



Rendiconti

Accademia Nazionale delle Scienze detta dei XL

Memorie di Matematica e Applicazioni

116° (1998), Vol. XXII, fasc. 1, pagg. 85-128

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Stochastic Integral of Process Measures in Banach Spaces I. Process Measures with Integrable Variation (**)

ABSTRACT. — In this paper we study the stochastic integral $H \cdot X$ of certain two parameter processes H with respect to a p -summable process measure X . We prove that a right continuous process measure X with integrable variation is summable and that the stochastic integral $H \cdot X$ can be computed pathwise, as a Stieltjes integral of a certain kind.

Integrazione stocastica rispetto a una misura-processo in uno spazio di Banach.

Parte I: caso di una misura-processo con variazione integrabile

SUNTO. — Nella presente memoria si costruisce e si studia l'integrale stocastico $H \cdot X$ di un processo H dipendente da due parametri, rispetto a una misura-processo X sommabile. Si prova che, se una misura-processo X è continua a destra e con variazione integrabile, allora essa è sommabile, e il corrispondente integrale stocastico $H \cdot X$ può essere calcolato «per traiettorie» come un particolare integrale di Stieltjes.

INTRODUCTION

The framework for this paper consists of a probability space (Ω, \mathcal{F}, P) , a Lusin space L endowed with its Borel σ -algebra \mathcal{L} , $1 \leq p < \infty$ and E, F, G Banach spaces with $E \subset L(F, G)$ isometrically. We study process measures $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$, that is, for each $t \in \mathbb{R}_+$ and $B \in \mathcal{L}$, the mapping $\omega \mapsto X(\omega, t, B)$ belongs to L_E^p , and for each $t \in \mathbb{R}_+$, the mapping $B \mapsto X(\cdot, t, B)$ of \mathcal{L} into L_E^p is σ -additive.

The process measures have been used for the first time by J. Walsh [W] in the important particular case of scalar valued, orthogonal, square integrable martingale measures.

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(**) Memoria presentata il 25 novembre 1998 da Giorgio Letta, uno di XL.

The main part of the paper is Sect. 3. Here we define the p -summable process measures X and construct their stochastic integral $H \cdot X$, for some two parameter processes $H: \Omega \times \mathbb{R}_+ \times L \rightarrow F$, using the general integration theory presented in [B-D.1]. The stochastic integral $H \cdot X$ is again a process measure, with values in G . The following are the main results of the paper:

1) The right continuous, adapted process measures X with integrable variation are summable (Corollary 3.6), hence their stochastic integral $H \cdot X$ is defined;

2) The stochastic integral $H \cdot X$ for such process measures X can be computed pathwise, as a Stieltjes integral of a certain kind (Theorem 3.10):

$$(H \cdot X)(\omega, t, B) = \int_{[0, t] \times B} H(t, \omega, x) X(\omega, dt, dx).$$

Similar results were obtained in [D3] for usual, one parameter, right continuous, adapted processes $X: \Omega \times \mathbb{R}_+ \rightarrow E$ with integrable variation, and in [D4] for two parameter, right continuous, adapted processes $X: \Omega \times \mathbb{R}_+^2 \rightarrow E$ with integrable variation.

There is a close relationship between process measures and two parameter processes, which is studied in Sect. 2. Here we show that if X is a right continuous process measure with integrable variation $|X|$, which is pathwise σ -additive in E (in addition of being σ -additive in L_E^p), then we can associate to X a two parameter, right continuous process $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ with integrable variation $|F|$, by the equality

$$F(\omega, t, x) = X(\omega, t, (-\infty, x]),$$

and then we have also

$$|F|(\omega, t, x) = |X|(\omega, t, (-\infty, x]).$$

This establishes a 1-1 correspondence between such process measures X and two parameter processes F . However, if X is only σ -additive in L_E^p (without being pathwise σ -additive in E), then this is no longer possible. But we can still associate to X a two parameter, right continuous process F that satisfies only

$$F(\omega, t, x) = X(\omega, t, (-\infty, x]), \quad \text{a.s.},$$

the negligible set depending on x , and only an inequality between the variations:

$$|F|(\omega, t, x) \leq |X|(\omega, t, (-\infty, x]).$$

(See Theorem 2.4). This is one of the main results of Sect. 2, which allows us to reduce the study of process measures to that of two parameter processes. The relationship be-

tween the Stochastic integrals $H \cdot X$ and $H \cdot F$ is given by an equality of the same kind:

$$(H \cdot F)(\omega, t, x) = (H \cdot X)(\omega, t, (-\infty, x]).$$

Another important result in Sect. 2 is Theorem 2.6 which states the existence of a P -measure $\mu_X: \mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \rightarrow E$ with integrable variation, satisfying

$$\int H d\mu_X = E \left(\int H(\omega, t, x) X(\omega, dt, dx) \right) = E \left(\int H(\omega, t, x) F(\omega, dt, dx) \right),$$

for certain processes $H: \Omega \times \mathbb{R}_+ \times L \rightarrow F$. Here again, the integral with respect to X is a Stieltjes integral (of a certain kind), and the integral with respect to F is the usual Stieltjes integral in the plane $\mathbb{R} \times L$.

This leads us to the need of defining the Stieltjes integral of a «certain kind» $\int f dg$ with respect to a right continuous function measure $g: \mathbb{R} \times \mathcal{L} \rightarrow E$ with bounded variation $|g|$.

One of the main results in Sect. 1 is Theorem 1.3 which states that if g is a right continuous function measure with bounded variation, then its variation $|g|$ is also a right continuous function measure. To a right continuous function measure g with bounded variation $|g|$, we associate a σ -additive measure $m_g: \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \rightarrow E$ with bounded variation, satisfying

$$m_g((s, t] \times B) = g(t, B) - g(s, B), \quad \text{for } s \leq t \text{ in } \mathbb{R} \text{ and } B \in \mathcal{L}.$$

Then we have $|m_g| = m_{|g|}$ (Theorem 1.6). We denote by $L_F^1(g)$ the space $L_F^1(m_g)$ of Bochner integrable functions $f: \mathbb{R} \times L \rightarrow F$, with respect to $|m_g|$. For a function $f \in L_F^1(g)$ we define the Stieltjes integral $\int f dg$, also denoted $\int f(t, x) g(dt, dx)$, by the equality

$$\int f dg = \int f dm_g.$$

Here, again, we have a close relationship between right continuous function measures $g: \mathbb{R} \times \mathcal{L} \rightarrow E$ with bounded variation $|g|$, and right continuous functions of two variables $G: \mathbb{R} \times L \rightarrow E$ with bounded variation $|G|$, given by the equality

$$G(t, x) = g(t, (-\infty, x]).$$

Moreover

$$|G|(t, x) = |g|(t, (-\infty, x]),$$

and we have $m_G = m_g$ (Theorem 1.6). It turns out that the Stieltjes integrals with respect to g and G are equal:

$$\int f(t, x) g(dt, dx) = \int f(t, x) G(dt, dx).$$

In [D3] and [D4] we have shown that there is a new class of processes which are summable, namely the right continuous, adapted (one or two parameter) processes with integrable semivariation (rather than integrable variation).

This class can only be evidenced for processes with values in an infinite dimensional Banach space, since for finite dimensional Banach spaces, the variation and the semivariation are equal.

Correspondingly we defined in [D3] and [D4] a new kind of Stieltjes integral $\int f dg$, for right continuous functions g of one or two variables, with bounded *semivariation* (rather than bounded variation).

In a forthcoming paper [D6] we shall extend these results for process measures, by showing that right continuous, adapted process measures with integrable *semivariation* are also summable.

Finally, we shall show that right continuous, orthogonal, square integrable martingales with values in a *Hilbert* space are 2-summable [Di-Mu]. This extends the result presented in [W] for scalar valued, orthogonal, square integrable martingale measures.

NOTATIONS

Throughout the paper, we shall use the following notations. In general, we adopt the notations and the definitions in [D-M].

1) (Ω, \mathcal{F}, P) is a probability space and $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is a filtration satisfying the usual conditions. For $t < 0$ we set $\mathcal{F}_t = \mathcal{F}_0$.

\mathcal{R} is the ring generated by the subsets of $\Omega \times \mathbb{R}_+$ of the form $A \times \{0\}$ with $A \in \mathcal{F}_0$ and $A \times (s, t]$ with $A \in \mathcal{F}_s$. The σ -algebra generated by \mathcal{R} is the σ -algebra \mathcal{P} of predictable subsets of $\Omega \times \mathbb{R}_+$. However, in Section 1.2, we shall denote by \mathcal{R} the ring generated by the intervals $(s, t]$ in \mathbb{R} .

2) (L, \mathcal{L}) is a Lusin space endowed with its Borel σ -algebra \mathcal{L} . Without loss of generality, we shall consider $L = \mathbb{R}$ and $\mathcal{L} = \mathcal{B}(\mathbb{R})$, but we shall maintain the notations L and \mathcal{L} .

We shall assume that any measure μ on \mathcal{L} has its support contained in $(0, +\infty)$, and that any function defined on L vanishes on $(-\infty, 0]$. This is justified in the following way: we embed first L into $(0, \infty)$ and $\mathcal{L} \subset \mathcal{B}(0, \infty)$; then any measure μ on \mathcal{L} can be extended to $\mathcal{B}(\mathbb{R})$ by $\mu(B) = \mu(B \cap L)$ for $B \in \mathcal{B}(\mathbb{R})$; and any function on L is extended with 0 on $(-\infty, 0]$.

\mathcal{S} is the ring generated by the intervals $(x, y]$ with $x \leq y$ in L and \mathcal{S}_Q is the subring of \mathcal{S} generated by the intervals $(x, y]$ with x, y rationals. The σ -algebra generated by \mathcal{S}_Q is \mathcal{L} . \mathcal{T} is the ring $r(\mathcal{R} \times \mathcal{S})$ generated by the «cubes» of the form $A \times (s, t] \times (x, y]$ with $A \in \mathcal{F}_s$, $s \leq t$ in \mathbb{R} and $x \leq y$ in L .

3) E, F, G are Banach spaces with $E \subset L(F, G)$ isometrically; $1 \leq p < \infty$ and $L_E^p = L_E^p(P)$.

We shall use also the letters F and G to represent functions of two variables, or two parameter processes. It will be clear from the context and from the notations the precise meaning of the letters F and G .

4) We shall consider the functions defined on $\mathbb{R}_+ \times L$ automatically extended to $\mathbb{R} \times L$, with 0 outside $\mathbb{R}_+ \times L$. Similarly, two parameter processes defined on $\Omega \times \mathbb{R}_+ \times L$ will be considered extended with 0 outside $\Omega \times \mathbb{R}_+ \times L$.

1. - FUNCTION MEASURES WITH FINITE VARIATION

The purpose of this paragraph is to define the Stieltjes integral $\int f dg$ with respect to a function measure g with bounded variation $|g|$. For this purpose we associate to g a function G of two variables with bounded variation $|G|$ and define $\int f dg = \int f dG$.

One of the main result is Theorem 1.3 which states that if g is a function measure, then $|g|$ is also a function measure. Then (Theorem 1.6) $|G|$ is the function of two variables associated to $|g|$. We associate also to a right continuous function measure g with bounded variation $|g|$, a measure m_g on $\mathcal{B}(\mathbb{R}) \times \mathcal{L}$ with finite variation $|m_g| = m_{|g|}$ (Theorem 1.6).

1.1. Function measures.

We define the variation of a function defined on $\mathbb{R} \times \mathcal{X}$, where \mathcal{X} is a ring of subsets of L with $\mathcal{S}_Q \subset \mathcal{X} \subset \mathcal{L}$.

DEFINITION 1.1: Let $g: \mathbb{R} \times \mathcal{X} \rightarrow E$ be a function. For every interval $I \subset \overline{\mathbb{R}}$ and every set $B \in \mathcal{X}$ we define the variation

$$\text{var}(g, I \times B) = \sup \sum_j \sum_i |g(t_{i+1}, B_j) - g(t_i, B_j)|$$

where the supremum is taken for all divisions $t_0 < t_1 < \dots < t_n$ of finite points from I and all finite families $(B_j)_{j \in J}$ of disjoint sets from \mathcal{X} contained in B .

The variation function $|g|: \mathbb{R} \times \mathcal{X} \rightarrow \mathbb{R}_+$ is defined by

$$|g|(t, B) = \text{var}(g, (-\infty, t] \times B), \quad \text{for } t \in \mathbb{R} \text{ and } B \in \mathcal{X}.$$

We say g has finite (respectively bounded) variation if $|g|$ is finite (respectively bounded).

REMARK: The variation $|g|(t, B)$ depends on the ring \mathcal{X} . If we want to emphasize this dependence, we write $|g|_{\mathcal{X}}(t, B)$. For the restriction of g to $\mathbb{R} \times \mathcal{S}$, we have $|g|_{\mathcal{S}}(t, B) \leq |g|_{\mathcal{X}}(t, B)$, for $B \in \mathcal{S}$. However, if for each $t \in \mathbb{R}_+$, the mapping $B \mapsto |g|_{\mathcal{X}}(t, B)$ of \mathcal{X} into E is σ -additive, we have equality: $|g|_{\mathcal{S}}(t, B) = |g|_{\mathcal{X}}(t, B)$ for $B \in \mathcal{S}$.

We can extend the above definition of $|g|$ for $t = -\infty$ and $t = +\infty$. We have then

$$|g|(-\infty, B) = 0 \quad \text{and} \quad |g|(\infty, B) = \text{var}(g, (-\infty, \infty] \times B).$$

We note that $\text{var}(g, I \times B)$ is an additive set function with respect to $I \times B$. We have then

$$\text{var}(g, (s, t] \times B) = |g|(t, B) - |g|(s, B),$$

for $s \leq t$ in \mathbb{R} and $B \in \mathcal{L}$.

Next we define the function measures.

DEFINITION 1.2: Let $g: \mathbb{R} \times \mathcal{X} \rightarrow E$ be a function.

We say that g is right continuous, if for every set $B \in \mathcal{X}$, the function $t \mapsto |g|(t, B)$ is right continuous.

We say that g is a function measure, if for every $t \in \mathbb{R}$, the set function $B \mapsto |g|(t, B)$ is σ -additive in E on \mathcal{X} .

We show next that the properties in the above definitions are inherited by the variation. This is the main result of this paragraph.

THEOREM 1.3: Let $g: \mathbb{R} \times \mathcal{X} \rightarrow E$ be a function with finite variation $|g|$.

a) The function g is right continuous iff its variation $|g|$ is right continuous.

b) If g is a function measure, then its variation $|g|$ is also a function measure.

PROOF: a) From the inequality

$$|g(t, B) - g(s, B)| \leq |g|(t, B) - |g|(s, B)$$

valid for $s \leq t$ in \mathbb{R} and $B \in \mathcal{X}$, it follows that if $|g|$ is right continuous, then g is also right continuous.

Assume now g is right continuous, but that there is a point $a \in \mathbb{R}$ and a set $B \in \mathcal{X}$ such that $|g|$ is not right continuous at (a, B) , that is,

$$M = |g|(a+, B) - |g|(a, B) > 0,$$

and show that this leads to a contradiction. Let $\varepsilon > 0$ and $b > a$. We shall show that there is a decreasing sequence $x_n \downarrow a$ in \mathbb{R} with $x_1 = b$ and a family $D = (B_j)_{1 \leq j \leq m}$ of disjoint sets from \mathcal{X} contained in B and for each n a division d_n of points from

$[x_{n+1}, x_n]$ such that

$$\text{var}_{d_n, D}(g) := \sum_{1 \leq j \leq m} \sum_{t_i \in d_n} |g(t_{i+1}, B_j) - g(t_i, B_j)| \geq M - \frac{\varepsilon}{2^n}.$$

In fact, let Δ_1 be a division of $[a, b]$ and $D = (B_j)_{1 \leq j \leq m}$ a finite family of disjoint sets from \mathcal{X} contained in B , such that

$$\begin{aligned} \text{var}_{\Delta_1, D}(g) &= \sum_{1 \leq j \leq m} \sum_{t_i \in \Delta_1} |g(t_{i+1}, B_j) - g(t_i, B_j)| > \\ &> \text{var}(g, [a, b] \times B) - \frac{\varepsilon}{4} = |g|(b, B) - |g|(a, B) - \frac{\varepsilon}{4} \geq M - \frac{\varepsilon}{4}. \end{aligned}$$

We take $x_1 = b$. Since for each B_j , the function $t \mapsto g(t, B_j)$ is right continuous, there is a point $x_2 > a$ such that for every $x \in [a, x_2]$ and $1 \leq j \leq m$ we have

$$|g(x, B_j) - g(a, B_j)| < \frac{\varepsilon}{4m}.$$

We can choose $x_2 = \inf \{t \in \Delta_1, t > a\}$ and add x_2 to the division Δ_1 , which does not change the above inequalities. Denote $d_1 = \Delta_1 - \{a\}$; then d_1 is a division of the interval $[x_2, x_1]$ and

$$\begin{aligned} \text{var}_{d_1, D}(g) &= \sum_{1 \leq j \leq m} \sum_{t_i \in d_1} |g(t_{i+1}, B_j) - g(t_i, B_j)| = \text{var}_{\Delta_1, D}(g) - \sum_{1 \leq j \leq m} |g(x_2, B_j) - \\ &- g(a, B_j)| \geq M - \frac{\varepsilon}{4} + \sum_{1 \leq j \leq m} \frac{\varepsilon}{4m} = M - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = M - \frac{\varepsilon}{2}. \end{aligned}$$

The second step and the n -th step are proved similarly. By induction, we have, for every n ,

$$|g|(b, B) - |g|(a, B) \geq \sum_{1 \leq i \leq n} [|g|(x_i, B) - |g|(x_{i+1}, B)] \geq \sum_{1 \leq i \leq n} \text{var}_{d_i, D}(g) \geq nM - \varepsilon,$$

hence $|g|(b, B) - |g|(a, B) = +\infty$, which contradicts the hypothesis that the variation $|g|$ is finite. This proves assertion *a*).

b) Assume g is a function measure. Let $t \in \mathbb{R}$ be fixed and prove that the set function $B \mapsto |g|(t, B)$ is σ -additive on \mathcal{X} . We prove first that $|g|$ is additive. Let $B, B' \in \mathcal{X}$ be disjoint sets. There are sequences of divisions $(c_n), (d_n), (d'_n)$ of points from $(-\infty, t]$ and sequences $(C_n), (D_n), (D'_n)$ of finite families of disjoint sets from \mathcal{X} contained in $B \cup B', B, B'$ respectively, such that

$$|g|(t, B \cup B') = \lim \text{var}_{c_n, C_n}(g),$$

$$|g|(t, B) = \lim \text{var}_{d_n, D_n}(g),$$

$$|g|(t, B') = \lim \text{var}_{d'_n, D'_n}(g),$$

For each n we can take refinements of C_n , D_n and D'_n and the above equalities remain valid. We can therefore assume that $C_n = D_n \cup D'_n$. For each n there is a division b_n in $(-\infty, t]$, finer than c_n , d_n and d'_n . Then

$$|g|(t, B \cup B') = \lim \text{var}_{b_n, C_n}(g),$$

$$|g|(t, B) = \lim \text{var}_{b_n, D_n}(g),$$

$$|g|(t, B') = \lim_{b_n, D'_n}(g).$$

For each n we have

$$\text{var}_{b_n, C_n}(g) = \text{var}_{b_n, D_n}(g) + \text{var}_{b_n, D'_n}(g).$$

Passing to the limit we obtain

$$|g|(t, B \cup B') = |g|(t, B) + |g|(t, B').$$

Then this equality remains valid for finitely many disjoint sets from \mathcal{X} .

Let now (B_n) be a sequence of disjoint sets from \mathcal{X} with union B . Let d be a division in $(-\infty, t]$ and $(B'_j)_{1 \leq j \leq m}$ a family of disjoint sets from \mathcal{X} with union B . For each n , the family $(B_n \cap B'_j)_{1 \leq j \leq m}$ of disjoint sets has union B_n , and for each $j \leq m$, the sequence $(B_n \cap B'_j)_{1 \leq n < \infty}$ of disjoint sets has union B'_j . Then

$$\begin{aligned} \sum_{t_i \in d} \sum_{1 \leq j \leq m} |g(t_{i+1}, B'_j) - g(t_i, B'_j)| &= \\ &= \sum_{t_i} \sum_j \left| \sum_{1 \leq n < \infty} g(t_{i+1}, B_n \cap B'_j) - \sum_{1 \leq n < \infty} g(t_i, B_n \cap B'_j) \right| \leq \\ &\leq \sum_i \sum_j \sum_n |g(t_{i+1}, B_n \cap B'_j) - g(t_i, B_n \cap B'_j)| = \\ &= \sum_n \sum_i \sum_j |g(t_{i+1}, B_n \cap B'_j) - g(t_i, B_n \cap B'_j)| \leq \sum_{1 \leq n < \infty} |g|(t, B_n), \end{aligned}$$

therefore

$$|g|(t, \cup_n B_n) \leq \sum_{1 \leq n < \infty} |g|(t, B_n),$$

that is, the set function $B \mapsto |g|(t, B)$ is σ -subadditive.

To prove the converse inequality, let

$$0 < \theta < |g|(t, B_1) + |g|(t, B_2)$$

and let $\theta = \alpha_1 + \alpha_2$ with

$$0 < \alpha_1 < |g|(t, B_1) \text{ and } 0 < \alpha_2 < |g|(t, B_2).$$

There is a division d' in $(-\infty, t]$ and a finite partition (B_j') of B_1 such that

$$\alpha_1 < \sum_j \sum_{t_i \in d'} |g(t_{i+1}, B_j') - g(t_i, B_j')|.$$

There is also a division d'' in $(-\infty, t]$ and a finite partition (B_j'') of B_2 such that

$$\alpha_2 < \sum_j \sum_{t_i \in d''} |g(t_{i+1}, B_j'') - g(t_i, B_j'')|.$$

The sets B_j' and B_j'' are mutually disjoint and form a partition of $B_1 \cup B_2$. Let d be a division in $(-\infty, t]$ finer than d' and d'' . Then

$$\begin{aligned} \theta = \alpha_1 + \alpha_2 &< \sum_j \sum_{t_i \in d} |g(t_{i+1}, B_j') - g(t_i, B_j')| + \\ &+ \sum_j \sum_{t_i \in d} |g(t_{i+1}, B_j'') - g(t_i, B_j'')| \leq |g|(t, B_1 \cup B_2), \end{aligned}$$

therefore

$$|g|(t, B_1) + |g|(t, B_2) \leq |g|(t, B_1 \cup B_2).$$

By induction we have, for every n ,

$$\sum_{1 \leq i \leq n} |g|(t, B_i) \leq |g|\left(t, \bigcup_{i=1}^n B_i\right) \leq |g|\left(t, \bigcup_{i=1}^{\infty} B_i\right).$$

Then

$$\sum_{i=1}^{\infty} |g|(t, B_i) \leq |g|\left(t, \sum_{i=1}^{\infty} B_i\right),$$

and this proves that the set function $B \mapsto |g|(t, B)$ is σ -additive.

1.2. Measures associated to function measures.

In this paragraph we denote by \mathcal{R} the ring generated by the intervals $(s, t]$ in \mathbb{R} , and by \mathcal{K} any ring of subsets of L such that $\mathcal{S}_Q \subset \mathcal{K} \subset \mathcal{L}$.

Let $g: \mathbb{R} \times \mathcal{K} \rightarrow E$ be a function. We define the additive measure $m_g: \mathcal{R} \times \mathcal{K} \rightarrow E$ by

$$m_g((s, t] \times B) = g(t, B) - g(s, B), \quad \text{for } s \leq t \text{ in } \mathbb{R} \text{ and } B \in \mathcal{K},$$

and then we extend m_g to an additive measure on the ring $r(\mathcal{R} \times \mathcal{K})$ generated by the semiring $\mathcal{R} \times \mathcal{K}$ of rectangular sets $(s, t] \times B$.

The relationship between the variations of g and m_g is given by the following proposition.

PROPOSITION 1.4: Consider the variation $|m_g|$ of m_g on the ring $r(\mathcal{R} \times \mathcal{K})$. Then for every interval $I \subset \mathbb{R}$ and set $B \in \mathcal{K}$, we have

$$|m_g|(I \times B) = \text{var}(g, I \times B).$$

In particular, for $I = (-\infty, t]$, we have

$$|m_g|((-\infty, t] \times B) = |g|(t, B)$$

and for $s \leq t$ in \mathbb{R} we have

$$|m_g|((s, t] \times B) = m_{|g|}((s, t] \times B).$$

PROOF: Let $(A_i)_{1 \leq i \leq n}$ be a family of disjoint sets from \mathcal{R} contained in I and $(B_j)_{1 \leq j \leq m}$ a family of disjoint sets from \mathcal{K} contained in B .

Each set A_i is a finite union of disjoint intervals,

$$A_i = \bigcup_{k \in K_i} (s_{ik}, t_{ik}).$$

Arrange all points s_{ik}, t_{ik} in increasing order $u_0 < u_1 < \dots < u_r$. Then

$$\begin{aligned} \sum_{i,j} |m_g(A_i \times B_j)| &\leq \sum_{j,r} |m_g(u_r, u_{r+1}] \times B_j| = \\ &= \sum_{j,r} |g(u_{r+1}, B_j) - g(u_r, B_j)| \leq \text{var}(g, I \times B) \leq \infty, \end{aligned}$$

therefore $|m_g|(I \times B) \leq \text{var}(g, I \times B)$.

For the converse inequality let $d: t_0 < t_1 < \dots < t_n$ be a division of points from I and $(B_j)_{1 \leq j \leq m}$ a family of disjoint sets from \mathcal{K} contained in B . Then

$$\sum_{i,j} |g(t_{i+1}, B_j) - g(t_i, B_j)| = \sum_{i,j} |m_g((t_i, t_{i+1}] \times B_j)| \leq |m_g|(I \times B),$$

therefore $\text{var}(g, I \times B) \leq |m_g|(I \times B)$; hence the equality $|m_g|(I \times B) = \text{var}(g, I \times B)$, and the proposition is proved.

The next question is whether m_g can be extended to a σ -additive measure on the product σ -algebra $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$. We answer this question in theorem 1.6 in the next section.

1.3. Function measures and functions of two variables.

Let $g: \mathbb{R} \times S \rightarrow E$ be a function. We associate to g a function of two variables $G: \mathbb{R} \times L \rightarrow E$ by the following equality:

$$G(t, x) = g(t, (-\infty, x]), \quad \text{for } t \in \mathbb{R} \text{ and } x \in L.$$

This allows us to reduce the study of the function measures g to that of the functions of two variables G , which are studied in detail in ([D4], Sect. 1).

The variation of G on a rectangle $I \times J \subset \mathbb{R} \times L$, where I and J are intervals, bounded or not, is defined by

$$\begin{aligned} \text{var}(G, I \times J) &= \sup_{i,j} \sum |\Delta_{(t_i, t_{i+1}] \times (x_j, x_{j+1}]}(G)| = \\ &= \sup_{ij} \sum |G(t_{i+1}, x_{j+1}) + G(t_i, x_j) - G(t_i, x_{j+1}) - G(t_{i+1}, x_j)|, \end{aligned}$$

where the supremum is taken for all divisions $t_0 < t_1 < \dots < t_n$ of points from I and all divisions $x_0 < x_1 < \dots < x_m$ of points from J .

It turns out that the variations of g and G are equal:

PROPOSITION 1.5: *For any intervals $I \subset \mathbb{R}$ and $J \subset L$ we have*

$$\text{var}(G, I \times J) = \text{var}(g, I \times J).$$

PROOF: Denote by m_G the additive measure defined on $\mathcal{R} \times \mathcal{S}$ by

$$m_G((s, t] \times (x, y]) = \Delta_{(s, t] \times (x, y]}(G),$$

for $s \leq t$ in \mathbb{R} and $x \leq y$ in L , where

$$\Delta_{(s, t] \times (x, y]}(G) = G(s, x) + G(t, y) - G(s, y) - G(t, x).$$

Then

$$m_G((s, t] \times (x, y]) = m_g((s, t] \times (x, y]).$$

By proposition 1.4 we have

$$|m_g|(I \times B) = \text{var}(g, I \times B), \quad \text{for any } B \in \mathcal{S},$$

in particular for $B = J$. From ([D4], Sect. 1.6, property 10), with g replaced by G , we have

$$|m_G|(I \times J) = \text{var}(G, I \times J).$$

Since $|m_G|(I \times J) = |m_g|(I \times J)$ we deduce that

$$\text{var}(G, I \times J) = \text{var}(g, I \times J),$$

and the proposition is proved.

The following theorem gives conditions to insure that G is right continuous on $\mathbb{R} \times L$ and has bounded variation, and answers the question whether m_g can be extended to a σ -additive measure on $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$.

THEOREM 1.6: Let $g: \mathbb{R} \times \mathcal{S} \rightarrow E$ be a right continuous function measure with bounded variation $|g|$. Let $G: \mathbb{R} \times L \rightarrow E$ be the function defined by

$$G(t, x) = g(t, (-\infty, x]), \quad \text{for } t \in \mathbb{R} \text{ and } x \in L.$$

Then:

a) G is right continuous and has bounded variation $|G|$ satisfying

$$|G|(t, x) = |g|(t, (-\infty, x]), \quad \text{for } t \in \mathbb{R} \text{ and } x \in L.$$

b) g can be extended uniquely to a right continuous function measure with bounded variations from $\mathbb{R} \times \mathcal{L}$ into E , still denoted by g .

c) The measures m_g and m_G can be extended to σ -additive measures from $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$ into E , with finite variation $|m_g|$ and $|m_G|$ respectively, and we have

$$m_G = m_g \text{ and } |m_G| = |m_g| = m_{|g|} = m_{|G|}, \quad \text{on } \mathcal{B}(\mathbb{R}) \otimes \mathcal{L}.$$

PROOF: The function G is separately right continuous in t (since g is right continuous in t) and right continuous in x (since the measure $B \mapsto g(t, B)$ is σ -additive). Moreover, G has bounded variation on $\mathbb{R} \times L$. In fact, let $I \subset \mathbb{R}$ and $J \subset L$ be two intervals and consider a grid

$$Q: t_0 < t_1 < \dots < t_n, \quad x_0 < x_1 < \dots < x_m$$

consisting of points t_i from I and x_j from J . Then

$$\sum_{ij} |\Delta_{(t_i, t_{i+1}] \times (x_j, x_{j+1}]}(G)| = \sum_{i,j} |g(t_{i+1}, (x_j, x_{j+1}]) - g(t_i, (x_j, x_{j+1}])| \leq \text{var}(g, I \times J),$$

hence G has finite variation on $I \times J$ and

$$\text{var}(G, I \times J) \leq \text{var}(g, I \times J).$$

From ([D4]), Corollary 1.3) it follows that G is right continuous.

The equality $|G|(t, x) = |g|(t, (-\infty, x])$ will be proved below.

Consider G extended with 0 for every point (s, t) with s or t equal to $-\infty$. By Radu's theorem [R] (see also [D4], Theorem 1.8), there is a σ -additive measure $m_G: \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \rightarrow E$ with finite variation satisfying

$$m_G(R) = \Delta_R(G), \quad \text{for any rectangle } R,$$

where, if $R = (s, t] \times (x, y]$, then

$$\Delta_R(G) = \Delta_{(s, t] \times (x, y]}(G) = G(s, x) + G(t, y) - G(s, y) - G(t, x).$$

In particular, if $R = (-\infty, t] \times (-\infty, x]$, then

$$m_G(R) = \Delta_R(G) = G(t, x) = g(t, (-\infty, x]),$$

hence

$$m_G((-\infty, t] \times B) = g(t, B), \quad \text{for } t \in \mathbb{R} \text{ and } B \in \mathcal{S}.$$

Then we can extend g from $\mathbb{R} \times \mathcal{L}$ into E by setting

$$g(t, B) = m_G((-\infty, t] \times B), \quad \text{for } t \in \mathbb{R} \text{ and } B \in \mathcal{L}.$$

The extended function g is a right continuous function measure on \mathcal{L} . From the above equality we deduce, for $t \in \mathbb{R}$ and $B \in \mathcal{L}$,

$$|g|(t, B) = \text{var}(g, (-\infty, t] \times B) \leq |m_G|((-\infty, t] \times B) < \infty,$$

hence the extended function measure g has bounded variation $|g|$. This proves assertion *b*).

Consider now the additive measure $m_g: \mathcal{R} \times \mathcal{L} \rightarrow E$ satisfying

$$m_g((s, t] \times B) = g(t, B) - g(s, B), \quad \text{for } s, t \in \mathbb{R} \text{ and } B \in \mathcal{L}.$$

Then m_g has bounded variation $|m_g|$ on $\mathcal{R} \times \mathcal{L}$ and we have

$$|m_g|((s, t] \times B) = |g|(t, B) - |g|(s, B), \quad \text{for } s, t \in \mathbb{R} \text{ and } B \in \mathcal{L}.$$

We deduce that

$$m_g((s, t] \times (-\infty, x]) = g(t, (-\infty, x]) - g(s, (-\infty, x]) = m_G((s, t] \times (-\infty, x]),$$

therefore $m_g = m_G$ on $\mathcal{R} \times \mathcal{S}$, consequently $|m_g| = |m_G|$, that is $m_{|g|} = m_{|G|}$ on $\mathcal{R} \times \mathcal{S}$. Since m_G is σ -additive on $B(\mathbb{R}) \otimes \mathcal{L}$, we extend m_g to a σ -additive measure on $B(\mathbb{R}) \otimes \mathcal{L}$ by the equality

$$m_g(C) = m_G(C), \quad \text{for } C \in B(\mathbb{R}) \otimes \mathcal{L}.$$

Then m_g has finite variation $|m_g|$ on $B(\mathbb{R}) \otimes \mathcal{L}$ and we have $|m_g| = |m_G|$ on $B(\mathbb{R}) \otimes \mathcal{L}$. The equalities $|m_g| = m_{|g|}$ and $|m_G| = m_{|G|}$ follow from Proposition 1.4 and ([D.4], property 11 of section 1.5). This proves assertion *c*). Finally, for $t \in \mathbb{R}$ and $x \in L$ we have

$$\begin{aligned} |G|(t, x) &= \Delta_{(-\infty, t] \times (-\infty, x]} |G| = m_{|G|}((-\infty, t] \times (-\infty, x]) = \\ &= m_{|g|}((-\infty, t] \times (-\infty, x]) = \text{var}(g, (-\infty, t] \times (-\infty, x]) = |g|(t, (-\infty, x]), \end{aligned}$$

and this proves assertion *a*) and the theorem.

1.4. The Stieltjes integral with respect to a function measure.

Let $g: \mathbb{R} \times \mathcal{L} \rightarrow E$ be a right continuous function measure with bounded variation $|g|$. Let $m_g: \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \rightarrow E$ be the σ -additive measure with bounded variation $|m_g| = m_{|g|}$ associated to g by Theorem 1.6 and satisfying

$$m_g((s, t] \times B) = g(t, B) - g(s, B), \quad \text{for } s \leq t \text{ and } B \in \mathcal{L}.$$

Let F, G be Banach spaces with $E \subset L(F, G)$ and consider the space $L_F^1(m_g) = L_F^1(|m_g|)$ of functions $f: \mathbb{R} \times L \rightarrow F$ which are Bochner integrable with respect to $|m_g|$. Then, for any $f \in L_F^1(m_g)$, the integral $\int f dm_g$ is defined and belongs to G .

By analogy with the definition of the Stieltjes integral with respect to a *function of two variables* with bounded variation ([D4], Sect. 1.6) we are led to denote $L_F^1(m_g)$ by $L_F^1(g)$ or $L_F^1(dg)$, and for $f \in L_F^1(g)$ to call $\int f dm_g$ the Stieltjes integral of f with respect to g and to denote it by $\int f dg$ or $\int f(t, x) g(dt, dx)$:

$$\int f(t, x) g(dt, dx) = \int f dm_g.$$

Then we have

$$\left| \int f(t, x) g(dt, dx) \right| \leq \int |f(t, x)| |g|(dt, dx).$$

This convention is further justified in the following way. Let $G: \mathbb{R} \times L \rightarrow E$ be the function of two variables defined by

$$G(t, x) = g(t, (-\infty, x]), \quad \text{for } t \in \mathbb{R} \text{ and } x \in L.$$

By Theorem 1.6, G is jointly right continuous and has bounded variation $|G|$ satisfying

$$|G|(t, x) = |g|(t, (-\infty, x]), \quad \text{for } t \in \mathbb{R} \text{ and } x \in L.$$

Consider the σ -additive measure $m_G: \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \rightarrow E$ with bounded variation $|m_G|$ associated to G by Radu's theorem ([L]; [D4] Theorem 1.8) and satisfying

$$m_G((s, t] \times (x, y]) = \Delta_{(s, t] \times (x, y]}(G).$$

The space $L_F^1(m_G)$ of functions $f: \mathbb{R} \times L \rightarrow F$ which are Bochner integrable with respect to $|m_G|$ is denoted in ([D4], Sect. 1.6) by $L_F^1(G)$ or $L_F^1(dG)$, and for $f \in L_F^1(G)$, the Stieltjes integral $\int f dG$ or $\int f(t, x) G(dt, dx)$ is defined by

$$\int f(t, x) G(dt, dx) = \int f dm_G.$$

Now, for each rectangle $(s, t] \times (x, y]$ we have

$$m_g((s, t] \times (x, y]) = m_G((s, t] \times (x, y]) .$$

Since both measures m_g and m_G are σ -additive on $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$ and since the rectangles $(s, t] \times (x, y]$ generate $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$, we deduce that

$$m_g = m_G \text{ and } |m_g| = |m_G| , \quad \text{on } \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} .$$

It follows that $L_F^1(m_g) = L_F^1(m_G)$, and for $f \in L_F^1(m_g)$ we have

$$\int f dm_g = \int f dm_G = \int f(t, x) G(dt, dx) .$$

This means that $\int f dm_g$ is indeed a Stieltjes integral with respect to the function G of two variables. If we identify g and G , it is justified to call $\int f dm_g$ the Stieltjes integral of f with respect to g and to denote it by $\int f dg$ or $\int f(t, x) g(dt, dx)$, as we did above. We have then

$$\int f(t, x) g(dt, dx) = \int f(f, x) G(dt, dx)$$

for $f \in L_F^1(g) = L_F^1(G)$.

2. - P -MEASURES INDUCED BY PROCESS MEASURES WITH INTEGRABLE VARIATION

2.1. Definitions and properties.

Let (Ω, \mathcal{F}, P) be a probability space, \mathcal{X} a ring of subsets of L satisfying $S_Q \subset \mathcal{X} \subset \mathcal{L}$ and $X: \Omega \times \mathbb{R}_+ \times \mathcal{X} \rightarrow E$ a function. Its values $X(\omega, t, B)$ for $\omega \in \Omega$, $t \in \mathbb{R}_+$ and $B \in \mathcal{X}$ will be also denoted by $X_t(\omega, B)$. As we mentioned before, the function X will be automatically extended to $\Omega \times \mathbb{R} \times \mathcal{X}$, with 0 outside $\Omega \times \mathbb{R}_+ \times \mathcal{X}$. Furthermore we shall define $X_{-\infty}(\omega, B) = 0$, for every $\omega \in \Omega$ and $B \in \mathcal{X}$.

We gather in the following definition all the terms about X that will be needed later on.

DEFINITION 2.1: a) A function $X: \Omega \times \mathbb{R}_+ \times \mathcal{X} \rightarrow E$ is called a process set function.

b) A process set function X is said to be measurable, if for every $B \in \mathcal{X}$, the process $(\omega, t) \mapsto X_t(\omega, B)$ is measurable with respect to $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$.

c) A process set function X is said to be adapted to a filtration (\mathcal{F}_t) , if for every $t \in \mathbb{R}_+$, and every $B \in \mathcal{X}$, the random variable $\omega \mapsto X_t(\omega, B)$ is \mathcal{F}_t -measurable.

d) We say a process set function X is right continuous, if for every $\omega \in \Omega$ and $B \in \mathcal{X}$, the function $t \mapsto X_t(\omega, B)$ is right continuous.

e) A process set function X is said to be pathwise σ -additive in E on \mathcal{X} , if for

every $\omega \in \Omega$ and $t \in \mathbb{R}_+$, the set function $B \mapsto X_t(\omega, B)$, from \mathcal{X} into E is σ -additive.

f) Let $1 \leq p < \infty$. We say a process set function X is a p -process measure, if it is σ -additive in L_E^p , that is, if for every $t \in \mathbb{R}$ and $B \in \mathcal{X}$, the random variable $X_t(\cdot, B)$ belongs to L_E^p and for every $t \in \mathbb{R}_+$, the set function $B \mapsto X_t(\cdot, B)$ from \mathcal{X} into L_E^p is σ -additive.

g) We say a process set function X has finite (respectively bounded) variation, if for every $\omega \in \Omega$, the function $(t, B) \mapsto X_t(\omega, B)$, has finite (respectively bounded) variation $|X(\omega)|_t(B)$, in the sense of Definition 1.1.

We shall also denote $|X(\omega)|_t(B)$ by $|X|_t(\omega, B)$ and we shall denote the process set function $(\omega, t, B) \mapsto |X|_t(\omega, B)$ by $|X|$.

h) Let $1 \leq p < \infty$. We say a process set function X has p -integrable variation, if the function $\omega \mapsto |X|_\infty(\omega, L) = \sup_{t \geq 0} |X|_t(\omega, L)$ belongs to L^p . If $p = 1$, we say that X has integrable variation.

We list now a series of properties of interrelationship between the above notions.

Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{X} \rightarrow E$ be a process set function.

1) If X is right continuous and adapted, then X is measurable.

2) If X is measurable and right continuous and has p -integrable variation $|X|$, then X is right continuous in L_E^p , that is, for each $B \in \mathcal{X}$, the mapping $t \mapsto X_t(\cdot, B)$ from \mathbb{R}_+ into L_E^p is right continuous.

In fact, if $t_n \downarrow t_0$, we have $X_{t_n}(\omega, B) \rightarrow X_{t_0}(\omega, B)$ and $|X_{t_n}(\omega, B)| \leq |X|_{t_n}(\omega, B) \leq |X|_\infty(\omega, L)$ and the last function belongs to L^p . We apply then Lebesgue's theorem and deduce that $X_{t_n}(\omega, B) \rightarrow X_{t_0}(\omega, B)$ in L_E^p .

REMARK: The converse is not true.

3) If X is measurable and pathwise σ -additive in E on \mathcal{X} and has p -integrable variation, then X is σ -additive in L_E^p on \mathcal{X} , that is, X is a p -process measure.

In fact, $B \mapsto X_t(\cdot, B)$ is additive; if $B_n \downarrow \phi$, then $X_t(\omega, B_n) \rightarrow 0$ in E for each $\omega \in \Omega$, and $|X_t(\omega, B_n)| \leq |X|_\infty(\omega, L)$. Since $|X|_\infty(\cdot, L) \in L^p$, we can apply Lebesgue's theorem and deduce that $X_t(\cdot, B_n) \rightarrow 0$ in L_E^p .

REMARK: The converse is not true.

4) If X is right continuous and has finite variation, then the variation process set function $|X|$ is also right continuous.

See Theorem 1.3a.

5) If X is pathwise σ -additive in E on \mathcal{X} and has finite variation, then the variation $|X|$ is also pathwise σ -additive in \mathbb{R} on \mathcal{X} .

See Theorem 1.3b.

REMARK: If X is pathwise σ -additive in E on \mathcal{X} and has finite variation (but not p -integrable variation), then the variation $|X|$ is not necessarily σ -additive in L^p .

6) If X is measurable and pathwise σ -additive in E on \mathcal{X} and has p -integrable variation $|X|$, then $|X|$ is σ -additive in L^p on \mathcal{X} .

Use Properties 5 and 3.

2.2. Process measures and two parameter processes.

Let \mathcal{S} be the ring generated by the intervals $(x, y]$ in L . The σ -algebra generated by \mathcal{S} is \mathcal{L} .

Let $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ be a two parameter process. We associate to F a process set function $X: \Omega \times \mathbb{R}_+ \times \mathcal{S} \rightarrow E$, defined by

$$X_t(\omega, (-\infty, x]) = F(\omega, t, x), \quad \text{for } \omega \in \Omega, \quad t \in \mathbb{R}_+ \text{ and } x \in X$$

and then extend X by additivity to $\Omega \times \mathbb{R}_+ \times \mathcal{S}$. We call X the process set function induced by (or associated with) the two parameter process F . The set function $B \mapsto X(\omega, B)$ is finitely additive on \mathcal{S} , for each $\omega \in \Omega$ and $t \in \mathbb{R}_+$.

Conversely, any process set function $X: \Omega \times \mathbb{R}_+ \times \mathcal{S} \rightarrow E$ which is finitely additive on \mathcal{S} , is induced by the two parameter process $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ defined by

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{for } \omega \in \Omega, \quad t \in \mathbb{R}_+ \text{ and } x \in L.$$

Let F and X be associated to each other as above. We list some properties of relationship between F and X . For each $\omega \in \Omega$, we denote by $F(\omega)$ the function defined on $\mathbb{R}_+ \times L$ by $F(\omega)(t, x) = F(\omega, t, x)$. Similar definition for $X(\omega)$.

We state first a property about the variations of F and X .

7) For any intervals $I \subset \mathbb{R}$ and $J \subset L$ and for every $\omega \in \Omega$ we have

$$\text{var}(X(\omega), I \times J) = \text{var}(F(\omega), I \times J).$$

See Proposition 1.5.

7') In particular, taking $I = (-\infty, t]$ and $J = (-\infty, x]$ we get

$$|X|_t(\omega, (-\infty, x]) = |F|(\omega, t, x).$$

Next we state a property about right continuity in t .

8) F is separately right continuous in the variable t on \mathbb{R} iff X is right continuous on \mathbb{R} .

See Definition 2.1.d.

The relationship between separate right continuity of F in x and σ -additivity of X is given by the following property:

9) If X is pathwise σ -additive in E on S , then F is separately right continuous in x on L .

In fact, if $x_n \downarrow x_0$ in L , then $(-\infty, x_n] \downarrow (-\infty, x_0]$, hence, for any ω and t we have $X_t(\omega, (-\infty, x_n]) \rightarrow X_t(\omega, (-\infty, x_0])$, therefore $F(\omega, t, x_n) \rightarrow F(\omega, t, x_0)$ in E .

Conversely:

10) Assume F is separately right continuous in the variable $x \in L$ and has bounded variation $|F|$. Then X has bounded variation on $\Omega \times \mathbb{R}_+ \times S$ and is pathwise σ -additive in E on S . Moreover, X can be extended to a process set function $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ which is pathwise σ -additive in E on \mathcal{L} .

If, in addition, F is measurable and has p -integrable variation, then X has p -integrable variation and is σ -additive in L_E^p on S .

In fact, for each $\omega \in \Omega$ and $t \in \mathbb{R}_+$, the function $x \mapsto F(\omega, t, x)$ is right continuous and has bounded variation $|F(\omega, t)|(x) \leq |F|(\omega, t, x)$; therefore it induces a σ -additive Stieltjes measure with finite variation $\mu_{\omega, t}: \mathcal{L} \rightarrow E$, satisfying

$$\mu_{\omega, t}(x, y] = F(\omega, t, y) - F(\omega, t, x).$$

For every $\omega \in \Omega$, $t \in \mathbb{R}_+$ and $B \in \mathcal{L}$ we set

$$X_t(\omega, B) = \mu_{\omega, t}(B)$$

Then X is a process set function, which is pathwise σ -additive in E on \mathcal{L} and satisfies

$$X_t(\omega, (x, y]) = F(\omega, t, y) - F(\omega, t, x),$$

hence X is an extension of the process set function associated to F . By property 7', the restriction of X to $\Omega \times \mathbb{R}_+ \times S$ has bounded variation.

For the last part we use Property 3.

REMARK: For the variation $|X_S|$ of X on $\Omega \times \mathbb{R}_+ \times S$ we have, by property 7',

$$|X_S|_t(\omega, (-\infty, x]) = |F|(\omega, t, x).$$

But the variation $|X_{\mathcal{L}}|$ of X on $\Omega \times \mathbb{R}_+ \times \mathcal{L}$ satisfies the inequality $|X_{\mathcal{L}}| \geq |X_S|$, and $|X_{\mathcal{L}}|$ is not necessarily finite. For this reason, although X is pathwise σ -additive in E on \mathcal{L} , X is not necessarily σ -additive in L_E^p on \mathcal{L} , since the variation $|X_{\mathcal{L}}|$ of the extension $X_{\mathcal{L}}$ is not necessarily finite or p -integrable.

If, in property 10 we impose F to be right continuous jointly in t and x , then X has bounded variation on $\Omega \times \mathbb{R}_+ \times \mathcal{L}$.

THEOREM 2.2: Assume the function $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ is right continuous jointly in t and x and has bounded variation $|F|$. Then there exists a process set function $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ which is right continuous, pathwise σ -additive in E on \mathcal{L} and has bounded variation $|X|$ on $\Omega \times \mathbb{R}_+ \times \mathcal{L}$, which is itself right continuous, pathwise σ -additive on \mathcal{L} and satisfies

$$X_t(\omega, (-\infty, x]) = F(\omega, t, x), \quad \text{everywhere}$$

and

$$|X|_t(\omega, (-\infty, x]) = |F|(\omega, t, x), \quad \text{everywhere}.$$

If, in addition, F has p -integrable variation, then X has p -integrable variation, is pathwise σ -additive in E on \mathcal{L} and σ -additive in L_E^p on \mathcal{L} .

PROOF: Since F is right continuous and has bounded variation, by Radu's theorem ([R], and [D4] Theorem 1.8), for each $\omega \in \Omega$ there is a σ -additive measure $m_{F(\omega)}: \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \rightarrow E$ with finite variation $|m_{F(\omega)}|$ satisfying

$$m_{F(\omega)}((-\infty, t] \times (-\infty, x]) = F(\omega, t, x)$$

and

$$|m_{F(\omega)}|((-\infty, t] \times (-\infty, x]) = m_{|F(\omega)|}((-\infty, t] \times (-\infty, x]) = |F|(\omega, t, x).$$

We define $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ by

$$X_t(\omega, B) = m_{F(\omega)}((-\infty, t] \times B), \quad \text{for } \omega \in \Omega, \quad t \in \mathbb{R}_+ \text{ and } B \in \mathcal{L}.$$

For $\omega \in \Omega$, $t \in \mathbb{R}_+$ and $x \in L$, taking $B = (-\infty, x]$ we obtain

$$X_t(\omega, (-\infty, x]) = F(\omega, t, x),$$

therefore X is the extension to $\Omega \times \mathbb{R}_+ \times \mathcal{L}$ of the process set function induced by F .

Since $m_{F(\omega)}$ is σ -additive, it follows that X is right continuous and pathwise σ -additive in E on \mathcal{L} . To prove that X has bounded variation $|X|$, let $\omega \in \Omega$, $t \in \mathbb{R}$ and $B \in \mathcal{L}$. Let $t_1 < t_2 < \dots < t_n$ be a division of points in $(-\infty, t]$ and $(B_j)_{1 \leq j \leq m}$ a family of disjoint sets from \mathcal{L} contained in B . Then

$$\sum_{ij} |X_{t_{i+1}}(\omega, B_j) - X_{t_i}(\omega, B_j)| = \sum_{ij} |m_{F(\omega)}((t_i, t_{i+1}] \times B_j)| \leq$$

$$\leq |m_{F(\omega)}|((-\infty, t] \times B) = m_{|F(\omega)|}((-\infty, t] \times B),$$

therefore

$$|X|_t(\omega, B) \leq m_{|F(\omega)|}((-\infty, t] \times B) \leq |F|_\infty(\omega, \infty, \infty) < \infty.$$

For $B = (-\infty, x]$ we get

$$|X|_t(\omega, (-\infty, x]) \leq m_{|F(\omega)|}((-\infty, t] \times (-\infty, x]) = |F|(\omega, t, x).$$

To prove the converse inequality, let $\omega \in \mathbb{R}$, $t \in \mathbb{R}_+$ and $x \in L$. Let $t_1 < t_2 < \dots < t_n$ be a division of points in $(-\infty, t]$ and $x_1 < x_2 < \dots < x_m$ a division of points in $(-\infty, x]$. Denoting $R_{ij} = (t_i, t_{i+1}] \times (x_j, x_{j+1}]$, we have

$$\begin{aligned} \sum_{ij} |\Delta_{R_{ij}}(F(\omega))| &= \sum_{ij} |m_{F(\omega)}((t_i, t_{i+1}] \times (x_j, x_{j+1}])| = \\ &= \sum_{ij} |X_{t_{i+1}}(\omega, (x_j, x_{j+1}]) - X_{t_i}(\omega, (x_j, x_{j+1}])| \leq \\ &\leq \text{var}(X(\omega), (-\infty, t] \times (-\infty, x]) = |X|_t(\omega, (-\infty, x]), \end{aligned}$$

therefore

$$|F|(\omega, t, x) = \text{var}(F(\omega), (-\infty, t] \times (-\infty, x]) \leq |X|_t(\omega, (-\infty, x]);$$

consequently

$$|X|_t(\omega, (-\infty, x]) = |F|(\omega, t, x), \quad \text{everywhere.}$$

In particular, for $t = \infty$ and $x = \infty$, we get

$$|X|_\infty(\omega, L) = |F|(\omega, \infty, \infty),$$

therefore if F has p -integrable variation, i.e., if $|F|(\cdot, \infty, \infty) \in L^p$, then $|X|_\infty(\cdot, L) \in L^p$, that is, X has p -integrable variation. We use then property 3 to deduce that X is σ -additive in L^p on \mathcal{L} . This proves the theorem.

The converse of Theorem 2.2 is also true for pathwise σ -additive process measures.

THEOREM 2.3: *Let \mathcal{K} be a ring such that $\mathcal{S} \subset \mathcal{K} \subset \mathcal{L}$. Assume $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ is a right continuous process measure with bounded variation $|X|$ and pathwise σ -additive in E on \mathcal{K} . Then the two parameter process $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ defined by*

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{for } \omega \in \Omega, \quad t \in \mathbb{R}_+ \text{ and } x \in L$$

is right continuous jointly in t and x and has bounded variation.

The process measure X can be extended uniquely to a right continuous process measure $X': \omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ with bounded variation $|X|$ and pathwise σ -additive in E on \mathcal{L} .

PROOF: Since X is right continuous and pathwise σ -additive in E , by properties 8 and 9, F is separately right continuous in t and x . Since X has bounded variation, F has also bounded variation, by property 7'. Then, by ([D4], Theorem 1.4),

F is right continuous, jointly in t and x . For the existence of X' we apply Theorem 2.2.

REMARK: Theorem 2.3 is no longer true if we assume that X is σ -additive in L_E^p , but not pathwise σ -additive in E . Such a process is no longer induced by a two parameter right continuous process with p -integrable variation; however, we shall see below (Theorem 2.4) that there is a right continuous two parameter process $F: \Omega \times \mathbb{R}_+ \times \times L \rightarrow E$ with p -integrable variation, satisfying

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{a.s.},$$

but not everywhere, the negligible set depending on x . We close this series of properties with the property of being adapted to a filtration.

11) Let (\mathcal{F}_t) be a filtration satisfying the usual conditions. Consider also the constant filtration $(\mathcal{F}_x)_{x \in L}$ with $\mathcal{F}_x = \mathcal{F}$ for every $x \in L$, and the double filtration $(\mathcal{F}_{t,x})$ with $\mathcal{F}_{t,x} = \mathcal{F}_t \cap \mathcal{F}_x = \mathcal{F}_t$.

If F is right continuous in the variable x , has bounded variation $|F|$ and is adapted to the filtration $(\mathcal{F}_{t,x})$, then $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ is pathwise σ -additive in E on \mathcal{L} and is adapted to the filtration (\mathcal{F}_t) .

The extension of X to $\Omega \times \mathbb{R}_+ \times \mathcal{L}$ is insured by property 10. Let $t \in \mathbb{R}_+$ and $B \in \mathcal{L}$ and prove that the path $\omega \mapsto X_t(\omega, B)$ is \mathcal{F}_t -measurable. This is true first for $B = (-\infty, x]$, from the definition of X :

$$X_t(\omega, (-\infty, x]) = F(\omega, t, x).$$

Then $X_t(\cdot, B)$ is \mathcal{F}_t -measurable for B in the ring \mathcal{S} generated by the intervals $(-\infty, x]$. Since these intervals generate \mathcal{L} and since $B \mapsto X_t(\omega, B)$ is σ -additive in E , for each ω and t , it follows by a monotone class argument that $X_t(\cdot, B)$ is \mathcal{F}_t -measurable for every $B \in \mathcal{L}$.

We state now and prove the property mentioned in the Remark following Theorem 2.3, and which is one of the main results of this paragraph.

THEOREM 2.4: Let \mathcal{K} be a ring such that $\mathcal{S} \subset \mathcal{K} \subset \mathcal{L}$. Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{K} \rightarrow E$ be a right continuous p -process measure with p -integrable variation, i.e. $|X|_\infty(L) \in L^p$. Then there exists a two parameter, right continuous process $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ with p -integrable variation $|F|$, satisfying, for every $t \in \mathbb{R}_+$ and $x \in L$,

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{a.s.}$$

and

$$|F|(\omega, t, x) \leq |X|_t(\omega, (-\infty, x]), \quad \text{a.s.}$$

If X is adapted to a filtration (\mathcal{F}_t) satisfying the usual conditions, then F is also adapted to (\mathcal{F}_t) .

PROOF: As usual, we shall consider the process measure X extended with 0 for $t < 0$, and the two parameter process F , to be defined below, will be also considered extended with 0 for (t, x) with one or both coordinates equal to $-\infty$.

We shall divide the proof into several steps.

a) We define first the two parameter process $G: \Omega \times \mathbb{R} \times L \rightarrow E$ by

$$G(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{for } \omega \in \Omega, \quad t \in \mathbb{R} \text{ and } x \in L.$$

Since $X_t: \mathcal{S} \rightarrow L_E^p$ is σ -additive, it is finitely additive, hence, for $A, B \in \mathcal{S}$ disjoint, we have

$$X_t(A \cup B) = X_t(A) + X_t(B), \quad \text{in } L_E^p,$$

hence

$$X_t(\omega, A \cup B) = X_t(\omega, A) + X_t(\omega, B), \quad \text{a.s.},$$

the negligible set $N(t, A, B)$ depending on t, A and B . Since X is right continuous, for ω outside the negligible set $N(A, B) = \bigcup_{t \in \mathbb{Q}} N(t, A, B)$ we have

$$X_t(\omega, A \cup B) = X_t(\omega, A) + X_t(\omega, B), \quad \text{for all } t \in \mathbb{R}.$$

It follows that if $A \subset B$, then

$$X_t(\omega, B \setminus A) = X_t(\omega, B) - X_t(\omega, A), \quad \text{for } \omega \notin N(A, B) \text{ and } t \in \mathbb{R}.$$

Taking $A = (-\infty, x]$ and $B = (-\infty, y]$ with $x < y$, we denote $N(x, y) = N(A, B)$ and we get

$$(*) \quad X_t(\omega, (x, y]) = G(\omega, t, y) - G(\omega, t, x), \quad \text{for } \omega \notin N(x, y) \text{ and } t \in \mathbb{R}.$$

If we set $N = \bigcup \{N(x, y); x, y \text{ rational}\}$, then N is negligible and the equality (*) is true for $\omega \notin N$, all $t \in \mathbb{R}$ and all x, y rational.

b) Denote by G_0 the restriction of G to the set Q^2 of points (t, x) with t and x rational. Then $X_t(\omega, (x, y]) = G_0(\omega, t, y) - G_0(\omega, t, x)$, for $\omega \notin N$, $t \in \mathbb{R}$ and $x < y$ rational. For any rectangle $R = (s, t] \times (x, y]$ with $s < t, x < y$ reals, consider a grid Q consisting of rational points $t_0 < t_1 < \dots < t_n$ contained in $(s, t]$ and rational points $x_0 < x_1 < \dots < x_m$ contained in $(x, y]$. Then, for $\omega \notin N$ we have

$$\begin{aligned} \sum_{i,j} |\Delta_{R_{ij}}|(G_0)| &= \sum_{i,j} |G_0(t_i, x_j) + G_0(t_{i+1}, x_{j+1}) - G_0(t_i, x_{j+1}) - G_0(t_{i+1}, x_j)| \leq \\ &\leq \sum_{i,j} |X_{t_{i+1}}((x_j, x_{j+1}]) - X_{t_i}((x_j, x_{j+1}])| \leq \\ &\leq \text{var}(X, (s, t] \times (x, y]) \leq \text{var}(X, \mathbb{R} \times L) < \infty, \end{aligned}$$

therefore the variation of G_0 on Q^2 is bounded. By ([D4], Theorem 1.1) it follows that for $\omega \notin N$, the function G_0 has right limits at every point $(t, x) \in \mathbb{R} \times L$, with t, x rational or not. We set then, for every $(t, x) \in \mathbb{R} \times L$,

$$F(\omega, t, x) = \lim_{\substack{t' \downarrow t \\ x' \downarrow x \\ t', x' \in Q}} G_0(\omega, t', x'), \quad \text{if } \omega \notin N,$$

and $F(\omega, t, x) = 0$, for $\omega \in N$. Then F is right continuous at every point $(t, x) \in \mathbb{R} \times L$, for every $\omega \in \Omega$.

c) For every $(t, x) \in \mathbb{R} \times L$ we have

$$F(t, x) = G(t, x) = X_t((-\infty, x]), \quad \text{a.s.}$$

In fact, since for $\omega \notin N$ the limit of G_0 at (t, x) exists and is equal to $F(t, x)$, the iterated limits of G_0 at (t, x) also exist and are equal to $F(t, x)$:

$$F(t, x) = \lim_{\substack{t' \downarrow t \\ x' \downarrow x \\ t', x' \in Q}} G_0(t', x') = \lim_{x' \downarrow x} \lim_{t' \downarrow t} G_0(t', x') = \lim_{x' \downarrow x} \lim_{t' \downarrow t} G(t', x') = \lim_{t' \downarrow x} G(t, x'),$$

since $t \mapsto G(t, x) = X_t((-\infty, x])$ is right continuous. Let $x'_n \downarrow x$ with x'_n rational. Since $B \rightarrow X_t(B)$ is σ -additive in L_E^p , we have $X_t((-\infty, x'_n]) \rightarrow X_t((-\infty, x])$ in L_E^p , that is $G(t, x'_n) \rightarrow G(t, x)$ in L_E^p . Since, by the above, we have also $G(t, x'_n) \rightarrow F(t, x)$ pathwise a.s., we deduce that the two limits are equal a.s.:

$$F(t, x) = G(t, x) = X_t((-\infty, x]), \quad \text{a.s.}$$

d) F has bounded variation. Let R be any rectangle in $\mathbb{R} \times L$. Since F is right continuous, we can use only grids consisting of rational points $t_0 < t_1 < \dots < t_n$ and $x_0 < x_1 < \dots < x_m$, with $R_{ij} = (t_i, t_{i+1}] \times (x_j, x_{j+1}] \subset R$. Then, for ω outside a negligible set we have

$$\begin{aligned} \sum_{i,j} |\Delta_{R_{ij}}(F)| &= \sum_{ij} |\Delta_{R_{ij}}(G)| = \sum_{i,j} |X_{t_{i+1}}((x_j, x_{j+1}]) - X_{t_i}((x_j, x_{j+1}])| \leq \\ &\leq \text{var}(X, R) \leq \text{var}(X, \mathbb{R} \times L) = |X|_\infty(L) < \infty \end{aligned}$$

(since $|X|_\infty(L)$ is p -integrable); it follows that

$$\text{var}(F, R) \leq \text{var}(X, R) \leq |X|_\infty(L), \quad \text{a.s.}$$

Since F is right continuous, its variation function $|F|(t, x) = \text{var}(F, (-\infty, t] \times (-\infty, x])$ is right continuous ([L] Theorem II.3.2 and [D4], Sect. 1.4, property 5), hence $|F|$ is \mathcal{F} -measurable. From

$$|F|(\infty, \infty) = \text{var}(F, \mathbb{R} \times L) \leq |X|_\infty(L) \in L^p,$$

we deduce that F has p -integrable variation.

e) From the inequality $\text{var}(F, R) \leq \text{var}(X, R)$, a.s., proved above, taking $R = (-\infty, t] \times (-\infty, x]$ we get

$$\begin{aligned} |F|(t, x) &= \text{var}(F, (-\infty, t] \times (-\infty, x]) \leq \\ &\leq \text{var}(X, (-\infty, t] \times (-\infty, x]) = |X|_t((-\infty, x]), \quad \text{a.s.} \end{aligned}$$

f) Let now (\mathcal{F}_t) be a filtration satisfying the usual conditions and assume X is adapted to (\mathcal{F}_t) , i.e., for each $t \in \mathbb{R}$ and $B \in \mathcal{S}$, the random variable $X_t(\cdot, B)$ is \mathcal{F}_t -measurable and prove that F is adapted to $(\mathcal{F}_{t,x})$ with $\mathcal{F}_{t,x} = \mathcal{F}_t$. This follows from the equality

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{a.s.}$$

and the fact that \mathcal{F}_t contains all the negligible sets of Ω .

This proves the theorem.

REMARKS: 1°. We are able to prove only the inequality $|F|(\omega, t, x) \leq |X|_t(\omega, (-\infty, x])$, a.s. By Theorem 2.2, we have the equality $|F|(\omega, t, x) = |X|_t(\omega, (-\infty, x])$, provided that we have $F(\omega, t, x) = X_t(\omega, (-\infty, x])$ everywhere or outside a negligible set independent on t and x ; and this is realized if X is pathwise σ -additive in E .

2°. Under the hypothesis of Theorem 2.4, we do not know whether or not, the variation $|X|$ is σ -additive in L^p , that is, we do not know whether or not $|X|$ is itself a p -process measure.

From Theorem 2.2 it follows that $|X|$ is a p -process measure if X is induced by a right continuous, two parameter process F with p -integrable variation $|F|$. In this case $|X|$ is itself induced by $|F|$.

Combining Theorems 2.4 and 2.2 we obtain the following theorem.

THEOREM 2.5: Let \mathcal{K} be a ring such that $\mathcal{S} \subset \mathcal{K} \subset \mathcal{L}$ and let $X: \Omega \times \mathbb{R}_+ \times \mathcal{K} \rightarrow E$ be a right continuous process measure with p -integrable variation $|X|$. Then

1) There is a right continuous process measure $X': \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ which is pathwise σ -additive in E on \mathcal{L} and has p -integrable variation $|X'|$, and which is a modification of X , that is,

$$X'_t(\omega, B) = X_t(\omega, B), \quad \text{a.s., for } t \in \mathbb{R}_+ \text{ and } B \in \mathcal{K},$$

the negligible set depending on B .

2) The variation $|X'|$ is right continuous and pathwise σ -additive in E , and satisfies

$$|X'|_t(\omega, B) \leq |X|_t(\omega, B), \quad \text{a.s. for } B \in \mathcal{S}.$$

3) If (\mathcal{F}_t) is a filtration satisfying the usual conditions and if X is adapted to (\mathcal{F}_t) , then X' is also adapted to (\mathcal{F}_t) .

PROOF: Let $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ be a right continuous two parameter process with p -integrable variation $|F|$ associated to X by Theorem 2.4 and satisfying

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{a.s.}$$

and

$$|F|(\omega, t, x) \leq |X|_t(\omega, (-\infty, x]), \quad \text{a.s.}$$

for each $t \in \mathbb{R}_+$ and $x \in L$.

Let $X': \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ be the right continuous process measure with p -integrable variation, associated to F by Theorem 2.2 and satisfying

$$X'_t(\omega, (-\infty, x]) = F(\omega, t, x), \quad \text{everywhere}$$

and

$$|X'_t|(\omega, (-\infty, x]) = |F|(\omega, t, x), \quad \text{everywhere.}$$

Then

$$|X'_t|_t(\omega, (-\infty, x]) \leq |X|_t(\omega, (-\infty, x]), \quad \text{a.s.,}$$

the negligible set depending on x .

From

$$X'_t(\omega, (-\infty, x]) = X_t(\omega, (-\infty, x]), \quad \text{a.s.,}$$

we deduce

$$X'_t(\cdot, (-\infty, x]) = X_t(\cdot, (-\infty, x]), \quad \text{in } L_E^p,$$

therefore

$$X'_t(\cdot, B) = X_t(\cdot, B), \quad \text{in } L_E^p$$

for every $B \in \mathcal{S}$. Since the set functions $B \mapsto X'_t(\cdot, B)$ and $B \mapsto X_t(\cdot, B)$ from \mathcal{X} into L_E^p are both σ -additive and coincide on \mathcal{S} , they are equal on \mathcal{X} :

$$X'_t(\cdot, B) = X_t(\cdot, B), \quad \text{in } L_E^p, \quad \text{for } t \in \mathbb{R}_+ \text{ and } B \in \mathcal{X}$$

that is

$$X'_t(\omega, B) = X_t(\omega, B), \quad \text{a.s., for } t \in \mathbb{R}_+, \quad \text{and } B \in \mathcal{X},$$

the negligible set depending on B , because of the right continuity of X' and X . This proves assertion 1).

The right continuity and pathwise σ -additivity of $|X'|$ follows from Theorem 1.3, for each $\omega \in \Omega$. To prove the inequality in condition 2), let $t \in \mathbb{R}_+$ and $B \in \mathcal{S}$. Since X'

is right continuous and pathwise σ -additive in E , we can compute the variation $|X'|_t(\omega, B)$ by taking divisions $d: t_1 < t_2 < \dots < t_n$ consisting of rational points and a family $(B_j)_{1 \leq j \leq m}$ consisting of intervals $B_j = (x_j, y_j]$ with x_j, y_j rational. Then, for ω outside a negligible set which is the same for all rational points, we have

$$\begin{aligned} \sum_{i,j} |X'_{t_{i+1}}(\omega, B_j) - X'_{t_i}(\omega, B_j)| &= \sum_{i,j} |X_{t_{i+1}}(\omega, B_j) - X_{t_i}(\omega, B_j)| \leq \\ &\leq \text{var}(X(\omega), (-\infty, t] \times B) = |X|_t(\omega, B), \end{aligned}$$

therefore

$$|X'|_t(\omega, B) \leq |X|_t(\omega, B).$$

This proves assertion 2).

Finally, assertion 3) follows from assertion 1).

2.3. Pathwise σ -additive process measures and their Stieltjes integral.

We define first the Stieltjes integral for process measures which are pathwise σ -additive. By Theorem 2.3, we can assume these process measure to be pathwise σ -additive on the whole σ -algebra \mathcal{L} .

Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ be a right continuous process measure, pathwise σ -additive in E on \mathcal{L} and with bounded variation $|X(\omega)|$ for each $\omega \in \Omega$. For each $\omega \in \Omega$ consider the measure $m_{X(\omega)}: \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \rightarrow E$ with finite variation, associated to the function measure $X(\omega)$ by Theorem 1.6 and satisfying $m_{X(\omega)}((s, t] \times B) = X_t(\omega, B) - X_s(\omega, B)$, for $s \leq t$ and $B \in \mathcal{L}$.

Let F, G be Banach spaces with $E \subset L(F, G)$ and let $H: \Omega \times \mathbb{R}_+ \times L \rightarrow F$ be a two parameter, measurable process with $\int |H(\omega, t, x)| |m_{X(\omega)}|(dt, dx) < \infty$ for each $\omega \in \Omega$. This means that for each $\omega \in \Omega$, the function $(t, x) \mapsto H(\omega, t, x)$ belongs to $L^1_F(m_{X(\omega)})$, therefore the integral

$$\int H(\omega, t, x) m_{X(\omega)}(dt, dx)$$

is defined and belongs to G . According to Sect. 1.4, this is the Stieltjes integral $\int H(\omega) dX(\omega)$ which was also denoted

$$\int H(\omega, t, x) X(\omega)(dt, dx) = \int H(\omega, t, x) X(\omega, dt, dx).$$

So

$$\int H(\omega, t, x) X(\omega, dt, dx) = \int H(\omega, t, x) m_{X(\omega)}(dt, dx).$$

Consider now the right continuous, two parameter process $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ with

bounded variation, associated to X by Theorem 2.3 and satisfying

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{for } t \in \mathbb{R}_+ \text{ and } x \in L.$$

For each $\omega \in \Omega$, let $F(\omega): \mathbb{R}_+ \times L \rightarrow E$ be the function defined by $F(\omega)(t, x) = F(\omega, t, x)$.

Then $F(\omega)$ is the function of two variables associated to the function measure $X(\omega)$ by Theorem 1.6. The measures $m_{F(\omega)}$ and $m_{X(\omega)}$ corresponding to $F(\omega)$ and $X(\omega)$ respectively are σ -additive on $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$ and are equal on the rectangles $(s, t] \times (x, y]$:

$$\begin{aligned} m_{X(\omega)}((s, t] \times (x, y]) &= X_t(\omega, (x, y]) - X_s(\omega, (x, y]) = \\ &= \Delta_{(s, t] \times (x, y]}(F(\omega)) = m_{F(\omega)}((s, t] \times (x, y]). \end{aligned}$$

It follows that $m_{X(\omega)} = m_{F(\omega)}$ and $|m_{X(\omega)}| = |m_{F(\omega)}|$; therefore $L_F^1(m_{X(\omega)}) = L_F^1(m_{F(\omega)})$ and for any two parameter process $H: \Omega \times \mathbb{R}_+ \times L \rightarrow F$ with

$$\int |H(\omega, t, x)| d|m_{X(\omega)}| < \infty,$$

we have

$$\int |H(\omega, t, x)| d|m_{F(\omega)}| < \infty,$$

hence the following two integrals are defined and are equal:

$$\int H(\omega, t, x) m_{X(\omega)}(dt, dx) = \int H(\omega, t, x) m_{F(\omega)}(dt, dx),$$

which, translated in terms of Stieltjes integrals, gives

$$\int H(\omega, t, x) X(\omega, dt, dx) = \int H(\omega, t, x) F(\omega, dt, dx).$$

This equality justifies the name of Stieltjes integral given to the first integral, since it is equal to a genuine Stieltjes integral.

2.4. General process measures and their Stieltjes integral.

We define now the Stieltjes integral for an arbitrary process measure.

Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ be a right continuous p -process measure with p -integrable variation $|X|$. The process measure X is σ -additive in L_E^p , but not necessarily pathwise σ -additive in E . For each $\omega \in \Omega$, consider, as in the preceding section, the measure $m_{X(\omega)}: \mathbb{R} \times \mathcal{L} \rightarrow E$ defined by

$$m_{X(\omega)}((s, t] \times B) = X_t(B) - X_s(B), \quad \text{for } s \leq t \text{ and } B \in \mathcal{L}.$$

The measure $m_{X(\omega)}$ is finitely additive, but not necessarily σ -additive in E on $\mathbb{R} \times \mathcal{L}$.

For this reason, the integral $\int H(\omega, t, x) m_{X(\omega)}(dt, dx)$ cannot be defined as a usual integral. It will be defined below just as a notation for a genuine Stieltjes integral (in the sense of Section 2.3) with respect to modifications of X that are pathwise σ -additive in E .

Indeed, by Theorem 2.5, there is a right continuous process measure $X': \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$, pathwise σ -additive in E on \mathcal{L} , with p -integrable variation $|X'|$, such that X' is a modification of X , that is,

$$X'_t(\omega, B) = X_t(\omega, B), \quad \text{a.s., for } B \in \mathcal{L},$$

the negligible set depending on B only. Then

$$m_{X'(\omega)}((s, t] \times B) = m_{X(\omega)}((s, t] \times B), \quad \text{a.s., for } B \in \mathcal{L},$$

outside a negligible set depending on B .

If $X'': \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ is another right continuous process measure, pathwise σ -additive in E on \mathcal{L} , with p -integrable variation, such that X'' is a modification of X , then, because of right continuity and σ -additivity in E of X' and X'' , there is a negligible set N such that for $\omega \notin N$ we have

$$X'_t(\omega, B) = X''_t(\omega, B), \quad \text{for every } t \in \mathbb{R}_+ \text{ and } B \in \mathcal{L}.$$

Then

$$m_{X'(\omega)}((s, t] \times B) = m_{X''(\omega)}((s, t] \times B), \quad \text{for } \omega \notin N, \quad s \leq t \text{ and } B \in \mathcal{L}.$$

Since $m_{X'(\omega)}$ and $m_{X''(\omega)}$ are σ -additive in E on $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$, we deduce that

$$m_{X'(\omega)} = m_{X''(\omega)}, \quad \text{on } \mathcal{B}(\mathbb{R}) \otimes \mathcal{L}, \quad \text{for } \omega \notin N.$$

Then, for $\omega \notin N$ we have $L_F^1(m_{X'(\omega)}) = L_F^1(m_{X''(\omega)})$; and for a measurable process $H: \Omega \times \mathbb{R}_+ \times L \rightarrow F$ we have

$$\int |H(\omega, t, x)| |m_{X'}|(dt, dx) < \infty \quad \text{iff} \quad \int |H(\omega, t, x)| |m_{X''}|(dt, dx) < \infty.$$

It follows that for $\omega \notin N$, the following integrals are defined and are equal:

$$\int H(\omega, t, x) m_{X'(\omega)}(dt, dx) = \int H(\omega, t, x) m_{X''(\omega)}(dt, dx).$$

These are Stieltjes integrals in the sense of Section 2.3:

$$\int H(\omega, t, x) X'(\omega, dt, dx) = \int H(\omega, t, x) X''(\omega, dt, dx), \quad \text{for } \omega \notin N.$$

This means that these two integrals are in the same equivalence class modulo P , and this equivalence class is determined by H and the process measure X .

By analogy with the above Stieltjes integrals, we denote their equivalence class by

$$\int H(\omega, t, x) m_{X(\omega)}(dt, dx)$$

and

$$\int H(\omega, t, x) X(\omega, dt, dx),$$

and call it the Stieltjes integral of H with respect to X :

$$\int H(\omega, t, x) m_{X(\omega)}(dt, dx) = \int H(\omega, t, x) m_{X'(\omega)}(dt, dx)$$

and

$$\int H(\omega, t, x) X(\omega, dt, dx) = \int H(\omega, t, x) X'(\omega, dt, dx).$$

We emphasize again that $\int H(\omega, t, x) m_{X(\omega)}(dt, dx)$ and $\int H(\omega, t, x) X(\omega, dt, dx)$ do not have a meaning as integrals with respect to $m_{X(\omega)}$, but that they are just notations for the equivalence class, of the meaningful integral with respect to X' :

$$\int H(\omega, t, x) X'(\omega, dt, dx) = \int H(\omega, t, x) m_{X'(\omega)}(dt, dx).$$

We define now the expectation of the equivalence class to be the common value of the expectations of its representatives:

$$E\left(\int H(\omega, t, x) X(\omega, dt, dx)\right) = E\left(\int H(\omega, t, x) X'(\omega, dt, dx)\right).$$

Using the considerations of the preceding section, we have also

$$\int H(\omega, t, x) m_{X'(\omega)}(dt, dx) = \int H(\omega, t, x) F(\omega, dt, dx)$$

or

$$\int H(\omega, t, x) X'(\omega, dt, dx) = \int H(\omega, t, x) F(\omega, dt, dx),$$

where $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ is a right continuous two parameter process with p -integrable variation $|F|$, associated to X by Theorem 2.4 and satisfying

$$F(\omega, t, x) = X_t(\omega, t, (-\infty, x]), \quad \text{a.s.}$$

Then we have

$$E\left(\int H(\omega, t, x) X'(\omega, dt, dx)\right) = E\left(\int H(\omega, t, x) F(\omega, dt, dx)\right),$$

therefore with the above convention about the Stieltjes integral with respect to X , we have

$$E\left(\int H(\omega, t, x) X(\omega, dt, dx)\right) = E\left(\int H(\omega, t, x) F(\omega, dt, dx)\right).$$

2.5. P -measures associated to process measures.

The next theorem, which is one of the main results of this paragraph, associates to a process measure X , a P -measure μ_X by means of the Stieltjes integral. We have to distinguish the measure μ_X from the measures $m_{X(\omega)}$ considered in the preceding two sections.

THEOREM 2.6: *Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ be a right continuous process measure with integrable variation $|X|$, i.e. the function $\omega \mapsto |X|_\infty(\omega, L)$ is integrable.*

There exists a P -measure $\mu_X: \mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \rightarrow E$ with finite variation $|\mu_X|$, satisfying the following conditions:

Let F, G be Banach spaces with $E \subset L(F, G)$ and let $X': \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ be a right continuous process measure, pathwise σ -additive in E , with integrable variation and a modification of X . If $H: \Omega \times \mathbb{R}_+ \times L \rightarrow F$ is measurable, then $H \in L_F^1(\mu_X)$ iff H is μ_X -almost separably valued and $E\left(\int |H(\omega, t, x)| |X'|(\omega, dt, dx)\right) < \infty$. In this case $E\left(\int H(\omega, t, x) X(\omega, dt, dx)\right)$ is defined (in the sense of Section 2.4) and we have

$$\int H d\mu_X = E\left(\int H(\omega, t, x) X(\omega, dt, dx)\right)$$

and

$$\int |H| d|\mu_X| = E\left(\int |H(\omega, t, x)| |X'|(\omega, dt, dx)\right),$$

that is $|\mu_X| = \mu_{|X'|}$.

PROOF: Let $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ be a right continuous two parameter process with integrable variation $|F|$, associated to X by theorem 2.4 and satisfying

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{a.s.}$$

By ([L], theorem IV.2.1; see also [D4], Theorem 3.1), there is a P -measure $\mu_F: \mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \rightarrow E$ with finite variation $|\mu_F|$ such that, if $H: \Omega \times \mathbb{R}_+ \times L \rightarrow F$ is measurable, (distinguish between the Banach space F and the process F !), then $H \in L_F^1(\mu_F)$ iff H is μ_F -separably valued and

$$E\left(\int |H(\omega, t, x)| |F|(\omega, dt, dx)\right) < \infty.$$

In this case, $E\left(\int H(\omega, t, x) F(\omega, dt, dx)\right)$ is defined and we have

$$\int H d\mu_F = E\left(\int H(\omega, t, x) F(\omega, dt, dx)\right)$$

and

$$\int |H| d|\mu_F| = E\left(\int |H(\omega, t, x)| |F|(\omega, dt, dx)\right),$$

that is, $|\mu_F| = \mu_{|F|}$. We notice that for the modification X' of X we have

$$F(\omega, t, x) = X'_t(\omega, (-\infty, x]), \quad \text{a.s.},$$

the negligible set being independent of t and x , and

$$|F|(\omega, t, x) = |X'|_t(\omega, (-\infty, x]), \quad \text{a.s.},$$

hence

$$\int |H(\omega, t, x)| |F|(\omega, dt, dx) = \int |H(\omega, t, x)| |X'|(\omega, dt, dx), \quad \text{a.s.}$$

and $\mu_F = \mu_{X'}$, and also $\mu_{|F|} = \mu_{|X'|}$. We take $\mu_X := \mu_F = \mu_{X'}$ and the theorem is proved.

REMARK: If X is itself pathwise σ -additive in E , then

$$\int H d\mu_X = E\left(\int H(\omega, t, x) X(\omega, dt, dx)\right)$$

and $|\mu_X| = \mu_{|X|}$.

The following theorem is a converse of Theorem 2.6.

THEOREM 2.7: Let $\mu: \mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \rightarrow E$ be a P -measure with finite variation $|\mu|$. Assume E has the Radon Nikodym Property. Then there exists a right continuous process measure $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$, pathwise σ -additive in E and with integrable variation $|X|$, such that $\mu = \mu_X$.

PROOF: By ([L], Theorem IV. 2.2) there is a right continuous two parameter process $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ with integrable variation $|F|$, such that $\mu = \mu_F$. Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ be the right continuous process measure, pathwise σ -additive in E , with integrable variation, associated to F by Theorem 2.2, and satisfying

$$X_t(\omega, (-\infty, x]) = F(\omega, t, x), \quad \text{everywhere.}$$

Then we have $\mu_X = \mu_F$, therefore $\mu = \mu_X$, and the theorem is proved.

3. - THE STOCHASTIC INTEGRAL FOR PROCESS MEASURES WITH INTEGRABLE VARIATION

This paragraph contains the main results of the paper.

We define first the summable process measures and prove that the right continuous process measures with integrable variation are summable (Corollary 3.6). Then we define the stochastic integral for summable process measures and show that for right continuous process measures with integrable variation, the stochastic integral can be computed pathwise, as a Stieltjes integral (Theorem 3.10).

3.1. Summable process set functions.

Let (Ω, \mathcal{F}, P) be a probability space and (\mathcal{F}_t) a filtration satisfying the usual conditions. In this paragraph we denote by \mathcal{R} the ring of subsets of $\Omega \times \mathbb{R}_+$, generated by the sets $A \times \{0\}$ with $A \in \mathcal{F}_0$ and $A \times (s, t]$ with $A \in \mathcal{F}_s$. We also denote by \mathcal{S} the ring generated by the intervals $(x, y]$ in L , and by \mathcal{T} or $r(\mathcal{R} \times \mathcal{S})$, the ring generated by the semiring

$$\mathcal{R} \times \mathcal{S} = \{A \times B; A \in \mathcal{R}, B \in \mathcal{S}\}.$$

The σ -algebra generated by \mathcal{T} is the product σ -algebra $\mathcal{P} \otimes \mathcal{L}$, where \mathcal{P} is the predictable σ -algebra of $\Omega \times \mathbb{R}_+$.

Let F, G be Banach spaces with $E \subset L(F, G)$ and $1 \leq p < \infty$. Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ be a right continuous, adapted p -process measure. We define the finitely additive measure $I_X: \mathcal{T} \rightarrow L_E^p \subset L(F, L_G^p)$ by

$$I_X(A \times \{0\} \times (x, y]) = 1_A X_0((x, y]), \quad \text{for } A \in \mathcal{F}_0,$$

and

$$I_X(A \times (s, t] \times (x, y]) = 1_A (X_t((x, y]) - X_s((x, y])), \quad \text{for } A \in \mathcal{F}_s,$$

and extended by additivity to \mathcal{T} .

DEFINITION 3.1: We say that an adapted, right continuous process measure $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ is p -summable, relative to (F, G) , if I_X can be extended to a σ -additive measure $I_X: \mathcal{P} \otimes \mathcal{L} \rightarrow L_E^p \subset L(F, L_G^p)$ with finite semivariation $(\tilde{I}_X)_{F, L_G^p}$.

We shall prove below (Corollary 3.6) that if X is adapted, right continuous, σ -additive in L_E^p on \mathcal{L} and with integrable variation $|X|$, then X is summable.

Let now $X': \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ be another adapted, right continuous process measure, which is a modification of X , that is,

$$X'_t(\omega, B) = X_t(\omega, B), \quad \text{a.s. for } t \in \mathbb{R}_+ \text{ and } B \in \mathcal{L},$$

the negligible set depending on B . Then we have

$$I_{X'}(C) = I_X(C), \quad \text{a.s., for each } C \in \mathcal{T},$$

the negligible set depending on C . As elements in L_E^p , we have

$$I_{X'}(C) = I_X(C), \quad \text{for every } C \in \mathcal{I}.$$

It follows that, as L_E^p -valued additive measures, $I_{X'}$ and I_X have the same variation:

$$|I_{X'}|_{L_E^p} = |I_X|_{L_E^p}, \quad \text{on } \mathcal{I},$$

and the same semivariation relative to (F, L_G^p) ,

$$(\tilde{I}_{X'})_{F, L_G^p} = (\tilde{I}_X)_{F, L_G^p}, \quad \text{on } \mathcal{I}.$$

We deduce that X is p -summable iff X' is p -summable.

3.2. The Stochastic integral with respect to a summable process measure.

Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ be a right continuous, adapted process measure. Assume that $E \subset L(F, G)$ and that X is p -summable relative to (F, G) . Then we shall use the general theory of integration presented in [B-D.1] to define the Stochastic integral with respect to X .

Consider the σ -additive measure $I_X: \mathcal{P} \otimes \mathcal{L} \rightarrow L_E^p \subset L(F, L_G^p)$ with finite semivariation $(\tilde{I}_X)_{F, L_G^p}$. Let $1/p + 1/q = 1$ and let $Z \subset L_G^{q*}$ be a norming space for L_G^p . For each $z \in Z$, consider the measure $(I_X)_z: \mathcal{P} \otimes \mathcal{L} \rightarrow F^*$ defined by

$$\langle x, (I_X)_z(M) \rangle = \langle I_X(M)x, z \rangle, \quad \text{for } x \in F \text{ and } M \in \mathcal{P} \otimes \mathcal{L},$$

where the first bracket represents the duality between F and F^* , and the second bracket represents the duality between L_G^p and L_G^{q*} .

Each measure $(I_X)_z$ has finite variation $|(I_X)_z|$ and we have

$$(\tilde{I}_X)_{F, L_G^p}(M) = \sup_{\|z\|_q \leq 1} |(I_X)_z|(M), \quad \text{for } M \in \mathcal{P} \otimes \mathcal{L}.$$

Let D be a Banach space. For each $\mathcal{P} \otimes \mathcal{L}$ -measurable two parameter process $H: \Omega \times \mathbb{R}_+ \times L \rightarrow D$, we define

$$(\tilde{I}_X)(H) = \sup_{\|z\| \leq 1} \int |H| d|(I_X)_z| \leq +\infty.$$

We denote by $\mathcal{F}_D((I_X)_{F, G})$ the set of $\mathcal{P} \otimes \mathcal{L}$ measurable process $H: \Omega \times \mathbb{R}_+ \times L \rightarrow D$ with $(\tilde{I}_X)(H) < \infty$. Then $\mathcal{F}_D((I_X)_{F, G})$ is a vector space, complete for the seminorm $(\tilde{I}_X)(H)$.

If $F = D$ we denote $\mathcal{F}_{F, G}(I_X) = \mathcal{F}_F((I_X)_{F, G})$. For $H \in \mathcal{F}_{F, G}(I_X)$ we define the integral $\int H dI_X$ in the following way: we have

$$\mathcal{F}_{F, G}(I_X) \subset \prod_{z \in Z} L_F^1((I_X)_z).$$

Let $H \in \mathcal{F}_{F,G}(I_X)$. Then for every $z \in Z$ we have $H \in L_F^1((I_X)_z)$, hence the integral $\int H d(I_X)_z$ is defined and is a scalar. The mapping $z \mapsto \int H d(I_X)_z$ is a continuous linear functional on Z :

$$\left| \int H d(I_X)_z \right| \leq \tilde{I}_X(H) |z|, \quad \text{for } z \in Z.$$

We denote this linear functional by $\int H dI_X$ and call it the integral of H with respect to I_X . We have then $\int H dI_X \in Z^*$,

$$\left\langle \int H dI_X, z \right\rangle = \int H d(I_X)_z, \quad \text{for } z \in Z,$$

and

$$\left| \int H dI_X \right| \leq \tilde{I}_X(H).$$

If $H \in \mathcal{F}_{F,G}(I_X)$, then for every $t \geq 0$ and $B \in \mathcal{L}$ we have $1_{[0,t] \times B} H \in \mathcal{F}_{F,G}(I_X)$. We denote

$$\int_{[0,t] \times B} H dI_X = \int 1_{[0,t] \times B} H dI_X.$$

We are interested in processes for which $\int_{[0,t] \times B} H dI_X \in L_G^p$ for every $t \geq 0$ and $B \in \mathcal{L}$.

We denote by the same symbol the equivalence class $\int_{[0,t] \times B} H dI_X$ in L_G^p as well as any random variable belonging to this equivalence class. If we choose a representative from each equivalence class, we obtain a process set function

$$\left[\left(\int_{[0,t] \times B} H dI_X \right) (\omega) \right]_{t \geq 0, B \in \mathcal{L}}$$

with values in G . This process set function is automatically adapted and σ -additive in L_G^p on \mathcal{L} :

THEOREM 3.2: *Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E \subset L(F, G)$ be an adapted, right continuous p -process measure, and assume X is p -summable relative to (F, G) . Let $H \in \mathcal{F}_{F,G}(I_X)$*

be such that $\int_{[0, t] \times B} H dI_X \in L_G^p$ for every $t \geq 0$ and $B \in \mathcal{L}$. Then

$$\left[\left(\int_{[0, t] \times B} H dI_X \right) (\omega) \right]_{t \geq 0, B \in \mathcal{L}}$$

is an adapted p -process measure with values in G .

PROOF: a) We prove first that the mapping $B \mapsto \int_B H dI_X$ from \mathcal{L} into L_G^p is σ -additive. Let $z \in (L_G^p)^*$. Then $1_M H \in L_F^1((I_X)_z)$ for every $M \in \mathcal{P} \otimes \mathcal{L}$, and the indefinite integral $\int_M H d(I_X)_z = \left\langle \int_M H dI_X, z \right\rangle$ is σ -additive on $\mathcal{P} \otimes \mathcal{L}$. Taking $M = \Omega \times [0, t] \times B$ with t fix and $B \in \mathcal{L}$, it follows that the indefinite integral $\int_{[0, t] \times B} H dI_X$ is weakly σ -additive in L_G^p on \mathcal{L} , therefore, by the Pettis theorem, it is strongly σ -additive in L_G^p on \mathcal{L} , hence, $\left(\int_{[0, t] \times B} H dI_X \right)_{t \geq 0, B \in \mathcal{L}}$ is a p -process measure with values in G .

We prove now that this process is adapted. Let $t \geq 0$ and prove that $\int_{[0, t]} H dI_X$ is \mathcal{F}_t -measurable. Replacing H with $1_B H$ for $B \in \mathcal{L}$, it will follow that $\int_{[0, t] \times B} H dI_X$ is \mathcal{F}_t -measurable.

b) Assume that $H = 1_{A \times (u, v] \times B} x$ with $A \in \mathcal{F}_u$, $B \in \mathcal{L}$ and $x \in F$. By the definition of the integral we have

$$\int_{[0, t]} H dI_X = \int 1_{A \times (u \wedge t, v \wedge t] \times B} x dI_X = 1_A (X_{v \wedge t}(B) - X_{u \wedge t}(B)) x,$$

and the last term is equal to $1_A (X_{v \wedge t} - X_u) x$ if $u \leq t$, and to 0 if $t < u$; in both cases it is \mathcal{F}_t -measurable.

It follows that $\int_{[0, t]} 1_M x dI_X$ is \mathcal{F}_t -measurable for $x \in F$ and M in the ring $r(\mathcal{R} \times \mathcal{S})$ generated by the rectangular sets $A \times B$ with $A \in \mathcal{R}$ and $B \in \mathcal{S}$.

c) We prove now that $\int_{[0, t]} 1_M x dI_X$ is \mathcal{F}_t -measurable for $x \in F$ and $M \in \mathcal{P} \otimes \mathcal{L}$. Let $x \in F$ and denote by \mathcal{M}_0 the class of sets $M \in \mathcal{P} \otimes \mathcal{L}$ for which $\int_{[0, t]} 1_M x dI_X$ is \mathcal{F}_t -measurable. By step b), \mathcal{M}_0 contains the ring $r(\mathcal{R} \times \mathcal{S})$. We show that \mathcal{M}_0 is a monotone class. Let (M_n) be a monotone sequence in \mathcal{M}_0 with limit M . Let $z \in (L_G^p)^*$. Then $1_{[0, t]} 1_{M_n} x \rightarrow 1_{[0, t]} 1_M x$ in $L_F^p((I_X)_z)$, therefore

$$\int_{[0, t]} 1_{M_n} x d(I_X)_z \rightarrow \int_{[0, t]} 1_M x d(I_X)_z,$$

that is

$$\left\langle \int_{[0, t]} 1_{M_n} x dI_X, z \right\rangle \rightarrow \left\langle \int_{[0, t]} 1_M x dI_X, z \right\rangle.$$

It follows that

$$\int_{[0, t]} 1_{M_n} x dI_X \rightarrow \int_{[0, t]} 1_M x dI_X, \quad \text{weakly in } L_G^p = L_G^p(\mathcal{F}, P).$$

Now, by hypothesis, for each n we have $\int_{[0, t]} 1_{M_n} x dI_X \in L_G^p(\mathcal{F}_t, P)$. It follows that $\int_{[0, t]} 1_M x dI_X$ belongs to the weak closure in $L_G^p(\mathcal{F}, P)$ of the convex set $L_G^p(\mathcal{F}_t, P)$; and the weak closure of this convex set is equal to its strong closure, which, in turn, is $L_G^p(\mathcal{F}_t, P)$ itself. It follows that $\int_{[0, t]} 1_M x dI_X$ is \mathcal{F}_t -measurable, hence $M \in \mathcal{M}_0$. We deduce that $\mathcal{M}_0 = \mathcal{P} \otimes \mathcal{L}$ and this proves assertion c).

It follows that $\int_{[0, t]} H dI_X$ is \mathcal{F}_t -measurable for any $\mathcal{P} \otimes \mathcal{L}$ -step function

$$H: \Omega \times \mathbb{R}_+ \times L \rightarrow F.$$

d) Assume now that H is as in the statement. Since H is $\mathcal{P} \otimes \mathcal{L}$ -measurable, it is separably valued (by definition of measurability in Banach spaces), therefore there is a sequence (H^n) of F -valued $\mathcal{P} \otimes \mathcal{L}$ -step processes such that $H^n \rightarrow H$ pointwise and $|H^n| \leq |H|$.

For each $z \in (L_G^p)^*$ we have $H \in L_F^p((I_X)_z)$, hence, by Lebesgue's theorem, $1_{[0, t]} H^n \rightarrow 1_{[0, t]} H$ in $L_F^p((I_X)_z)$, therefore $\int_{[0, t]} H^n d(I_X)_z \rightarrow \int_{[0, t]} H d(I_X)_z$, consequently $\int_{[0, t]} H^n dI_X \rightarrow \int_{[0, t]} H dI_X$, weakly in $L_G^p(\mathcal{F}, P)$. Since, by step c), we have $\int_{[0, t]} H^n dI_X \in L_G^p(\mathcal{F}_t, P)$, it follows by the same argument in step c) that the limit $\int_{[0, t]} H dI_X$ also belongs to $L_G^p(\mathcal{F}_t, P)$ hence it is \mathcal{F}_t -measurable. This proves the theorem.

REMARK: If X and H are as in the statement of Theorem 3.2, it does not follow that an arbitrary choice of representatives

$$\left[\left(\int_{[0, t] \times B} H dI_X \right) (\omega) \right]_{t \geq 0, B \in \mathcal{L}}$$

is cadlag or that there is a cadlag choice. This leads us to the following definition.

DEFINITION 3.3: We denote by $L_{F, G}^1(X)$ the subspace of $\mathcal{F}_{F, G}(I_X)$ consisting of processes H satisfying the following conditions:

$$1) \int_{[0, t] \times B} H dI_X \in L_G^p, \text{ for every } t \in \mathbb{R}_+ \text{ and } B \in \mathcal{L};$$

2) The process set function $\left(\int_{[0, t] \times B} H dI_X \right)_{t \geq 0, B \in \mathcal{L}}$ has a cadlag modification.

The processes $H \in L_{F, G}^1(X)$ are said to be integrable with respect to X . Any right continuous process measure which is a modification of $\left(\int_{[0, t] \times B} H dI_X \right)_{t \geq 0, B \in \mathcal{L}}$ is called the stochastic integral of H with respect to X and is denoted $H \cdot X$:

$$(H \cdot X)_t(\omega, B) = \left(\int_{[0, t] \times B} H dI_X \right)(\omega), \quad \text{a.s.}$$

It follows that the stochastic integral $H \cdot X$ is defined up to an evanescent set, i.e. a subset of $N \times \mathbb{R}_+ \times L$ with N negligible.

From Theorem 3.2 and Definition 3.3 we deduce the following theorem.

THEOREM 3.4: For every $H \in L_{F, G}^1(X)$, the stochastic integral $H \cdot X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow G$ is a cadlag, adapted p -process measure.

REMARK: Assume X is p -summable and X' is a modification of X . Then we have $I_{X'} = I_X$ in L_E^p and $(I_{X'})_{F, L_G^p} = (\tilde{I}_X)_{F, L_G^p}$, $\mathcal{F}_{F, G}(I_{X'}) = \mathcal{F}_{F, G}(I_X)$ and $L_{F, G}^1(X') = L_{F, G}^1(X)$. Moreover, for $H \in L_{F, G}^1(X)$ we have $H \cdot X' = H \cdot X$.

It follows that the stochastic integration for process measures is invariant with respect to modifications. Then, for right continuous process measures with p -integrable variation, from the point of view of Stochastic integration, we can consider only process measures which are pathwise σ -additive in E , according to theorem 2.5.

3.3. Two parameter processes associated to process measures with integrable variation.

For right continuous, adapted process measures with integrable variation, their stochastic integral can be reduced to that of two parameter processes.

Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ be a right continuous, adapted p -process measure with p -integrable variation and let $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ be a right continuous, two parameter process with p -integrable variation, related to X by Theorem 2.4 and satisfying the following equality:

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{a.s. for } t \in \mathbb{R}_+ \text{ and } x \in L.$$

Then F is adapted to the filtration $(\mathcal{F}_{t, x})$ with $\mathcal{F}_{t, x} = \mathcal{F}_t$.

Let I_F be the L_E^p -valued additive measure defined on the ring \mathcal{T} by

$$I_F(A \times (s, t] \times (x, y]) = 1_A \Delta_{(s, t] \times (x, y]}(F) =$$

$$= 1_A (F(s, x) + F(t, y) - F(s, y) - F(t, x)), \quad \text{for } A \in \mathcal{F}_s,$$

and

$$I_F(A \times \{0\} \times (x, y]) = 1_A (F(\cdot, 0, y) - F(\cdot, 0, x)), \quad \text{for } A \in \mathcal{F}_0.$$

Then we have

$$I_F = I_X \quad \text{in } L_E^p \text{ on } \mathcal{T},$$

therefore, their variations as L_E^p -valued additive measures on \mathcal{T} are equal:

$$|I_F|_{L_E^p} = |I_X|_{L_E^p} \leq +\infty, \quad \text{on } \mathcal{T}.$$

In particular

$$|I_F|_{L_E^1} = |I_X|_{L_E^1} \leq +\infty, \quad \text{on } \mathcal{T}.$$

If F, G are Banach spaces with $E \subset L(F, G)$, hence $L_E^p \subset L(F, L_G^p)$, then the semivariations of I_F and I_X relative to (F, L_G^p) are also equal:

$$(\tilde{I}_F)_{F, L_G^p} = (\tilde{I}_X)_{F, L_G^p} \leq +\infty, \quad \text{on } \mathcal{T}.$$

In particular

$$(\tilde{I}_F)_{F, L_G^1} = (\tilde{I}_X)_{F, L_G^1} \leq +\infty, \quad \text{on } \mathcal{T}.$$

It follows that X is p -summable relative to (F, G) iff F is p -summable relative to (F, G) , and in this case we have

$$I_F = I_X, \quad \text{on } \mathcal{P} \otimes \mathcal{L}$$

and

$$\mathcal{F}_{F, G}(I_F) = \mathcal{F}_{F, G}(I_X).$$

For $H \in \mathcal{F}_{F, G}(I_F)$ we have $\int H dI_F = \int H dI_X$, in L_G^p (rather than in L_G^{p**} , since the simple processes are dense in $\mathcal{F}_{F, G}(I_F)$, see [D4], Theorem 4.7).

By ([D4], Theorem 4.6), I_F can be extended to a σ -additive measure $I_F: \mathcal{P} \otimes \mathcal{L} \rightarrow L_G^1$ with finite variation. We obtain then the following theorem.

THEOREM 3.5: *Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ be a right continuous, adapted process measure with integrable variation $|X|$, i.e. $|X|_\infty(\cdot, L) \in L^1$ and let $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ be a right continuous, adapted, two parameter process with integrable variation $|F|$, as-*

sociated to X by theorem 2.4, and satisfying

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{a.s., for } t \in \mathbb{R}_+ \text{ and } x \in L.$$

Then the measures I_X and I_F can be extended to σ -additive measures $I_X, I_F: \mathcal{P} \otimes \mathcal{L} \rightarrow L_E^1$ with finite variation $|I_X|$ and $|I_F|$, and we have

$$I_X = I_F \text{ and } |I_X| = |I_F| = I_{|F|}, \quad \text{on } \mathcal{P} \otimes \mathcal{L}.$$

If, in addition, X is pathwise σ -additive in E , then $|I_X| = I_{|X|}$.

PROOF: It remains only to prove the last equality. This follows from Theorems 2.2 and 2.3:

$$|X|_t(\omega, (-\infty, x]) = |F|(\omega, t, x), \quad \text{a.s.,}$$

the negligible set being independent of t and x , because of the right continuity of F and X and σ -additivity in E of X . Then, applying the first part of this theorem to $|X|$ and $|F|$, we deduce that $I_{|X|} = I_{|F|} = |I_F| = |I_X|$.

COROLLARY 3.6: If $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ is a right continuous, adapted process measure with integrable variation, then X is 1-summable relative to any embedding $E \subset L(F, G)$.

The stochastic integrals $(H \cdot F)$ and $(H \cdot X)$ are in the same kind of relationship as F and X :

COROLLARY 3.7: If X and F are as in the statement of Theorem 3.5, for $H \in L_{F, G}^1(X) \cap L_{F, G}^1(F)$ we have

$$(H \cdot F)(\omega, t, x) = (H \cdot X)(\omega, t, (-\infty, x]), \quad \text{a.s.}$$

for $t \geq 0$ and $x \in L$.

PROOF: Consider H, F and X extended with 0 for $t < 0$ or $x < 0$. Then we have, a.s.

$$\begin{aligned} (H \cdot F)(\omega, t, x) &= \left(\int_{[0, t] \times [0, x]} H dI_F \right)(\omega) = \left(\int_{(-\infty, t] \times (-\infty, x]} H dI_F \right)(\omega) = \\ &= \left(\int_{[-\infty, t] \times (-\infty, x]} H dI_X \right)(\omega) = (H \cdot X)(\omega, t, (-\infty, x]). \end{aligned}$$

The following theorem gives the relationship between the measures I_X and μ_X .

THEOREM 3.8: Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ be a right continuous, adapted process measure with integrable variation $|X|$ and let $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ be a right continuous,

adapted, two parameter process with integrable variation associated to X by Theorem 2.4 and satisfying

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{a.s., for } t \in \mathbb{R}_+ \text{ and } x \in L.$$

Let also $\mu_X: \mathcal{P} \otimes \mathcal{L} \rightarrow E$ be the σ -additive measure associated to X by theorem 2.6 and $\mu_F: \mathcal{P} \otimes \mathcal{L} \rightarrow E$ the σ -additive measure associated to F by ([L], Theorem IV.2.1; see also [D4], Theorem 3.1).

Then, for every $C \in \mathcal{P} \otimes \mathcal{L}$ we have

$$\mu_X(C) = \mu_F(C) = E(I_X(C)) = E(I_F(C))$$

and

$$|I_X|(C) = |\mu_X|(C) = |I_F|(C) = |\mu_F|(C) = \mu_{|F|}(C) = E(I_{|F|}(C)).$$

If, in addition, X is pathwise σ -additive in E , then

$$|I_X|(C) = |\mu_X|(C) = \mu_{|X|}(C) = E(I_{|X|}(C)),$$

for $C \in \mathcal{P} \otimes \mathcal{L}$.

PROOF: Let $C = A \times (s, t] \times (x, y]$ with $A \in \mathcal{F}_s$, with $s \leq t$ in \mathbb{R}_+ and $x \leq y$ in L . Then, by the definition of the Stieltjes measure in Sect. 2.4 we have

$$\begin{aligned} I_X(C)(\omega) &= 1_A(\omega) (X_t(\omega, (x, y]) - X_s(\omega, (x, y])) = \\ &= \int 1_C(\omega, t, x) m_{X(\omega)}(dt, dx) = \int 1_C(\omega, t, x) X(\omega, dt, dx), \end{aligned}$$

therefore

$$\mu_X(C) = E\left(\int 1_C(\omega, t, x) X(\omega, dt, dx)\right) = E(I_X(C)).$$

Since both μ_X and $E(I_X)$ are σ -additive on $\mathcal{P} \otimes \mathcal{L}$ and are equal on \mathcal{T} which generates $\mathcal{P} \otimes \mathcal{L}$, we deduce that

$$\mu_X(C) = E(I_X(C)), \quad \text{for } C \in \mathcal{P} \otimes \mathcal{L}.$$

Similar considerations for F and μ_F to deduce that

$$\mu_F(C) = E(I_F(C)), \quad \text{for } C \in \mathcal{P} \otimes \mathcal{L}.$$

From $I_F = I_X$ on $\mathcal{P} \otimes \mathcal{L}$ we deduce that

$$E(I_F(C)) = E(I_X(C)), \quad \text{for } C \in \mathcal{P} \otimes \mathcal{L},$$

and this proves the first series of equalities. To prove the second series of equalities, we notice that from $\mu_X = \mu_F$ we deduce that $|\mu_X| = |\mu_F|$. We have also $|\mu_F| = \mu_{|F|}$. Applying the first part of the theorem to $|F|$, we deduce that

$$\mu_{|F|}(C) = E(I_{|F|}(C)), \quad \text{for } C \in \mathcal{P} \otimes \mathcal{L}.$$

Finally, from $I_X = I_F$ we deduce that

$$|I_X| = |I_F| = |\mu_F| = |\mu_X|.$$

The last series of equalities follow from Theorem 2.6: $|\mu_X| = \mu_{|X|}$, in case X is pathwise σ -additive in E . This completes the proof of the theorem.

We give now the relationship between the Bochner integrable processes with respect to $|I_X|$, and the processes which are «stochastic» integrable with respect to $(\tilde{I}_X)_{F, L_G^1}$.

THEOREM 3.9: *Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ be a right continuous, adapted process measure with integrable variation $|X|$. Then the family of measures*

$$(I_X)_{F, L_G^1} := \{ |(I_X)_z| : z \in L_G^{\infty*}, \|z\|_{\infty} \leq 1 \}$$

is uniformly σ -additive, and we have

$$L_F^1(\mathcal{P} \otimes \mathcal{L}, |\mu_X|) = L_F^1(\mathcal{P} \otimes \mathcal{L}, |I_X|) \subset L_{F, L_G^1}^1(X),$$

and

$$\mathcal{F}_{F, L_G^1}(S(\mathcal{R} \times \mathcal{L}), X) = \mathcal{F}_{F, L_G^1}(X),$$

where the left hand side is the closure in $\mathcal{F}_{F, L_G^1}(X)$ of the space of $\mathcal{R} \times \mathcal{L}$ step processes.

PROOF: The first equality of spaces follows from $|\mu_X| = |I_X|$ and the inclusion follows from $(\tilde{I}_X)_{F, L_G^1} \leq |I_X|$.

To prove the uniform σ -additivity, let $z \in L_G^{\infty*}$ with $\|z\|_{\infty} \leq 1$. Then the variation $|(I_X)_z|$ of the measure $(I_X)_z$ satisfies $|(I_X)_z| \leq |I_X|$, hence the family $(I_X)_{F, L_G^1}$ is uniformly σ -additive. Then by ([B-D.1], A18a and b) we have $\mathcal{F}_{F, L_G^1}(\mathcal{B}, C) = \mathcal{F}_{F, L_G^1}(X)$, where $\mathcal{F}_{F, L_G^1}(\mathcal{B}, X)$ is the closure in $\mathcal{F}_{F, L_G^1}(X)$ of the bounded processes. From ([B-D.1], AI 11b) it follows that

$$\mathcal{F}_{F, L_G^1}(S(\mathcal{R} \times \mathcal{L}), X) = \mathcal{F}_{F, L_G^1}(\mathcal{B}, X),$$

and this proves the theorem.

3.4. The Stochastic Integral as a Stieltjes Integral.

One of the main results of this paper is the following theorem which shows that the stochastic integral of right continuous, adapted process measures with integrable variation can be computed pathwise, as a Stieltjes integral, like in the case of two parameter, or one parameter right continuous, adapted processes with integrable variation ([B-D.1], Theorem 3.3 and [D4], Theorem 4.9).

THEOREM 3.10: *Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E \subset L(F, G)$ be a right continuous, adapted process measure with integrable variation $|X|$ and let $X': \Omega \times \mathbb{R}_+ \times \mathcal{L} \rightarrow E$ be a right*

continuous process measure with integrable variation $|X'|$, pathwise σ -additive in E and a modification of X .

Let $H \in \mathcal{F}_{F, G}(X)$ be a process with separable range and with

$$\int |H(\omega, t, x)| |X'|(\omega, dt, dx) < \infty.$$

Then $H \in L_{F, G}^1(X)$ and the stochastic integral $H \cdot X$ can be computed pathwise as a Stieltjes integral (in the sense of Section 2.4):

$$(H \cdot X)_t(\omega, B) = \int_{[0, t] \times B} H(\omega, s, x) X(\omega, ds, dx),$$

for $\omega \in \Omega$, $t \in \mathbb{R}_+$ and $B \in \mathcal{L}$.

PROOF: Let $F: \Omega \times \mathbb{R}_+ \times L \rightarrow E$ be the right continuous, two parameter process with integrable variation $|F|$, defined by

$$F(\omega, t, x) = X'_t(\omega, (-\infty, x]), \quad \text{for } t \in \mathbb{R}_+ \text{ and } x \in L.$$

(See Theorems 2.2 and 2.3). Then $I_F = I_{X'} = I_X$ on $\mathcal{P} \otimes \mathcal{L}$, hence $\mathcal{F}_{F, G}(I_F) = \mathcal{F}_{F, G}(I_{X'}) = \mathcal{F}_{F, G}(I_X)$. We have also

$$|F|(\omega, t, x) = |X'|_t(\omega, (-\infty, x]), \quad \text{everywhere,}$$

hence

$$|I_X| = |I_{X'}| = |I_F| = I_{|F|} = I_{|X'|}.$$

Let $C \in \mathcal{P} \otimes \mathcal{L}$. Then

$$\int_C |H(\omega, t, x)| |F|(\omega, dt, dx) = \int_C |H(\omega, t, x)| |X'|(\omega, dt, dx) < \infty.$$

By ([D4], Theorem 4.8) we have $1_C H \in L_{F, G}^1(F)$, hence $\int_C H dI_F \in L_G^1$ and the stochastic integral $H \cdot F$ can be computed pathwise, as a Stieltjes integral:

$$\int_C H dI_F = (1_C H \cdot F)_\infty(\omega, \infty) = \int_{\mathbb{R}_+ \times L} (1_C H)(\omega, t, x) F(\omega, dt, dx).$$

Since $I_X = I_F$ and $1_C H \in \mathcal{F}_{F, G}(I_X) = \mathcal{F}_{F, G}(I_F)$, we have

$$\int_C H dI_X = \int_C H dI_F = \int_C H(\omega, t, x) F(\omega, dt, dx).$$

In particular, for $C = [0, t] \times B$ with $t \in \mathbb{R}_+$ and $B \in \mathcal{L}$, we have

$$\int_{[0, t] \times B} H dI_X = \int_{[0, t] \times B} H(\omega, t, x) F(\omega, dt, dx) \in L_G^1.$$

The last Stieltjes integral is right continuous in t and σ -additive in B on \mathcal{L} . It follows

that $H \in L^1_{F,G}(X)$, that is, H is integrable with respect to X , and

$$\begin{aligned} (H \cdot X)_t(\omega, B) &= \int_{[0, t] \times B} H(\omega, t, x) F(\omega, dt, dx) = \\ &= \int_{[0, t] \times B} H(\omega, t, x) X'(\omega, dt, dx) = \int_{[0, t] \times B} H(\omega, t, x) X(\omega, dt, dx), \end{aligned}$$

according to Section 2.4, and the theorem is proved.

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