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Asymptotic Behaviour of Solutions to Second-Order Hyperbolic Equations with non-Linear Damping Term (**)

Abstract. — A second-order hyperbolic equation with non-linear damping term is considered in a cylinder $\Omega \times \mathbb{R}_t$, where $\Omega \subset \mathbb{R}_x^n$ is a bounded domain. Sufficient conditions are found for the existence, uniqueness, and asymptotic stability of a time–bounded solution. Under the additional condition that the right-hand side of the equation is an almost periodic function in time, the almost periodicity of this solution is proved. An example of a right-hand side for which the problem in question has no almost periodic solution is constructed.

Comportamento asintotico delle soluzioni di equazioni iperboliche del secondo ordine con termine dissipativo non lineare

Sunto. — Si considera un'equazione iperbolica del secondo ordine, con termine dissipativo non lineare, in un insieme cilindrico $\Omega \times \mathbb{R}_t$, con Ω dominio limitato contenuto in $\mathbb{R}_x^n$. Per una tale equazione si danno condizioni sufficienti a garantire l'esistenza, l'unicità e la stabilità asintotica di una soluzione limitata rispetto al tempo. Si prova inoltre che questa soluzione è quasi periodica rispetto al tempo se tale è il secondo membro dell'equazione. Si costruisce infine un esempio di secondo membro per il quale il problema non possiede soluzioni quasi periodiche.

0. - Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Consider the problem

$$u_{tt} + g(u_t) + Lu = h(x, t), \tag{0.1}$$

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$$(0.2) \qquad u|_{\partial\Omega} = 0\,,$$

$$(0.3) \qquad u(x,\tau) = u_0(x)\,, \qquad u_t(x,\tau) = u_1(x)\,,$$

where $\tau \in \mathbb{R}$, $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $g\colon \mathbb{R} \to \mathbb{R}$ is a non-decreasing function, and L is a second-order partial differential operator of the form

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)\,u\,.$$

We assume that the operator L satisfies the following two conditions:

(L1) $a_{ij}, a_0 \in L^\infty(\Omega)$; $a_{ij}(x) = a_{ji}(x)$, $a_0(x) \geq 0$ for $1 \leq i, j \leq n$ and almost all $x \in \Omega$;

(L2) $\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j \geq \lambda|\xi|^2$ for $\xi \in \mathbb{R}^n$ and almost all $x \in \Omega$, where $\lambda > 0$.

It is well known that under the above conditions the problem (0.1)-(0.3) has a unique weak solution on the half-line $\mathbb{R}_\tau := [\tau, +\infty)$ for any right-hand side $h \in L^1_{\mathrm{loc}}(\mathbb{R}_\tau, L^2(\Omega))$ (see [4], Chapter 2). The present paper is devoted to studying the asymptotic behaviour of solutions to the problem (0.1)-(0.3) as $t \to +\infty$ in case the right-hand side $h(x,t)$ is a bounded or almost periodic (a.p.) function of the variable t with range in $L^2(\Omega)$. This problem was first studied by Prouse [11] (see also [1]). He proved that if $h(x,t)$ is a Bohr a.p. function, then under some conditions on $g(p)$ there is a unique a.p. solution, which is asymptotically stable as $t \to +\infty$. Prouse's investigations were continued by many mathematicians (see the references in [4,15]). According to one of the most general results [5], the above assertion is true if $n \geq 3$, $h(x,t)$ is a Stepanov a.p. function (see Section 2), $g(p)$ satisfies the inequality

$$(0.4) \qquad |g(p)| \leq C(1+|p|)^k\,,$$

where $k < (n+2)/(n-2)$, and the inverse function $g^{-1}(p)$ is defined and uniformly continuous on $\mathbb{R}$. In the present paper, a similar result is established for the case $k = (n+2)/(n-2)$. For instance, it can be applied to an arbitrary monotone–increasing continuous function $g(p)$ such that

$$g(p) = \sum_{i=1}^{N} c_i\,|p-p_i|^{k_i-1}(p-p_i) \qquad \text{for } |p| \geq p_0 \gg 1\,,$$

where $N \geq 1$, $p_i \in \mathbb{R}$ and $c_i > 0$ for $1 \leq i \leq N$, $1 \leq k_1 \leq (n+2)/(n-2)$, and $0 < k_i \leq (n+2)/(n-2)$ for $2 \leq i \leq N$. We also consider the case in which the right-hand side of (0.1) is a Levitan a.p. function.

Let us briefly describe the structure of the paper. In § 1 we prove the existence, uniqueness, and asymptotic stability of a uniformly bounded trajectory (defined throughout the time axis) for abstract processes in a Banach space. In the case of a.p. processes, the almost periodicity of the constructed trajectory is established. In § 2 we apply these results to study the asymptotic behaviour of solutions to the problem

(0.1)-(0.3). Theorem 2.2 is the main result of this paper. In § 3 we construct an example of an unbounded Levitan a.p. function $h(x, t)$ for which the problem (0.1), (0.2) has no a.p. solution. In Appendix we prove a variant of the Gronwall inequality.

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NOTATION: Let B be a Banach space with a norm $\|\cdot\|_B$ and let $J \subset \mathbb{R}$ be a closed interval. We shall use the following function spaces:

$L^p(J, B)$ is the space of Lebesgue–measurable functions $f: J \to B$ such that $\int_J \|f(t)\|_B^p \, dt < \infty$ if $1 \leq p < \infty$ and $\operatorname{ess\,sup}_{t \in J} \|f(t)\|_B < \infty$ if $p = \infty$;

$C^k(J, B)$ is the space of k times continuously differentiable functions on J with range in B; if $k = 0$, the corresponding superscript will be omitted.

We denote by C_i and c_i unessential positive constants.

1. - Asymptotic behaviour of trajectories for abstract processes

1.1. *Existence, uniqueness, and asymptotic stability of a bounded trajectory.*

Let $\mathbb{E}$ be a Banach space with a norm $\|\cdot\|_{\mathbb{E}}$. Suppose that $\mathcal{U}_\sigma(t, s): \mathbb{E} \to \mathbb{E}$, $t \geq s$, is a family of processes depending on a parameter $\sigma \in \Sigma$, where Σ is a metric space with a metric d_Σ. (For the definition of a process and related notions see [2], § 1, 2). It is assumed that a group of continuous operators $T(s)$, $s \in \mathbb{R}$, acts on Σ and that the following conditions hold.

(H1) The map $T(s)\sigma: \mathbb{R} \to \Sigma$ is continuous with respect to $s \in \mathbb{R}$ for any $\sigma \in \Sigma$.

(H2) The translation identity holds (see [2], p. 171), that is,

$$\mathcal{U}_{T(r)\sigma}(t, s) = \mathcal{U}_\sigma(t + r, s + r) \quad \text{for } \sigma \in \Sigma, \quad r \in \mathbb{R}, \quad t \geq s. \tag{1.1}$$

(H3) For any positive numbers R and μ there are uniformly bounded functions $a_{\mu, R}(t) \geq 0$ and $b_{\mu, R}(t, s; r) \geq 0$ defined for $t \geq s$ and $r \geq 0$ such that

$$\|\mathcal{U}_\sigma(t, s)\, U - \mathcal{U}_\varrho(t, s)\, V\|_{\mathbb{E}} \leq a_{\mu, R}(t - s)\|U - V\|_{\mathbb{E}} + b_{\mu, R}(t, s; d_\Sigma(\sigma, \varrho)) + \mu\,, \tag{1.2}$$

where $\sigma, \varrho \in \Sigma$, $U, V \in \mathbb{E}$, $\|U\|_{\mathbb{E}}, \|V\|_{\mathbb{E}} \leq R$, and $t \geq s$. Moreover,

$$a_{\mu, R}(t) \to 0 \quad \text{as} \quad t \to +\infty, \tag{1.3}$$

$$\sup_{-T \leq s \leq t \leq T} b_{\mu, R}(t, s; r) \to 0 \quad \text{as} \quad r \to 0 \tag{1.4}$$

for any $T > 0$.

(H4) There is $\sigma_0 \in \Sigma$ such that the process $\mathcal{U}_{\sigma_0}(t, s)$ has a uniformly bounded semi-trajectory $\{V_0(t), t \geq 0\}$ in $\mathbb{E}$.

THEOREM 1.1: *Let Conditions* (H1)-(H4) *hold. Then for any $\sigma \in \Sigma$ there is a unique uniformly bounded trajectory $U_\sigma(t)$ of the process $\mathcal{U}_\sigma(t, s)$. Moreover, the following assertions take place.*

(i) $U_{T(r)\sigma}(t) = U_\sigma(t + r)$ *for* $t, r \in \mathbb{R}$ *and* $\sigma \in \Sigma$.

(ii) *The* map $U_\sigma(t)$: $\Sigma \times \mathbb{R} \to \mathbb{E}$ *is continuous with respect to* $(\sigma, t) \in \Sigma \times \mathbb{R}$.

(iii) *The trajectory $U_\sigma(t)$ is asymptotically stable as $t \to +\infty$. Moreover,*

$$\sup_{\sigma \in \Sigma,\, V \in \mathbb{B}_R} \|\mathcal{U}_\sigma(t, \tau) V - U_\sigma(t)\|_{\mathbb{E}} \to 0 \quad \text{as } t - \tau \to +\infty \tag{1.5}$$

for any $R > 0$, where $\mathbb{B}_R \subset \mathbb{E}$ is a ball of radius R centred at zero.

PROOF: 1) We first show that $U_m(t) = \mathcal{U}_\sigma(t, -m)\mathbf{0}$ (where $\mathbf{0}$ is the zero element in $\mathbb{E}$) is a convergent sequence in $\mathbb{E}$ for any $\sigma \in \Sigma$ and $t \in \mathbb{R}$. Indeed, by virtue of identity (1.1) and inequality (1.2) with $\mu = 1$, we have

$$\begin{aligned} \|U_m(t)\|_{\mathbb{E}} \leq \|\mathcal{U}_{T(-m)\sigma}(t+m, 0)\mathbf{0} - \mathcal{U}_{\sigma_0}(t+m, 0)\, V_0(0)\|_{\mathbb{E}} + \|V_0(t+m)\|_{\mathbb{E}} \leq \\ \leq a_{1, R}(t+m)\|V_0(0)\|_{\mathbb{E}} + b_{1, R}(t+m, 0, d_\Sigma(T(-m)\sigma, \sigma_0)) + \\ + 1 + \sup_{r \geq 0}\|V_0(r)\|_{\mathbb{E}}, \end{aligned}$$

where $t \geq -m$ and $R = \|V_0(0)\|_{\mathbb{E}}$. Since $a_{1, R}$ and $b_{1, R}$ are bounded functions, we conclude that

$$\|U_m(t)\|_{\mathbb{E}} \leq R_1 \quad \text{for all } m \text{ and } t \geq -m, \tag{1.6}$$

where $R_1 > 0$ is a constant. Let us fix an arbitrary $\varepsilon > 0$. According to inequalities (1.6) and (1.2) with $R = R_1$, for any $\mu > 0$ we have

$$\|U_m(t) - U_k(t)\|_{\mathbb{E}} \leq a_{\mu, R_1}(t+k)\|\mathcal{U}_\sigma(-k, -m)\mathbf{0}\|_{\mathbb{E}} + \mu \leq R_1 a_{\mu, R_1}(t+k) + \mu, \tag{1.7}$$

where $m \geq k \geq -t$. Set $\mu = \varepsilon/2$. By (1.3), there is $k_0 > 0$ such that $a_{\mu, R_1}(t+k) \leq \leq \varepsilon(2R_1)^{-1}$ for $k \geq k_0$. Inequality (1.7) implies that $\|U_m(t) - U_k(t)\|_{\mathbb{E}} \leq \varepsilon$ for $k \geq k_0$. Thus, for each $\sigma \in \Sigma$ the sequence $\mathcal{U}_\sigma(t, -m)\mathbf{0}$ converges to a function $U_\sigma(t)$ contained in $\mathbb{B}_{R_1}$. It is easily seen that $U_\sigma(t)$ is a trajectory of the process $\mathcal{U}_\sigma(t, s)$.

2) We now show that (1.5) holds and that $U_\sigma(t)$ is the only uniformly bounded trajectory. Let $V \in \mathbb{B}_R$, where $R \geq R_1$. In view of (1.2), we have

$$\|\mathcal{U}_\sigma(t, \tau)\, V - U_\sigma(t)\|_{\mathbb{E}} \leq \|\mathcal{U}_\sigma(t, \tau)\, V - \mathcal{U}_\sigma(t, \tau)\, U_\sigma(\tau)\|_{\mathbb{E}} \leq$$
$$\leq a_{\mu, R}(t - \tau) \|V - U_\sigma(\tau)\|_{\mathbb{E}} + \mu \leq 2R a_{\mu, R}(t - \tau) + \mu\,.$$

Combining this with (1.3) we arrive at (1.5).

To prove the uniqueness, assume that $V_\sigma(t)$ is another bounded trajectory for $\mathcal{U}_\sigma(t, s)$. In this case, relation (1.5) implies

$$\|U_\sigma(t) - V_\sigma(t)\|_{\mathbb{E}} \leq \|\mathcal{U}_\sigma(t, \tau)\, V_\sigma(\tau) - U_\sigma(t)\|_{\mathbb{E}} \to 0 \quad \text{as } \tau \to -\infty\,,$$

whence follows that $U_\sigma \equiv V_\sigma$.

3) Thus, it remains to prove assertions (i) and (ii). To this end, we note that

$$\mathcal{U}_{T(r)\,\sigma}(t, s)\, U_\sigma(s + r) = \mathcal{U}_\sigma(t + r, s + r)\, U_\sigma(s + r) = U_\sigma(t + r)\,.$$

This means that $U_\sigma(t + r)$ is a bounded trajectory for $\mathcal{U}_{T(r)\,\sigma}(t, s)$ and therefore, by the uniqueness, it coincides with $U_{T(r)\sigma}$.

We now show that $U_\sigma(t)$ is continuous with respect to (t, σ). Assume that sequences $\{t_k\} \subset \mathbb{R}$ and $\{\sigma_k\} \subset \Sigma$ converge to $t \in \mathbb{R}$ and $\sigma \in \Sigma$, respectively. In this case, according to (H3), for any $\mu > 0$ we have

$$\text{(1.8)} \quad \|U_{\sigma_k}(t_k) - U_\sigma(t)\|_{\mathbb{E}} \leq \|\mathcal{U}_{\sigma_k}(t_k, s)\, U_{\sigma_k}(s) - \mathcal{U}_{T(t - t_k)\,\sigma}(t_k, s)\, U_\sigma(s + t - t_k)\|_{\mathbb{E}} \leq$$
$$\leq a_{\mu, R_1}(t_k - s) \|U_{\sigma_k}(s) - U_\sigma(s + t - t_k)\|_{\mathbb{E}} + \mathfrak{b}_{\mu, R_1}(t_k, s, d_\Sigma(\sigma_k, T(t - t_k)\,\sigma)) + \mu\,.$$

It follows from (1.3), (1.4), and Condition (H1) that the first and second terms on the right-hand side of (1.8) tend to zero as $s \to -\infty$ and $k \to \infty$, respectively. The theorem is proved.

1.2. *Existence of almost periodic trajectories.*

We now consider the case in which the parameter space Σ coincides with the hull of an a.p. function. Before stating the corresponding results, recall some notions in the theory of a.p. functions (see [1, 4, 10, 15]).

Let $\mathfrak{M} = \{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}$ be a countable module [10, Chapter III, § 2]. We set

$$D_{\mathfrak{M}}(t, s) = \sum_{k=1}^{\infty} 2^{-k} |\exp(i\lambda_k(t - s)) - 1|$$

for $t, s \in \mathbb{R}$. In what follows we assume without stipulation that the module $\mathfrak{M}$ has no basis consisting of a single element, that is, it cannot be represented in the form $\mathfrak{M} = \{\lambda j : j \in \mathbb{Z}\}$ with $\lambda \neq 0$. In this case, $D_{\mathfrak{M}}$ defines a new metric on the real line $\mathbb{R}$. Denote by $\mathbb{R}_{\mathfrak{M}}$ the set of real numbers endowed with the metric $D_{\mathfrak{M}}$.

Let $\mathbb{M}$ be a complete metric space with a metric $d_{\mathbb{M}}$. Since $\mathbb{R}$ and $\mathbb{R}_{\mathfrak{M}}$ coincide in the set-theoretical sense, any function $f(t)$: $\mathbb{R} \to \mathbb{M}$ can be regarded as a map from $\mathbb{R}_{\mathfrak{M}}$ into $\mathbb{M}$.

Definition 1.2: A function f: $\mathbb{R} \to \mathbb{M}$ is said to be *Levitan a.p.* (*Bohr a.p.*) *with a module contained in* $\mathfrak{M}$ if $f(t)$ is continuous (uniformly continuous) as a map from $\mathbb{R}_{\mathfrak{M}}$ into $\mathbb{M}$. The set of these functions will be denoted by $LAP(\mathbb{M}, \mathfrak{M})$ (accordingly, $AP(\mathbb{M}, \mathfrak{M})$).

We now assume that the above-mentioned parameter space Σ and the group of operators $T(s)$ satisfy one of the following conditions.

(H5) There is a metric space $\mathbb{M}$ and a Bohr a.p. function $\varrho_0 \in AP(\mathbb{M}, \mathfrak{M})$ such that Σ coincides with the hull $\mathcal{H}(\varrho_0)$ of ϱ_0 (see [10], Chapter I, § 3), and $T(s)$ has the form

$$(T(s)\sigma)(t) = \sigma(t+s) \quad \text{for } \sigma \in \Sigma\,. \tag{1.9}$$

The metric on Σ is defined by the formula

$$d_\Sigma(\sigma_1, \sigma_2) = \sup_{t \in \mathbb{R}} d_{\mathbb{M}}(\sigma_1(t), \sigma_2(t))\,.$$

(H6) There is a metric space $\mathbb{M}$ and a Levitan a.p. function $\varrho_0 \in LAP(\mathbb{M}, \mathfrak{M})$ such that Σ coincides with the set $\mathcal{S}(\varrho_0) \equiv \{\varrho_0(\cdot + s), s \in \mathbb{R}\}$ of the shifts of ϱ_0 and the group $T(s)$ is defined by (1.9). The metric on Σ has the form

$$d_\Sigma(\sigma_1, \sigma_2) = \sum_{j=1}^{\infty} 2^{-j} \kappa(d_{\mathbb{M},j}(\sigma_1, \sigma_2))\,, \qquad d_{\mathbb{M},j}(\sigma_1, \sigma_2) = \sup_{|t| \leq j} d_{\mathbb{M}}(\sigma_1(t), \sigma_2(t))\,,$$

where $\kappa(s) = s(s+1)^{-1}$, $s \geq 0$.

Definition 1.2 easily implies that in both the cases Condition (H1) is satisfied. The two assertions below refine the conclusion of Theorem 1.1 in the case of a.p. processes (cf. [13]).

Proposition 1.3: *Let Conditions* (H2)-(H5) *hold. Then the trajectories* $U_\sigma(t)$, $\sigma \in \Sigma$, *constructed in Theorem* 1.1 *belong to* $AP(\mathbb{E}, \mathfrak{M})$.

Proposition 1.4: *Let Conditions* (H2)-(H4) *and* (H6) *hold. Then the trajectories* $U_\sigma(t)$, $\sigma \in \Sigma$, *constructed in Theorem* 1.1 *belong to* $LAP(\mathbb{E}, \mathfrak{M})$.

Proof of Proposition 1.3: We must prove that the function $U_\sigma(t)\colon \mathbb{R}_{\mathfrak{M}} \to X$ is uniformly continuous for any $\sigma \in \Sigma$. To this end, it suffices to show that if a sequence $\{t_k\} \subset \mathbb{R}$ is fundamental, then so is the sequence $\{U_\sigma(t_k)\} \subset \mathbb{E}$.

Let $\{t_k\} \subset \mathbb{R}_{\mathfrak{M}}$ be a fundamental sequence. In view of Condition (H5), $T(t_k)\,\sigma = \sigma(\cdot + t_k)$ converges to a function σ_1 in Σ. Therefore, by virtue assertions (i) and (ii) in Theorem 1.1, we have

$$\lim_{k\to\infty} U_\sigma(t_k) = \lim_{k\to\infty} U_{T(t_k)\,\sigma}(0) = U_{\sigma_1}(0)\,,$$

whence follows that $\{U_\sigma(t_k)\}$ is a fundamental sequence.

Proof of Proposition 1.4: We must prove that the map $U_\sigma(t)\colon \mathbb{R}_{\mathfrak{M}} \to X$ is continuous for an arbitrary fixed $\sigma \in \Sigma$. Let $t \in \mathbb{R}$, $\{t_k\} \subset \mathbb{R}$, and $t_k \to t$ in $\mathbb{R}_{\mathfrak{M}}$ as $k \to \infty$. In view of Condition (H6), we have $T(t_k)\sigma \to T(t)\sigma$ in Σ as $k \to \infty$. Combining this with assertions (i) and (ii) in Theorem 1.1 we obtain

$$\lim_{k\to\infty} U_\sigma(t_k) = \lim_{k\to\infty} U_{T(t_k)\,\sigma}(0) = U_{T(t)\,\sigma}(0) = U_\sigma(t)\,,$$

which means that $U_\sigma(t)$ is continuous at the point $t \in \mathbb{R}_{\mathfrak{M}}$.

2. - Asymptotic behaviour of solutions to the problem (0.1)-(0.3) as $t \to +\infty$

In this section, we study the asymptotic behaviour of solutions to the problem (0.1)-(0.3) as $t \to +\infty$ in case the right-hand side $h(x, t)$ is a bounded or a.p. function with range in $L^2(\Omega)$.

2.1. *Statement of the main result.*

We introduce some notation. Given a Banach space B and an interval $J \subset \mathbb{R}$, we denote by $L^p_{\mathrm{loc}}(J, B)$ the space of measurable functions $f(t)\colon J \to B$ such that $f \in L^p(I, B)$ for any finite subinterval $I \subset J$. We also define the space $S(J, B)$ consisting of the functions $f \in L^1_{\mathrm{loc}}(J, B)$ such that

$$\|f\|_{S(J,\,B)} := \sup_{t\in J} \int_{J\,\cap\,[t,\,t+1]} \|f(s)\|_B\, ds < \infty\ .$$

Recall the notion of almost periodicity in the sense of Stepanov and Levitan-Stepanov (see [4], Chapter 2; [10]).

Definition 2.1: Let $\mathfrak{M} \subset \mathbb{R}$ be a countable module. A function $f(t) \in L^1_{\mathrm{loc}}(\mathbb{R}, B)$ is said to be *Levitan a.p. (Levitan–Stepanov a.p.) with module contained in* $\mathfrak{M}$ if the function $f(t+\eta)$, $\eta \in [0, 1]$, belongs to $AP(L^1([0, 1], B), \mathfrak{M})$ (accordingly, $LAP(L^1([0, 1], B), \mathfrak{M})$). The set of these functions will be denoted by $SAP(B, \mathfrak{M})$ (accordingly, $LSAP(B, \mathfrak{M})$).

We shall denote by H and H_1 the spaces $L^2(\Omega)$ and $H_0^1(\Omega)$ with the norms

$$\|u\| = \left(\int_\Omega u^2\, dx \right)^{1/2}, \qquad \|u\|_1 = \left(\int_\Omega \left(\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + a_0 u^2 \right) dx \right)^{1/2},$$

respectively. For a function $u(x, t)$, set $E_u(t) = (1/2)(\|u_t(\cdot, t)\|^2 + \|u(\cdot, t)\|_1^2)$.

We now turn to the investigation of the problem (0.1)-(0.3). Suppose that the coefficients of L satisfy Conditions (L1) and (L2) (see Introduction) and that $g(p)$ is a continuous non-decreasing function on $\mathbb{R}$. It can be assumed without loss of generality that $g(0) = 0$. As is shown in [4, Chapter 2], if $h \in L^1_{\mathrm{loc}}(\mathbb{R}_\tau, H)$, then for arbitrary two functions $u_0 \in H_1$ and $u_1 \in H$ the problem (0.1)-(0.3) has a unique weak solution $u \in C(\mathbb{R}_\tau, H_1) \cap C^1(\mathbb{R}_\tau, H)$. Moreover, if $u(x, t)$ and $v(x, t)$ are two solutions, then $E_{u-v}(t) \leqslant E_{u-v}(s)$ for all $t \geqslant s \geqslant \tau$. Denote by $\mathbb{E}$ the space of the vector functions $U = [u_0, u_1] \in H_1 \times H$ with the norm $\|U\|_E = (\|u_0\|_1^2 + \|u_1\|^2)^{1/2}$. Thus, for each right-hand side $h \in L^1_{\mathrm{loc}}(\mathbb{R}, H)$ we can define a process of non-expanding operators $\mathcal{U}(t, s)\colon \mathbb{E} \to \mathbb{E}$, $t \geqslant s$, mapping a vector function $[u_0, u_1]$ to $[u(\cdot, t), u_t(\cdot, t)]$, where $u(x, t)$ is the solution to the problem (0.1)-(0.3) with $\tau = s$.

The asymptotic behaviour of solutions to the problem (0.1)-(0.3) will be obtained for the case in which the function $g(p)$ satisfies the following assumptions (cf. [7]).

(G1) The function $g(p)$ is monotone–increasing and continuous on $\mathbb{R}$; the inverse function $g^{-1}(p)$ is defined and uniformly continuous on $\mathbb{R}$.

(G2) Depending on the dimension n of the domain Ω, the function g satisfies one of the following conditions:

(a) if $n \geqslant 3$, then g can be represented as a sum of non-decreasing functions $g_0, \dots, g_N$ on $\mathbb{R}$, where g_0 is uniformly continuous on $\mathbb{R}$ and the inequality

(2.1) $$|g_i(p) - g_i(q)| \leqslant C_1 (1 + g(p)\, p + g(q)\, q)^{k_i} |p - q|^{\alpha_i}, \; p, q \in \mathbb{R},$$

with $0 < \alpha_i \leqslant 1$ and $k_i = (n + 2 - \alpha_i(n-2))/2n$ holds for $i = 1, \dots, N$;

(b) if $n = 2$, then g is representable as a sum of two non-decreasing functions g_0 and g_1,, where g_0 is uniformly continuous on $\mathbb{R}$ and g_1 satisfies the inequalities

(2.2) $$|g_1(p)| \leqslant C_1 \exp(c_1 |p|^\gamma), \qquad p \in \mathbb{R},$$

(2.3) $$|g_1(p) - g_1(q)| \leqslant C_2 (1 + g(p)\, pf(p) + g(q)\, qf(q)) |p - q|^\alpha, \; |p - q| \leqslant 1,$$

with $\gamma < 2$, $0 < \alpha \leqslant 1$, and $f(p) = (\ln(2 + p^2))^{-\varepsilon} (\ln(2 + g(p)\, p)^{-1/2}$, $\varepsilon > 0$;

(c) if $n = 1$, then g can be represented as a sum of two non-decreasing functions g_0 and g_1, where g_0 is uniformly continuous on $\mathbb{R}$ and g_1 satisfies inequality (2.3) with $f \equiv 1$ and $0 < \alpha \leqslant 1$.

We shall denote by $C_b(\mathbb{R}, \mathbb{E})$ the space of uniformly bounded continuous functions $U(\cdot, t)\colon \mathbb{R} \to \mathbb{E}$.

THEOREM 2.2: *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Suppose that L and g satisfy Conditions* (L1), (L2), (G1), *and* (G2) *and* $g(0) = 0$. *Then for any right-hand side $h \in S(\mathbb{R}, H)$ the problem* (0.1), (0.2) *has a unique solution $u(x, t)$ such that $[u, u_t] \in C_b(\mathbb{R}, \mathbb{E})$. Moreover, the following assertions take place.*

(i) *The solution $u(x, t)$ is globally asymptotically stable as $t \to +\infty$, that is,*

$$\sup_{[u_0, u_1] \in \mathbb{B}_R} (\|u(\cdot, t) - v(\cdot, t; \tau)\|_1 + \|u_t(\cdot, t) - v_t(\cdot, t; \tau)\|) \to 0 \quad \text{as } t - \tau \to +\infty \tag{2.4}$$

for any $R > 0$, where $v(x, t; \tau)$ is the solution to the problem (0.1)-(0.3) *and $\mathbb{B}_R \subset \mathbb{E}$ is a ball of radius R centred at zero.*

(ii) *If the right-hand side $h(x, t)$ belongs to $SAP(H, \mathfrak{M})$ or $LSAP(H, \mathfrak{M})$ for some countable module $\mathfrak{M} \subset \mathbb{R}$, then the vector function $[u, u_t]$ is an element of $AP(\mathbb{E}, \mathfrak{M})$ or $LAP(\mathbb{E}, \mathfrak{M})$, respectively.*

REMARKS: 1) Under some additional conditions on g, an estimate for the rate of convergence in (2.4) is obtained in [4], Chapter 5; [7].

2) In the case $n = 1$ and $h \in AP(H, \mathfrak{M})$, Haraux established the existence and asymptotic stability of a Bohr a.p. solution for the problem (0.1), (0.2) assuming only that Conditions (L1), (L2), and (G1) hold (see [4], Chapter 4; [5]).

3) In the case where $n \geqslant 3$, g is a locally Lipschitz function, and its derivative satisfies the inequality $0 < \gamma \leqslant g'(p) \leqslant C(1 + g(p)p)^{2/n}$ almost everywhere on $\mathbb{R}$, the existence, uniqueness, and asymptotic stability of a bounded (Bohr a.p.) solution are proved in [4], pp. 206–211; [7].

4) It is easy to see that if Condition (G2) holds, then g satisfies inequality (0.4) with $k = (n+2)/(n-2)$ for $n \geqslant 3$. Some sufficient conditions ensuring the existence of a Bohr a.p. solution for the problem (0.1), (0.2) and allowing a faster growth of the function g at infinity are obtained in [6] and [12].

5) In Section 3, we shall construct an example of an a.p. function $h \in LAP(H, \mathfrak{M})$ not belonging to $S(\mathbb{R}, H)$ for which the problem (0.1), (0.2) has no a.p. solution.

2.2. PROOF OF THEOREM 2.2: We shall show that the solving process $\mathcal{U}(t, s)$ of the problem (0.1), (0.2) is contained in a family of processes that satisfies the conditions of Theorem 1.1. This fact will imply the existence, uniqueness, and asymptotic stability of a bounded solution $u(x, t)$ such that $[u, u_t] \in C_b(\mathbb{R}, \mathbb{E})$. To prove assertion (ii), we shall use Propositions 1.3 and 1.4.

Set $\mathbb{E} = H_1 \times H$, $\Sigma = \{h(x, t+s): s \in \mathbb{R}\}$,

$$
(2.5) \quad \begin{cases} (T(s)\,\sigma)(x, t) = \sigma(x, t+s)\,, & \sigma \in \Sigma\,, \\ \mathcal{U}_\sigma(t, s)[u_0, u_1] = [u(x, t), u_t(x, t)]\,, & [u_0, u_1] \in \mathbb{E}\,, \\ d_\Sigma(\sigma_1, \sigma_2) = \sum_{j=1}^{\infty} 2^{-j} \kappa(\|\sigma_1 - \sigma_2\|_{S([-j, j], H)})\,, & \sigma_1, \sigma_2 \in \Sigma\,, \end{cases}
$$

where $\kappa(s) = s(s+1)^{-1}$ and $u(x, t)$ is the solution to the problem (0.1)-(0.3) with $h = \sigma$ and $\tau = s$. It is easy to see that the family of processes $\mathcal{U}_\sigma(t, s)$ satisfies Conditions (H1) and (H2). To verify (H3) and (H4) we need some auxiliary assertions.

Lemma 2.3: *(a) Let g be a continuous function on $\mathbb{R}$ satisfying Condition* (G1). *Then for any $\nu > 0$ there is $\gamma(\nu) > 0$ such that*

$$
(g(p) - g(q))(p - q) \geq \gamma(\nu)\,|p - q|^2 - \nu \qquad \text{for } p, q \in \mathbb{R},
$$

where $\nu/\gamma(\nu) \to 0$ as $\nu \to +0$.

(b) Let g_0 be a uniformly continuous function on $\mathbb{R}$. Then for any $\beta > 0$ there is $\Gamma(\beta) > 0$ such that

$$
|g_0(p) - g_0(q)| \leq \Gamma(\beta)\,|p - q| + \beta \qquad \text{for } p, q \in \mathbb{R}.
$$

Lemma 2.3 is a simple consequence of the definition of the uniform continuity (e.g., see [4], p. 162 for the proof of assertion (a)).

Lemma 2.4: *Let a function g satisfy Condition* (G2) *and let $g(0) = 0$. Then for any $\beta > 0$ there is $C_3 = C_3(\beta) > 0$ such that*

$$
(2.6) \qquad \|g(u)\|_{-1} \leq \beta \int_\Omega g(u)\, u\, dx + C_3(\beta) \qquad \text{for } u \in H_1\,,
$$

where $\|\cdot\|_{-1}$ is the norm in the dual space $H_1^* = H^{-1}(\Omega)$ of H_1.

Proof: Inequality (2.6) is proved in ([3], Proposition 4, p. 101) for $n = 2$ and in [4], p. 181 for $n = 1$. Let $n \geq 3$. It follows from (2.1) that inequality (0.4) with $k = (n+2)/(n-2)$ holds for g. Therefore (2.6) is a consequence of Proposition IV.4.1.1 in ([4], pp. 181-182).

Lemma 2.5: *Under the conditions of Theorem 2.2, for any $R > 0$ there is $C_4 = C_4(R) > 0$ such that the solution $u(x, t)$ to the problem* (0.1)-(0.3) *with arbitrary $h = \sigma \in \Sigma$ and $[u_0, u_1] \in \mathbb{B}_R$ satisfies the inequality $E_u(t) \leq C_4$ for $t \geq \tau$. In particular, Condition (H4) holds for the family of the processes $\mathcal{U}_\sigma(t, s)$.*

Proof: The definition of the metric space Σ and the conditions of Theorem 2.2

imply that $\|\sigma\|_{S(\mathbf{R},\, H)} \leqslant C_5$ for every $\sigma \in \Sigma$, where the constant $C_5 > 0$ does not depend on σ. Therefore Lemma 2.5 follows from Theorem IV.2.1.1 in ([4], p. 149) and Lemma 2.4.

To simplify notation, the integral $\int_\Omega f_1(x)\, f_2(x)\, dx$ will be denoted by $(f_1,\, f_2)$ for any two functions $f_1(x)$ and $f_2(x)$, and the dependence of functions on x and t will not often be indicated.

LEMMA 2.6: *Let a function g satisfy Condition* (G2) *and let* $g(0) = 0$. *Then there is a continuous increasing function* $b(r) \geqslant 0$ *defined for* $r \geqslant 0$ *such that*

$$\int_\Omega |g(u) - g(v)\|w|\, dx \leqslant \beta b(\|w\|_1)\, F(u,\, v) + \tag{2.7}$$

$$+ C_6\left(\|u - v\|^2 + \int_\Omega (g(u) - g(v))(u - v)\, dx\right)$$

for any $\beta > 0$, *where* $u,\, v,\, w \in H_1$, $F(u,\, v) = \int_\Omega (1 + g(u)\, u + g(v)\, v)\, dx$, *and the constant* $C_6 = C_6(\beta) > 0$ *does not depend on* u, v, *and* w.

PROOF: We first assume that $n \geqslant 3$. Let us fix an arbitrary $\beta > 0$. In view of Condition (G2), Lemma 2.3, and the Schwarz inequality, we have

$$\int_\Omega |g_0(u) - g_0(v)\|w|\, dx \leqslant \Gamma(\beta)\, \|u - v\|\, \|w\| + \beta \|w\|_{L^1(\Omega)} \leqslant \tag{2.8}$$

$$\leqslant \beta C_7\, (1 + \|w\|_1^2) + C_8(\beta)\, \|u - v\|^2,$$

$$\int_\Omega |g_j(u) - g_j(v)\|w|\, dx \leqslant \int_\Omega \frac{|g_j(u) - g_j(v)|}{|u - v|^{\alpha_j}}\, |u - v|^{\alpha_j}\, |w|\, dx \leqslant \tag{2.9}$$

$$\leqslant \int_\Omega \frac{|g_j(u) - g_j(v)|}{|u - v|^{\alpha_j}}\, (|u - v|^{1 + \alpha_j} \beta^{-1/\alpha_j} + \beta |w|^{1 + \alpha_j})\, dx \leqslant$$

$$\leqslant \beta^{-\frac{1}{\alpha_j}}\, (g_j(u) - g_j(v),\, u - v) + \beta C_1 \int_\Omega (1 + g(u)\, u + g(v)\, v)^{k_j}\, |w|^{1 + \alpha_j} dx \leqslant$$

$$\leqslant \beta^{-\frac{1}{\alpha_j}}\, (g(u) - g(v),\, u - v) + \beta C_1\, F(u,\, v)^{k_j}\, \|w\|_{L^m(\Omega)}^{1 + \alpha_j}, \qquad 1 \leqslant j \leqslant N,$$

where $m = 2n/(n - 2)$. To estimate the second term on the right-hand side of (2.9), we apply the inequality $\|w\|_{L^m(\Omega)} \leqslant \text{const}\, \|w\|_1$, where $w \in H_1$ (for instance, see [4], Theorem I.1.3.3). Its substitution into (2.9) results in

$$\int_{\Omega} |g_j(u) - g_j(v)| |w| \, dx \leqslant \beta^{-1/\alpha_j}(g(u) - g(v),\, u - v) + \beta C_9 F(u,\, v)^{k_j} \|w\|_1^{1+\alpha_j},$$

where $1 \leqslant j \leqslant N$. Combining this with (2.8) we derive (2.7).

Consider now the case $n = 2$. Let us fix an arbitrary $\beta > 0$. Since (2.8) holds for any n, it suffices to estimate the integral

$$I = \int_{\Omega} |g_1(u) - g_1(v)| \, |w| \, dx \, .$$

Set $\varphi(p) = p(\ln(1+p))^{\varepsilon}$ and $A_\beta(p) = \beta\varphi(p)$, $p \geqslant 0$, where $\varepsilon > 0$ is defined in (2.3). Denote by $A_\beta^*(q)$ the Legendre transform of $A_\beta(p)$, that is,

$$A_\beta^*(q) = q \sup\{p \geqslant 0\colon\ A_\beta'(p) = q\}, \qquad q \geqslant 0,$$

where A_β' is the derivative of A_β. By the Young inequality,

$$\text{(2.10)} \quad I \leqslant \int_{\Omega_\nu} |g_1(u) - g_1(v)| |w| \, dx + \int_{\Omega_\nu'} \frac{|g_1(u) - g_1(v)|}{|u - v|^{\alpha}} |u - v|^{\alpha} |w| \, dx \leqslant$$

$$\leqslant \int_{\Omega_\nu} |g(u) - g(v)| (A_\beta^*(1) + A_\beta(|w|)) \, dx +$$

$$+ \int_{\Omega_\nu'} \frac{|g_1(u) - g_1(v)|}{|u - v|^{\alpha}} (A_\beta^*(|u - v|^{\alpha}) + A_\beta(|w|)) \, dx \, ,$$

where $\Omega_\nu = \{x \in \Omega\colon\ |u(x) - v(x)|^{\alpha} \geqslant \nu\}$, $\Omega_\nu' = \Omega \backslash \Omega_\nu$, and the positive constant $\nu = \nu(\beta) \leqslant 1$ is so small that

$$\text{(2.11)} \qquad A_\beta^*(p) \leqslant \beta \quad \text{for } 0 \leqslant p \leqslant \nu \, .$$

Denote by $I(\Omega_\nu)$ and $I(\Omega_\nu')$ the integrals on the right-hand sides of (2.10). According to (2.3) and (2.11), we have

$$\text{(2.12)} \quad I(\Omega_\nu) \leqslant$$

$$\leqslant A_\beta^*(1)\nu^{-1/\alpha} \int_{\Omega_\nu} (g(u) - g(v),\, u - v) \, dx + \beta \int_{\Omega_\nu} (|g(u)| + |g(v)|)\varphi(|w|) \, dx \, ,$$

$$\text{(2.13)} \quad I(\Omega_\nu') \leqslant \beta \int_{\Omega_\nu'} (1 + g(u)\, uf(u) + g(v)\, vf(v))(1 + \varphi(|w|)) \, dx \, .$$

Furthermore, it follows from (2.2) and the definition of $f(p)$ that

$$\text{(2.14)} \qquad |f(p)| \leqslant C_{10}\, , \qquad |pf(p)| \geqslant c_2 > 0 \quad \text{for } p \in \mathbb{R} \, .$$

Substituting (2.12) and (2.13) into (2.10) and taking into account (2.14), we derive

(2.15) $$I \leqslant C_{11}\beta(F(u, v) + I_1) + C_{12}(\beta)(g(u) - g(v), u - v)\,,$$

where

$$I_1 = \int_\Omega (1 + g(u)\, uf(u) + g(v)\, vf(v))\varphi(|w|)\, dx\,.$$

To estimate I_1 we need the Pokhozhaev-Trudinger inequality ([14], Theorem 2): there are positive constants κ and C_{13} such that

(2.16) $$\int_\Omega \exp(\kappa z^2)\, dx \leqslant C_{13} \qquad \text{for } z \in H_1\,,\ \|z\|_1 \leqslant 1\,.$$

Denote by $\psi(p)$ the inverse function of φ and set

$$B_r(p) = \exp(\kappa\psi^2(p)/(1 + r^2))\,, \qquad p \geqslant 0\,,$$

$$B_r^*(q) = q \sup\{p \geqslant 0\colon B_r'(p) = q\}\,, \qquad q \geqslant 0\,,$$

where $r = \|w\|_1$ and B_r' is the derivative of B_r. It is easily seen that B_r^* is well defined. Moreover, $B_r^*(q)$ is an increasing function on $\mathbb{R}_+ = [0, +\infty)$, and the inequalities

(2.17) $$|B_r^*(q)| \leqslant b_1(r)\, q(\ln q)^{1/2}(\ln\ln q)^\varepsilon, \qquad q \geqslant q_0 \gg 1\,,$$

(2.18) $$pq \leqslant B_r(p) + B_r^*(q)\,, \qquad p, q \geqslant 0\,,$$

hold. Here and henceforth $b_i(r)$, $i = 1, 2, \ldots$, symbolise positive increasing functions of $r \geqslant 0$. Inequalities (2.2) and (2.17) with $q = g(u)\, uf(u)$ imply that

$$|B_r^*(g(u)\, uf(u))| \leqslant C_{14}\, b_1(r)(1 + g(u)\, u)\,.$$

Combining this with (2.16) (where $z = w(1 + r^2)^{-1/2}$) and (2.18), we obtain

$$\int_\Omega g(u)\, uf(u)\, \varphi(|w|)\, dx \leqslant \int_\Omega [B_r(\varphi(|w|)) + B_r^*(g(u)\, uf(u))]\, dx \leqslant$$

$$\leqslant \int_\Omega \exp(\kappa w^2(1 + \|w\|_1^2)^{-1})\, dx + C_{14}\, b_1(r)\int_\Omega (1 + g(u)\, u)\, dx \leqslant$$

$$\leqslant b_2(r)\int_\Omega (1 + g(u)u)\, dx\,,$$

$$\int_\Omega \varphi(|w|)\, dx \leqslant \int_\Omega [B_r^*(1) + B_r(\varphi(|w|))]\, dx \leqslant b_3(r)\,.$$

Thus, we have

$$I_1 \leqslant b_4(\|w\|_1)\, F(u, v)\,.$$

The required inequality (2.7) follows now from (2.15).

The proof of (2.7) in the one-dimensional case is simpler than for $n = 2$. The distinction is that we use the continuous embedding $H_1 \subseteq C(\overline{\Omega})$ when estimating I. (Here $C(\overline{\Omega})$ is the space of continuous functions on $\overline{\Omega}$). For brevity, we omit the proof of the lemma in the case $n = 1$.

For $J = \mathbb{R}$ or $\mathbb{R}_\tau$, we denote by $W^{k,\,p}_{\mathrm{loc}}(J, B)$ (where $k \in \mathbb{Z}$, $k \geqslant 0$, and $p = 1$ or ∞) the space of functions $f \in L^p_{\mathrm{loc}}(J, B)$ whose generalised derivatives up to the order k belong to $L^p_{\mathrm{loc}}(J, B)$.

LEMMA 2.7: *For any $\mu > 0$ and $R > 0$ there are $\delta(\mu, R) > 0$ and $C_{15}(\mu, R) > 0$ such that if $u(x, t)$ and $v(x, t)$ are two solutions to the problem* (0.1)-(0.3) *with right-hand sides $\sigma, \varrho \in \Sigma$ and initial data $[u_0, u_1], [v_0, v_1] \in \mathrm{B}_R$, respectively, then the inequality*

$$E_{u-v}(t) \leqslant C_{15} \exp(-2\delta(t-s))\, E_{u-v}(s) + C_{15} \|\sigma - \varrho\|^2_{S([s,\,t],\,H)} + \mu \tag{2.19}$$

holds for $t \geqslant s \geqslant \tau$. In particular, the family of processes $\mathcal{U}_\sigma(t, s)$ satisfies Condition (H3).

PROOF: We shall prove (2.19) for the case in which the right-hand sides and the solutions possess some additional smoothness, namely, $\sigma, \varrho \in W^{1,\,1}_{\mathrm{loc}}(\mathbb{R}, H)$ and $u, v \in W^{1,\,\infty}_{\mathrm{loc}}(\mathbb{R}_\tau, H_1) \cap W^{2,\,\infty}_{\mathrm{loc}}(\mathbb{R}_\tau, H)$. The general case can be obtained by passing to the limit in the inequality for smooth solutions (see [4], p. 155).

Let us fix an arbitrary $\mu > 0$. Consider the functional

$$z(t) = E_w(t) + \eta(w, w_t)\,, \tag{2.20}$$

where $w = u - v$ and the constant $\eta > 0$ is sufficiently small and will be chosen later. Let $\lambda_1 > 0$ be the first eigenvalue of L in the domain Ω with Dirichlet boundary condition. It is easy to see that if $\eta \leqslant \sqrt{\lambda_1}/2$, then

$$E_w(t)/2 \leqslant z(t) \leqslant 3\, E_w(t)/2\,, \tag{2.21}$$

$$\|w_t + \eta w\| \leqslant 2\,(E_w(t))^{1/2} \leqslant 2\sqrt{2}(z(t))^{1/2} \tag{2.22}$$

for any t. Since u and v are solutions to (0.1) with right-hand sides σ and ϱ, the function $z(t)$ satisfies the differential equation

$$\begin{aligned} z'(t) + (g(u_t) - g(v_t), w_t) - \eta\|w_t\|^2 + \eta\|w\|_1^2 + \eta(g(u_t) - g(v_t), w) = \\ = (\sigma - \varrho, w_t + \eta w)\,, \end{aligned} \tag{2.23}$$

where $z' = dz/dt$. In view of Lemmas 2.3 (*a*) and 2.6, for any positive numbers ν and β, we have

$$(2.24)\qquad (g(u_t)-g(v_t),\, w_t) \geqslant \gamma(\nu)\|w_t\|^2 - \nu\,\mathrm{meas}\,(\Omega)\,,$$

$$(2.25)\qquad |(g(u_t)-g(v_t),\, w)| \leqslant C_6(\beta)\,(\|w_t\|^2 + (g(u_t)-g(v_t),\, w_t)) + \\ + \beta b(\|w(\cdot,\, t)\|_1)\, f(t)\,,$$

where meas (Ω) is the measure of Ω and $f(t) = \int_\Omega (1+g(u_t)\,u_t + g(v_t)\,v_t)\,dx$. Let us estimate $b(\|w(\cdot,\, t)\|_1)$. Lemma 2.5 implies $\|w(\cdot,\, t)\|_1 \leqslant 2\,(E_u(t)+E_v(t)) \leqslant 4\,C_4(R)$. Since $b(r)$ is an increasing function on $\mathbb{R}_+$, we have $b(\|w(\cdot,\, t)\|_1) \leqslant b(4\,C_4) =: C_{16}$ for $t \geqslant \tau$. Combining this with (2.21)-(2.25) and the Schwarz inequality, we derive

$$(2.26)\qquad z'(t) + [\gamma(\nu)/2 - \eta(C_6(\beta)+1)]\,\|w_t\|^2 + \eta\|w\|_1^2 + \\ + (1/2 - \eta C_6(\beta))(g(u_t)-g(v_t),\, w_t) \leqslant \psi(t)\,(z(t))^{1/2} + \varphi(t)\,,$$

where

$$(2.27)\quad \varphi(t) = \nu\,\mathrm{meas}\,(\Omega)/2 + \eta\beta C_{16} f(t)\,, \qquad \psi(t) = 2\,\sqrt{2}\|\sigma(\cdot,\, t) - \varrho(\cdot,\, t)\|\,.$$

It can be assumed without loss of generality that

$$(2.28)\qquad \gamma(\nu) \leqslant \min\{2\,\sqrt{\lambda_1},\, 1\}\,.$$

Set

$$(2.29)\quad \eta := \frac{\gamma(\nu)}{2\,(C_6(\beta)+2)} \leqslant \frac{\sqrt{\lambda_1}}{2}\,, \qquad \delta := \frac{2}{3}\,\eta = \frac{\gamma(\nu)}{3\,(C_6(\beta)+2)} \leqslant \frac{1}{6}\,.$$

It follows from (2.21), (2.26), and (2.28) that

$$(2.30)\qquad z'(t) + 2\,\delta z(t) \leqslant \psi(t)\,(z(t))^{1/2} + \varphi(t)\,.$$

To estimate $z(t)$ we apply the following lemma whose proof is given in Appendix (see § 4).

LEMMA 2.8: *Suppose that an absolutely continuous non-negative function $z(t)$ satisfies inequality (2.30) for almost all $t \geqslant \tau$, where $\delta > 0$, φ, $\psi \in L^1_{\mathrm{loc}}(\mathbb{R}_\tau)$, and φ, $\psi \geqslant 0$ almost everywhere on $\mathbb{R}_\tau$. Then the inequality*

$$(2.31)\quad z(t) \leqslant \frac{5\,e^{2\delta}}{4}\left\{e^{-2\delta(t-s)}\,z(s) + (e^{2\delta}-1)^{-1}\|\varphi\|_{S([s,\,t],\,\mathbb{R})} + (e^{\delta}-1)^{-2}\|\psi\|^2_{S([s,\,t],\,\mathbb{R})}\right\}$$

holds for $t \geqslant s \geqslant \tau$.

It is easily seen that $(e^r-1)^{-1} \leqslant r^{-1}$ for $r > 0$. Therefore, in view of (2.27), (2.29), and (2.31), the function $z(t)$ defined in (2.20) satisfies the inequality

$$(2.32)\quad z(t) \leqslant C_{17}\,(\exp(-\delta(t-s))\,z(s) + 2\delta^{-2}\|\sigma-\varrho\|^2_{S([s,\,t],\,\mathbb{R})}) + \Phi(s,\,t)\,, \qquad t \geqslant s\,,$$

where $\Phi(s,\,t) = C_{17}\delta^{-1}\|\varphi\|_{S([s,\,t],\,\mathbb{R})}$ and the constant $C_{17} > 0$ does not depend on ν and β. We claim that

$$(2.33)\qquad \Phi(s,\,t) \leqslant \mu/2 \quad \text{for } t \geqslant s \geqslant \tau$$

under suitable choice of ν and β. Indeed, it follows from (2.27), (2.29), and the definition of Φ that

$$(2.34)\quad \Phi(s,\,t) \leqslant \frac{3}{2}\,C_{17}\left\{4\ \mathrm{meas}\,(\Omega)\,(C_6(\beta)+1)\,\frac{\nu}{\gamma(\nu)} + C_{16}\beta\|f(\cdot)\|_{S([s,\,t],\,\mathbb{R})}\right\}.$$

Let us estimate the second term in the brackets on the right-hand side of (2.34). Because $u(x,\,t)$ is a solution to the problem (0.1), (0.2) with $h=\sigma$, for any $r \geqslant \tau$ we have (see [4], Proposition II.1.2.1)

$$\int_r^{r+1}\int_\Omega g(u_t)\,u_t\,dx\,dt \leqslant E_u(r) - E_u(r+1) + \int_r^{r+1}(\sigma,\,u_t)\,dt \leqslant$$

$$\leqslant C_4(R) + \int_r^{r+1}\|u_t\|\,\|\sigma\|\,dt \leqslant C_4(R) + \sqrt{2C_4(R)}\,\|h\|_{S(\mathbb{R},\,H)} =: C_{18}(R)\,.$$

The second and third inequalities are consequences of Lemma 2.5 and the definition of Σ. A similar estimate holds for $v(x,\,t)$. Hence

$$\|f(\cdot)\|_{S([s,\,t],\,\mathbb{R})} \leqslant \mathrm{meas}\,(\Omega) + 2C_{18}(R) \qquad \text{for } t \geqslant s \geqslant \tau\,.$$

Comparing this with (2.34) we obtain

$$(2.35)\qquad \Phi(s,\,t) \leqslant C_{19}\,\frac{\nu}{\gamma(\nu)} + C_{20}\beta \quad \text{for } t \geqslant s \geqslant \tau\,,$$

where $C_{19} > 0$ and $C_{20} > 0$ depend only on β and R, respectively. By assertion (*a*) in Lemma 2.2, $\nu/\gamma(\nu) \to 0$ as $\nu \to +0$. If ν and β are so small that $C_{19}\nu/\gamma(\nu) \leqslant \mu/4$ and $C_{20}\beta \leqslant \mu/4$, then we derive (2.33) from (2.35). Inequality (2.19) with $C_{15} = C_{17}\max\{3,\,2\delta^{-2}\}$ follows now from (2.21), (2.32), and (2.33). Lemma 2.7 is proved.

We can now complete the proof of Theorem 2.2. By Lemmas 2.5 and 2.7, the family $\{\mathcal{U}_\sigma(t,\,s),\,\sigma \in \Sigma\}$ of solving processes for the problem (0.1), (0.2) with $h=\sigma$ satisfies the conditions of Theorem 1.1. Hence, this problem has a unique solution $u(x,\,t)$, $[u,\,u_t] \in C_b(\mathbb{R},\,\mathbb{E})$, for which (2.4) holds. Let us prove that this solution is almost periodic if so is the right-hand side $h(x,\,t)$.

We confine ourselves to the case $h \in SAP(H,\,\mathfrak{M})$. Set $\mathbb{M} = L^1([0,\,1],\,H)$ and consider a function $\varrho_0 \in AP(\mathbb{M}, \mathfrak{M})$ defined by the formula

$$\varrho_0(x, t; \eta) = h(x, t+\eta), \qquad x \in \Omega, \quad t \in \mathbb{R}, \quad \eta \in [0, 1].$$

According to the definition of $SAP(H, \mathfrak{M})$, we have $\varrho_0 \in AP(\mathbb{M}, \mathfrak{M})$. Let us denote by Σ the hull of the a.p. function ϱ_0 and endow it with the metric (cf. (2.5))

$$d_\Sigma(\sigma_1, \sigma_2) = \sup_{t \in \mathbb{R}} \|\sigma_1 - \sigma_2\|_{S(\mathbb{R}, H)} .$$

Obviously, this metric is stronger than the one defined in (2.5). Thus, the above-mentioned family $\{\mathcal{U}_\sigma(t, s), \sigma \in \Sigma\}$ satisfies Conditions (H2)-(H5). Hence, by Proposition 1.3, the function $U = [u, u_t]$ belongs to $AP(\mathbb{R}, \mathbb{E})$.

3. - Example of an equation without a.p. solution

In this section we prove that if the right-hand side of Equation (0.1) is an unbounded Levitan a.p. function, then the problem (0.1), (0.2) generally has no a.p. solution.

Consider the ordinary differential equation

$$u'' + 2u' + u = h(t), \tag{3.1}$$

where $u' = du/dt$. For a Banach space B, denote by $LAP(B)$ the set of all Levitan a.p. functions, that is, the union of the spaces $LAP(B, \mathfrak{M})$ over all countable modules $\mathfrak{M} \subset \mathbb{R}$.

Theorem 3.1: *For any increasing function $\varphi(r)$: $\mathbb{R}_+ \to \mathbb{R}_+$ tending to $+\infty$ as $r \to +\infty$, there is a Levitan a.p. scalar function $h(t) \in LAP(\mathbb{R})$ such that*

$$|h(t)| \leqslant \varphi(|t|) \quad \textit{for all } t \in \mathbb{R} \tag{3.2}$$

and Equation (3.1) has no solution $u(t)$ such that $u, u' \in LAP(\mathbb{R})$.

Remark: Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, let $\lambda_1 > 0$ be the first eigenvalue of the operator $(-\Delta)$ in Ω with Dirichlet boundary condition, and let $e_1(x)$ be the corresponding eigenfunction. It is easy to see that if $h(t)$ is the function constructed in Theorem 3.1, then the problem

$$u_{tt} + 2\sqrt{\lambda_1} u_t - \Delta u = h(t\sqrt{\lambda_1})\, e_1(x), \; u|_{\partial\Omega} = 0$$

has no solution $u(x, t)$ satisfying the inclusion $[u, u_t] \in LAP(\mathbb{E})$.

Proof of Theorem 3.1: Suppose that a non-negative function $h \in LAP(\mathbb{R})$ satisfies the following condition:

(H) there are sequences $\{t_k\}$ and $\{h_k\} \subset \mathbb{R}$ and a number $\delta > 0$ such that $h_k \to +\infty$ as $k \to \infty$ and $h(t) \geqslant h_k$ for $t \in [t_k - \delta, t_k + \delta]$.

We claim that Equation (3.1) with the right-hand side $h(t)$ has no solution $u(t)$ such that $u, u' \in LAP(\mathbb{R})$.

Indeed, it is easy to prove that any solution $u(t)$ to (3.1) can be represented in the form

$$(3.3)\quad u(t) = e^{s-t}(u(s) + (t-s)(u'(s) + u(s))) + \int_s^t e^{\tau - t}(t-\tau)\, h(\tau)\, d\tau\,, \qquad t, s \in \mathbb{R}\,.$$

Suppose that $u, u' \in LAP(\mathbb{R})$. In this case there is a sequence $\{s_k\} \subset \mathbb{R}$ tending to $-\infty$ such that $|u(s_k)| + |u'(s_k)| \leqslant C$ for all k. We set $s = s_k$ in (3.3) and pass to the limit as $k \to +\infty$. Since the integrand in (3.3) is non-negative for $\tau \leqslant t$, we derive

$$(3.4)\qquad u(t) = \int_{-\infty}^{t} e^{\tau - t}(t-\tau)\, h(\tau)\, d\tau\,.$$

It follows from the inclusion $u \in LAP(\mathbb{R})$ that there are constants $L > 0$ and $C > 0$ such that for any k the interval $[t_k + \delta, t_k + \delta + L]$ contains a point T_k for which $|u(T_k)| \leqslant C$. On the other hand, since $h(t) \geqslant 0$, Condition (H) and relation (3.4) imply

$$u(T_k) \geqslant \int_{t_k - \delta}^{t_k} e^{\tau - T_k}(T_k - \tau)\, h(\tau)\, d\tau \geqslant \delta e^{-(L+2\delta)} h_k \to +\infty \quad \text{as } k \to +\infty\,.$$

This contradiction proves Theorem 3.1. Thus, it remains to establish the existence of a function $h(t) \in LAP(\mathbb{R})$ satisfying (H) and (3.2).

Denote by $\mathbb{T}$ a two-dimensional torus represented on the plane $\mathbb{R}^2_{(x,y)}$ as the square $\{(x, y)\colon -\pi \leqslant x, y \leqslant \pi\}$ with identified opposite sides. Let us endow $\mathbb{T}$ with the metric $d(q_1, q_2) = |e^{i(x_1 - x_2)} - 1| + |e^{i(y_1 - y_2)} - 1|$, where $q_i = (x_i, y_i) \in \mathbb{T}$, $i = 1, 2$. Let $\lambda \in (0, 1)$ be an irrational number. In this case the curve

$$(3.5)\qquad \xi\colon \mathbb{R} \to \mathbb{T}\,, \qquad t \mapsto (t, \lambda t) \pmod{2\pi}$$

has no self-intersections, and its image $\xi(\mathbb{R})$ is everywhere dense in $\mathbb{T}$. Denote by $\mathbb{S} \subset \mathbb{T}$ the set $\xi(\mathbb{R})$ with the induced metric. Let $\mathfrak{M} \subset \mathbb{R}$ be the smallest module generated by the numbers 1 and λ (see [10], Chapter III, § 2). Since the metric space $\mathbb{R}_{\mathfrak{M}}$ (see § 1) and the real line $\mathbb{R}$ coincide in the set-theoretical sense, the map $\xi(t)$ defined by (3.5) can be regarded as a function from $\mathbb{R}_{\mathfrak{M}}$ to $\mathbb{S}$. It is easy to show that ξ is a homeomorphism of the metric spaces $\mathbb{R}_{\mathfrak{M}}$ and $\mathbb{S}$. Therefore, in view of the definition of the almost periodicity in the Levitan sense (see [10], Chapter IV, § 1; [9]), to any continuous function $\tilde{h}(x, y)$ on $\mathbb{S}$ there corresponds a Levitan a.p. function $h(t)$ on $\mathbb{R}$ that is defined by the formula

$$(3.6)\qquad h(t) = \tilde{h}(\xi(t))\,.$$

To construct the function $h(t) \in LAP(\mathbb{R})$, we first define a non-negative continuous function $\tilde{h}$: $\mathbb{S} \to \mathbb{R}$ and then show that (3.6) possesses the desired properties.

For any integer $j \geq 0$, we set $J_j = [\tau_j - \pi, \tau_j + \pi]$, where $\tau_j = -\pi(j+1)$ $(\tau_j = \pi j)$ if j odd (even). Denote by $q_j = (0, y_j)$, $-\pi < y_j < \pi$ the intersection point of the set $\xi(J_j)$ and the straight line $\{(x, y): x = 0\}$. Let $j_0 = 0$ and let $\{j_k, k \geq 1\}$ be an increasing sequence of positive integers such that $0 < y_{j_k} < \pi(1-\lambda)$ for any k, $y_{j_k} < y_{j_m}$ for $k < m$, and $y_{j_k} \to \pi(1-\lambda)$ as $k \to \infty$. For $k \geq 1$ denote by α_k an arbitrary positive number that satisfies the following inequalities:

$$(3.7) \qquad \alpha_k \leq (y_{j_{k+1}} - y_{j_k})/3\,, \qquad \alpha_k \leq (y_{j_k} - y_{j_{k-1}})/3\,,$$

$$(3.8) \qquad \alpha_k \leq |y_{j_k} - y_j|/2 \quad \text{for } 0 \leq j \leq j_k - 1 \text{ and } j = j_k + 1\,.$$

Set

$$(3.9) \qquad P_k = \{(x, y) \in \mathbb{S}: |x| < \pi/2,\ |y - y_{j_k} - \lambda x| < \alpha_k\}, \qquad k \geq 1\,.$$

Inequality (3.7) implies that the sets P_k are mutually disjoint. Let $\chi(r)$ be a continuous function on $\mathbb{R}$ such that $0 \leq \chi \leq 1$, $\chi(r) = 0$ for $|r| \geq 1$, and $\chi(r) = 1$ for $|r| \leq 1/2$. Set

$$(3.10)\ \tilde{h}(x, y) = \begin{cases} \varphi(|\tau_{j_k} + x|)\,\chi(2x\pi^{-1})\,\chi(\alpha_k^{-1}(y - y_{j_k} - \lambda x))\,, & (x, y) \in P_k\,, \\ 0, & (x, y) \in \mathbb{S} \setminus \bigcup_{k=1}^{\infty} P_k\,. \end{cases}$$

Clearly, $\tilde{h}$ is a non-negative function. Since

$$\{(x, y) \in \mathbb{T}: y = \lambda x + \pi(1-\lambda),\ |x| \leq \pi\} \cap \mathbb{S} = \emptyset\,,$$

we see that $\tilde{h}(x, y)$ is continuous on $\mathbb{S}$. Consequently, the function $h(t)$ defined by (3.6) is a.p. in the Levitan sense.

Let us show that $h(t)$ satisfies Condition (H) with $\delta = \pi/4$ and $t_k = \tau_{j_k}$. Set $I_k = [t_k - \delta, t_k + \delta]$. Since

$$\xi(I_k) = \{(x, y) \in \mathbb{S}: |x| \leq \pi/4,\ y - y_{j_k} = \lambda x\}$$

we conclude from (3.9), (3.10), and the definition of $\chi(s)$ that

$$(3.11) \qquad h(t) = \varphi(|t|)\,\chi(2(t - t_k)/\pi)\,\chi(0) = \varphi(|t|) \qquad \text{for } t \in I_k\,.$$

It remains to note that $t_k \to \infty$ as $k \to \infty$ and $\varphi(r) \to +\infty$ as $r \to +\infty$, and hence Condition (H) with $h_k = \min_{t \in I_k} \varphi(|t|)$ holds for $h(t)$.

We now prove (3.2). Let $t \in \mathbb{R}$. If $\xi(t) \in \mathbb{S} \setminus \bigcup_{k=1}^{\infty} P_k$, then, by (3.10), we have $h(t) = 0 \leq \varphi(|t|)$. Assume that $\xi(t) \in P_k$ for some k. If $t \in I_k$, then (3.2) is a consequence of (3.11). Let $t \in J_j$ for some $j \neq j_k$. In this case, by (3.9) and (3.8), we have $|t| \geq |t_k| + \pi$. Since $\varphi(r)$ is a non-decreasing function, we conclude from (3.10) and the definition of

$\chi(s)$ that

$$|h(t)| = |\tilde{h}(\xi(t))| \leqslant \varphi(|t_k| + \pi) \leqslant \varphi(|t|).$$

The proof of Theorem 3.1 is complete.

4. - Appendix

Proof of Lemma 2.7: Lemma 2.7 is a variant of the well-known Gronwall inequality. Therefore we only outline the proof.

Let us fix an arbitrary $T \geqslant s$ and consider the function

$$w(t) = e^{2\delta(t-s)} z(t) - \int_s^t e^{2\delta(\theta - s)} \varphi(\theta)\, d\theta, \qquad s \leqslant t \leqslant T. \tag{4.1}$$

It is easy to see that

$$z(t) \leqslant (w(t) + K) e^{-2\delta(t-s)} \qquad \text{for } s \leqslant t \leqslant T, \tag{4.2}$$

where

$$K = K(T) = \int_s^T e^{2\delta(\theta - s)} \varphi(\theta)\, d\theta. \tag{4.3}$$

Now note that

$$w'(t) = e^{2\delta(t-s)} (z'(t) + 2\delta z(t) - \varphi(t)) \leqslant$$

$$\leqslant e^{2\delta(t-s)} z(t)^{1/2} \psi(t) \leqslant e^{\delta(t-s)} \psi(t) (w(t) + K)^{1/2}, \qquad s \leqslant t \leqslant T,$$

whence it follows that

$$(w(t) + K)^{1/2} \leqslant (w(s) + K)^{1/2} + \frac{1}{2} \int_s^t e^{\delta(\theta - s)} \psi(\theta)\, d\theta, \qquad s \leqslant t \leqslant T.$$

Combining this with (4.1) and (4.2), we arrive at the inequality

$$z(t)^{1/2} \leqslant e^{-\delta(t-s)} (z(s) + K)^{1/2} + \frac{1}{2} \int_s^t e^{-\delta(t-\theta)} \psi(\theta)\, d\theta, \qquad s \leqslant t \leqslant T. \tag{4.4}$$

Squaring both sides of (4.4) and setting $T = t$, we derive

$$z(t) \leqslant \frac{5}{4} \left\{ e^{-2\delta(t-s)} z(s) + \int_s^t e^{-2\delta(t-\theta)} \varphi(\theta)\, d\theta + \left(\int_s^t e^{-\delta(t-s)} \psi(\theta)\, d\theta \right)^2 \right\}, \qquad t \geqslant s.$$

It remains to note that

$$\int_s^t e^{-2\delta(t-\theta)} \varphi(\theta)\, d\theta \leqslant e^{2\delta}(e^{2\delta}-1)^{-1} \|\varphi\|_{S([s,\,t],\,\mathbf{R})},$$

$$\int_s^t e^{-\delta(t-\theta)} \psi(\theta)\, d\theta \leqslant e^{\delta}(e^{\delta}-1)^{-1} \|\psi\|_{S([s,\,t],\,\mathbf{R})},$$

where $t \geqslant s$. The proof of Lemma 2.7 is complete.

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