ be such that $|f(x)| < 1$ at all $x \in M$. For $\zeta \in \bar{\Delta}$ and $u \in \Theta(M)$, let $g_\zeta \in C(M)$ be defined by

$$g_\zeta = \frac{\zeta u + f}{1 + \zeta \bar{f} u} , \tag{9}$$

where $\bar{f}: x \mapsto \overline{f(x)}$. If $\zeta \in \partial \Delta$,

$$g_\zeta = \frac{\zeta u + f}{\zeta u (\overline{\zeta u + f})} ,$$

showing that $g_\zeta \in \Theta(M)$ if $\zeta \in \partial \Delta$. Thus, by (8),

$$\langle P_n(g_\zeta), \lambda \rangle = 0, \qquad \forall \zeta \in \partial \Delta , \qquad \forall n > 1 . \tag{10}$$

It will be shown now that (10) holds for all $\zeta \in \bar{\Delta}$.

If $\zeta \in \Delta$, the Cauchy integral formula yields

$$\langle P_n(g_\zeta), \lambda \rangle = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{1}{\tau^{n+1}} \langle F(\tau g_\zeta), \lambda \rangle \, d\tau ,$$

when $\partial\Delta$ is oriented counterclockwise.

Let l be an oriented closed path in Δ. Since $\zeta \mapsto \langle F(\tau g_\zeta), \lambda \rangle$ is holomorphic in Δ, Fubini's theorem and Cauchy's integral theorem yield

$$\int_l \langle P_n(g_\zeta), \lambda \rangle \, d\zeta = \frac{1}{2\pi i} \int_l \left(\int_{\partial\Delta} \frac{1}{\tau^{n+1}} \langle F(\tau g_\zeta), \lambda \rangle \, d\tau \right) d\zeta =$$

$$= \frac{1}{2\pi i} \int_{\partial\Delta} \frac{1}{\tau^{n+1}} \left(\int_l \langle F(\tau g_\zeta), \lambda \rangle \, d\zeta \right) d\tau = 0 .$$

Hence, by Morera's theorem, $\zeta \mapsto \langle P_n(g_\zeta), \lambda \rangle$ is holomorphic in Δ. In view of (10),

$$\langle P_n(g_\zeta), \lambda \rangle = 0 , \qquad \forall \zeta \in \bar{\Delta}, \quad \forall n > 1 .$$

For $\zeta = 0$, (9) yields then

$$\langle P_n f, \lambda \rangle = 0 \qquad \text{for } n = 2, 3, \ldots \tag{11}$$

and for all bounded functions $f \in C(M)$. Since bounded continuous functions are dense in $C(M)$ and $f \mapsto \langle P_n f, \lambda \rangle$ is continuous, then (11) holds for all $f \in C(M)$ and all $\lambda \in C(M)'$. Hence,

$$P_n = 0 , \qquad \forall f \in C(M) , \qquad n = 2, 3, \ldots . \qquad \blacksquare$$

As a consequence, the following theorem holds, which can be seen as a Fréchet space-version of the Schwarz lemma (2).

THEOREM 3: *If the holomorphic map* $F: B(M) \to C(N)$ *is such that:* $F(B(M)) \subset \overline{B(N)}$, $F(0) = 0$ *and, for every* $u \in \Theta(M)$ *there is some* $\varrho \in \Delta \setminus \{0\}$

$$\frac{1}{\varrho} F(\varrho u) \in \Theta(N) ,$$

then F is the restriction to $B(M)$ *of a continuous linear map* $C(M) \to C(N)$.

COROLLARY 1: *The same conclusion holds if F is a holomorphic map of a neighbourhood of* $B(M) \cup \Theta(M)$ *in* $C(M)$, *and if moreover* $F(0) = 0$, $F(B(M)) \subset \overline{B(N)}$ *and* $F(\Theta(M)) \subset \Theta(N)$.

(2) See [3] for a similar result in the case of J^*-algebras.

4. Let M be connected and satisfy the hypotheses stated at the beginning of n. 2, and let $T\colon \boldsymbol{R}_+ \to \mathcal{L}(C(M))$ be a semigroup such that $T(t)$ is injective and

$$T(t)\,\Theta(M) \subset \Theta(M)\,, \qquad \forall t \in \boldsymbol{R}_+\,. \tag{12}$$

By Theorem 1, there exist a unique $\alpha_t \in \Theta(M)$ and a unique continuous map $\phi_t\colon M \to M$, with $\overline{\phi_t(M)} = M$, such that

$$T(t)\,f = \alpha_t \cdot (f \circ \phi_t)\,, \qquad \forall f \in C(M)\,, \qquad \forall t \in \boldsymbol{R}_+\,. \tag{13}$$

The fact that T is a semigroup implies that $\alpha_0 = 1$, $\phi_0 =$ identity and

$$\alpha_{t_1+t_2} \cdot (f \circ \phi_{t_1+t_2}) = \alpha_{t_1} \cdot (\alpha_{t_2} \circ \phi_{t_1}) \cdot (f \circ \phi_{t_2} \circ \phi_{t_1})$$

for all $t_1, t_2 \in \boldsymbol{R}_+$ and all $f \in C(M)$. Hence,

$$\alpha_{t_1+t_2} = \alpha_{t_1} \cdot (\alpha_{t_2} \circ \phi_{t_1}) \tag{14}$$

and

$$\phi_{t_1+t_2} = \phi_{t_1} \circ \phi_{t_2} \tag{15}$$

for all $t_1, t_2 \in \boldsymbol{R}_+$.

Suppose moreover that the semigroup T is continuous:

$$\lim_{t \to t_0} T(t)\,f = T(t_0)\,f\,, \tag{16}$$

uniformly on compact sets in M, *i.e.*

$$\lim_{t \to t_0} p_H(T(t)\,f - T(t_0)\,f) = 0 \tag{17}$$

for all $f \in C(M)$, all $t_0 \geqslant 0$ and every compact set $H \subset M$.

Because, by (13), $\alpha_t = T(t)(1)$, then, by (17),

$$\lim_{t \to t_0} p_H(\alpha_t - \alpha_{t_0}) = 0 \tag{18}$$

for all $t_0 \in \boldsymbol{R}_+$ and every compact set $H \subset M$. Thus $t \mapsto \alpha_t$ is a continuous co-cycle.

Since

$$\begin{aligned} p_H(f \circ \phi_t - f \circ \phi_{t_0}) = \sup\{\,|\alpha_t(x)(f(\phi_t(x)) - f(\phi_{t_0}(x)))|: x \in H\} &\leqslant \\ \leqslant \sup\{\,|(T(t)\,f)(x) - (T(t_0)\,f)(x)\,|: x \in H\} &+ \\ + \sup\{\,|\alpha_t(x) - \alpha_{t_0}(x)\,|\,|f(\phi_{t_0}(x))|: x \in H\} &\leqslant \\ \leqslant p_H(T(t)\,f - T(t_0)\,f) + p_H(\alpha_t - \alpha_{t_0})\,p_H(f \circ \phi_{t_0})\,, & \end{aligned}$$

(17) and (18) yield

$$\lim_{t \to t_0} p_H(f \circ \phi_t - f \circ \phi_{t_0}) = 0$$

for all $t_0 \in \boldsymbol{R}_+$, $f \in C(M)$ and every compact set $H \subset M$. Thus, setting

$$S(t)\, f = f \circ \phi_t, \tag{19}$$

the above arguments yield the «only if» part of the following lemma.

LEMMA 7: *The semigroup $S: \boldsymbol{R}_+ \to \mathcal{L}(C(M))$ is continuous if, and only if, the semiflow $\phi: \boldsymbol{R}_+ \times M \to M$ is continuous.*

PROOF: If ϕ is continuous, for any $s > 0$ and any compact set $H \subset M$, $K := \phi([0, s] \times H)$ is compact in M. Hence, for any $f \in C(M)$ and all $t \in [0, s]$,

$$\sup\{|f(\phi_t(x))| : x \in H\} \leq \sup\{|f(y)| : y \in K\},$$

i.e.,

$$p_H(S(t)\, f) \leq p_K(f), \qquad \forall f \in C(M), \qquad \forall t \in [0, s]. \tag{20}$$ ■

A continuous semigroup T acting on a locally convex space $\mathcal{E}$ is said to be locally equicontinuous [3] if, for every $s > 0$ and every continuous seminorm p on $\mathcal{E}$ there is a continuous seminorm q on $\mathcal{E}$ such that

$$p(T(t)\, f) \leq q(f), \qquad \forall f \in \mathcal{E}, \qquad \forall t \in [0, s]. \tag{21}$$

If s may be chosen equal to $+\infty$, T is said to be equicontinuous, (see [12] and also [16]).

Since $C(M)$ is a barreled space, [16], every continuous semigroup $\boldsymbol{R}_+ \to \mathcal{L}(C(M))$ is locally equicontinuous ([6], Proposition 1.1). Hence (21) and Lemma 6 yield

LEMMA 8: *If the continuous semigroup $T: \boldsymbol{R}_+ \to \mathcal{L}(C(M))$ satisfies* (12), *for every compact set $H \subset M$ and every $s > 0$ there exist a compact set $K \subset M$ and a constant $c > 0$ such that*

$$p_H(T(t)\, f) \leq c p_K(f), \qquad \forall f \in C(M), \qquad \forall t \in [0, s]. \tag{22}$$

Hence

$$p_H(f \circ \phi_t) = \sup\{|f(\phi_t(x))| : x \in H\} = \sup\{|\alpha_t(x)\, f(\phi_t(x))| : x \in H\} = \tag{23}$$

$$= \sup\{|(T(t)\, f)(x)| : x \in H\} = p_H(T(t)\, f) \leq c p_K(f), \quad \forall f \in C(M), \quad \forall t \in [0, s],$$

(3) For the theory of locally equicontinuous semigroups, see, *e.g.*, [6], [7], [11].

and therefore

$$\sup\left\{ |f(x)| : x \in \bigcup_{t \in [0, s]} \phi_t(H) \right\} \leqslant c \sup \{ |f(x)| : x \in K \}$$

for all $f \in C(M)$.

If $\phi_t(H) \not\subset K$ for some $t \in [0, s]$, there is $x \in \phi_t(H)$ such that $x \notin K$. Since $\phi_t(H)$ is closed, there is an open neighbourhood U of x disjoint from K. Thus there exists a function $f \in C(M)$ for which

$$|f(x)| > cp_K(f).$$

This contradiction proves

LEMMA 9: *If the continuous semigroup satisfies* (12), *for every* $s > 0$ *and every compact set* $H \subset M$ *there is a compact set* $K \subset M$ *such that*

$$\bigcup_{t \in [0, s]} \phi_t(H) \subset K.$$

PROPOSITION 2: *If the continuous semigroup* $T: \boldsymbol{R}_+ \to \mathcal{L}(C(M))$ *satisfies* (12), *the semiflow* $\phi: \boldsymbol{R}_+ \times M \to M$ *is continuous.*

PROOF: For $t_0 \in \boldsymbol{R}_+$, $x_0 \in M$, $f \in C(M)$ and $\varepsilon > 0$, let H be a compact neighbourhood of x_0 in M such that

$$|f(\phi_{t_0}(x)) - f(\phi_{t_0}(x_0))| < \frac{\varepsilon}{2}, \qquad \forall x \in K.$$

The fact that T is continuous, and therefore S is continuous, hence locally equicontinuous, implies that there is $\delta > 0$ such that, if $t \in \boldsymbol{R}_+$ and $|t - t_0| \leqslant \delta$, then

$$p_H(f \circ \phi_t - f \circ \phi_{t_0}| < \frac{\varepsilon}{2}.$$

Since

$$|f(\phi_t(x)) - f(\phi_{t_0}(x_0))| \leqslant |f(\phi_t(x)) - f(\phi_{t_0}(x))| + |f(\phi_{t_0}(x)) - f(\phi_{t_0}(x_0))| \leqslant$$

$$\leqslant p_H(S(t)f - S(t_0)f) + |f(\phi_{t_0}(x)) - f(\phi_{t_0}(x_0))| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

the map $(t, x) \mapsto f(\phi_t(x))$ is continuous on $\boldsymbol{R}_+ \times M$ for every choice of $f \in C(M)$. That shows that the semiflow ϕ is continuous. ■

REMARK: The fact that, if the semiflow ϕ is continuous, the semigroup S is locally equincontinuous follows directly from (20), without appealing to [6].

Let T: $\boldsymbol{R} \to \mathcal{L}(C(M))$ be a group which is continuous, *i.e.* such that (17) holds for all $t_0 \in \boldsymbol{R}$ and all $f \in C(M)$.

If

$$T(t)\ \Theta(M) = \Theta(M)\,, \qquad \forall t \in \boldsymbol{R}\,, \tag{24}$$

then T is expressed, for all $f \in C(M)$ and all $t \in \boldsymbol{R}$, by (13), where $\alpha_t \in \Theta(M)$ and ϕ_t is a homeomorphism of M; (15), holding now for all $t_1, t_2 \in \boldsymbol{R}$ together with (14), shows that $(t, x) \mapsto \phi_t(x)$ is a flow on M.

Let S: $\boldsymbol{R} \to \mathcal{L}(C(M))$ be the group defined by (19) for all $t \in \boldsymbol{R}$. Proposition 0 applied to the semigroups $\boldsymbol{R}_+ \to \mathcal{L}(C(M))$ defined by: $t \mapsto T(t)$, $t \mapsto T(-t)$, $t \mapsto S(t)$ and $t \mapsto S(-t)$, yields

THEOREM 4: *If the continuous group T satisfies* (24), *the flow ϕ: $\boldsymbol{R} \times M \to M$ is continuous.*

If the flow ϕ is continuous, the group S is continuous.

That implies that, for every continuous seminorm p on $C(M)$ and for all $s > 0$, a continuous seminorm q on $C(M)$ exists satisfying (21) for all $f \in C(M)$ and all $t \in [-s, s]$.

If T is continuous and, for every compact set $H \subset M$ there exists a compact set $K \subset M$ such that $\phi_t(H) \subset K$ for all $t \in \boldsymbol{R}$, then T is equicontinuous, *i.e.* s may be chosen equal to $+\infty$.

5. Let T: $\boldsymbol{R}_+ \to \mathcal{L}(C(M))$ be a continuous semigroup satisfying (12), and expressed therefore by (13) for all $t \in \boldsymbol{R}_+$.

The infinitesimal generator of T is a linear closed operator X: $\mathcal{D}(X) \subset C(M) \to$ $\to C(M)$, whose domain $\mathcal{D}(X)$ is dense in $C(M)$. The infinitesimal generator of the continuous semigroup S expressed by (19) is a derivation D: $\mathcal{D}(D) \subset C(M) \to C(M)$. In particular, $1 \in \mathcal{D}(D)$.

For $t > 0$ and $f \in C(M))$,

$$\frac{1}{t}(T(t)\ f - f)(x) = \frac{\alpha_t(x) - 1}{t}(S(t)\ f)(x) + \frac{1}{t}(S(t)\ f - f)(x)\,. \tag{25}$$

Letting $t \downarrow 0$ and arguing as in the proof of Theorem 2 of [15], one shows that, if $\mathcal{D}(X) \cap \mathcal{D}(D)$ contains a function $f \in C(M)$ such that $f(x) \neq 0$ at all points $x \in M$ - in particular, if $1 \in \mathcal{D}(X)$ - then $\mathcal{D}(X) = \mathcal{D}(D)$ and $t \mapsto \alpha_t(x)$ is a function $\boldsymbol{R}_+ \to \partial\Delta$ of class C^1 for all $x \in M$. Letting

$$i\beta(x) := \dot{\alpha}_t(x)_{|t=0}\,, \tag{26}$$

then $\beta \in C_{\boldsymbol{R}}(M)$ and (25) yields

$$X = i\beta I + D\,. \tag{27}$$

Since, by (14),

$$T(r)\,\alpha_t = \alpha_r \cdot (\alpha_t \circ \phi_r) = \alpha_{t+r},$$

then:

LEMMA 10: *The function* $t \mapsto \alpha_t(x)$ *is of class* C^1 *on* $\mathbf{R}_+$ *for all* $x \in M$ *if, and only if,* $\alpha_t \in \mathcal{D}(X)$ *for all* $t \in \mathbf{R}_+$.

As a consequence of the above observations, the following lemma holds.

LEMMA 11: *There exists* $\beta \in C_{\mathbf{R}}(M)$ *for which* (27) *holds if, and only if,*

$$\mathcal{D}(X) \cap \mathcal{D}(D) \cap \Theta(M) \neq \emptyset\,.$$

In which case, $\mathcal{D}(X) = \mathcal{D}(D)$.

Now, let D be the infinitesimal generator of a continuous semigroup S expressed by (19), and, for a given function $\beta \in C_{\mathbf{R}}(M)$, let $X\colon \mathcal{D}(D) \to C(M)$ be the continuous perturbation of D expressed by (27). For any $x \in M$, the integral

$$\alpha_t(x) := e^{\,i\int_0^t \beta(\phi_r(x))\,dr}$$

solves the differential equation (26) with initial condition $\alpha_0(x) = 1$, and thus defines a C^1 cocycle. The continuous semigroup defined by (13) satisfies (12) and is generated by X.

The problem arises now to characterize the continuous perturbations of D which generate semigroups T satisfying (12).

Let $E \in \mathcal{L}(C(M))$, let $X\colon \mathcal{D}(D) \to C(M)$ be defined by

$$X = E + D\,,$$

and let $T\colon \mathbf{R}_+ \to \mathcal{L}(C(M))$ be the locally equicontinuous semigroup generated by X, [2].

Since $1 \in \mathcal{D}(X)$, if T satisfies (12) there exist a function $\gamma \in C_{\mathbf{R}}(M)$ and a derivation

$$D_0\colon \mathcal{D}(D_0) = \mathcal{D}(X) = \mathcal{D}(D) \to C(M)$$

such that

$$X = i\gamma I + D_0.$$

The derivation $D - D_0 = i\gamma I - E$ is a continuous operator on $C(M)$, and therefore vanishes (see, *e.g.*, [15]). Thus, $E = i\gamma I$, and the following theorem holds.

THEOREM 5: *If* X *generates a continuous semigroup* T *satisfying* (12), *all continuous* *perturbations of X generating semigroups R: $\boldsymbol{R}_+ \to \mathcal{L}(C(M))$ such that*

$$R(t)\ \Theta(M) \subset \Theta(M)\,, \qquad \forall t \in \boldsymbol{R}_+\,,$$

are given by $i\beta I + X$, for any choice of $\beta \in C_{\boldsymbol{R}}(M)$.

6. Under the same hypotheses on M and N stated in n. 2, let U be a non-empty domain in C and let F: $U \to \mathcal{L}(C(M), C(N))$ be a holomorphic map such that

$$F(w)\ \overline{B(M)} \subset \overline{B(N)}\,, \qquad \forall w \in U\,. \tag{28}$$

and

$$F(w)\ \Theta(M) \subset \Theta(N)\,, \qquad \forall w \in U\,. \tag{29}$$

Let $r > 0$ be such that $\Delta(w, r) = \{z \in \boldsymbol{C}\colon |z - w| < r\} \subset U$, and let

$$F(z) = F_0 + (z - w)\,F_1 + (z - w)^2 F_2 + \ldots$$

be the power-series expansion of F in $\Delta(w, r)$, with $F_n \in \mathcal{L}(C(M), C(N))$ for $n = 0, 1, 2, \ldots$. By the strong maximum principle for holomorphic functions with values in locally convex spaces, [13], if $u \in \Theta(M)$, and therefore $F(z)\ u \in \Theta(N)$, then $F_n u = 0$ for $n = 1, 2, \ldots$, *i.e.*,

$$\langle u, F_n'\,\delta_y\rangle = \langle F_n u, \delta_y\rangle = (F_n u)(y) = 0$$

for all $y \in N$ and $n = 1, 2, \ldots$.

For $n \geqslant 1$, the continuous linear form $F_n'\,\delta_y$ is represented by a regular, complex, compactly supported, Borel measure μ_n on M. If $\sigma \in C_{\boldsymbol{R}}(M)$ and $t \in \boldsymbol{R}$, then $u = e^{it\sigma} \in \Theta(M)$. Hence $\langle u, \mu_n\rangle = 0$, that is

$$\sum_{\nu=0}^{+\infty} \frac{(it)^\nu}{\nu!} \langle \sigma^\nu, \mu_n\rangle = 0$$

for all $t \in \boldsymbol{R}$. Thus $\langle \sigma, \mu_n\rangle = 0$ for all $\sigma \in C_{\boldsymbol{R}}(M)$, and therefore $F_n'\,\delta_y = 0$, or also $(F_n f)(y) = 0$ for all $f \in C(M)$ and all $y \in N$. In conclusion $F_n = 0$ for $n = 1, 2, 3, \ldots$, and the following theorem holds.

THEOREM 6: *If the holomorphic map F: $U \to \mathcal{L}(C(M), C(N))$ satisfies* (28) *and* (29), *F is constant.*

COROLLARY 2: *For any domain $U \subset \boldsymbol{C}$ containing $\boldsymbol{R}_+^*$, there is no non-trivial semigroup T: $\boldsymbol{R}_+ \to \mathcal{L}(C(M))$ whose restriction to $\boldsymbol{R}_+^*$ is the restriction to $\boldsymbol{R}_+^*$ of a holomorphic map F: $U \to \mathcal{L}(C(M))$ satisfying* (28) *and* (29).

7. Theorem 4 of [15] will now be extended to the case in which M is a connected, n-dimensional (paracompact) complete Riemannian manifold of class C^∞. Let $\langle , \rangle$ be the Riemannian metric and let v be a C^∞ vector field on M for which there exists a po-

sitive constant k such that

(30) $$\langle v(x), v(x)\rangle \leq k^2, \qquad \forall x \in M.$$

Given $x_0 \in M$, let ξ be the C^∞ integral curve of v with initial condition x_0; *i.e.*

$$\dot{\xi}(t) = v(\xi(t)), \qquad \forall t \in [0, a]$$

for some $a > 0$, and $\xi(0) = x_0$.

The set

$$C_+(x_0) = \{t \in \boldsymbol{R}_+ : \exists \xi : [0, t] \to M \text{ of class } C^1, \\ \text{with } \dot{\xi}(r) = v(\xi(r)),\ \forall r \in [0, t] \text{ and } \xi(0) = x_0\}$$

is open and non-empty.

Lemma 12: *The set $C_+(x_0)$ is closed.*

Proof: Let $t_0 \in \overline{C_+(x_0)}$ and let $\{t_\nu\}$ be an increasing sequence in $C_+(x_0)$ such that $t_\nu \uparrow t_0$. If ξ_ν is the integral curve on $[0, t_\nu]$ of v, with initial condition x_0, then

$$\nu_1 < \nu_2 \Rightarrow \xi_{\nu_2 | [0, t_{\nu_1}]} = \xi_{\nu_1}.$$

That defines an integral curve ξ: $[0, t_0) \to M$ of v with initial condition x_0 such that $\xi_{|[0, t_\nu]} = \xi_\nu$.

Let d: $M \times M \to \boldsymbol{R}_+$ be the distance defined by the Riemannian metric of M.

a) It will be shown that $\{\xi_\nu(t_\nu)\}$ is a Cauchy sequence for d. If $\nu_1 < \nu_2$,

$$d(\xi_{\nu_1}(t_{\nu_1}), \xi_{\nu_2}(t_{\nu_2})) \leq \int_{t_{\nu_1}}^{t_{\nu_2}} (\langle \dot{\xi}(t), \dot{\xi}(t)\rangle_{\xi(t)})^{1/2}\, dt =$$

$$= \int_{t_{\nu_1}}^{t_{\nu_2}} (\langle v(\xi(t)), v(\xi(t))\rangle_{\xi(t)})^{1/2}\, dt \leq k \int_{t_{\nu_1}}^{t_{\nu_2}} dt \leq k(t_{\nu_2} - t_{\nu_1}).$$

Since $\{t_\nu\}$ is a Cauchy sequence $\{\xi_\nu(t_\nu)\}$ is a Cauchy sequence.

b) Let $y_0 = \lim\limits_{\nu \to +\infty} \xi_\nu(t_\nu)$. For any $\varepsilon > 0$ there exists an index $\nu_0 \geq 1$ such that, whenever $\nu \geq \nu_0$,

$$t_\nu \in (t_0 - \varepsilon, t_0) \quad \text{and} \quad d(\xi_{t_\nu}, y_0) < \varepsilon.$$

Let $t \in (t_0 - \varepsilon, t_0)$ and let $\nu > \nu_0$ be such that $t < t_\nu < t_0$. Then

$$d(\xi(t), y_0) \leq d(\xi(t), \xi_\nu(t)) + d(\xi_\nu(t), \xi_\nu(t_\nu)) + d(\xi_\nu(t_\nu), y_0) <$$

$$< \int_t^{t_\nu} (\langle \dot{\xi}_\nu(r), \dot{\xi}_\nu(r)\rangle_{\xi_\nu(r)})^{1/2}\, dr + \varepsilon \leq k(t_\nu - t) + \varepsilon < (k+1)\,\varepsilon.$$

c) Letting $\xi(t_0) = y_0$, the map $\xi : [0, t_0] \to M$ is continuous. The proof will be complete once we show that $\xi : [0, t_0] \to M$ is an integral curve of v.

By *b)* there exist an open coordinate neighbourhood U of y_0, with local coordinates $r_1, \dots, r_n$, and an index $\nu_1 \geqslant \nu_0$ such that

$$\xi(t) \in U\,, \qquad \forall t \geqslant t_{\nu_1}.$$

The neighbourhood U can be chosen so small that there exists a positive constant c such that, denoting with $\eta_1, \dots, \eta_n$ the components, with respect to $r_1, \dots, r_n$, of a tangent vector η to M at a point $x \in U$ and setting

$$|\eta|^2 = \eta_1^2 + \dots + \eta_n^2,$$

then

$$\frac{1}{c^2}\,|\eta|^2 \leqslant \langle \eta, \eta \rangle_x \leqslant c^2\,|\eta|^2, \qquad \forall x \in U\,.$$

Since

$$\xi(t_0) - \xi_{\nu_1}(t_{\nu_1}) - \int_{t_{\nu_1}}^{t_0} v(\xi(r))\,dr = \xi(t_0) - \xi(t) + \xi_{\nu_1}(t) - \xi_{\nu_1}(t_{\nu_1}) -$$

$$- \int_{t_{\nu_1}}^{t} v(\xi(r))\,dr - \int_{t}^{t_0} v(\xi(r))\,dr = \xi(t_0) - \xi(t) - \int_{t}^{t_0} v(\xi(r))\,dr\,,$$

then

$$\left| \xi(t_0) - \xi_{\nu_1}(t_{\nu_1}) - \int_{t_{\nu_1}}^{t_0} v(\xi(r))\,dr \right| \leqslant |\xi(t_0) - \xi(t)| + \int_{t}^{t_0} |v(\xi(r))|\,dr \leqslant$$

$$\leqslant |\xi(t_0) - \xi(t)| + c \int_{t}^{t_0} (\langle v(\xi(r)), v(\xi(r))\rangle)^{1/2}\,dr \leqslant |\xi(t_0) - \xi(t)| + ck(t - t_0) \to 0$$

as $t \uparrow t_0$. Hence

$$\xi(t_0) = \xi_{\nu_1}(t_{\nu_1}) - \int_{t_{\nu_1}}^{t_0} v(\xi(r))\,dr\,,$$

and therefore $t_0 \in C_+(x_0)$. ■

In conclusion, the following theorem holds.

THEOREM 7: *If the C^∞ Riemannian manifold M is connected and complete, and if v is a C^∞ vector field on M for which* (30) *holds, there is a C^∞ flow ϕ: $\mathbf{R} \times M \to M$ such that*

$$\frac{d}{dt}\phi(t, x) = v(\phi(t, x))$$

and

$$\phi(0, x) = x$$

for all $t \in \mathbf{R}$ ans all $x \in M$.

At this point, since Lemmas 6 and 7 and Corollary 1 of [15] do not require the hypothesis that M is compact, proceeding as in [15] one proves

THEOREM 8: *If the locally equicontinuous group T: $\mathbf{R} \to \mathcal{L}(C(M))$ is such that $T(t)\,\Theta(M) \subset \Theta(M)$ and $T(t)\,C^\infty(M) \subset C^\infty(M)$ for all $t \in \mathbf{R}$, the infinitesimal generator X of T is given by* (27), *where D is a C^∞ vector field on M and $\beta \in C^\infty_{\mathbf{R}}(M)$.*

Viceversa, let D: $\mathcal{D}(D) \subset C(M) \to C(M)$ be a closed operator such that $C^\infty(M) \subset \mathcal{D}(D)$ and that $D_{|\mathcal{D}(D)}$ is a derivation defined by a C^∞ vector field v on M satisfying (30). *Let $t \mapsto \alpha_t$ be a continuous cocycle associated to the flow defined by D, such that $\alpha_t \in C^\infty(M) \cap \Theta(M)$ for all $t \in \mathbf{R}$, and let $\beta \in C^\infty_{\mathbf{R}}(M)$ be given by* (26).

Then the operator X: $\mathcal{D}(D) \to C(M)$ defined by (27) *is the infinitesimal generator of a locally equicontinuous group T: $\mathbf{R} \to \mathcal{L}(C(M))$ such that $T(t)\,\Theta(M) \subset \Theta(M)$ and $T(t)\,C^\infty(M) \subset C^\infty(M)$ for all $t \in \mathbf{R}$.*

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