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Global Averaging of Dissipative Dynamical Systems (**)(***)

ABSTRACT. — Non-autonomous equations with rapidly oscillating (with frequency ω) and almost periodic non-linear interaction terms and right-hand sides are considered. The following global generalization of the Bogolyubov averaging principle is proved. Attractors of non-autonomous equations tend to the attractor of the averaged autonomous equation as $\omega \to \infty$. This global averaging principle is applied to a reaction-diffusion system, two-dimensional Navier–Stokes system and damped wave equation.

Medie globali di sistemi dinamici dissipativi

SUNTO. — Si studiano equazioni non-autonome, dotate di secondo membro, nelle quali figurano termini d'interazione non-lineari, quasi periodici e rapidamente oscillanti (con frequenza ω). Per siffatte equazioni si prova la seguente generalizzazione globale del principio di media di Bogolyubov: al tendere di ω all'infinito, gli attrattori delle equazioni non autonome tendono agli attrattori dell'equazione autonoma risultante dall'operazione di media. Si applica poi questo principio a sistemi di reazione-diffusione, a sistemi di Navier-Stokes in dimensione 2 e ad un'equazione delle onde con smorzamento.

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0. - Introduction

The aim of this paper is to provide a connection between the long time dynamics of autonomous and non-autonomous partial differential equations via the Bogolyubov averaging principle [2].

If an autonomous differential equation $\partial_t u = N(u)$, $u(0) = u_0$ is well posed in a Banach space E, then the long time behaviour of the semigroup of its solution operators $S_t\colon E \to E$, $S_t u_0 = u(t)$ is described by the attractor $\mathcal{A}$ of this semigroup (of course, when it exists) which, by definition, is a compact ($\mathcal{A} \Subset E$), strictly invariant ($S_t \mathcal{A} = \mathcal{A}$) and globally attracting ($\mathrm{dist}_E (S_t B, \mathcal{A}) \to 0$, $t \to \infty$) set [1], [5], [12], [17]. The global attractor $\mathcal{A}$ is the ω-limit set of a ball $B_E(R)$ with sufficiently large radius: $\mathcal{A} = \omega(B_E(R)) = \bigcap_{\tau \geq 0} \left[\bigcup_{t \geq \tau} S_t B_E(R) \right]_E$.

A non-autonomous equation in E will be written in the form [4]

$$\partial_t u = N_{\sigma(t)}(u), \qquad u(\tau) = u_\tau, \qquad t \geq \tau \in \mathbb{R}, \tag{0.1}$$

where $\sigma(t)$ is the collection of all time-dependent terms of the equation and is called the time symbol. For instance, $\sigma(t) = \{F(\cdot, t), f(t)\}$ for an equation of the form

$$\partial_t u = -Au + F(u, t) + f(t). \tag{0.2}$$

In general, $\sigma(t)$ takes values in a Banach space $\mathfrak{M}$ defined by a particular partial differential equation and as a function of t, σ belongs to a symbol space Σ.

We suppose that equation (0.1) has a unique solution $u(t)$, $u(t) = U_\sigma(t, \tau) u_\tau$, defining thereby a family of solution operators $U_\sigma(t, \tau)$, $t \geq \tau \in \mathbb{R}$, which is called the process generated by (0.1). The uniqueness of the solution implies the following two characteristic properties of $U_\sigma(t, \tau)$:

$$\begin{cases} U_\sigma(t, s) \circ U_\sigma(s, \tau) = U_\sigma(t, \tau), & t \geq s \geq \tau, \\ U_{T(h)\sigma}(t, \tau) = U_\sigma(t + h, \tau + h), \end{cases} \tag{0.3}$$

where $T(h)$ is the translation operator $T(h)\sigma(\cdot) = \sigma(\cdot + h)$.

By a formal application of (0.3) we see that the operator

$$S_t(u, \sigma) = (U_\sigma(t, 0)\, u,\ T(t)\, \sigma), \qquad t \geq 0, \qquad u \in E, \qquad \sigma \in \Sigma$$

acting in the extended phase space $E \times \Sigma$ [15] enjoys the semigroup property: $S_{t_1} \circ S_{t_2} = S_{t_1 + t_2}$. To be able to construct the attractor of S_t as an ω-limit set, the symbol space Σ must be invariant with respect to $T(h)$, $T(h)\Sigma \subseteq \Sigma$, and compact in a certain topological space. The simplest way to ensure this (although, not the only possible way) is to choose this topological space to be $C_b(\mathbb{R}; \mathfrak{M})$ and suppose that σ is an almost periodic function with values in $\mathfrak{M}$. Then we can set $\Sigma = \mathcal{H}(\sigma)$, where $\mathcal{H}(\sigma)$ is the hull of the a.p. function σ: $\mathcal{H}(\sigma) = [\sigma(\cdot + h), h \in \mathbb{R}]_{C_b(\mathbb{R};\, \mathfrak{M})}$. By Bochner's criterion (see, for instance, [14]), $\mathcal{H}(\sigma) \Subset C_b(\mathbb{R}; \mathfrak{M})$, the embedding being compact.

If the semigroup S_t: $E \times \Sigma \to E \times \Sigma$ has a global attractor $\mathcal{A}_{E \times \Sigma}$ in the usual sense, then its projection on E, $\mathcal{A}_\Sigma := \Pi_1 \mathcal{A}_{E \times \Sigma}$ is minimal with respect to inclusion among all closed sets $\mathcal{A}_1$ that are uniformly attracting, that is, which have the property that

$$\lim_{t \to \infty} \sup_{\delta \in \Sigma} \operatorname{dist}_E (U_\delta(t, \tau) B, \mathcal{A}_1) = 0 , \qquad \forall B \in \mathcal{B}(E) .$$

The set $\mathcal{A}_\Sigma$ is called the uniform attractor of the equation (0.1) [3], [4], [8].

From the theory of almost periodic functions (see, for instance, [14]) it follows that $\sigma(t)$ has the average $\sigma_0 \in \mathfrak{M}$:

$$\lim_{t \to \infty} \left\| \frac{1}{t} \int_\tau^{t+\tau} \sigma(s)\, ds - \sigma_0 \right\|_{\mathfrak{M}} = 0 ,$$

uniformly with respect to $\tau \in \mathbb{R}$. Thus, to equation (0.1) there naturally corresponds the autonomous averaged equation

$$\partial_t \bar{u} = N_{\sigma_0}(\bar{u}) . \tag{0.4}$$

We suppose that equation (0.1) has the form

$$\partial_t u = N_{\sigma(\omega t)}(u) , \tag{0.5}$$

where $\omega \gg 1$ (in other words, we suppose that the equation is written in the so-called standard form of Bogolyubov). The main result of the paper can now be formulated in this abstract setting as follows.

Suppose that the averaged equation has an attractor $\overline{\mathcal{A}}$ which is stable in the sense of Lyapunov. Then the solutions of (0.1) are absorbed by an infinitesimally thin (as $\omega \to \infty$)) neighbourhood of $\overline{\mathcal{A}}$. If, in addition, the non-autonomous equation (0.5) has an attractor $\mathcal{A}_{\Sigma(\omega)}$ which is bounded in E uniformly with respect to ω, then $\mathcal{A}_{\Sigma(\omega)}$ depends upper semicontinuously on ω as $\omega \to \infty$:

$$\operatorname{dist}_E (\mathcal{A}_{\Sigma(\omega)}, \overline{\mathcal{A}}) \to 0 , \qquad \omega \to \infty ,$$

where $\operatorname{dist}_E (X, Y) := \sup_{x \in X} \inf_{y \in Y} \|x - y\|_E$.

We call this result a global averaging theorem.

The outlined strategy is realized in the abstract setting for the equation (0.2) in Sect. 1 and Sect. 2. Examples of application of global averaging to evolution equations of mathematical physics are given in Sect. 3. They include a reaction-diffusion system, a two-dimensional Navier-Stokes system and a damped wave equation.

The present paper may be regarded as a continuation of [10] where equations of the type (0.2) with $F(u, t) = F_0(u)$ were considered.

Finally, we compare our results with those of [6], where equations of the type (0.5), (0.2) with unbounded A and bounded periodic functions F and f were considered. In particular, it was shown in [6] that the local attractors $\mathcal{A}(\omega)$ of the Poincaré map corresponding to (0.5), (0.2) converge to the attractor $\overline{\mathcal{A}}$ of the equation (0.4) as $\omega \to \infty$. In our work we construct global (not local) attractors. Moreover, we give applications to the equations of mathematical physics mentioned above.

1. - Abstract averaging theorem

We shall be dealing with an equation of the form

$$\partial_t u + Au = F(u, \omega t) + f(\omega t), \quad u(0) = u_0, \tag{1.1}$$

where A is a linear and F a non-linear operator, f is a right-hand side.

Let Banach spaces E, F, X, $\mathcal{E}$ satisfy

$$E \subset F; \qquad E, F, X \subset \mathcal{E},$$

each embedding being dense and continuous.

We further suppose that the linear operator A is densely defined in $\mathcal{E}$ and is such that the linear equation

$$\partial_t u + Au = 0, \quad u(0) = u_0,$$

generates the semigroup of linear bounded operators

$$e^{-At}: \mathcal{E} \to \mathcal{E}, \quad u(t) = e^{-At} u_0,$$

which for $t > 0$ can be extended to the linear bounded operators from F to E satisfying the following estimates

$$\|e^{-At}\|_{E \to E} \leq K e^{-at}, \tag{1.2}$$

$$\|e^{-At}\|_{F \to E} \leq K t^{-\alpha_1} e^{-at}, \quad 0 \leq \alpha_1 < 1, \tag{1.3}$$

$$\|A e^{-At}\|_{F \to E} \leq K t^{-\alpha_2} e^{-at}, \quad 0 \leq \alpha_2 < 2. \tag{1.4}$$

We also suppose that the following natural condition is satisfied

$$A e^{-At} = e^{-At} A, \tag{1.5}$$

in the sense of $\mathcal{L}(F, E)$.

Function F. The non-linear function F is a Lipschitz map from F to E in the following sense:

$$\|F(u_1, t) - F(u_2, t)\|_F \leq L(R) \|u_1 - u_2\|_E, \quad u_1, u_2 \in B_E(R), \tag{1.6}$$

where $L(R)$ is an increasing continuous function. For $u \in E$, $F(u, \cdot) \in L_1^{\mathrm{loc}}(\mathbb{R}, F)$.

Function f. The right-hand side $f \in L_1^{\mathrm{loc}}(\mathbb{R}, X)$ and the operators e^{-At}, $t > 0$ can be extended to the linear bounded operators from X to E satisfying the estimates

$$\|e^{-At}\|_{X\to E} \leqslant Kt^{-\beta_1}e^{-at}, \qquad 0 \leqslant \beta_1 < 1, \tag{1.7}$$

$$\|Ae^{-At}\|_{X\to E} \leqslant Kt^{-\beta_2}e^{-at}, \qquad 0 \leqslant \beta_2 < 2, \tag{1.8}$$

and the equation (1.5), this time in the sense of $\mathcal{L}(X, E)$.

Existence of average. The non-linear function $F(u, t)$ has the average $F_0(u)$, that is, for $u \in B_E(R)$

$$\left\| \frac{1}{t} \int_\tau^{\tau+t} F(u, s)\, ds - F_0(u) \right\|_F \leqslant \min(M_R, \mu_R(t)), \tag{1.9}$$

uniformly with respect to $u \in B_E(R)$ and $\tau \in \mathbb{R}$, where $M_R > 0$ is a constant and $\mu_R(t) \to 0$ monotonely as $t \to \infty$.

Note that by (1.6) and (1.9), $F_0(u)$: $E \to F$ is a bounded Lipschitz map with the same Lipschitz constant $L(R)$:

$$\|F_0(u_1) - F_0(u_2)\|_F \leqslant L(R)\|u_1 - u_2\|_E. \tag{1.10}$$

The right-hand side f has the average. There exists $f_0 \in X$ for which

$$\left\| \frac{1}{t} \int_\tau^{\tau+t} f(s)\, ds - f_0 \right\|_X \leqslant \min(M, \mu(t)), \tag{1.11}$$

where $M > 0$ and $\mu(t) \to 0$ monotonely as $t \to \infty$.

If $F(u, t)$ and $f(t)$ are periodic, then (1.9) and (1.11) can be made more precise:

$$\left\| \frac{1}{t} \int_\tau^{\tau+t} F(u, s)\, ds - F_0(u) \right\|_F \leqslant M_R \min(1, 1/t), \tag{1.9'}$$

$$\left\| \frac{1}{t} \int_\tau^{\tau+t} f(s)\, ds - f_0 \right\|_X \leqslant M \min(1, 1/t), \tag{1.11'}$$

with, in general, other M and M_R.

Along with equation (1.1) we consider the autonomous averaged equation

$$\partial_t \bar{u} + A\bar{u} = F_0(\bar{u}) + f_0, \qquad \bar{u}(0) = \bar{u}_0, \tag{1.12}$$

What do we mean by the solution of (1.1) and (1.12) will become clear in the analysis of the particular examples of partial differential equations in Sect. 3; for the moment, however, we suppose that $u, \bar{u} \in C([0, T]; E)$ and satisfy the following integral equations

$$(1.13) \qquad u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}(F(u(s), \omega s) + f(\omega s))\, ds\,,$$

$$(1.14) \qquad \bar{u}(t) = e^{-At}\bar{u}_0 + \int_0^t e^{-A(t-s)}(F_0(\bar{u}(s)) + f_0)\, ds\,.$$

We suppose that the autonomous equation (1.12) has a unique solution and thereby generates a semigroup of non-linear continuous operators acting in E:

$$(1.15) \qquad S_t\bar{u}_0 = \bar{u}(t)\,, \qquad S_t\colon E \to E\,.$$

We suppose that S_t has the following properties.

Dissipativity and uniform boundedness. The semigroup S_t has an absorbing ball $B_E(R_0)$. This means that for every ball $B_E(R)$

$$(1.16) \qquad S_t B_E(R) \subset B_E(R_0)\,, \qquad \text{for } t \geqslant t_0(R_0, R)\,.$$

The semigroup S_t is uniformly bounded for $t \geqslant 0$, that is, for every ball $B_E(R_1)$ there exists a ball $B_E(R_2)$, $R_2 = R_2(R_1)$ such that

$$S_t B_E(R_1) \subset B_E(R_2)\,, \qquad \text{for all } t \geqslant 0\,.$$

This inclusion is valid, in particular, for the absorbing ball $B_E(R_0)$. Hence we may assume that

$$(1.17) \qquad S_t B_E(R_0) \subset B_E(R - \varrho)\,, \qquad t \geqslant 0\,, \qquad \varrho > 0\,, \qquad R = R_2(R_0) + \varrho\,.$$

We fix R_0 and R.

Estimate of derivative. We suppose that for any $T > 0$ the time derivative of the solution $\bar{u}(t) = S_t u_0$ satisfies

$$(1.18) \qquad \|\partial_t \bar{u}(t)\|_E \leqslant t^{-1} D(T, \|u_0\|_E)\,, \qquad 0 < t \leqslant T\,,$$

where $D(\cdot, \cdot)$ is a continuous increasing function.

THEOREM 1.1: *Let all the assumptions listed above be true and let $T > 0$ be arbitrary but fixed. If $u(0) = \bar{u}(0) = u_0 \in B_E(R_0)$, that is, the initial points coincide and belong to the absorbing ball, then the solutions of the initial and averaged equation satisfy for $t \in [0, T]$ the following proximity estimate*

$$(1.19) \qquad \|u(t) - \bar{u}(t)\|_E \leqslant \eta_{T, R_0}(\omega) \to 0 \quad \text{as} \quad \omega \to \infty\,.$$

REMARK 1.1: The assumtion that the initial points be taken from the absorbing ball involves no loss of generality. We can increase R_0 if necessary.

THEOREM 1.2: *If equation* (1.1) *has the form*

$$\partial_t u + Au = F_0(u) + f(\omega t), \tag{1.20}$$

then Theorem 1.1 *holds without the assumptions* (1.4) *and* (1.18).

If, in addition, (1.11′) *holds, then the proximity estimate has the following explicit form*:

$$\|\bar{u}(t) - u(t)\|_E \leqslant \frac{1}{\omega}\,(\omega^{\beta_1} + \omega^{\beta_2 - 1} + \delta_{\beta_2,\,1} \ln \omega + 1)\; C(R,\, T), \tag{1.21}$$

where δ_{ij} is the Kronecker delta.

The proof of Theorems 1.1 and 1.2 will be given in the Appendix and we now turn to the global averaging.

2. - GLOBAL AVERAGING

The main result of this section is the theorem on the proximity of the attractor of the initial equation to that of the averaged equation. We recall the definition of the attractor of an autonomous equation [1], [5], [12], [17].

Let a semigroup of non-linear continuous operators act in a Banach space E, $S_t\colon E \to E$, $t \geqslant 0$. (For instance, S_t may be the solution operator (1.15).) A compact set $\mathcal{A} \Subset E$ is called a global attractor of the semigroup S_t if $\mathcal{A}$ is strictly invariant $S_t \mathcal{A} = \mathcal{A}$, $t \geqslant 0$, and globally attracting, that is, for every bounded set $B \in \mathcal{B}(E)$, $\mathrm{dist}_E(S_t B,\, \mathcal{A}) \to 0$ as $t \to \infty$.

The attractor $\mathcal{A}$ is stable in the Lyapunov sense if for every ε-neighbourhood $\mathcal{O}_\varepsilon(\mathcal{A}) = \{u \in E,\ \mathrm{dist}_E(u,\, \mathcal{A}) < \varepsilon\}$ there exists a δ-neighbourhood $\mathcal{O}_\delta(\mathcal{A})$ such that

$$S_t \mathcal{O}_\delta(\mathcal{A}) \subset \mathcal{O}_\varepsilon(\mathcal{A}), \qquad \forall t \geqslant 0. \tag{2.1}$$

The assertion of the following lemma is the same as that of [1], Proposition II.1.3 but the hypothesis is slightly different.

LEMMA 2.1: *If S_t has an attractor $\mathcal{A}$ and S_t is continuous in E in the following sense*:

$$\|S_t u_1 - S_t u_2\|_E \leqslant c(T,\, R)\|u_1 - u_2\|_E \quad \text{for } t \in [0,\, T], \qquad u_1,\, u_2 \in B_E(R), \tag{2.2}$$

then $\mathcal{A}$ is stable.

PROOF: Let a neighbourhood $\mathcal{O}_\varepsilon(\mathcal{A})$, $\varepsilon \leqslant 1$, be given. By the attraction property there is a $T(\varepsilon)$ such that $S_t \mathcal{O}_1(\mathcal{A}) \subset \mathcal{O}_\varepsilon(\mathcal{A})$ for all $t \geqslant T(\varepsilon)$. We choose δ, $0 < \delta \leqslant \varepsilon$ so small that $c(T(\varepsilon),\, R)\,\delta < \varepsilon$, where R is defined by $\mathcal{A} \subset B_E(R-1)$. Then $\mathcal{O}_\delta(\mathcal{A})$ is the desired neighbourhood.

We now consider a non-autonomous equation (1.1) written in the form (0.1) (the value of ω does not matter for the moment, so we put $\omega = 1$)

(2.3) $$\partial_t u = N_{\sigma(t)}(u), \qquad u|_{t=\tau} = u_\tau, \qquad t \geqslant \tau \in \mathbb{R}.$$

Here $\sigma(t)$ is the collection of all time-dependent terms of the equation (right-hand sides, non-linear interaction coefficients etc). For instance, $\sigma(t) = \{F(u, t), f(t)\}$ for equation (1.1). Following [4] we call $\sigma(t)$ the time symbol of the equation (2.3).

Suppose that in a Banach space E the equation (2.3) has a unique solution $u(t)$, $u \in C([\tau, t]; E)$ and thereby generates the evolution operator $U_\sigma(t, \tau)$ which we shall call the process

(2.4) $$U_\sigma(t, \tau)\colon E \to E, \qquad t \geqslant \tau, \qquad \tau \in \mathbb{R}, \qquad U_\sigma(t, \tau) u_\tau = u(t).$$

The process $U_\sigma(t, \tau)$ has two characteristic properties which follow from the uniqueness of the solution $u(t)$:

(2.5) $$U_\sigma(t, s) \circ U_\sigma(s, \tau) = U_\sigma(t, \tau), \qquad t \geqslant s \geqslant \tau, \qquad U_\sigma(\tau, \tau) = E,$$

and

(2.6) $$U_{T(h)\sigma}(t, \tau) = U_\sigma(t + h, \tau + h),$$

where $T(h)$ is the translation operator

$$T(h)\sigma(\cdot) = \sigma(\cdot + h).$$

The symbol $\sigma(t)$ is defined for $t \in \mathbb{R}$ and takes values in a Banach space $\mathcal{M}$,

$$\sigma(t) \in \mathcal{M}, \qquad t \in \mathbb{R}.$$

For instance, for equation (1.1)

$$\sigma(t) \in \mathcal{M} = \mathcal{M}_1 \times X,$$

where X is the same as in (1.7), (1.8) and the Banach space $\mathcal{M}_1$ characterizes the rate of growth of the non-linear function $F(u, t)$ (see Sect. 3). (If we are dealing with equation (1.20), then $\mathcal{M} = X$.)

Suppose that $\sigma(t)$ is almost periodic (a.p.) with values in $\mathcal{M}$. By Bochner's criterion [14] the set of all translations $T(h)\sigma(\cdot) = \sigma(\cdot + h)$, $h \in \mathbb{R}$, is precompact in $C_b(\mathbb{R}; \mathcal{M})$. The closure of this set in $C_b(\mathbb{R}; \mathcal{M})$ is called the hull of σ and is denoted by

$$\mathcal{H}(\sigma) = [\sigma(\cdot + h), h \in \mathbb{R}]_{C_b(\mathbb{R};\, \mathcal{M})}.$$

Moreover, if $\delta \in \mathcal{H}(\sigma)$, then

(2.7) $$\mathcal{H}(\delta) = \mathcal{H}(\sigma) =: \Sigma.$$

Along with an individual process (2.4) we shall consider the family of processes generated by (2.3) where σ is replaced by an arbitrary element δ, $\delta \in \Sigma = \mathcal{H}(\sigma)$.

DEFINITION 2.1 (see [4],[8]): A compact set $\mathcal{A}_\Sigma \Subset E$ is called the uniform (with respect to $\delta \in \Sigma$) attractor of the family of processes $U_\delta(t, \tau)$: $E \to E$, $\delta \in \Sigma$, if

(1) the set $\mathcal{A}_\Sigma$ is uniformly attracting, that is,

$$\lim_{t \to \infty} \sup_{\delta \in \Sigma} \operatorname{dist}_E (U_\delta(t, \tau) B, \mathcal{A}_\Sigma) = 0 , \qquad \forall B \in \mathcal{B}(E) ;$$

(2) if $\mathcal{A}_1$ is another closed uniformly attracting set, then $\mathcal{A}_\Sigma \subset \mathcal{A}_1$.

THEOREM 2.1: *Let the symbol σ be a.p. in $\mathcal{M}$ with hull $\Sigma = \mathcal{H}(\sigma)$ and let the family of processes $U_\delta(t, \tau)$, $\delta \in \Sigma$ satisfy:*

(1) *there exists a compact uniformly attracting set in E;*

(2) *the operators $U_\delta(t, \tau)$: $E \times \Sigma \to E$ are continuous with respect to $(u, \delta) \in E \times \Sigma$.*

Then the semigroup S_t acting in the extended phase space $E \times \Sigma \subset E \times C_b(\mathbb{R}; \mathcal{M})$ according to the rule

$$S_t(u, \delta) = (U_\delta(t, 0)\, u, T(t)\, \delta) ,$$

has an attractor $\mathcal{A}_{E \times \Sigma}$ in this phase space. If we denote $\Pi_1(u, \delta) = u$, then the uniform attractor of the family of processes in the sense of Definition 2.1 is the projection

$$\mathcal{A}_\Sigma = \Pi_1 \mathcal{A}_{E \times \Sigma} .$$

For the proof see [4], Theorem 3.2.

We now return to the problems connected with averaging in this more abstract setting. We introduce a large parameter ω in (2.3):

$$\partial_t u = N_{\sigma(\omega t)}(u) , \qquad u|_{t=\tau} = u_\tau , \qquad \tau \in \mathbb{R} . \tag{2.8}$$

In fact, (2.8) is just a short way of writing (1.1). Since we have supposed that σ is a.p. with values in $\mathcal{M}$, it follows [14] that σ has the average $\sigma_0 \in \mathcal{M}$:

$$\left\| \frac{1}{t} \int_\tau^{t+\tau} \sigma(s)\, ds - \sigma_0 \right\|_{\mathcal{M}} \leq \min (M, \mu(t)) , \qquad M = \text{const}, \qquad \mu(t) \to 0 , \qquad t \to \infty ,$$

uniformly with respect to $\tau \in \mathbb{R}$. We further suppose that (1.9) and (1.11) follow from the existence of the average σ_0. Of course, this will be verified in each particular example of a PDE (for (1.11) this is obvious).

Associated with (2.8) in a natural way is the averaged equation (see (1.12))

$$\partial_t \bar{u} = N_{\sigma_0}(\bar{u}) , \qquad \bar{u}(0) = u_0 . \tag{2.9}$$

We suppose that (2.9) generates in E a semigroup of solution operators

$$S_t \colon E \to E , \; S_t u_0 = \bar{u}(t) .$$

THEOREM 2.2: *Suppose that S_t is uniformly bounded in E and has an absorbing ball $B_E(R_0)$. Suppose further that S_t has an attractor $\overline{\mathcal{A}}$ in E which is stable in the sense of Lyapunov. Finally, suppose that the theorem on averaging on a finite time interval is valid. In other words, if $u(\tau) = \overline{u}(\tau) \in B_E(R_0)$, then for $t \in [\tau, \tau + T]$*

$$\|U_{\delta(\omega\cdot)}(t, \tau)\, u(\tau) - S_{t-\tau}\overline{u}(\tau)\|_E \leq \eta_{T, R_0}(\omega) \to 0\,, \quad \omega \to \infty\,,$$

uniformly with respect to $\delta \in \Sigma$.

Then for arbitrary large R_1 and arbitrary small $\mu > 0$ there exists $\omega_0(R_1, \mu)$ such that for $\omega \geq \omega_0$ the μ-neighbourhood of $\overline{\mathcal{A}}$ uniformly (with respect to $\delta \in \Sigma$) absorbs the ball $B_E(R_1)$:

$$U_{\delta(\omega\cdot)}(t, \tau)\, B_E(R_1) \subset \mathcal{O}_\mu(\overline{\mathcal{A}})\,, \quad \textit{for } t - \tau \geq T(R_1, \mu)\,. \tag{2.10}$$

If, in addition, the family of processes $U_{\delta(\omega\cdot)}(t, \tau)$, $\delta \in \Sigma$ has an attractor $\mathcal{A}_{\Sigma(\omega)}$ which is bounded in E uniformly in ω, then the attractor $\mathcal{A}_{\Sigma(\omega)}$ depends upper semicontinuously on ω as $\omega \to \infty$:

$$\operatorname{dist}_E(\mathcal{A}_{\Sigma(\omega)}, \overline{\mathcal{A}}) \to 0 \quad \textit{as } \omega \to \infty\,.$$

PROOF: Since $\overline{\mathcal{A}}$ is stable, we can choose ε, $0 < \varepsilon < \mu/2$ such that

$$S_t \mathcal{O}_\varepsilon(\overline{\mathcal{A}}) \subset \mathcal{O}_{\mu/2}(\overline{\mathcal{A}})\,, \quad t \geq 0\,. \tag{2.11}$$

We consider the absorbing ball $B_E(R_0)$ for the semigroup S_t. Taking if necessary a larger R_0, we may assume that $R_0 = R_1$.

Let $T = T(R_0, \varepsilon)$ be so large that

$$S_t B_E(R_0) \subset \mathcal{O}_{\varepsilon/2}(\overline{\mathcal{A}})\,, \quad t > T\,. \tag{2.12}$$

We now fix this $T > 0$.

We consider a point $u_0 \in B_E(R_0)$. By the averaging theorem there exists an $\omega_0 = \omega_0(R_0, T, \varepsilon)$, such that for $\omega > \omega_0$ the inequality $\eta_{T, R_0}(\omega) < \varepsilon/2$ is valid. Let two trajectories of (2.8) and (2.9) start at u_0: $u(t) = U_{\delta(\omega\cdot)}(t, 0)\, u_0$, $\overline{u}(t) = S_t u_0$. These trajectories will diverge on the interval $t \in [0, T]$ by a distance less than $\varepsilon/2$ and the endpoint $\overline{u}(T) \in \mathcal{O}_{\varepsilon/2}(\overline{\mathcal{A}})$, hence, $u(T) \in \mathcal{O}_\varepsilon(\overline{\mathcal{A}})$.

From this moment $u(t)$ will never leave $\mathcal{O}_\mu(\overline{\mathcal{A}})$. To see this, we take $u_1 = u(T) \in \mathcal{O}_\varepsilon(\overline{\mathcal{A}})$ as the initial point, consider the trajectory $S_t u_1$, $t \in [0, T]$, starting from it, and continue the trajectory $u(t)$ to the interval $[T, 2T]$:

$$u(t + T) = U_{\delta(\omega\cdot)}(t + T, 0)\, u_0.$$

By (2.12), $S_T u_1 \in \mathcal{O}_{\varepsilon/2}(\overline{\mathcal{A}})$, and for $t \in [0, T]$, $S_t u_1 \in \mathcal{O}_{\mu/2}(\overline{\mathcal{A}})$, by (2.11).

Again, by the averaging theorem we see that $\|U_{\delta(\omega\cdot)}(t + T, 0)\, u_0 - S_t u_1\|_E \leq \varepsilon/2$,

$t \in [0, T]$ and therefore the end-point $U_{\delta(\omega\cdot)}(2T, 0)\, u_0 \in \mathcal{O}_\varepsilon(\overline{\mathcal{A}})$, while on the interval $t \in [0, T]$

$$U_{\delta(\omega\cdot)}(T+t, 0)\, u_0 \in \mathcal{O}_{\mu/2+\varepsilon/2}(\overline{\mathcal{A}}) \subset \mathcal{O}_\mu(\overline{\mathcal{A}})\,.$$

Repeating this procedure we see that $u(t) \in \mathcal{O}_\mu(\overline{\mathcal{A}})$ for $t \geq T^* \in [0, T]$.

However, neither T nor ω_0 depends on the choice of the particular initial point $u_0 \in B_E(R_0)$. It is also clear that the above construction is applicable to the trajectory $U_{\delta(\omega\cdot)}(t, \tau)\, u_0$, $t \geq \tau \in \mathbb{R}$, as well. Finally, all our estimates are uniform with respect to $\delta \in \Sigma$.

Thus, we have proved that for $\omega > \omega_0$ and $t - \tau > T^* \in [0, T]$

$$\bigcup_{\delta \in \Sigma} U_{\delta(\omega\cdot)}(t, \tau)\, B_E(R_0) \subset \mathcal{O}_\mu(\overline{\mathcal{A}})\,,$$

which gives (2.10).

The last statement of the theorem now follows from Definition 2.1 if we choose the absorbing ball $B_E(R_0)$ so large that it contains the attractors $\mathcal{A}_{\Sigma(\omega)}$ for all $\omega > 0$. The proof is complete.

3. - Examples

1. *Reaction-diffusion system.* We consider a non-autonomous reaction-diffusion system:

$$(3.1) \qquad \partial_t u = d\Delta u - F(u, \omega t) + f(\omega t)\,, \qquad u|_{\partial\Omega} = 0\,,$$

where $\Omega \Subset \mathbb{R}^n$, $d \geq d_0 I$ is a positive diagonal $N \times N$-matrix and u, F, f are N-dimensional vector-functions. The non-linear function F and its derivatives with respect to u_j are of class $C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^N)$ and satisfy the following growth conditions:

$$(3.2) \qquad -C_1 \leq (F(u, t), u)\,, \qquad \gamma > 0\,,$$

$$(3.3) \qquad (F'_u v, v) \geq -C_2(v, v)\,, \qquad \forall v \in \mathbb{R}^N,$$

$$(3.4) \qquad |F(u, t)| \leq C_3(|u|^{p-1} + 1)\,, \qquad |F'_u(u, t)| \leq C_4(|u|^{p-2} + 1)\,,$$

where

$$(3.5) \qquad 2 \leq p \leq \frac{2n-2}{n-2} \ \text{ if } n \geq 3\,; \qquad 2 \leq p < \infty \ \text{ otherwise}\,.$$

The analysis of the equation

$$(3.1') \qquad \partial_t u = d\Delta u - F_0(u) + f(\omega t)\,, \qquad u|_{\partial\Omega} = 0\,,$$

where F_0 satisfies (3.2)–(3.4) is, of course, included in that of (3.1). However, in this case we may impose less restrictive growth conditions:

$$(3.5') \qquad 2 \leqslant p < \frac{2n}{n-2} \text{ if } n \geqslant 3 \, ; \qquad 2 \leqslant p < \infty \text{ otherwise} \, .$$

(Condition (3.5) is used only in Lemma 3.1 below.)

We suppose that f is an a.p. function with values in $L_2(\Omega)^N$. Next we define the Banach space $\mathfrak{M}_1$ of functions $\Psi(u)$, Ψ: $\mathbb{R}^N \to \mathbb{R}^N$ with norm (see [4])

$$(3.6) \qquad \|\Psi\|_{\mathfrak{M}_1} = \sup_{u \in \mathbb{R}^N} \left(\frac{|\Psi(u)|}{1 + |u|^{p-1}} + \frac{|\Psi'_u(u)|}{1 + |u|^{p-2}} \right),$$

and suppose that $F(\cdot, t)$ is an a.p. function with values in $\mathfrak{M}_1$.

We shall now define the spaces X, E, F from Sect. 1. We set $A = -d\Delta$ supplemented with Dirichlet boundary conditions. Then A is an unbounded self-adjoint positive operator in $(L_2(\Omega))^N$ with compact resolvent. We define the Hilbert spaces $D(A^\alpha)$, $\alpha \in \mathbb{R}$, as the domains of the powers of A in the standard way. Then $D(A^0) = (L_2)^N$, $D(A^{1/2}) = (H_0^1)^N$ and by interpolation

$$D(A^{\theta/2}) = [(L_2)^N, (H_0^1)^N]_\theta = (H_0^\theta)^N \, , \qquad \theta \neq \frac{1}{2} \, , \qquad \theta \in [0, 1] \, ,$$

which gives, by duality, that for $0 \leqslant \varepsilon < 1/2$, $D(A^{(-1+\varepsilon)/2}) = (H^{-1+\varepsilon}(\Omega))^N$. We now set in the notation of Sect. 1 $A = -d\Delta$ and

$$(3.7) \qquad \begin{cases} E = (H_0^1(\Omega))^N = D(A^{1/2}) \, , \\ X = (L_2(\Omega))^N \, , \\ F = (H^{-1+\varepsilon}(\Omega))^N = D(A^{(-1+\varepsilon)/2}) \, , \end{cases}$$

where $0 < \varepsilon = \varepsilon(p)$ is small and will be defined later.

Setting $\mathcal{E} = F$ we find (see, for instance, [9]) that the (analytic) semigroup e^{-At} is well defined and satisfies (1.5) and (1.2)-(1.4), (1.7), (1.8) with

$$\alpha_1 = 1 - \varepsilon/2 \, , \qquad \alpha_2 = 2 - \varepsilon/2 \, , \qquad \beta_1 = 1/2 \, , \qquad \beta_2 = 3/2 \, .$$

Let us verify other conditions from Sect. 1. Since by our assumption f is a.p. in X, an average $f_0 \in X$ exists and (1.11) holds. Moreover, (1.11) holds if we replace f by an arbitrary element $g \in \mathcal{H}(f)$. Analogously, there exists the average $F_0 \in \mathfrak{M}_1$ such that

$$(3.8) \quad \left\| \frac{1}{t} \int_\tau^{t+\tau} F(\cdot, s) \, ds - F_0(\cdot) \right\|_{\mathfrak{M}_1} \leqslant \min(M, \mu(t)) \, , \qquad \mu(t) \to 0 \, , \qquad t \to \infty \, ,$$

where F can be replaced by an arbitrary $G \in \mathcal{H}(F)$.

This implies (1.9). In fact, by (3.6), for a smooth $u = u(x) \in (C_0^\infty(\Omega))^N$

$$\left| \frac{1}{t} \int_\tau^{t+\tau} F(u(x), s)\, ds - F_0(u(x)) \right| \leq \min(M, \mu(t))(1 + |u(x)|^{p-1}).$$

We now note that by the Sobolev embedding theorem $H_0^{1-\varepsilon} \subset L_q$, $q = 2n/(n-2) - \delta_1(\varepsilon)$. (Here and in what follows $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$, ..., are certain uniquely defined functions of ε such that $\lim_{\varepsilon \to 0} \delta_k(\varepsilon) = 0$.) By duality, for $q' = 2n/(n+2) + \delta_2(\varepsilon)$, $L_{q'} \subset H^{-1+\varepsilon} = F$. Hence

$$\left\| \frac{1}{t} \int_\tau^{t+\tau} F(u, s)\, ds - F_0(u) \right\|_{L_{q'}} \leq \min(M, \mu(t))\, c(|\Omega|)(1 + \|u\|_{L_{q'(p-1)}}^{(p-1)}),$$

By (3.5′), $q'(p-1) < 2n/(n-2)$ for all sufficiently small ε. Taking the closure in $(H_0^1)^N$ and using the Sobolev embedding $E \subset L_{q'(p-1)}$ we obtain (1.9).

The Lipschitz condition (1.6) is verified in a similar way. For smooth u_1, u_2 by (3.6), the mean value theorem and Hölder's inequality we have

$$\|F(u_1, s) - F(u_2, s)\|_{L_{q'}} \leq \|F\|_{C_b(\mathbb{R};\, \mathfrak{M}_1)} \|(1 + |u_1|^{p-2} + |u_2|^{p-2})\, |u_1 - u_2|\,\|_{L_{q'}} \leq$$

$$\leq \|F\|_{C_b(\mathbb{R};\, \mathfrak{M}_1)} \|(1 + |u_1|^{p-2} + |u_2|^{p-2})\|_{L_{q'r'}} \|u_1 - u_2\|_{L_{q'r}},$$

which gives (1.6) if we choose $q'r = 2n/(n-2)$. Indeed, then $q'r' = n/2 + \delta_3(\varepsilon)$, giving that $(p-2)\, q'r' < 2n/(n-2)$, when ε is small enough.

Thus, to (3.1) there corresponds the averaged autonomous equation

$$\partial_t \bar{u} = d\Delta \bar{u} - F_0(\bar{u}) + f_0, \quad \bar{u}(0) = u_0, \tag{3.9}$$

where $f_0 \in X$, $F_0 \in \mathfrak{M}_1$. Clearly, F_0 satisfies conditions (3.2)-(3.5) with the same constants C_i and γ.

We now turn to the analysis of the averaged equation (3.9). We invoke the following results concerning (3.9) (see [1], Theorems I.5.2, I.5.4 and Propositions I.2.1-I.2.3).

Equation (3.9) generates a semigroup of solution operators acting in E, $S_t u_0 = \bar{u}(t)$, $t \geq 0$, which is uniformly bounded in E and has an absorbing ball there; in other words, (1.16), (1.17) are satisfied. The semigroup S_t has also a compact absorbing set in E.

The semigroup S_t is continuous in E and estimate (2.2) holds. This will be shown in a more general context of non-autonomous equations (see (3.18), (3.19)). Now, by the well-known theorems on the existence of attractors for autonomous equations [1], [5], [12], [17] we obtain that the semigroup $S_t\colon E \to E$ has a global attractor $\bar{\mathfrak{A}} \Subset E$ which is stable in the Lyapunov sense.

LEMMA 3.1: *If condition* (3.5) *holds, then for equation* (3.9) *the estimate of the derivative*(1.18) *is valid.*

PROOF: We differentiate (3.9) with respect to t and set $\partial_t \bar{u} = \bar{u}'$:

$$\partial_t \bar{u}' = d\Delta \bar{u}' - F'_{0\bar{u}}(\bar{u})\, \bar{u}' \,. \tag{3.10}$$

Multiplying by $-\Delta \bar{u}'$, using Hölder's inequality and the Sobolev embeddings we obtain

$$\frac{1}{2}\, \partial_t \|\bar{u}'\|_E^2 + d_0 \|\Delta \bar{u}'\|^2 \leqslant \left| \int F'_{0\bar{u}}(\bar{u})\, \bar{u}' \Delta \bar{u}'\, dx \right| \leqslant \|\Delta \bar{u}'\|\, \|\bar{u}'\|_{L_{2n/(n-2)}} \|F'_{0\bar{u}}(\bar{u})\|_{L_n} \leqslant$$

$$\leqslant c\|\Delta \bar{u}'\|\, \|\bar{u}'\|_E \|1 + |\bar{u}|^{p-2}\|_{L_n} \leqslant c\|\Delta \bar{u}'\|\, \|\bar{u}'\|_E (1 + \|\bar{u}\|_E^{p-2}) \leqslant c(R)\, \|\Delta \bar{u}'\|\, \|\bar{u}'\|_E \,,$$

taking into account that $(p-2)n \leqslant 2n/(n-2)$ (see (3.5)). By Young's inequality

$$\partial_t \|\bar{u}'\|_E^2 \leqslant c_1(R) \|\bar{u}'\|_E^2 \,.$$

Multiplying by $t-\tau$, $0 < \tau \leqslant t \leqslant T$, we find

$$\partial_t ((t-\tau)\, \|\bar{u}'\|_E^2) \leqslant c(R,\, T)\, \|\bar{u}'\|_E^2 \,,$$

and integration from τ to t finally gives

$$(t-\tau)\, \|\bar{u}'(t)\|_E^2 \leqslant c(R,\, T) \int_\tau^T \|\bar{u}'\|_E^2\, ds \,. \tag{3.11}$$

To estimate the the integral we multiply (3.9) by $-\Delta u$. Since $F_0(\bar{u}) \in L_2$ for $\bar{u} \in E$, we obtain after simple transformations

$$\partial_t \|\bar{u}\|_E^2 + d_0 \|\Delta u\|^2 \leqslant c(d_0,\, R,\, \|f_0\|) \,,$$

which gives after integration

$$\int_0^T \|\Delta \bar{u}(s)\|_E^2\, ds \leqslant C_1(T,\, R) \,.$$

We now see that all the three terms on the right-hand side of (3.9) belong to $L_2(0,\, T;\, L_2(\Omega))$, therefore $\bar{u}'$ belongs to the same space, in other words,

$$\int_0^T \|\bar{u}'(s)\|^2\, ds \leqslant C_2(T,\, R) \,. \tag{3.12}$$

Let us multiply (3.10) by $\bar{u}'$. This gives

$$\frac{1}{2}\, \partial_t \|\bar{u}'\|^2 + d_0 \|\bar{u}'\|_E^2 \leqslant \|F'_{0\bar{u}}(\bar{u})\|_{L_n} \|\bar{u}'\|\, \|\bar{u}'\|_{L_{2n/(n-2)}} \leqslant c(d_0,\, R)\, \|\bar{u}'\|^2 + \frac{d_0}{2}\, \|\bar{u}'\|_E^2 \,.$$

Thus,

$$\partial_t \|\bar{u}'\|^2 + d_0 \|\bar{u}'\|_E^2 \leqslant c(R)\, \|\bar{u}'\|^2 \,. \tag{3.13}$$

Integrating and using (3.12) we obtain

$$(3.14) \qquad \int_\tau^T \|\bar{u}'(s)\|_E^2 \, ds \leq C_3(T, R) + \|\bar{u}'(\tau)\|^2 .$$

Inequality $\partial_t \|\bar{u}'\|^2 \leq c(R) \|\bar{u}'\|^2$ obviously follows from (3.13). Multiplying by t gives

$$\partial_t (t \|\bar{u}'\|^2) \leq C_4(T, R) \|\bar{u}'\|^2 ,$$

hence, integrating and using (3.12), we obtain

$$(3.15) \qquad t \|\bar{u}'(t)\|^2 \leq C_5(T, R), \qquad \text{or } \|\bar{u}'(\tau)\|^2 \leq \frac{1}{\tau} C_5(T, R).$$

Combining (3.11), (3.14), and (3.15) we finally obtain

$$\|\bar{u}'(t)\|_E^2 \leq \frac{C_6(T, R)}{t - \tau} (1 + 1/\tau) \leq \frac{C_7(T, R)}{\tau(t - \tau)} .$$

Setting $\tau = t/2$ we complete the proof.

Thus, all the hypotheses of Theorem 1.1 are verified and therefore this theorem is applicable to the reaction-diffusion system (3.1) (resp. (3.1'))), (3.2)-(3.4), (3.5) (resp. (3.5')). As far as the global averaging is concerned, we have to verify that in E there exists a uniform attractor of the non-autonomous equation which is bounded in E as $\omega \to \infty$.

In E there is a ball $B_E(R_0)$, $R_0 = R_0(\|F\|_{C_b(\mathbb{R}, \mathcal{M}_1)}, \|f\|_{C_b(\mathbb{R}, X)})$ which is uniformly absorbing for the family of processes generated by (3.1). This is proved in exactly the same way as for the autonomous case (see [1], Theorem I.5.4) by replacing in the estimates there the norms of the type $\|f_0\|_X$ by $\|f\|_{C_b(\mathbb{R}, X)}$ and similarly for F. This clearly shows that R_0 does not depend on ω. The same sort of argument shows that the family of processes is uniformly bounded in E:

$$(3.16) \qquad \bigcup_{\delta \in \mathcal{H}(F) \times \mathcal{H}(f)} \bigcup_{\tau \in \mathbb{R}} \bigcup_{t \geq \tau} U_\delta(t, \tau) B_E(R_0) \subset B_E(R), \qquad \text{for } R = R(R_0).$$

Then, uniformly with respect to $\tau \in \mathbb{R}$ and $\delta = \{G, g\} \in \mathcal{H}(F) \times \mathcal{H}(f)$

$$(3.17) \qquad \|U_\delta(t, \tau) B_E(R_0)\|_{D(A^{1/2+\mu})} \leq R_\mu(R_0, \delta', \|F\|_{C_b(\mathbb{R}; \mathcal{M}_1)}, \|f\|_{C_b(\mathbb{R}; X)}),$$

where $\varepsilon/2 > \mu > 0$, $\delta' > 0$, and $\tau + \delta' \leq t \leq \tau + 1$.

In fact, writing $u(t)$ as

$$u(t) = e^{-A(t-\tau)} u(\tau) + \int_\tau^t e^{-A(t-s)} (-G(u(s), s) + g(s)) \, ds ,$$

using the estimate $\|e^{-At}\|_{D(A^\alpha) \to D(A^\beta)} \leq t^{-(\beta - \alpha)} K e^{-at}$, $\beta \geq \alpha$, and the uniform (with respect to $s \in \mathbb{R}$ and $\{G, g\} \in \mathcal{H}(F) \times \mathcal{H}(f))$ boundedness of $\widetilde{G}(s) := G(u(s), s) + f(s)$ in $F = D(A^{(-1+\varepsilon)/2})$, we obtain

$$\|u(t)\|_{D(A^{1/2+\mu})} \leq K(t-\tau)^{-\mu} \|u(\tau)\|_E + K\|\widetilde{G}\|_{L_\infty(\mathbb{R};\, F)} \int_\tau^t (t-s)^{-(1+\mu-\varepsilon/2)}\, ds\,,$$

which implies (3.17). By compactness of the embedding $D(A^{1/2+\mu}) \subset D(A^{1/2})$, we see that the set

$$\bigcup_{\delta \in \mathcal{H}(F) \times \mathcal{H}(f)} \ \bigcup_{\tau \in \mathbb{R}} U_\delta(\tau+1, \tau)\, B_E(R_0)$$

meets all the requirements from point (1) of Theorem 2.1.

To prove the continuity we write for $\sigma_1 = \{F_1, f_1\}$, $\sigma_2 = \{F_2, f_2\}$

$$z(t) = u_1(t) - u_2(t) = U_{\sigma_1}(t, \tau)\, u_1(\tau) - U_{\sigma_2}(t, \tau)\, u_2(\tau)\,. \tag{3.18}$$

Then $z(t)$ satisfies the equation

$$z(t) = e^{-A(t-\tau)} z(\tau) -$$

$$-\int_\tau^t e^{-A(t-s)} (F_1(u_1(s), s) - F_2(u_1(s), s) + F_2(u_1(s), s) - F_2(u_2(s), s) - f_1(s) + f_2(s))\, ds\,,$$

and therefore

$$\|z(t)\|_E \leq KL(R) \int_\tau^t (t-s)^{\varepsilon/2-1} \|z(s)\|_E\, ds +$$

$$+ C(t-\tau, R)\,(\|z(\tau)\|_E + \|F_1 - F_2\|_{C_b(\mathbb{R};\, \mathcal{M}_1)} + \|f_1 - f_2\|_{C_b(\mathbb{R};\, X)})\,.$$

The desired continuity now follows from Lemma A.1 (see Appendix):

$$\|z(t)\|_E \leq C_1(t-\tau, R)\,(\|z(\tau)\|_E + \|F_1 - F_2\|_{C_b(\mathbb{R};\, \mathcal{M}_1)} + \|f_1 - f_2\|_{C_b(\mathbb{R};\, X)})\,. \tag{3.19}$$

Thus, we have verified all the hypotheses of Theorem 2.2.

THEOREM 3.1 (Global averaging of the reaction-diffusion system): *Suppose that in equation* (3.1) (*resp.*, (3.1′)), (3.2)-(3.4), (3.5)(*resp.*, (3.5′)) *the time symbol* $\sigma(t) = \{F(\cdot, t), f(t)\}$ *is a.p. in* $\mathcal{M} = \mathcal{M}_1 \times (L_2(\Omega))^N$, *where* $\mathcal{M}_1$ *is defined by* (3.6) (*resp.*, $\sigma(t) = f(t)$ *is a.p. in* $(L_2(\Omega))^N$). *Then there exists a uniform attractor* $\mathcal{A}_{\Sigma(\omega)} \Subset (H_0^1(\Omega))^N$ *in the sense of Definition* 2.1 *which depends upper semicontinuously on* ω as $\omega \to \infty$:

$$\mathrm{dist}_{(H_0^1)^N}(\mathcal{A}_{\Sigma(\omega)}, \overline{\mathcal{A}}) \to 0\,, \qquad \omega \to \infty\,,$$

where $\overline{\mathcal{A}}$ *is the global attractor of the averaged autonomous equation* (3.9).

2. *Navier-Stokes equations.* We consider the two-dimensional Navier-Stokes system

$$\begin{cases} \partial_t u + \sum_{i=1}^{2} u_i \partial_i u = \nu \Delta u - \nabla p + \varphi(\omega t), \\ \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \end{cases} \tag{3.20}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary.

The functional setting of the problem is well known [13], [16]. We denote by H and V the closures of the linear space $\{u \in C_0^\infty(\Omega)^2, \operatorname{div} u = 0\}$ in $L_2(\Omega)^2$ and $H_0^1(\Omega)^2$, respectively. Denote by P the corresponding orthogonal projection $P\colon L_2(\Omega)^2 \to H$. We further set

$$A = -P\Delta, \quad B(u, v) = P\left(\sum_{i=1}^{2} u_i \partial_i v\right).$$

The Stokes operator A is self-adjoint positive with domain $D(A)$ dense in H. The inverse operator is compact. We define the Hilbert spaces $D(A^\alpha)$, $\alpha \in [0, 1]$ as the domains of the powers of A in the standard way. Furthermore, $V = D(A^{1/2})$, and $\|u\|_{D(A^{1/2})} = \|\nabla u\| = \|\operatorname{rot} u\|$.

Applying P we write (3.20) as the evolution equation of the form (1.1)

$$\partial_t u + B(u, u) = -\nu A u + f(\omega t), \quad f = P\varphi. \tag{3.21}$$

We suppose that f is almost periodic with values in H. Then (1.11) (with $X = H$) obviously holds and to (3.21) there corresponds the averaged equation

$$\partial_t \bar{u} + B(\bar{u}, \bar{u}) = -\nu A \bar{u} + f_0. \tag{3.22}$$

We now set in the notation of Sect. 1

$$E = D(A^{1/2}), \quad X = H, \quad F = D(A^{-\delta})$$

and see that (1.2), (1.3), (1.7), (1.8) are valid with $\alpha_1 = 1/2 + \delta$, $\beta_1 = 1/2$, $\beta_2 = 3/2$.

The non-linear operator $B(u, u)$ is a bounded Lipschitz map in the following sense (see [10], Lemma 2.1):

$$\|B(u_1, u_1) - B(u_2, u_2)\|_{D(A^{-\delta})} \leq c_\delta \left(\|u_1\|_{D(A^{1/2})} + \|u_2\|_{D(A^{1/2})}\right) \|u_1 - u_2\|_{D(A^{1/2})},$$

where $\delta > 0$ can be taken arbitrarily small. This is the Lipschitz condition (1.10).

The averaged system (3.22) generates a semigroup S_t: $E \to E$ of solution operators which is dissipative and uniformly bounded in E (see [1], Theorem I.6.2 and [13], [16]), that is, (1.16) and (1.17) hold. Since (3.21) is of the form (1.20), the estimate (1.18) is not required (although it is valid). Thus, all the hypotheses of Theorem 1.1 are verified and we obtain the following theorem.

THEOREM 3.2: *Suppose that the right-hand side in the Navier-Stokes system* (3.21) *has the average in H in the sense of* (1.11). *If* $u(0) = \bar{u}(0) \in B_{D(A^{1/2})}(R_0)$ *and* $t \in [0, T]$, *then the solutions of the initial and averaged equation satisfy the following proximity estimate*

$$\|u(t) - \bar{u}(t)\|_{D(A^{1/2})} \leq \eta_{T, R_0}(\omega) \to 0 \quad as \quad \omega \to \infty \, .$$

Moreover, if (1.11′) *holds* (*this is the case, for instance, when f is a sum of finitely many functions periodic in H*), *then the estimate can be made more explicit*:

$$\|u(t) - \bar{u}(t)\|_{D(A^{1/2})} \leq \omega^{-1/2} C(R_0, T) \, .$$

PROOF: The proof follows from Theorems 1.1 and 1.2.

Turning to the global averaging we first observe that by [1], Theorem I.6.2, there exists a global attractor $\bar{\mathcal{A}} \Subset E$ of the the averaged equation (3.22). Then, similar to the reaction-diffusion system considered above, we see that the family of processes $U_{g(\omega\cdot)}(t, \tau)$, $g(\omega\cdot) \in \Sigma(\omega) = \mathcal{H}(f(\omega\cdot))$, generated by (3.21) with f replaced by g has a uniform attractor $\mathcal{A}_{\Sigma(\omega)}$ such that $\mathcal{A}_{\Sigma(\omega)} \subset B_E(R_0)$ uniformly for $\omega > 0$ (see details in [10], Example 4.2). By Theorem 2.2 we obtain as a result the following theorem.

THEOREM 3.3 (Global averaging of the Navier-Stokes system): *Suppose that the right-hand side f in equation* (3.21) *is a.p. in H. Then this equation has in* $D(A^{1/2})$ *a uniform attractor* $\mathcal{A}_{\Sigma(\omega)}$ *in the sense of Definition* 2.1 *and*

$$\mathrm{dist}_{D(A^{1/2})}(\mathcal{A}_{\Sigma(\omega)}, \bar{\mathcal{A}}) \to 0 \, , \qquad \omega \to \infty \, .$$

3. *Dissipative hyperbolic equation.* We consider a non-autonomous dissipative hyperbolic equation with rapidly oscillating right-hand side

$$\text{(3.23)} \qquad \begin{cases} \partial_t^2 u + \gamma \partial_t u = \Delta u - \mathcal{F}(u) + \varphi(\omega t) \, , \\ u|_{\partial\Omega} = 0 \, , \qquad x \in \Omega \Subset \mathbb{R}^n \, , \qquad n \leq 3 \, , \qquad \gamma > 0 \, . \end{cases}$$

Suppose that $\mathscr{F} \in C^2(\mathbb{R})$, $\mathscr{F}(0) = 0$ and for any $\eta > 0$

$$\int_0^u \mathscr{F}(v)\, dv + \eta u^2 \geqslant -C_1(\eta)\,, \tag{3.24}$$

$$u\mathscr{F}(u) - c_1 \int_0^u \mathscr{F}(v)\, dv + \eta u^2 \geqslant -C_2(\eta)\,, \tag{3.25}$$

$$|\mathscr{F}'(u)| \leqslant C(1 + |u|^{\beta(n)})\,, \quad \beta(1)\,, \quad \beta(2) < \infty\,, \quad \beta(3) < 2\,, \tag{3.26}$$

for some positive constants c_1, C, $C_1(\eta)$, $C_2(\eta)$ (see [1], [17]).

We shall write (3.23) in the form (1.1) (more precisely, (1.20))

$$\partial_t y = -Ay - F(y) + f(\omega t)\,, \tag{3.27}$$

where

$$A = -\begin{bmatrix} 0 & I \\ \Delta & -\gamma I \end{bmatrix}, \qquad \begin{aligned} &y = (u, p)^T\,, \quad p = \partial_t u\,, \\ &F(y) = (0, \mathscr{F}(u))^T\,, \\ &f(t) = (0, \varphi(t))^T\,. \end{aligned}$$

We shall now verify the hypotheses from Sect. 1 for $n = 3$, the cases $n = 1, 2$ are treated in a similar way. First we set

$$E = F = E_0\,, \qquad X = E_1\,,$$

where

$$E_0 = \{y = (u, p);\ u \in H_0^1(\Omega),\ p \in L_2(\Omega)\}\,, \qquad \|y\|_{E_0}^2 = \|\nabla u\|^2 + \|p\|^2\,,$$

$$E_1 = \{y = (u, p);\ u \in H^2(\Omega) \cap H_0^1(\Omega),\ p \in H_0^1(\Omega)\}\,, \qquad \|y\|_{E_1}^2 = \|\Delta u\|^2 + \|\nabla p\|^2\,.$$

The linear homogeneous equation $\partial_t y + Ay = 0$, $y(0) = y_0$ has a unique solution $y(t) = e^{-At} y_0$ and the semigroup of the solution operators satisfies the following estimates [1], [12], [17]:

$$\|e^{-At}\|_{E_i \to E_i} \leqslant K e^{-at}\,, \qquad i = 0\,,\ 1\,,\ a > 0\,. \tag{3.28}$$

Since A is an isomorphism from E_1 onto E_0, it follows that equation (1.5) is satisfied in the sense of $\mathscr{L}(E_1, E_0)$ and (3.28) implies that estimates (1.2), (1.3), (1.7), (1.8) are valid with $\alpha_1 = \beta_1 = \beta_2 = 0$.

The Lipschitz condition (1.10) follows from the following estimates:

$$\|F(y_1) - F(y_2)\|_{E_0} = \|\mathscr{F}(u_1) - \mathscr{F}(u_2)\|_{L_2} \leqslant C\|1 + |u_1|^{\beta(3)} + |u_2|^{\beta(3)}\|_{L_3} \|u_1 - u_2\|_{L_6} \leqslant$$

$$\leqslant L(R)\|u_1 - u_2\|_{H_0^1} \leqslant L(R)\|y_1 - y_2\|_{E_0}\,, \qquad y_1,\ y_2 \in B_{E_0}(R)\,,$$

where we have used the Sobolev embedding $H_0^1(\Omega) \subset L_6(\Omega)$.

Suppose that the average φ_0 exists in the following sense: φ_0, $\varphi(t) \in H_0^1$ and

$$(3.29) \qquad \left\| \frac{1}{t} \int_{\tau}^{\tau+t} \nabla(\varphi(s) - \varphi_0)\, ds \right\| \leqslant \min(M, \mu(t)), \quad \mu(t) \to 0, \quad t \to \infty,$$

which is obviously equivalent to the existence of the average $f_0 = (0, \varphi_0)^T$ of the right-hand side $f(t) = (0, \varphi(t))^T$ in $X = E_1$:

$$(3.30) \qquad \left\| \frac{1}{t} \int_{\tau}^{\tau+t} (f(s) - f_0)\, ds \right\|_{E_1} \leqslant \min(M, \mu(t)).$$

Thus, to (3.27) there naturally corresponds the averaged equation:

$$(3.31) \qquad \partial_t \bar{y} = -A\bar{y} - F(\bar{y}) + f_0.$$

The semigroup S_t: $E \to E$, $S_t \bar{y}_0 = \bar{y}(t)$ generated by (3.31) is continuous in the sense of (2.2), dissipative, uniformly bounded for $t \geqslant 0$ in E [1], Theorem I.8.1, [17], Lemma IV.3.2 and asymptotically compact [7] and therefore posesses a global attractor $\bar{\mathcal{A}}$ which is stable in the sense of Lyapunov.

Suppose that φ is almost periodic in $H_0^1(\Omega)$. Then (3.29), (3.30) hold. The family of equations

$$\partial_t y = -Ay - F(y) + g(\omega t), \qquad g(\omega \cdot) \in \Sigma(\omega) = \mathcal{H}(f(\omega \cdot)) = [f(\omega(\cdot + h)), h \in \mathbb{R}]_{C_b(\mathbb{R}, E_1)}$$

generates a family of processes $U_{g(\omega\cdot)}(t, \tau)$: $E \to E$, $t \geqslant \tau \in \mathbb{R}$, $g(\omega\cdot) \in \Sigma(\omega)$. This family is uniformly (with respect to $g \in \Sigma$) asymptotically compact and for each ω has a uniform attractor $\mathcal{A}_{\Sigma(\omega)} \Subset E$ in the sense of Definition 2.1 [3].

It remains to verify that the attractors $\mathcal{A}_{\Sigma(\omega)}$ are bounded in E uniformly with respect to $\omega > 0$. This follows from [17], Lemma IV.3.2, where an absorbing ball $B_E(R_0)$, $R_0 = R_0(\|\varphi_0\|_{L_2})$, was constructed for the autonomous equation. In fact, following the proof there, we see that it gives the desired result if the norm $\|\varphi_0\|_{L_2}$ is replaced by $\|\varphi\|_{L_\infty(\mathbb{R}; L_2)}$. Theferore for each ω, $\mathcal{A}_{\Sigma(\omega)} \subset B_E(R_0)$, where $R_0 = R_0(\|\varphi\|_{L_\infty(\mathbb{R}; L_2)})$. Hence the hypotheses of Theorems 1.2 and 2.2 are satisfied and we obtain the following result.

Theorem 3.4 (Global averaging of the hyperbolic equation): *Suppose that in the equation* (3.23), (3.24)-(3.26) *the right-hand side $\varphi(t)$ is almost periodic in $H_0^1(\Omega)$. Then*

$$\operatorname{dist}_{E_0}(\mathcal{A}_{\Sigma(\omega)}, \bar{\mathcal{A}}) \to 0, \qquad \omega \to \infty.$$

Remark 3.1: Averaging in the norm of $E_1 = \{(H^2 \cap H_0^1) \times H_0^1\}$ with $\beta(3) \leqslant 2$ in (3.26) has been considered in [10].

APPENDIX

PROOF OF THEOREM 1.1: The proof below goes along the same lines as the proofs of the corresponding results in [2], [6], and [11].

We set

$$\tau = \omega t\,, \qquad \varepsilon = \omega^{-1}$$

and write (1.1) in the so-called standard form of Bogolyubov

$$\partial_\tau u + \varepsilon A u = \varepsilon F(u, \tau) + \varepsilon f(\tau)\,, \qquad u(0) = u_0\,; \tag{A.1}$$

the averaged system becomes

$$\partial_\tau \bar{u} + \varepsilon A \bar{u} = \varepsilon F_0(\bar{u}) + \varepsilon f_0\,, \qquad \bar{u}(0) = \bar{u}_0\,, \tag{A.2}$$

where $\tau \in [0, T/\varepsilon]$.

In view of (1.17), $\|\bar{u}(\tau)\|_E \leq R - \varrho$ for all $\tau \geq 0$. We suppose for the moment that

$$\|u(\tau)\|_E \leq R\,, \qquad \tau \in [0, T/\varepsilon]\,. \tag{A.3}$$

We set $z(\tau) = u(\tau) - \bar{u}(\tau)$. By (1.13) and (1.14), $z(\tau)$ satisfies the inequality

$$\begin{aligned} \|z(\tau)\|_E \leq {} & \left\| \varepsilon \int_0^\tau e^{-\varepsilon A(\tau - s)} (F(u(s), s) - F(\bar{u}(s), s))\, ds \right\|_E + \\ & + \left\| \varepsilon \int_0^\tau e^{-\varepsilon A(\tau - s)} (F(\bar{u}(s), s) - F_0(\bar{u}(s)))\, ds \right\|_E + \\ & + \left\| \varepsilon \int_0^\tau e^{-\varepsilon A(\tau - s)} (f(s) - f_0)\, ds \right\|_E, \qquad \tau \in [0, T/\varepsilon]\,. \end{aligned} \tag{A.4}$$

By (1.6), (A.3), and (1.3) we see that the first term on the right-hand side of (A.4) is less than

$$\varepsilon^{1-\alpha_1} L(R) K \int_0^\tau e^{-a(\tau - s)} (\tau - s)^{-\alpha_1} \|z(s)\|_E\, ds\,. \tag{A.5}$$

We now show that the sum of the second and third terms in (A.4) tends to 0 as $\varepsilon \to 0$ uniformly with respect to $\tau \in [0, T/\varepsilon]$. We will show that it is dominated by $G_{R,T}(\varepsilon)$, where $G_{R,T}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover, we shall prove this estimate for the right-most point $\tau = T/\varepsilon$ only. It is clear that this involves no loss of generality provided that we can show that $G_{T,R}(\varepsilon)$ is increasing with respect to T and is bounded for $T = 0$.

We begin with the third term. Integrating by parts in s we find

$$\text{(A.6)}\quad \left\{\begin{aligned}
&\left\|\varepsilon\int_0^\tau e^{-\varepsilon A(\tau-s)}(f(s)-f_0)\,ds\right\|_E = \\
&\qquad = \left\|\varepsilon e^{-\varepsilon A\tau}\int_0^\tau (f(t)-f_0)\,dt + \varepsilon^2\int_0^\tau Ae^{-\varepsilon A(\tau-s)}\int_s^\tau (f(t)-f_0)\,dt\,ds\right\|_E \leqslant \\
&\text{(by (1.7), (1.8))}\\
&\leqslant \varepsilon Ke^{-\varepsilon a\tau}(\varepsilon\tau)^{-\beta_1}\tau\left\|\tau^{-1}\int_0^\tau (f(t)-f_0)\,dt\right\|_X + \\
&\qquad + \varepsilon^{2-\beta_2}K\int_0^\tau e^{-\varepsilon a(\tau-s)}(\tau-s)^{1-\beta_2}\left\|(\tau-s)^{-1}\int_s^\tau (f(t)-f_0)\,dt\right\|_X ds \leqslant \\
&\text{(taking into account that } \tau = T/\varepsilon \text{ and using (1.11))}\\
&\leqslant Ke^{-aT}T^{1-\beta_1}\min(M,\mu(T/\varepsilon)) + K\varepsilon^{2-\beta_2}\int_0^\tau e^{-\varepsilon as}s^{1-\beta_2}\min(M,\mu(s))\,ds \leqslant \\
&\text{(considering two cases } T<\varepsilon^{1/2} \text{ and } T\geqslant\varepsilon^{1/2} \text{ in the first term and splitting}\\
&\text{the integral with the point } \tau=\varepsilon^{-1/2} \text{ in the second)}\\
&\leqslant KM\varepsilon^{(1-\beta_1)/2} + \mu(\varepsilon^{-1/2})\sup_{0\leqslant T<\infty} e^{-aT}T^{1-\beta_1} + \\
&\quad + KM\varepsilon^{2-\beta_2}\int_0^{\varepsilon^{-1/2}} s^{1-\beta_2}\,ds + K\mu(\varepsilon^{-1/2})\,\varepsilon^{2-\beta_2}\int_0^\infty e^{-\varepsilon as}s^{1-\beta_2}\,ds \leqslant \\
&\leqslant c_{\beta_1,\beta_2}K(M+1)(\varepsilon^{(1-\beta_1)/2} + \varepsilon^{(2-\beta_2)/2} + \mu(\varepsilon^{-1/2})) =: \Phi(\varepsilon)\to 0\,,\ \varepsilon\to 0\,.
\end{aligned}\right.$$

If (1.11′) holds, then we can give a more explicit esimate for $\Phi(\varepsilon)$. In fact, a straightforward calculation shows that

$$\text{(A.7)}\quad \int_0^\infty e^{-\varepsilon as}s^{1-\beta}\min(1,1/s)\,ds \leqslant \mu_\beta(\varepsilon) := c(\beta)\begin{cases} 1\,, & \text{if } 2>\beta>1\,,\\ \ln(1/\varepsilon)\,, & \text{if } \beta=1\,,\\ \varepsilon^{\beta-1}\,, & \text{if } \beta<1\,.\end{cases}$$

Then

$$\text{(A.8)}\quad \begin{cases} \left\| \varepsilon \int\limits_0^\tau e^{-\varepsilon A(\tau - s)} (f(s) - f_0)\, ds \right\|_E \leqslant \\ \qquad \leqslant KM \min(T^{1-\beta_1}, \varepsilon T^{-\beta_1}) + KM\varepsilon^{2-\beta_2} \int\limits_0^\tau e^{-\varepsilon as} s^{1-\beta_2} \min(1, 1/s)\, ds \leqslant \\ \qquad \text{(considering two cases } T < \varepsilon \text{ and } T \geqslant \varepsilon \text{ and using (A.7))} \\ \qquad \leqslant KM(\varepsilon^{1-\beta_1} + \mu_{\beta_2}(\varepsilon)) \leqslant KMc_{\beta_2}(\varepsilon^{1-\beta_1} + \varepsilon^{2-\beta_2} + \delta_{\beta_2, 1}\varepsilon \ln(1/\varepsilon) + \varepsilon)\,, \end{cases}$$

where $\delta_{i,j}$ is the Kronecker delta. Note that estimates (A.6), (A.8) do not depend on T at all and therefore are valid not only for $\tau = T/\varepsilon$ but for any $\tau \in [0, T/\varepsilon]$.

We now consider the second term in (A.4). As before, $\tau = T/\varepsilon$. We split $[0, \tau]$ into m equal sub-intervals by points (cf. [11]) $\tau_0 = 0, \ldots, \tau_i = i\tau/m, \ldots, \tau_m = \tau$.

Then

$$\text{(A.9)}\qquad \left\| \varepsilon \int\limits_0^\tau e^{-\varepsilon A(\tau - s)} (F(\bar{u}(s), s) - F_0(\bar{u}(s)))\, ds \right\|_E \leqslant \Sigma_1 + \Sigma_2 + \Sigma_3\,,$$

where

$$\Sigma_1 = \sum_{i=0}^{m-1} \left\| \varepsilon \int\limits_{\tau_i}^{\tau_{i+1}} e^{-\varepsilon A(\tau - s)} (F(\bar{u}(s), s) - F(\bar{u}(\tau_i), s))\, ds \right\|_E,$$

$$\Sigma_2 = \sum_{i=0}^{m-1} \left\| \varepsilon \int\limits_{\tau_i}^{\tau_{i+1}} e^{-\varepsilon A(\tau - s)} (F(\bar{u}(\tau_i), s) - F_0(\bar{u}(\tau_i)))\, ds \right\|_E,$$

$$\Sigma_3 = \sum_{i=0}^{m-1} \left\| \varepsilon \int\limits_{\tau_i}^{\tau_{i+1}} e^{-\varepsilon A(\tau - s)} (F_0(\bar{u}(\tau_i)) - F_0(\bar{u}(s)))\, ds \right\|_E.$$

We first consider Σ_2. Setting $\tilde{F}(u, s) = F(u, s) - F_0(u)$ and $\bar{u}_i = \bar{u}(\tau_i)$ we have

$$\Sigma_2 \leqslant \sum_{i=0}^{m-1} \left\| \varepsilon \int\limits_0^{\tau_i} e^{-\varepsilon A(\tau - s)} \tilde{F}(\bar{u}_i, s)\, ds \right\|_E + \sum_{i=0}^{m-1} \left\| \varepsilon \int\limits_0^{\tau_{i+1}} e^{-\varepsilon A(\tau - s)} \tilde{F}(\bar{u}_i, s)\, ds \right\|_E.$$

Each term is estimated by (1.3), (1.4), (1.9) similarly to (A.6):

$$\left\| \varepsilon \int\limits_0^{\tau_i} e^{-\varepsilon A(\tau - s)} \tilde{F}(\bar{u}_i, s)\, ds \right\|_E \leqslant$$

$$\leqslant c_{\alpha_1, \alpha_2} K(M_R + 1)(\varepsilon^{(1-\alpha_1)/2} + \varepsilon^{(2-\alpha_2)/2} + \mu_R(\varepsilon^{-1/2})) =: \Phi_R(\varepsilon) \to 0\,, \qquad \varepsilon \to 0\,.$$

Since Σ_2 contains $2m-1$ terms of this type, it follows that

(A.10) $$\Sigma_2 \leqslant (2m-1)\Phi_R(\varepsilon),$$

The terms Σ_1 and Σ_3 in (A.9) are of the same type and we only consider Σ_1.

$$\Sigma_1 \leqslant \left\| \varepsilon \int_0^{\tau_1} e^{-\varepsilon A(\tau - s)} (F(\bar{u}(s), s) - F(\bar{u}(0), s))\, ds \right\|_E + $$
$$+ \sum_{i=1}^{m-1} \left\| \varepsilon \int_{\tau_i}^{\tau_{i+1}} e^{-\varepsilon A(\tau - s)} (F(\bar{u}(s), s) - F(\bar{u}_i, s))\, ds \right\|_E .$$

We estimate the first term integrating by parts using Lipschitz condition (1.6) and uniform boundedness (1.17)

(A.11) $$\left\| \varepsilon \int_0^{\tau_1} e^{-\varepsilon A(\tau - s)} (F(\bar{u}(s), s) - F(\bar{u}(0), s))\, ds \right\|_E \leqslant$$
$$\leqslant \left\| \varepsilon e^{-\varepsilon A \tau} \int_0^{\tau_1} (F(\bar{u}(t), t) - F(\bar{u}(0), t))\, dt \right\|_E +$$
$$+ \left\| \varepsilon^2 \int_0^{\tau_1} A e^{-\varepsilon A(\tau - s)} \int_s^{\tau_1} (F(\bar{u}(t), t) - F(\bar{u}(0), t))\, dt\, ds \right\|_E \leqslant$$
$$\leqslant 2KL(R)\, R \left((\varepsilon\tau)^{-\alpha_1} \varepsilon \tau_1 + \varepsilon^{2-\alpha_2} \int_0^{\tau_1} (\tau - s)^{-\alpha_2} (\tau_1 - s)\, ds \right) \leqslant$$
$$\leqslant c_{\alpha_2} KL(R)\, R \max(T^{1-\alpha_1}, T^{2-\alpha_2})(m^{-1} + m^{-(2-\alpha_2)}) \longrightarrow 0 \quad \text{as } m \longrightarrow \infty .$$

The i-th term ($i \geqslant 1$) is estimated by the mean value theorem. Integrating by parts and using (1.18) in the form $\|\bar{u}(t) - \bar{u}(\tau_i)\|_E \leqslant \varepsilon D(R, T)(Ti/m)^{-1}(t - \tau_i)$, $t \in (\tau_i, \tau_{i+1})$ and bearing in mind that $\tau = T/\varepsilon$ we obtain

$$\left\| \varepsilon \int_{\tau_i}^{\tau_{i+1}} e^{-\varepsilon A(\tau - s)} (F(\bar{u}(s), s) - F(\bar{u}(\tau_i), s)) ds \right\|_E \leqslant$$
$$\leqslant \left\| \varepsilon e^{-\varepsilon A(\tau - \tau_i)} \int_{\tau_i}^{\tau_{i+1}} (F(\bar{u}(t), t) - F(\bar{u}(\tau_i), t))\, dt \right\|_E +$$
$$+ \left\| \varepsilon^2 \int_{\tau_i}^{\tau_{i+1}} A e^{-\varepsilon A(\tau - s)} \int_s^{\tau_{i+1}} (F(\bar{u}(t), t) - F(\bar{u}(\tau_i), t))\, dt\, ds \right\|_E \leqslant$$

$$\leqslant \varepsilon^{1-\alpha_1} KL(R)(\tau-\tau_i)^{-\alpha_1} \int_{\tau_i}^{\tau_{i+1}} \|\bar{u}(t)-\bar{u}(\tau_i)\|_E \, dt +$$

$$+\varepsilon^{2-\alpha_2} KL(R) \int_{\tau_i}^{\tau_{i+1}} (\tau-s)^{-\alpha_2} \int_{s}^{\tau_{i+1}} \|\bar{u}(t)-\bar{u}(\tau_i)\| \, dt\, ds \leqslant$$

$$\leqslant c_{\alpha_2} KL(R)\, D(R, T)(T^{1-\alpha_1} m^{-(1-\alpha_1)} + T^{2-\alpha_2} m^{-(2-\alpha_2)})\, i^{-1} .$$

Summing these estimates with respect to i from 1 to $m-1$ and adding (A.11) to the result we obtain

$$\text{(A.12)} \qquad \Sigma_1 \leqslant \Psi_{R,T}(m) := c_{\alpha_1\alpha_2} KL(R)(R+1)(D(R,T)+1) \times$$
$$\times \max(T^{1-\alpha_1}, T^{2-\alpha_2})(m^{-1} + m^{-(1-\alpha_1)} + m^{-(2-\alpha_2)}) \ln m ,$$

where $\Psi_{R,T}(m) \to 0$ as $m \to \infty$. Since Σ_3 satisfies the same estimate, we see from (A.6), (A.9), (A.10), and (A.12) that the sum of the second and the third terms on the right-hand side of (A.4) is bounded by the expression

$$\Phi(\varepsilon) + (2m-1)\,\Phi_R(\varepsilon) + 2\Psi_{R,T}(m) =: \mathcal{G}_{R,T}(\varepsilon, m) .$$

With m being at our disposal we now set $m = m(\varepsilon) = (\Phi_R(\varepsilon))^{-1/2} \to \infty$, $\varepsilon \to 0$ and see that $\mathcal{G}_{R,T}(\varepsilon, m(\varepsilon)) =: G_{R,T}(\varepsilon) \to 0$ as $\varepsilon \to 0$. Observing that $G_{R,T}(\varepsilon)$ is continuous and monotone increasing with respect to T we obtain the following integral inequality

$$\text{(A.13)} \qquad \|z(\tau)\|_E \leqslant \varepsilon^{1-\alpha_1} KL(R) \int_0^{\tau} (\tau-s)^{-\alpha_1} \|z(s)\|_E \, ds + G_{R,T}(\varepsilon) ,$$

which is valid on the interval $\tau \in [0, T/\varepsilon]$.

We invoke the following generalization of the Gronwall inequality.

LEMMA A.1 (see [9], Lemma 7.1.2): *Let $\gamma \in (0, 1]$ and for $t \in [0, T]$*

$$u(t) \leqslant a + b \int_0^t (t-s)^{\gamma-1} u(s)\, ds .$$

Then

$$u(t) \leqslant a E_\gamma((b\Gamma(\gamma))^{1/\gamma} t) ,$$

where the function $E_\gamma(z)$ is monotone increasing and $E_\gamma(z) \sim \gamma^{-1} e^z$ as $z \to \infty$.

This lemma and (A.13) give that for $\tau \in [0, T/\varepsilon]$

$$\|z(\tau)\|_E \leqslant G_{R,T}(\varepsilon)\, E_{1-\alpha_1}(T(KL(R)\,\Gamma(1-\alpha_1))^{1/(1-\alpha_1)})) =: G_{R,T}(\varepsilon)\, C(R, T) ,$$

which proves the theorem if we revert in the equation to the original time $t = \varepsilon\tau$ and set $\eta_{R_0, T}(\omega) := G_{R, T}(1/\omega) C(R, T)$. (We recall that in (1.17) $R = R(R_0)$.)

Thus, we have proved the theorem assuming that $\|u(\tau)\|_E \leqslant R$, $\tau \in [0, T/\varepsilon]$. If this were not true, then let $\tau^* \in [0, T/\varepsilon]$ be the first moment when $\|u(\tau^*)\|_E = R$. Our proof then shows that on the interval $\tau \in [0, \tau^*]$, $\|u(\tau) - \overline{u}(\tau)\|_E \leqslant \varrho/2$ for all sufficiently small ε, which contradicts our assumption that $\|\overline{u}(\tau)\|_E \leqslant R - \varrho$ for all $\tau \geqslant 0$ including $\tau = \tau^*$ (see (1.17)). The proof is complete.

PROOF OF THEOREM 1.2: We observe that in this case (A.5) does not change and the second term on the right-hand side of (A.4) is zero. Hence $\Phi_R(\varepsilon) = \Psi_{R, T}(m) = 0$, and the proximity estimate takes the form

$$\|\overline{u}(t) - u(t)\|_E \leqslant \Phi(1/\omega)\, C(R, T), \qquad t \in [0, T], \tag{A.14}$$

where Φ is defined by (A.6). Finally, if (1.11′) holds, then (1.21) follows from (A.14) and (A.8).

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