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Stochastic Integral of Process Measures in Banach Spaces II. Process Measures with Integrable Semivariation (**)

SUMMARY. — In this paper we prove that a right continuous process measure X with integrable semivariation is summable and that the stochastic integral $H \cdot X$ can be computed pathwise as a Stieltjes integral of a special kind.

Integrazione stocastica rispetto a una misura-processo in uno spazio di Banach Parte II: caso di una misura-processo con semivariazione integrabile

SOMMARIO. — Nella presente memoria si prova che, se una misura-processo X è continua a destra e con semivariazione integrabile, allora essa è sommabile, e il corrispondente integrale stocastico $H \cdot X$ può essere calcolato «per traiettorie» come un particolare integrale di Stieltjes.

INTRODUCTION

This is a continuation of a previous paper [D5], in which we studied the stochastic integral of process measures with integrable variation in Banach spaces. In this paper we consider process measures with integrable *semivariation*, rather than integrable variation, and study their stochastic integral.

The framework for this paper consists of a probability space $(\Omega, \mathcal{F}, P)$, a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions, a Lusin space L with its Borel σ-algebra $\mathcal{L}$, $1 \leq p < \infty$ and E, F, G Banach spaces with $E \subset L(F, G)$ isometrically. Without loss of generality we take $L = \mathbb{R}$ and $\mathcal{L} = \mathcal{B}(\mathbb{R})$. We study process measures X: $\Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ which have finite pathwise semivariation $\widetilde{X}_{F,G}$ relative to (F, G).

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The summability of such a process measure and the stochastic integral $H \cdot X$ are defined in ([D5], Section 3.1 and 3.2). The main results of this paper are in Sect. 3, namely:

1) *If $c_0 \not\subset E$ and if X is a right continuous, adapted p-process measure with p-integrable semivariation, then X is p-summable* (*Theorem* 3.1).

2) *If $c_0 \not\subset E$ and if X is a right continuous, adapted p-process measure with p-integrable semivariation, then, for certain processes $H: \Omega \times \mathbb{R}_+ \times L \to F$, the stochastic integral $H \cdot X$ can be computed pathwise as a Stieltjes integral.*

$$(H \cdot X)(\omega, t, B) = \int_{[0,t] \times B} H(\omega, t, x)\, X(\omega, dt, dx).$$

(*See theorem* 3.4).

Similar results were obtained in [D5] for process measures with integrable *variation* — without assuming $c_0 \not\subset E$ — and in [D2] and [D3] for one or two parameter processes with integrable variation or p-integrable semivariation.

The condition $c_0 \not\subset E$ is imposed to insure that certain measures with bounded semivariation on a ring can be extended to σ-additive measures on the σ-ring generated by the ring (See [D2], Corollary 1.2). Such a condition is not necessary for measures with bounded variation.

There is a close relationship between process measures X and two parameter processes $F: \Omega \times \mathbb{R}_+ \times L \to E$. The two parameter processes are studied in ([D3], Sect. 3). The relationship between X and F is studied in this paper, in Sect. 2. The following are the main results of this paragraph:

3) *Assume $c_0 \not\subset E$ and E is separable. If X is a right continuous p-process measure with p-integrable semivariation, then there is a right continuous, two parameter process F with p-integrable semivariation, satisfying*

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \qquad a.s.$$

for $t \in \mathbb{R}_+$ and $x \in L$, the negligible set depending on x only. (*Theorem* 2.5).

4) *Assume $c_0 \not\subset E$ and that E and G are separable. If X is a right continuous process measure with integrable semivariation, then there exists a P-measure $\mu_X: \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{L} \to E$ with finite semivariation $\tilde{\mu}_X$ such that*

$$\int H d\mu_X = E\left(\int H(\omega, t, x)\, X(\omega, dt, dx)\right),$$

for certain processes $H \in L^1_{F,G}(\mu_X)$. (*Theorem* 2.7).

The Stieltjes-type integral appearing in 1) and 4) above, is of a special kind, and it is defined and studied in Sect. 1. To each right continuous function measure g: $\mathbb{R}_+ \times \mathcal{L} \to E$ with bounded semivariation $\tilde{g}$, we associate a measure m_g: $\mathcal{B}(\mathbb{R}_+) \otimes$

$\otimes\mathcal{L}\to E$ with bounded semivariation $\tilde{m}_g$. We apply then the integration theory presented in [B-D.1] and [B-D.2], to define the integral $\int f dm_g$ for certain functions f: $\mathbb{R}_+ \times L \to F$. The Stieltjes integral $\int f dg$ is defined by the equality $\int f dg = \int f dm_g$. In [D2] and [D3] we defined the Stieltjes integral for functions g of one or two variables, with bounded variation or bounded semivariation and in [D5] we defined the Stieltjes integral $\int f dg$ for function measures g with bounded variation.

There is a close relationship between the Stieltjes integral $\int f dg$ with respect to a function measure g, and the Stieltjes integral $\int f dG$ with respect to a function of two variables G, related to g by the equality

$$G(t, x) = g(t, (-\infty, x]), \qquad \text{for } t \in \mathbb{R}_+ \text{ and } x \in L.$$

(See Sect. 1.3). We mention also the following interesting result (Corollary 1.5): If $c_0 \not\subset E$ and if g is a function measure with bounded semivariation and weakly right continuous, then it is strongly right continuous.

In a forthcoming paper [Di-Mu] we shall study the Stochastic integral of square integrable martingale measures with values in a Hilbert space.

Notations

In general, we adopt the definitions and notations used in [D-M].

1) $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_t)_{t \geq 0}$ is a filtration satisfying the usual conditions. $\mathcal{R}$ is the ring generated by the subsets of $\Omega \times \mathbb{R}_+$ of the form $\{0\} \times A$ with $A \in \mathcal{F}_0$ and $(s, t] \times A$ with $A \in \mathcal{F}_s$. The σ-algebra generated by $\mathcal{R}$ is the σ-algebra $\mathcal{P}$ of predictable sets. Any process will be automatically considered to be extended to $\Omega \times \mathbb{R}$, with 0 for $t < 0$. In Sect. 1, $\mathcal{R}$ is the ring generated by the intervals $(s, t]$ in $\mathbb{R}$.

2) $(L, \mathcal{L})$ is a Lusin space endowed with its Borel σ-algebra $\mathcal{L}$. We shall take $L = \mathbb{R}$ and $\mathcal{L} = \mathcal{B}(\mathbb{R})$, but we shall maintain the notations L and $\mathcal{L}$. We assume that any measure μ on $\mathcal{L}$ has its support in $(0, \infty)$ and that any function on L vanishes on $(-\infty, 0]$.

$\mathcal{S}$ is the ring generated by the intervals $(x, y]$ in L. The σ-algebra generated by $\mathcal{S}$ is $\mathcal{L}$. We denote by $\mathcal{R} \times \mathcal{S}$ the semiring of rectangular sets $A \times B$ with $A \in \mathcal{R}$ and $B \in \mathcal{S}$, and by $\mathcal{T} = r(\mathcal{R} \times \mathcal{S})$ the ring generated by $\mathcal{R} \times \mathcal{S}$.

3) E, F, G are Banach spaces with $E \subset L(F, G)$ isometrically; $1 \leq p < \infty$ and $L_E^p = L_E^p(P)$.

We shall use the letters F and G also to represent functions of two variables, or two parameter processes. It will be clear from the context and from the notations the precise meaning of the letters F and G.

1. - Function measures with finite semivariation

The purpose of this paragraph is to define the Stieltjes integral $\int f dg$ with respect to a function measure g with bounded semivariation $\tilde{g}$. In this paragraph we denote by $\mathcal{R}$ the ring generated by the intervals $(s, t]$ in $\mathbb{R}$, and $\mathcal{K}$ is a ring of subsets of L such that $\mathcal{S} \subset \mathcal{K} \subset \mathcal{L}$.

1.1. *Function measures.*

We start with the definition of function measures.

Definition 1.1: *A function $g: \mathbb{R} \times \mathcal{K} \to E$ is said to be right continuous, if for every set $B \in \mathcal{K}$, the function $t \mapsto g(t, B)$ is right continuous.*

We say a function $g: \mathbb{R} \times \mathcal{K} \to E$ is a function measure on $\mathcal{K}$ if for every $t \in \mathbb{R}$, the set function $B \mapsto g(t, B)$ is σ-additive on $\mathcal{K}$.

Next, we define the semivariation of g (the variation of g was defined in ([D5], Definition 1.1.).

Definition 1.2: *Let $g: \mathbb{R} \times \mathcal{K} \to E \subset L(F, G)$ be a function. For any interval $I \subset \mathbb{R}$ and any set $B \in \mathcal{L}$, we define the semivariation of g on $I \times B$, relative to the pair (F, G), by the following equality:*

$$\operatorname{svar}_{F,G}(g, I \times B) = \sup \left| \sum_{i,j} [g(t_{i+1}, B_j) - g(t_i, B_j)] x_{ij} \right|$$

where the supremum is taken for all finite divisions d: $t_0 < t_1 < \ldots t_n$ of points from I, all finite families $(B_j)_{1 \leq j \leq m}$ of disjoint sets from $\mathcal{K}$ contained in B and all families $(x_{ij})_{0 \leq i < n,\, 1 \leq j \leq m}$ of elements from F with $|x_{ij}| \leq 1$.

The *semivariation function* $\tilde{g}_{F,G}: \overline{\mathbb{R}} \times \mathcal{L} \to \mathbb{R}_+$ is defined by

$$\tilde{g}_{F,G}(t, B) = \operatorname{svar}_{F,G}(g, (-\infty, t] \times B), \qquad \text{if } t < \infty,$$

and

$$\tilde{g}_{F,G}(\infty, B) = \operatorname{svar}_{F,G}(g, \mathbb{R} \times B).$$

In particular, for $t = -\infty$ we have $\tilde{g}_{F,G}(-\infty, B) = 0$.

We say that g has *finite* (respectively *bounded*) *semivariation* if $\tilde{g}_{F,G}(t, L) < \infty$ for every $t \in \mathbb{R}$ (respectively if $\tilde{g}_{F,G}(\infty, L) < \infty$).

There is an alternative way to define the semivariation.

Let $g: \mathbb{R} \times \mathcal{K} \to E \subset L(F, G)$ (isometrically) be a function and let $Z \subset G^*$ be a norming space for G, i.e.

$$|y| = \sup\{|\langle y, z\rangle| : z \in Z, \quad |z| \leq 1\}, \qquad \text{for each } y \in G.$$

For each $z \in Z$ we define the function g_z: $\mathbb{R} \times \mathcal{K} \to F^*$ by

$$\langle x, g_z(t, B)\rangle = \langle g(t, B)\, x, z\rangle, \qquad \text{for } x \in F,\ t \in \mathbb{R} \text{ and } B \in \mathcal{K}.$$

We denote by $|g_z|$ the variation function of g_z. Then for every interval $I \subset \mathbb{R}$ and any set $B \in \mathcal{L}$ we have

$$\operatorname{svar}_{F,G}(g, I \times B) = \sup\{\operatorname{var}(g_z, I \times B) \colon z \in Z,\ |z| \leq 1\},$$

and for every $t \in \mathbb{R}$ and $B \in \mathcal{L}$ we have

$$\tilde{g}_{F,G}(t, B) = \sup\{|g_z|(t, B) \colon z \in Z,\ \ |z| \leq 1\}.$$

If g has finite semivariation and is right continuous, then each g_z has finite variation and is right continuous, therefore by ([D5], Theorem 1.3), the variation $|g_z|$ is right continuous.

1.2. *Measures associated with function measures.*

Let g: $\mathbb{R} \times \mathcal{K} \to E$ be a function. We associate to it an additive measure m_g: $\mathcal{R} \times \mathcal{K} \to E$ by

$$m_g((s, t] \times B) = g(t, B) - g(s, B), \qquad \text{for } s \leq t \text{ in } \mathbb{R} \text{ and } B \in \mathcal{K},$$

and then we extend it by additivity to the ring $r(\mathcal{R} \times \mathcal{K})$ generated by the semiring of rectangles $(s, t] \times B$. We denote by $(\tilde{m}_g)_{F,G}$ the semivariation of m_g relative to the pair (F, G). The relationship between the semivariations of g and m_g is stated in the following proposition:

PROPOSITION 1.3: *For any interval $I \subset \mathbb{R}$ and any set $B \in \mathcal{L}$ we have*

$$(\tilde{m}_g)_{F,G}(I \times B) = \operatorname{svar}_{F,G}(g, I \times B).$$

In particular, for $I = (-\infty, t]$, we have

$$(\tilde{m}_g)_{F,G}((-\infty, t] \times B) = \tilde{g}_{F,G}(t, B).$$

PROOF: Let $I \subset \mathbb{R}$ be an interval and $B \in \mathcal{L}$. Let d: $t_0 < t_1 < \ldots t_n$ be a division consisting of points from I, $(B_j)_{1 \leq j \leq m}$ a family of disjoint sets from $\mathcal{K}$ contained in B, and $(x_{ij})_{0 \leq i < n,\, 1 \leq j \leq m}$ a family of points from F with $|x_{ij}| \leq 1$. Then

$$\left|\sum_{ij}[g(t_{i+1}, B_j) - g(t_i, B_j)]\, x_{ij}\right| = \left|\sum_{ij} m_g((t_i, t_{i+1}] \times B_j)\, x_{ij}\right| \leq (\tilde{m}_g)_{F,G}(I \times B),$$

hence

$$\operatorname{svar}_{F,G}(g, I \times B) \leqslant (\tilde{m}_g)_{F,G}(I \times B).$$

Conversely, let $(A_k)_{k \in K}$ be a finite family of disjoint sets from $\mathcal{R}$ contained in I and $(B_j)_{j \in J}$ a finite family of disjoint sets from $\mathcal{K}$ contained in B, and let $(x_{kj})_{(k,j) \in K \times J}$ be a family of elements from F with $|x_{kj}| \leqslant 1$. Each set A_k is a finite union of disjoint intervals,

$$A_k = \bigcup_{i \in I_k} (s_{k,i}, t_{k,i}].$$

We can arrange all the points $s_{k,i}$, $t_{k,i}$ in an increasing sequence $u_0 < u_1 < \ldots < u_m$. If the interval $(s_{k,i}, t_{k,i}]$ is equal to one of the intervals $(u_r, u_{r+1}]$, we set $y_{rj} = x_{kj}$ for every j; if $(s_{k,i}, t_{k,i}]$ is not among the intervals $(u_r, u_{r+1}]$, we set $y_{rj} = 0$ for every j. Then

$$\left| \sum_{k,j} m_g(A_k \times B_j)\, x_{kj} \right| = \left| \sum_{\substack{j \in J \\ 0 \leqslant r < m}} m_g((u_r, u_{r+1}] \times B_j)\, y_{kj} \right| =$$

$$= \left| \sum_{j,r} [g(u_{r+1}, B_j) - g(u_r, B_j)]\, y_{kj} \right| \leqslant \operatorname{svar}_{F,G}(g, I \times B),$$

therefore

$$(\tilde{m}_g)_{F,G}(I \times B) \leqslant \operatorname{svar}_{F,G}(g, I \times B),$$

and this proves the proposition.

Next we want to see whether m_g can be extended to a σ-additive measure on $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$.

For right continuous function measures with bounded *variation*, this was done in ([D5], Theorem 1.6).

We state now the analog for right continuous function measures with bounded *semivariation*.

THEOREM 1.4: *Assume $c_0 \not\subset E$ and let $Z \subset E^*$ be a norming space for E.*

a) Let $g: \mathbb{R} \times \mathcal{K} \to E$ be a function measure with bounded semivariation $\tilde{g}_{\mathbb{R},E}$. Then g is right continuous in E iff $x^* g$ is right continuous for $x^* \in Z$.

b) If $g: \mathbb{R} \times \mathcal{K} \to E \subset L(F, G)$ is a right continuous function measure with bounded semivariation $\tilde{g}_{F,G}$, then it can be extended to a right continuous function measure from $\mathbb{R} \times \mathcal{L}$ into E, with bounded semivariation relative to (F, G). The extension will be still denoted by g.

c) If $g: \mathbb{R} \times \mathcal{K} \to E \subset L(F, G)$ is a right continuous function measure with bounded semivariation $\tilde{g}_{F,G}$, then the additive measure $m_g: \mathcal{R} \times \mathcal{K} \to E$ defined by

$$m_g((s, t] \times B) = g(t, B) - g(s, B), \qquad \textit{for } s \leq t \textit{ in } \mathbb{R} \textit{ and } B \in \mathcal{K},$$

can be extended to a σ-additive measure $m: \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \to E$ with finite semivariation $(\tilde{m})_{F,G}$ and the semivariations $(\tilde{m}_g)_{F,G}$ and $\tilde{m}_{F,G}$ are equal on the ring $r(\mathcal{R} \times \mathcal{K})$. The measure m will be still denoted by m_g.

Proof: We shall prove *a)*, *b)* and *c)* simultaneously, with $F = \mathbb{R}$ and $G = E$ for assertion a). Since g has bounded semivariation $\tilde{g}_{F,G}$, by Proposition 1.3, m_g has bounded semivariation $(\tilde{m}_g)_{F,G}$ on $r(\mathcal{R} \times \mathcal{K})$. In fact, for any $C \in r(\mathcal{R} \times \mathcal{K})$ we have

$$(\tilde{m}_g)_{F,G}(C) \leq (\tilde{m}_g)_{F,G}(\mathbb{R} \times L) = \tilde{g}_{F,G}(\infty, L) < \infty.$$

Since $E \subset L(F, G)$ isometrically, we have

$$|m_g(C)| \leq (\tilde{m}_g)_{F,G}(C) \leq \tilde{g}_{F,G}(\infty, L) < \infty,$$

for $C \in r(\mathcal{R} \times \mathcal{K})$, hence m_g is bounded in $r(\mathcal{R} \times \mathcal{K})$. Assume that for each $x^* \in Z$, $x^* g$ is a right continuous scalar function measure with bounded variation $|x^* g| \leq \tilde{g}_{F,G}$. Then, by ([D5], Theorem 1.6), the corresponding measure$m_{x^* g}: r(\mathcal{R} \times \mathcal{K}) \to E$ can be extended to a σ-additive measure on $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$. In particular, $m_{x^* g}$ is σ-additive on $r(\mathcal{R} \times \mathcal{K})$. But $m_{x^* g} = x^* m_g$, hence $x^* m_g$ is σ-additive on $r(\mathcal{R} \times \mathcal{K})$. Since $c_0 \not\subset E$ and since m_g is bounded, by ([D2], Corollary 1.2), m_g can be extended to a σ-additive measure $m: \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \to E$. To prove that g is right continuous in E, let $t_n \downarrow t$ in $\mathbb{R}$ and $B \in \mathcal{K}$. Then $(t, t_n] \times B \downarrow \phi$, therefore $m_g((t, t_n] \times B) \to 0$, consequently

$$g(t_n, B) - g(t, B) = m_g((t, t_n] \times B) \to 0,$$

hence the function $t \mapsto g(t, B)$ is right continuous. Conversely, if g is right continuous, then $x^* g$ is right continuous for every $x^* \in Z$, and this proves assertion *a)*. To prove assertion *c)*, we notice that since m_g has bounded semivariation $(\tilde{m}_g)_{F,G}$ on $r(\mathcal{R} \times \mathcal{K})$, its extension m has bounded semivariation $\tilde{m}_{F,G}$ on $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$, and the two semivariations are equal on $r(\mathcal{R} \times \mathcal{K})$ ([D2], Theorem 1.4].

To prove *b)*, we define $g(t, B) = m_g((-\infty, t], B)$ for $t \in \mathbb{R}$ and $B \in \mathcal{L}$. Then $g: \mathbb{R} \times \mathcal{L} \to E$ is a right continuous function measure with bounded semivariation $\tilde{g}$, and we still have

$$m_g((s, t] \times B) = g(t, B) - g(s, B), \qquad \text{for } s \leq t \text{ in } \mathbb{R} \text{ and } B \in \mathcal{L}.$$

Remark: If $c_0 \not\subset E$, assertion *b)* of theorem 1.4 entitles us to consider the right continuous function measures with bounded variation defined on the whole set $\mathbb{R} \times \mathcal{L}$ (rather than on $\mathbb{R} \times \mathcal{K}$).

COROLLARY 1.5: *Assume $c_0 \not\subset E$. If a function measure $g: \mathbb{R} \times \mathcal{K} \to E$ with bounded semivariation $\tilde{g}_{\mathbb{R}, E}$ is weakly right continuous, then g is right continuous.*

1.3. *Functions of two variables associated to function measures.*

Let $g: \mathbb{R} \times \mathcal{S} \to E \subset L(F, G)$ be a function measure. We associate to g a function of two variables $G: \mathbb{R} \times L \to E$, by the equality

$$G(t, x) = g(t, (-\infty, x]), \qquad \text{for } t \in \mathbb{R} \text{ and } x \in L.$$

The same letter G is used above for a Banach space, but it will be clear from the context whether G represents a function or a Banach space. In this section we study the relationship between the properties of g and G.

The semivariation of G on a rectangle $I \times J \subset \mathbb{R} \times L$, relative to (F, G), where I and J are intervals, is defined by:

$$\operatorname{svar}_{F, G}(G, I \times J) = \sup \left| \sum_{i,j} \Delta_{(t_i, t_{i+1}] \times (x_j, x_{j+1}]}(G)\, x_{ij} \right| =$$

$$= \sup \left| \sum_{i,j} [G(t_i, x_j) + G(t_{i+1}, x_{j+1}) - G(t_i, x_{j+1}) - G(t_{i+1}, x_j)]\, x_{ij} \right|$$

where the supremum is taken for all divisions $t_0 < t_1 < \ldots t_n$ of points from I, all divisions $x_0 < x_1 < \ldots x_m$ of points from J, and all families $(x_{ij})_{0 \leq i < n,\, 0 \leq j < m}$ of elements of F with $|x_{ij}| \leq 1$.

For every $t \in \mathbb{R}$ and $x \in L$ we denote

$$\tilde{G}_{F, G}(t, x) = \operatorname{svar}_{F, G}(G, (-\infty, t] \times (-\infty, x])$$

and also

$$\tilde{G}_{F, G}(t, \infty) = \operatorname{svar}_{F, G}(G, (-\infty, t] \times L),$$

$$\tilde{G}_{F, G}(\infty, x) = \operatorname{svar}_{F, G}(G, \mathbb{R} \times (-\infty, x])$$

and

$$\tilde{G}_{F, G}(\infty, \infty) = \operatorname{svar}_{F, G}(G, \mathbb{R} \times L).$$

To the function G we associate a finitely additive measure $m_G: r(\mathcal{R} \times \mathcal{S}) \to \to E \subset L(F, G)$, defined by

$$m_G((s, t] \times (x, y]) = \Delta_{(s, t] \times (x, y]}(G) = G(s, x) + G(t, y) - G(s, y) - G(t, x),$$

for $s \leq t$ in $\mathbb{R}$ and $x \leq y$ in L, and then extended by additivity to $r(\mathcal{R} \times \mathcal{S})$.

The first property is that the semivariations of g and G are equal:

PROPOSITION 1.6: *For any intervals $I \subset \mathbb{R}$ and $J \subset L$ we have*

$$\operatorname{svar}_{F,G}(G, I \times J) = \operatorname{svar}_{F,G}(g, I \times J).$$

In particular,

$$\tilde{G}_{F,G}(t, x) = \tilde{g}_{F,G}(t, (-\infty, x]), \qquad \text{for } t \in \mathbb{R} \text{ and } x \in L.$$

We have also

$$m_G = m_g \quad \text{and} \quad (\tilde{m}_G)_{F,G} = (\tilde{m}_g)_{F,G} \text{ on } r(\mathcal{R} \times \mathcal{S}).$$

PROOF: From the above definition of G and m_G it follows that

$$m_G((s, t] \times (x, y]) = m_g((s, t] \times (x, y]),$$

therefore $m_G = m_g$ on $r(\mathcal{R} \times \mathcal{S})$, consequently $(\tilde{m}_G)_{F,G} = (\tilde{m}_g)_{F,G}$ on $r(\mathcal{R} \times \mathcal{S})$. By proposition 1.3 we have

$$(\tilde{m}_g)_{F,G}(I \times J) = \operatorname{svar}(g, I \times J).$$

By ([D3], Proposition 2.1), (with g replaced by G) we have

$$\tilde{m}_G(I \times J) = \operatorname{svar}_{F,G}(G, I \times J).$$

Since $m_G = m_g$ on $r(\mathcal{R} \times \mathcal{S})$, we deduce that

$$\operatorname{svar}_{F,G}(G, I \times J) = \operatorname{svar}_{F,G}(g, I \times J),$$

and the proposition is proved.

The following theorems state the relationship between the properties of right continuity and bounded semivariation of G and g.

THEOREM 1.7: *Assume $c_0 \not\subset E$. Let $g: \mathbb{R} \times \mathcal{S} \to E \subset L(F, G)$ be a right continuous function measure with bounded semivariation $\tilde{g}_{F,G}$ and let $G: \mathbb{R} \times L \to E$ be the function defined by*

$$G(t, x) = g(t, (-\infty, x]), \qquad \text{for } t \in \mathbb{R} \text{ and } x \in L.$$

Then G is right continuous and has bounded semivariation $\tilde{G}_{F,G}$ satisfying

$$\tilde{G}_{F,G}(t, x) = \tilde{g}_{F,G}(t, (-\infty, x]), \qquad \text{for } t \in \mathbb{R} \text{ and } x \in L.$$

Moreover, m_G and m_g can be extended to σ-additive measures on $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$ and we have

$$m_G = m_g \quad \text{and} \quad (\tilde{m}_G)_{F,G} = (\tilde{m}_g)_{F,G} \text{ on } \mathcal{B}(\mathbb{R}) \otimes \mathcal{L}.$$

PROOF: The function G is separately right continuous in t (since g is right continuous in t) and right continuous in x (since $B \mapsto g(t, B)$ is σ-additive on $\mathcal{L}$).

Since g has bounded semivariation $\tilde{g}_{F,G}$, from Proposition 1.6 we deduce that G has bounded semivariations $\tilde{G}_{F,G}$ and that

$$\tilde{G}_{F,G}(t, x) = \tilde{g}_{F,G}(t, (-\infty, x]), \qquad \text{for } t \in \mathbb{R} \text{ and } x \in L.$$

It remains to prove that G is right continuous. By Theorem 1.4, the measure m_g can be extended to a σ-additive measure on $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$; in particular m_g is σ-additive on $r(\mathcal{R} \times \mathcal{S})$. Then m_G is also σ-additive on $r(\mathcal{R} \times \mathcal{S})$. It follows then that G is right continuous: if $t_n \downarrow t$ and $x_n \downarrow x$, then $(t, t_n) \times (x, x_n] \downarrow \phi$ hence

$$G(t_n, x_n) - G(t, x) =$$

$$= m_G((t, t_n] \times (x, x_n]) - [G(t, x) - G(t, x_n)] - [G(t, x) - G(t_n, x)]$$

and the right hand side term has limit 0, since m_G is σ-additive, and G is separately right continuous at (t, x).

Since m_g is σ-additive and with bounded semivariation $(\tilde{m}_g)_{F,G}$ on $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$, and $m_G = m_g$ on $r(\mathcal{R} \times \mathcal{L})$, it follows that m_G can be extended to a σ-additive measure with bounded semivariation $(\tilde{m}_G)_{F,G}$ on $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$, namely $m_G = m_g$; and since $(\tilde{m}_G)_{F,G} = (\tilde{m}_g)_{F,G}$ on $r(\mathcal{R} \times \mathcal{L})$ we deduce the same equality on $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$.

1.4. *The Stieltjes Integral for function measures with bounded semivariation.*

Let $g: \mathbb{R} \times \mathcal{K} \to E \subset L(F, G)$ (isometrically) be a right continuous function measure with bounded semivariation $\tilde{g}_{F,G}$ and let $m_g: \mathcal{R} \times \mathcal{K} \to E$ be the finitely additive measure with bounded semivariation $(\tilde{m}_g)_{F,G}$, satisfying

$$m_g((s, t] \times B) = g(t, B) - g(s, B), \qquad \text{for } s \leq t \text{ and } B \in \mathcal{K}.$$

Assume m_g can be extended to a σ-additive measure $m: \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \to E$ with bounded semivariation $\tilde{m}_{F,G}$. This is the case, for example, if $c_0 \not\subset E$, according to theorem 1.4*c*. The extension m will still be denoted by m_g. Then we can apply the integration theory presented in [B-D.2]. Let $Z \subset G^*$ be a norming space for G and let D be a Banach space. For each $z \in Z$ consider the measure $(m_g)_z: \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \to F^*$ with finite variation $|(m_g)_z|$, defined by

$$\langle x, (m_g)_z(C) \rangle = \langle m_g(C)x, z \rangle,$$

for $x \in F$ and $C \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{L}$. For each measurable function $f: \mathbb{R} \times L \to D$, define

$$(\tilde{m}_g)_{F,G}(f) = \sup\left\{ \int |f| d|(m_g)_z| : z \in Z,\ |z| \leq 1 \right\}.$$

We denote by $\mathcal{F}_D((m_g)_{F,G})$ the set of measurable functions $f: \mathbb{R} \times L \to D$ with $(\tilde{m}_g)_{F,G}(f) < \infty$. Then $\mathcal{F}_D((m_g)_{F,G})$ is a complete vector space for the seminorm

$(\tilde{m}_g)_{F,G}(f)$, and we have

$$\mathcal{F}_D((m_g)_{F,G}) \subset \bigcap_{z \in Z} L_D^1((m_g)_z) .$$

If $D = F$ we can define the integral $\int f dm_g$ for any function $f \in \mathcal{F}_{F,G}(m_g) :=$ $:= \mathcal{F}_F((m_g)_{F,G})$, as follows: Let $f \in \mathcal{F}_{F,G}(m_G)$; then $f \in L_F^1((m_g)_z)$ for every $z \in Z$, hence the scalar integral $\int f d(m_g)_z$ is defined. The mapping $z \mapsto \int f d(m_g)_z$ is a continuous linear functional on Z, denoted by $\int f dm_g$. Then we have $\int f dm_g \in Z^*$,

$$\left\langle \int f dm_g, z \right\rangle = \int f d(m_g)_z, \qquad \text{for } z \in Z ,$$

and

$$\left| \int f dm_g \right| \leq (\tilde{m}_g)_{F,G}(f) .$$

For $f \in \mathcal{F}_{F,G}(g)$, we denote the integral $\int f dm_g$ by $\int f dg$ or $\int f(t, x)\, g(dt, dx)$, and we call it the Stieltjes integral of f with respect to g:

$$\int f(t, x)\, g(dt, dx) = \int f dm_g .$$

We denote by $L_{F,G}^1(m_g)$ or $L_{F,G}^1(g)$, the subspace of $\mathcal{F}_{F,G}(m_g)$ consisting of functions f such that $\int f dg \in G$ (rather than Z^*).

REMARK: Assume that g has bounded *variation* $|g|$. Then g has also bounded *semivariation* $\tilde{g}_{F,G}$, relative to any embedding $E \subset L(F, G)$, and $\tilde{g}_{F,G} \leq |g|$. The corresponding measure m_g: $\mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \to E$ has finite variation $|m_g| = m_{|g|}$, and finite semivariation $(\tilde{m}_g)_{F,G} \leq |m_g|$. It follows that $L_F^1(|m_g|) \subset L_{F,G}^1(m_g)$ and

$$\tilde{m}_{F,G}(f) \leq \int |f| d|m_g| , \qquad \text{for } f \in L_F^1(|m_g|) .$$

For a function $f \in L_F^1(|m_g|)$, the Stieltjes integral $\int f dg$ is the same, whether we consider $f \in L_F^1(m_g)$ or $f \in L_{F,G}^1(m_g)$.

2. - P-MEASURES INDUCED BY PROCESS MEASURES WITH INTEGRABLE SEMIVARIATION

In this paragraph we introduce the process measures, their relationship with two parameter processes and the P-measures associated to them.

2.1. *Definitions and properties.*

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{K}$ a ring of subsets of L, satisfying $\mathcal{S} \subset \mathcal{K} \subset \mathcal{L}$, and X: $\Omega \times \mathbb{R}_+ \times \mathcal{K} \to E$ a process set function. Its values at $\omega \in \Omega$, $t \in \mathbb{R}_+$ and $B \in \mathcal{K}$

will be denoted $X(\omega, t, B)$ or $X_t(\omega, B)$. We extend X automatically to $\Omega \times \mathbb{R} \times \mathcal{K}$ with 0 outside $\Omega \times \mathbb{R}_+ \times \mathcal{K}$, and we define also $X_{-\infty}(\omega, B) = 0$.

A series of properties have been defined in ([D5], Definition 2.1). We repeat some of them here and define new ones.

DEFINITION 2.1: *Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{K} \to E \subset L(F, G)$ (isometrically) be a process set function.*

a) We say X is measurable, if for every $B \in \mathcal{K}$, the process $(\omega, t) \mapsto X_t(\omega, B)$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

b) We say X is right continuous, if for every $\omega \in \Omega$ and $B \in \mathcal{K}$, the function $t \mapsto X_t(\omega, B)$ is right continuous.

c) X is said to be pathwise σ-additive on $\mathcal{K}$, if for every $\omega \in \Omega$ and $t \in \mathbb{R}_+$, the set function $B \mapsto X_t(\omega, B)$ is σ-additive in E on $\mathcal{K}$.

d) Let $1 \leq p < \infty$. X is said to be a p-process measure on $\mathcal{K}$, if for every $t \in \mathbb{R}_+$ and $B \in \mathcal{K}$, the random variable $\omega \mapsto X_t(\omega, B)$ belongs to L^p_E, and if for every $t \in \mathbb{R}_+$, the set function $B \mapsto X_t(\cdot, B)$ from $\mathcal{K}$ into L^p_E is σ-additive.

e) We say X has finite (respectively bounded) semivariation relative to (F, G), if for every $\omega \in \Omega$, the function $(t, B) \mapsto X_t(\omega, B)$ has finite (respectively bounded) semivariation relative to (F, G), (in the sense of definition 1.2), denoted $(\widetilde{X}(\omega))_{F,G}(t, B)$ or $\widetilde{X}_{F,G}(\omega, t, B)$.

f) Let $1 \leq p < \infty$. We say the process set function X has p-integrable semivariation relative to (F, G) if the function $\omega \mapsto \widetilde{X}_{F,G}(\omega, \infty, L)$ belongs to L^p. If $p = 1$, we say X has integrable semivariation:

We list now some immediate properties:

1) *If X is measurable and right continuous in E and has p-integrable semivariation $\widetilde{X}_{F,G}$, then X is right continuous in L^p_E.*

We have $|X_t(\omega, B)| \leq \widetilde{X}_{F,G}(\omega, t, B)$ and we use Lebesgue's theorem in L^p_E to deduce the right continuity of X in L^p_E.

2) *If X is measurable and pathwise σ-additive in E on $\mathcal{K}$ and has p-integrable semivariation relative to (F, G), then X is σ-additive in L^p_E on $\mathcal{K}$.*

This follows from the inequality $|X_t(\omega, B)| \leq \widetilde{X}_{F,G}(\omega, t, B)$, by applying Lebesgue's convergence theorem.

3) *If X is a right continuous p-process measure, then X is separably valued (a.s.).*

PROOF: For each $t \in \mathbb{R}_+$ and $B \in \mathcal{L}$, we have $X_t(\cdot, B) \in L^p_E$, hence $\omega \mapsto X_t(\omega, B)$ is $\mathcal{F}$-measurable, therefore it is almost separably valued. Let $N(t, B)$ be a negligible set such that the range $\{X_t(\omega, B): \omega \notin N(t, B)\}$ is separable.

The set $N(B) = \cup\{N(t, B): t \in Q\}$ is negligible. Since X is right continuous, the set $\{X_t(\omega, B): \omega \notin N(B), t \in \mathbb{R}_+\}$ is separable. Let $\mathcal{S}_0$ the ring generated by the intervals $(x, y]$ in L with x, y rational. The ring $\mathcal{S}_0$ is countable. The set $N = \cup\{N(B): B \in \mathcal{S}_0\}$ is negligible and the set

$$A = \{X_t(\omega, B): \omega \notin N, \quad t \in \mathbb{R}_+, \quad B \in \mathcal{S}_0\}$$

is separable. Denote by E_0 the closed vector space spanned by A. Let $\mathcal{L}_0$ be the class of sets $B \in \mathcal{L}$ such that the range of $X_t(\omega, B)$ is contained in E_0, for $\omega \notin N$ and show that $\mathcal{L}_0$ is a monotone class. It contains $\mathcal{S}_0$. Let (B_n) be a monotone sequence from $\mathcal{L}_0$ with limit B. Since X is σ-additive in B, we have $X_t(\omega, B_n) \to X_t(\omega, B)$ for every $\omega \in \Omega$ and $t \in \mathbb{R}_+$. Since $X_t(\omega, B_n) \in E_0$, for $\omega \notin N$, it follows that $X_t(\omega, B) \in E_0$ for $\omega \notin N$, hence $B \in \mathcal{L}_0$. It follows that $\mathcal{L}_0 = \mathcal{L}$, and the property is proved.

2.2. *Two parameter processes and process-measures.*

To a two parameter process $F: \Omega \times \mathbb{R}_+ \times L \to E$ we associate a process set function $X: \Omega \times \mathbb{R}_+ \times \mathcal{S} \to E$, by the equality

$$X_t(\omega, (-\infty, x]) = F(\omega, t, x), \qquad \text{for } \omega \in \Omega, \quad t \in \mathbb{R}_+ \text{ and } x \in L,$$

and, then we extend it by additivity to $\Omega \times \mathbb{R}_+ \times \mathcal{S}$. We have then

$$X_t(\omega, (x, y]) = X_t(\omega, (-\infty, y]) - X_t(\omega, (-\infty, x]) = F(\omega, t, y) - F(\omega, t, x).$$

Conversely, any process set function $X: \Omega \times \mathbb{R}_+ \times \mathcal{S} \to E$ which is finitely additive on $\mathcal{S}$ is induced by the two parameter process $F: \Omega \times \mathbb{R}_+ \times L \to E$ defined by

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \qquad \text{for } (\omega, t, x) \in \Omega \times \mathbb{R}_+ \times L.$$

Let F and X be associated to each other this way. We list some properties relating F and X. For each $\omega \in \Omega$, we denote by $F(\omega)$ the function defined on $\mathbb{R}_+ \times L$ by $F(\omega)(t, x) = F(\omega, t, x)$; we denote also by $X(\omega)$ the function defined on $\mathbb{R}_+ \times \mathcal{S}$ by $X(\omega)(t, B) = X_t(\omega, B)$.

4) *For any intervals $I \subset \mathbb{R}$ and $J \subset L$ we have*

$$\operatorname{svar}_{F,G}(X(\omega), I \times J) = \operatorname{svar}_{F,G}(F(\omega), I \times J).$$

See Proposition 1.6.

4′) *Taking $I = (-\infty, t]$ and $J = (-\infty, x]$ we obtain*

$$\tilde{X}_{F,G}(\omega, t, (-\infty, x]) = \tilde{F}_{F,G}(\omega, t, x).$$

5) *If X is pathwise σ-additive in E on $\mathcal{S}$, then F is separately right continuous in x on L.*

Conversely:

THEOREM 2.2: *Assume $c_0 \not\subset E$ and that F is separately right continuous in x on L and has bounded semivariation $\tilde{F}_{F,G}$. Then X has bounded semivariation $\tilde{X}_{F,G}$ on $\Omega \times \mathbb{R}_+ \times \mathcal{S}$ and is pathwise σ-additive in E on $\mathcal{S}$.*

Moreover, X can be extended to a process set function $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$, which is pathwise σ-additive in E on $\mathcal{L}$.

If, in addition, F has p-integrable semivariation $\tilde{F}_{F,G}$, then X has p-integrable semivariation $\tilde{X}_{F,G}$ on $\mathcal{S}$ and X is σ-additive in L_E^p on $\mathcal{S}$.

PROOF: For each $\omega \in \Omega$ and $t \in \mathbb{R}_+$, the function $x \mapsto F(t, \omega, x)$ is right continuous and has bounded semivariation $(\tilde{F}(\omega, t))_{F,G}(x) \leq \tilde{F}_{F,G}(\omega, t, x)$. By ([D2], Theorem 2.2), there is a Stieltjes measure $\mu_{\omega,t}: \mathcal{L} \to E$ satisfying

$$\mu_{\omega,t}(x, y] = F(\omega, t, y) - F(\omega, t, x).$$

For every $(\omega, t, B) \in \Omega \times \mathbb{R}_+ \times \mathcal{L}$ define

$$X_t(\omega, B) = \mu_{\omega,t}(B).$$

Then $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ is a process set function, pathwise σ-additive in E on $\mathcal{L}$ and satisfying

$$X_t(\omega, (x, y]) = F(\omega, t, y) - F(\omega, t, x),$$

hence X is an extension of the process set function association to F.

The last property about semivariations follows from property 4′. This proves the theorem.

We can improve the properties of X by imposing F to be jointly right continuous.

THEOREM 2.3: *Assume $c_0 \not\subset E$ and let F and X satisfy*

$$X(\omega, t, (-\infty, x]) = F(\omega, t, x).$$

Assume F is right continuous and has bounded semivariations $\tilde{F}_{F,G}$. Then X can be extended to a right continuous, pathwise σ-additive process measure $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ with bounded semivariation $\tilde{X}_{F,G}$, satisfying

$$\tilde{X}_{F,G}(\omega, t, (-\infty, x]) = \tilde{F}_{F,G}(\omega, t, x), \qquad \text{for } (\omega, t, x) \in \Omega \times \mathbb{R}_+ \times L.$$

If, in addition, F has p-integrable semivariation $\tilde{X}_{F,G}$, then X has p-integrable semivariation $\tilde{X}_{F,G}$ and X is σ-additive in L_E^p on $\mathcal{L}$.

PROOF: Since F is right continuous and has bounded semivariation $\tilde{F}_{F,G}$, for each $\omega \in \Omega$, the function $F(\omega)$ is right continuous on $\mathbb{R}_+ \times L$ and has bounded semivariation $(\tilde{F}(\omega))_{F,G}(t, x) = \tilde{F}_{F,G}(\omega, t, x)$. Then by ([D3], Theorem 2.2) there is a σ-addi-

tive measure $m_{F(\omega)}$: $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{L}\to E$ with bounded semivariation $(\tilde{m}_{F(\omega)})_{F,G}$ satisfying

$$m_{F(\omega)}((-\infty, t]\times(-\infty, x]) = F(\omega, t, x),$$

and

$$(\tilde{m}_{F(\omega)})_{F,G}((-\infty, t]\times(-\infty, x]) = \tilde{F}_{F,G}(\omega, t, x).$$

We define X: $\Omega\times\mathbb{R}_+\times\mathcal{L}\to E$ by

$$X_t(\omega, B) = m_{F(\omega)}((-\infty, t]\times B), \qquad \text{for } \omega\in\Omega,\ t\in\mathbb{R}_+ \text{ and } B\in\mathcal{L}.$$

Taking $B=(-\infty, x]$, we get

$$X_t(\omega, (-\infty, x]) = F(\omega, t, x),$$

therefore X is the extension to $\Omega\times\mathbb{R}_+\times\mathcal{L}$ of the process set function associated to F.

Since $m_{F(\omega)}$ is σ-additive, we deduce that X is right continuous and pathwise σ-additive in E on $\mathcal{L}$. By Theorem 2.2, X has bounded semivariation $\tilde{X}_{F,G}$ on $\Omega\times\mathbb{R}_+\times\mathcal{S}$. But we can prove that it has bounded semivariation $\tilde{X}_{F,G}$ on $\Omega\times\mathbb{R}_+\times\mathcal{L}$. In fact, let $\omega\in\Omega$, $t\in\mathbb{R}_+$ and $B\in\mathcal{L}$. Let $t_1<t_2<\ldots<t_n$ be a division of points from $(-\infty, t]$, $(B_j)_{1\leq j\leq m}$ a family of disjoint sets from $\mathcal{L}$ contained in B and $(x_{ij})_{1\leq i<n,\,1\leq j<m}$ a family of elements from F with $|x_{ij}|\leq 1$. Then

$$\left|\sum_{ij}[X_{t_{i+1}}(\omega, B_j) - X_{t_i}(\omega, (B_j))]\, x_{ij}\right| =$$

$$= \left|\sum_{ij} m_{F(\omega)}((t_i, t_{i+1}]\times B_j)\, x_{ij}\right| \leq (\tilde{m}_{F(\omega)})_{F,G}((-\infty, t]\times B),$$

therefore

$$(\tilde{X}_{F,G})(\omega, t, B) \leq (\tilde{m}_{F(\omega)})_{F,G}((-\infty, t]\times B),$$

hence $\tilde{X}_{F,G}$ is bounded on $\Omega\times\mathbb{R}_+\times\mathcal{L}$. For $B=(-\infty, x]$, we obtain

$$\tilde{X}_{F,G}(\omega, t, (-\infty, x]) \leq (\tilde{m}_{F(\omega)})_{F,G}((-\infty, t]\times(-\infty, x]) = \tilde{F}_{F,G}(\omega, t, x).$$

For the converse inequality, let $\omega\in\Omega$, $t\in\mathbb{R}_+$ and $x\in L$. Let $t_1<t_2<\ldots<t_n$ be a division of points in $(-\infty, t]$, $x_1<x_2<\ldots<x_m$ a division of points in $(-\infty, x]$ and $(x_{ij})_{1\leq i<n,\,1\leq j<m}$ a family of elements from F with $|x_{ij}|\leq 1$. Then, denoting

$R_{ij} = (t_i, t_{i+1}] \times (x_j, x_{j+1}]$, we have

$$\left| \sum_{ij} \Delta_{R_{ij}}(F(\omega))\, x_{ij} \right| = \left| \sum_{ij} m_{F(\omega)}((t_i, t_{i+1}] \times (x_j, x_{j+1}])\, x_{ij} \right| =$$

$$= \left| \sum_{ij} [X_{t_{i+1}}(\omega, (x_j, x_{j+1}]) - X_{t_i}(\omega, (x_j, x_{j+1}])]\, x_{ij} \right| \leqslant$$

$$\leqslant \operatorname{svar}_{F,G}(X(\omega), (-\infty, t] \times (-\infty, x]) = \tilde{X}_{F,G}(\omega, t, (-\infty, x]),$$

therefore

$$\tilde{F}_{F,G}(\omega, t, x) = \operatorname{svar}_{F,G}(F(\omega), (-\infty, t] \times (-\infty, x]) \leqslant \tilde{X}_{F,G}(\omega, t, (-\infty, x]);$$

consequently

$$\tilde{X}_{F,G}(\omega, t, (-\infty, x]) = \tilde{F}_{F,G}(\omega, t, x).$$

In particular, $\tilde{X}_{F,G}(\omega, \infty, L) = \tilde{F}_{F,G}(\omega, \infty, \infty)$. If F has p-integrable semivariation relative to (F, G), then $\tilde{F}_{F,G}(\omega, \infty, \infty) \in L^p$, hence $\tilde{X}_{F,G}(\cdot, \infty, L) \in L^p$, that is X has p-integrable semivariation relative to (F, G). The σ-additivity of X in L^p_E on $\mathcal{L}$ follows from property 3.

Conversely:

THEOREM 2.4: *Assume $c_0 \not\subset E$. Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{K} \to E$ be a right continuous p-process measure with bounded semivariation $\tilde{X}_{\mathbb{R},E}$ and pathwise σ-additive in E on $\mathcal{K}$. Then the two parameter process $F: \Omega \times \mathbb{R}_+ \times L \to E$ defined by*

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \qquad \textit{for } (\omega, t, x) \in \Omega \times \mathbb{R}_+ \times L,$$

is (jointly) right continuous and has bounded semivariation $\tilde{F}_{\mathbb{R},E}$.

If X has bounded (resp. p-integrable) semivariation $\tilde{X}_{F,G}$, then F has also bounded (resp. p-integrable) semivariation $\tilde{F}_{F,G}$, and

$$\tilde{F}_{F,G}(\omega, t, x) = \tilde{X}_{F,G}(\omega, t, (-\infty, x]), \qquad \textit{for } t \geqslant 0 \textit{ and } x \in L.$$

The process measure X can be extended uniquely to a right continuous process measure $X': \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ with bounded semivariation $\tilde{X}'_{F,G}$ and pathwise σ-additive in E.

PROOF: Let $x^* \in E^*$. Then $x^* X$ is a real valued, right continuous process measure with bounded variation $|x^* X| \leqslant \tilde{X}_{\mathbb{R},E}$, and pathwise σ-additive on $\mathcal{K}$, and satisfies

$$x^* F(\omega, t, x) = x^* X_t(\omega, (-\infty, x]), \qquad \text{for } (\omega, t, x) \in \Omega \times \mathbb{R}_+ \times L.$$

By ([D5], Theorem 2.3), $x^* F$ is right continuous. Then by ([D3], Theorem 2.3), F is right continuous.

The rest of the conclusion, concerning the semivariation of F, follows from property 4. The existence of the extension X' follows from Theorem 2.3.

REMARK: 1) If $c_0 \not\subset E$, Theorem 2.4 entitles us to consider all right continuous p-process measures with bounded semivariation defined on the whole set $\Omega \times \mathbb{R}_+ \times L$ (rather than on $\Omega \times \mathbb{R}_+ \times \mathcal{K}$).

2) If X is a p-process measure, σ-additive in L_E^p on $\mathcal{L}$, but not pathwise σ-additive in E on $\mathcal{L}$, then the above theorem is not necessarily true. However, we can still find a two parameter right continuous process F-satisfying the equality between F and X only a.s., if we impose E to be separable.

THEOREM 2.5: *Assume that E is separable and $c_0 \not\subset E$. Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ be a right continuous p-process measure with p-integrable semivariation $\tilde{X}_{\mathbb{R}, E}$. Then there is a two parameter, right continuous process $F: \Omega \times \mathbb{R}_+ \times L \to E$ with p-integrable semivariation $\tilde{F}_{\mathbb{R}, E}$ satisfying, for every $t \in \mathbb{R}_+$ and $x \in L$,*

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad a.s.$$

and

$$\tilde{F}_{\mathbb{R}, E}(\omega, t, x) \leq \tilde{X}_{\mathbb{R}, E}(\omega, t, (-\infty, x]).$$

If, in addition, G is separable and if X has p-integrable semivariation $\tilde{X}_{F, G}$, then F has p-integrable semivariation $\tilde{F}_{F, G}$ satisfying

$$\tilde{F}_{F, G}(\omega, t, x) \leq \tilde{X}_{F, G}(\omega, t, (-\infty, x]).$$

PROOF: Let $Z \subset E^*$ be a norming space for E and let $x^* \in Z$. Then $x^* X$ is a right continuous real valued p-process measure with p-integrable variation $|x^* X| \leq \tilde{X}_{\mathbb{R}, E}$. By ([D5], Theorem 2.4), there is a real valued, right continuous, two parameter process $F_{x^*}: \Omega \times \mathbb{R}_+ \times L \to \mathbb{R}$ with p-integrable variation $|F_{x^*}|$ satisfying

$$F_{x^*}(\omega, t, x) = x^* X_t(\omega, (-\infty, x]), \quad \text{a.s.}$$

and

$$|F_{x^*}|(\omega, t, x) \leq |x^* X|_t(\omega, (-\infty, x]) \leq \tilde{X}_{\mathbb{R}, E}(t, \omega, (-\infty, x])|x^*|, \quad \text{a.s.},$$

the negligible set $N(x^*, x)$ depending on x^* and x only, but not on t, because of right continuity of $x^* X$ and F_{x^*}.

Since E is separable, we can take Z separable. Let Z_0 be a countable set dense in Z, and still norming for E. We can assume that Z_0 is a vector space over the field of rational numbers. The set $N(x) = \cup\{N(x^*, x); x^* \in Z_0\}$ is negligible, and for $\omega \notin N(x)$ we

have

$$F_{x^*}(\omega, t, x) = x^* X_t(\omega, (-\infty, x])$$

and

$$|F_{x^*}|(\omega, t, x) \leq |x^* X|_t(\omega, (-\infty, x]) \leq \widetilde{X}_{\mathbb{R}, E}(\omega, t, (-\infty, x])|x^*|$$

for all $x^* \in Z_0$. For $\omega \notin N(x)$, the mapping $x^* \mapsto F_{x^*}(\omega, t, x)$ is a continuous linear functional (with respect to rational scalars) on Z:

$$|F_{x^*}(\omega, t, x)| \leq |F_{x^*}|(\omega, t, x) \leq \widetilde{X}_{\mathbb{R}, E}(\omega, t, (-\infty, x])|x^*|,$$

therefore, it can be extended to a continuous linear functional on Z, denoted $F(\omega, t, x)$. Therefore, $F(\omega, t, x) \in Z^*$ and we have

$$x^* F(\omega, t, x) = F_{x^*}(\omega, t, x) = x^* X_t(\omega, (-\infty, x])$$

for $\omega \notin N(x)$ and $x^* \in Z_0$. Since Z_0 is norming for E and for Z^*, we deduce that for $\omega \notin N(x)$ we have

$$F(\omega, t, x) = X_t(\omega, (-\infty, x])$$

and

$$x^* F(\omega, t, x) = F_{x^*}(\omega, t, x), \qquad \text{for } x^* \in Z, \text{ and } t \in \mathbb{R}.$$

We prove now that F has p-integrable semivariable $\widetilde{F}_{\mathbb{R}, E}$. Let $N = \cup\{N(x): x \in L$ rational$\}$. Then N is negligible. For $\omega \notin N$, $t \in \mathbb{R}$ and $x \in L$ rational, we have

$$F(\omega, t, x) = X_t(\omega, (-\infty, x])$$

and

$$|F(\omega, t, x)| \leq |X|_t(\omega, (-\infty, x]).$$

Let $t \in \mathbb{R}$ and $x \in L$ and let $x^* \in Z$ with $|x^*| \leq 1$. Let $t_1 < t_2 < \ldots < t_n$ be a division of points from $(-\infty, t]$, and $x_1 < x_2 < \ldots < x_m$ a division of *rational* points from $(-\infty, x]$. Then, for $\omega \notin N$ we have

$$\sum_{i,j} |\Delta_{(t_i, t_{i+1}] \times (x_j, x_{j+1}]}(x^* F)| = \sum_{i,j} |x^* X_{t_{i+1}}(x_j, x_{j+1}] - x^* X_{t_i}(x_j, x_{j+1}]| \leq$$

$$\leq |x^* X|(t, (-\infty, x]) \leq \widetilde{X}_{\mathbb{R}, E}(t, (-\infty, x]).$$

Since $x^* F$ is right continuous, its variation $|x^* F|$ can be computed using grids of rational points t_i and x_j. From the above inequality we deduce, for $\omega \notin N$,

$$|x^* F|(t, x) \leq \widetilde{X}_{\mathbb{R}, E}(t, (-\infty, x]),$$

therefore, for $\omega \notin N$,

$$\tilde{F}_{\mathbb{R},E}(t,x) \leq \tilde{X}_{\mathbb{R},E}(t,(-\infty,x]) \leq \tilde{X}_{\mathbb{R}\times E}(\infty,L).$$

Since for $\omega \notin N$, F has bounded semivariation $\tilde{F}_{\mathbb{R},E}$ and x^*F is right continuous for every $x^* \in Z$, and since $c_0 \not\subset E$, by ([D3], Theorem 2.3), we deduce that F is right continuous, for $\omega \notin N$. Redefining $F(\omega,t,x)=0$ for $\omega \in N$, we obtain a two parameter process F which is right continuous everywhere and satisfies

$$F(\omega,t,x) = X_t(\omega,(-\infty,x]), \quad \text{a.s.}$$

and

$$\tilde{F}_{\mathbb{R},E}(\omega,t,x) \leq \tilde{X}_{\mathbb{R},E}(\omega,t,(-\infty,x]), \quad \text{a.s.}$$

We have to show that $\tilde{F}_{\mathbb{R},E}(\omega,\infty,\infty)$ is p-integrable. We have to prove first that it is measurable. Let $t \in \mathbb{R}$ and $x \in L$. For $t_1 < \ldots < t_n \leq t$ rational in $\mathbb{R}$ and $x_1 < x_2 < \ldots < x_m \leq x$ rational in L, the function

$$\sum_{i,j} |\Delta_{t_i,t_{i+1}]\times(x_j,x_{j+1}]}(x^*F)|$$

is measurable. Taking the supremum for all such divisions of rational points, we deduce that the variation $|x^*F|(\omega,t,x)$ is $\mathcal{F}$-measurable. Finally, taking the supremum for $x^* \in Z_0$ with $|x^*| \leq 1$, it follows that the semivariation $\tilde{F}_{\mathbb{R},E}(\omega,t,x)$ is $\mathcal{F}$-measurable. Since $\tilde{X}_{\mathbb{R},E}(\cdot,\infty,L) \in L^p$, we deduce that $\tilde{F}_{\mathbb{R},E}(\cdot,\infty,\infty) \in L^p$, i.e., F has p-integrable semivariation relative to $(\mathbb{R},E)$.

Assume now that X has p-integrable semivariation $\tilde{X}_{F,G}$ and prove that F has also p-integrable semivariation $\tilde{F}_{F,G}$. Let N be a negligible set such that for $\omega \notin N$, $t \in \mathbb{R}$ and $x \in L$ *rational*, we have

$$F(\omega,t,x) = X_t(\omega,(-\infty,x]).$$

Let $Z \subset G^*$ be a norming space for G. For each $z \in Z$ we have

$$(F_z)(\omega,t,x) = (X_z)_t(\omega,(-\infty,x])$$

for $\omega \notin N$, $t \in \mathbb{R}$ and x rational in L.

Let $\omega \notin N$, $t \in \mathbb{R}$ and $x \in L$. Let $t_1 < t_2 < t \ldots t_n$ be a division of rational points in $(-\infty,t]$ and $x_1 < x_2, \ldots < x_m$ a division of rational points in $(-\infty,x]$. Then

$$\sum_{i,j} |\Delta_{(t_i,t_{i+1}]\times(x_j,x_{j+1}]}(F_z)| = \sum_{ij} |(X_z)_{t_{i+1}}(x_j,x_{j+1}] - (X_z)_{t_i}(x_j,x_{j+1}]]| \leq$$

$$\leq |X_z|(t,(-\infty,x]) \leq \tilde{X}_{F,G}(t,(-\infty,x]),$$

therefore, since F_z is right continuous,

$$|F_z|(t,x) \leq \tilde{X}_{F,G}(t,(-\infty,x]).$$

Taking the supremum for $z \in Z$ with $|z| \leq 1$ we obtain

$$\tilde{F}_{F,G}(t, x) \leq \tilde{X}_{F,G}(t, (-\infty, x]).$$

Since G is separable, we can take Z countable and prove, as above, that $\tilde{F}_{F,G}(t, x)$ is $\mathcal{F}$-measureable, for each t and x. Since

$$\tilde{F}_{F,G}(\infty, \infty) \leq \tilde{X}_{F,G}(\infty, L) \in L^p,$$

we deduce that $\tilde{F}_{F,G}(\infty, \infty) \in L^p$, i.e. F has p-integrable semivariation $\tilde{F}_{F,G}$, and the theorem is completely proved.

From Theorem 2.3 and 2.5 we deduce the following theorem.

THEOREM 2.6: *Assume $c_0 \not\subset E$ and that E is separable. Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{K} \to E$ be a right continuous p-process measure with p-integrable semivariation $\tilde{X}_{\mathbb{R},E}$. Then there is a right continuous p-process measure $X': \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$, pathwise σ-additive in E on $\mathcal{L}$ and with p-integrable semivariation $\tilde{X}_{\mathbb{R},E}$, and which is a modification of X, that is*

$$X'_t(\omega, B) = X_t(\omega, B), \quad a.s., \qquad for\ t \in \mathbb{R}_+ \ and\ B \in \mathcal{K},$$

the negligible set depending on B only. We have also

$$\tilde{X}'_{\mathbb{R},E}(\omega, t, B) \leq \tilde{X}_{\mathbb{R},E}((\omega, t, B)), \quad a.s.$$

If, in addition, $E \subset L(F, G)$ with G separable and if X has p-integrable semivariation $\tilde{X}_{F,G}$ then X has p-integrable semivariation $\tilde{X}_{F,G}$ and we have

$$\tilde{X}'_{F,G}(\omega, t, B) \leq \tilde{X}_{F,G}(\omega, t, B), \quad a.s.$$

PROOF: Let $F: \Omega \times \mathbb{R}_+ \times L \to E$ be a right continuous, two parameter process with p-integrable semivariation $\tilde{F}_{\mathbb{R},E}$ associated to X by Theorem 2.5 and satisfying

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{a.s.}$$

and

$$\tilde{F}_{F,G}(\omega, t, x) \leq \tilde{X}_{F,G}(\omega, t, (-\infty, x]), \quad \text{a.s.}$$

Let $X': \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ be the right continuous process measure with p-integrable semivariation and pathwise σ-additive in E on $\mathcal{L}$, associated to E by Theorem 2.3 and satisfying

$$X'(\omega, t, (-\infty, x]) = F(\omega, t, x), \qquad \text{everywhere}$$

and

$$\widetilde{X}'_{F,G}(\omega, t, (-\infty, x]) = \widetilde{F}_{F,G}(\omega, t, x), \qquad \text{everywhere}.$$

Then, by property 3), X' is σ-additive in L^p_E on $\mathcal{L}$. Then we have

$$X'_t(\omega, (-\infty, x]) = X_t(\omega, (-\infty, x]), \quad \text{a.s.}, \qquad \text{for } t \in \mathbb{R}_+ \text{ and } x \in L$$

and

$$\widetilde{X}'_{\mathbb{R},E}(\omega, t, (-\infty, x]) \leq \widetilde{X}_{\mathbb{R},E}(\omega, t, (-\infty, x]), \quad \text{a.s.}$$

that is

$$X'_t(\cdot, (\infty, x]) = X_t(\cdot, (-\infty, x]), \qquad \text{in } L^p_E$$

for each $t \in \mathbb{R}_+$ and $x \in L$. It follows that

$$X'_t(\cdot, B) = X_t(\cdot, B), \qquad \text{in } L^p_E, \qquad \text{for } B \in \mathcal{S}.$$

Since both measures $B \mapsto X'_t(\cdot, B)$ and $B \mapsto X_t(\cdot, B)$ are σ-additive in L^p_E on $\mathcal{K}$ and coincide on $\mathcal{S}$, we have

$$X'_t(\cdot, B) = X_t(\cdot, B) \qquad \text{in } L^p_E, \qquad \text{for every } t \in \mathbb{R}_+ \text{ and } B \in \mathcal{K},$$

that is

$$X'_t(\omega, B) = X_t(\omega, B), \quad \text{a.s. for } t \in \mathbb{R}_+ \text{ and } B \in \mathcal{K},$$

the negligible set depending on B only, because of right continuity of X' and X.

2.3. *Pathwise σ-additive process measures and their Stieltjes integral.*

Assume $c_0 \not\subset E$.

Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ be a right continuous process measure, pathwise σ-additive in E on $\mathcal{L}$ and with bounded semivariation $\widetilde{X}_{F,G}$. For each $\omega \in \Omega$, consider the σ-additive measure $m_{X(\omega)}: \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{L} \to E$ with bounded semivariation $(\widetilde{m}_{X(\omega)})_{F,G}$, associated to $X(\omega)$ by theorem 1.4, and satisfying

$$m_X(\omega)((s, t] \times (x, y]) = X_t(\omega, (x, y]) - X_s(\omega, (x, y]).$$

Let F, G be Banach spaces with $E \subset L(F, G)$ isometrically and let $H: \Omega \times \mathbb{R}_+ \times \times L \to F$ be a two parameter, measurable process with

$$\widetilde{m}_{X(\omega)}(H(\omega)) < \infty,$$

where $H(\omega): \mathbb{R}_+ \times L$ is defined by $H(\omega)(t, x) = H(\omega, t, x)$, for $t \in \mathbb{R}_+$ and $x \in L$. This means that $H(\omega) \in \mathcal{F}_{F,G}(m_{X(\omega)})$, therefore the integral

$$\int H(\omega)(t, x)\, m_{X(\omega)}(dt, dx)$$

is defined and belongs to G^{**}, in general. According to section 1.4, this is the Stieltjes integral $\int H(\omega)\, dX(\omega)$, denoted also by

$$\int H(\omega, t, x)\, X(\omega, dt, dx).$$

Therefore

$$\int H(\omega, t, x)\, X(\omega, dt, dx) = \int H(\omega, t, x)\, m_{X(\omega)}(dt, dx).$$

This Stieltjes integral can also be defined in terms of the right continuous two parameter function $F\colon \Omega \times \mathbb{R}_+ \times L \to E$ with bounded semivariation $\tilde{F}_{F,G}$, associated to X by theorem 2.4 and satisfying

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \qquad \text{for } t \in \mathbb{R}_+ \text{ and } x \in L.$$

For each $\omega \in \Omega$, let $F(\omega)\colon \mathbb{R}_+ \times L \to E$ be the function defined by $F(\omega)(t, x) = F(\omega, t, x)$. Then $F(\omega)$ is the function of two variables associated to the function measure $X(\omega)$, by Theorem 1.7. The measures $m_{X(\omega)}$ and $m_{F(\omega)}$ are σ-additive and equal on the σ-algebra $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{L}$, hence

$$(\tilde{m}_{X(\omega)})_{F,G} = (\tilde{m}_{F(\omega)})_{F,G},$$

therefore

$$\mathcal{F}_{F,G}(m_{X(\omega)}) = \mathcal{F}_{F,G}(m_{F(\omega)})$$

and

$$\int H(\omega, t, x)\, m_{X(\omega)}(dt, dx) = \int H(\omega, t, x)\, m_{F(\omega)}(dt, dx)$$

for any measurable process $H\colon \Omega \times \mathbb{R}_+ \times L \to F$ such that $H(\omega) \in \mathcal{F}_{F,G}(m_{G(\omega)})$ for every $\omega \in \Omega$.

This last equality can be expressed in terms of Stieltjes integrals:

$$\int H(\omega, t, x)\, X(\omega, dt, dx) = \int H(\omega, t, x)\, F(\omega, dt, dx).$$

2.4. *General process measures and their Stieltjes integral.*

Assume $c_0 \not\subset E$ and that E and G are separable. Let $X\colon \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ be a right continuous p-process with p-integrable semivariation $\tilde{X}_{F,G}$. The process X is σ-additive in L^p_E, but is not necessarily pathwise σ-additive in E. For each $\omega \in \Omega$ we consider the additive measure $m_{X(\omega)}\colon \mathcal{R} \times \mathcal{L} \to E$ defined by

$$m_{X(\omega)}((s, t] \times B) = X_t(B) - X_s(B), \qquad \text{for } s \leq t \text{ and } B \in \mathcal{L}.$$

The measure $m_{X(\omega)}$ is not necessarily σ-additive in E on $\mathcal{R} \times \mathcal{L}$, therefore the integral $\int H(\omega, t, x)\, m_{X(\omega)}(dt, dx)$ cannot be defined as a usual integral. It will be defined as a

notation for a genuine Stieltjes integral defined in section 2.3 with respect to modifications of X which are pathwise σ-additive in E.

By Theorem 2.6, there is a right continuous process measure $X': \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$, pathwise σ-additive in E on $\mathcal{L}$, with p-integrable semivariation $\tilde{X}'_{F,G}$, and a modification of X, that is

$$X'_t(\omega, B) = X_t(\omega, B), \quad \text{a.s.},$$

the negligible set depending on B only. Then

$$m_{X'(\omega)}((s, t] \times B) = m_{X(\omega)}((s, t] \times B), \quad \text{a.s.}$$

outside a negligible set depending on B. If X'' is another right continuous, pathwise σ-additive modification of X, then there exists a negligible set $N \subset \Omega$ such that for every $\omega \notin N$ we have

$$X'_t(\omega, B) = X''_t(\omega, B), \qquad \text{for every } t \in \mathbb{R}_+ \text{ and } B \in \mathcal{L}.$$

Then, for $\omega \notin N$ we have also

$$m_{X'(\omega)}((s, t] \times B) = m_{X''(\omega)}((s, t] \times B), \qquad \text{for every } t \in \mathbb{R}_+ \text{ and } B \in \mathcal{L}.$$

Since $m_{X'(\omega)}$ and $m_{X''(\omega)}$ are σ-additive in E on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{L}$, we deduce that for $\omega \notin N$ we have

$$m_{X'(\omega)} = m_{X''(\omega)}, \qquad \text{on } \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{L}$$

and $(\tilde{m}_{X'(\omega)})_{F,G} = (m_{X''(\omega)})_{F,G}$. For $\omega \notin N$ and for a measurable process H: $\Omega \times \mathbb{R}_+ \times L \to F$ we have

$$(\tilde{m}_{X'(\omega)})_{F,G}(H(\omega)) = (\tilde{m}_{X''(\omega)})_{F,G}(H(\omega)),$$

therefore, if $\tilde{m}_{X'(\omega)}(H(\omega)) < \infty$, then $H(\omega) \in \mathcal{F}_{F,G}(m_{X'(\omega)}) = \mathcal{F}_{F,G}(m_{X''(\omega)})$, and the following integrals are defined and are equal:

$$\int H(\omega, t, x)\, m_{X'(\omega)}(dt, dx) = \int H(\omega, t, x)\, m_{X''(\omega)}(dt, dx).$$

According to Section 2.3, these are Stieltjes integrals,

$$\int H(\omega, t, x)\, X'(\omega, dt, dx) = \int H(\omega, t, x)\, X''(\omega, dt, dx), \qquad \text{for } \omega \notin N.$$

This means that the two Stieltjes integrals are in the same equivalence class modulo P, and this equivalence class is determined by H and the process measure X.

By analogy with the above Stieltjes integrals, and with the definition in ([D5], Section 2.4) of the Stieltjes integral when X has p-integrable *variation*, we denote the equivalence class of $\int H(\omega, t, x)\, X'(\omega, dt, dx)$ by $\int H(\omega, t, x)\, X(\omega, dt, dx)$, and call it the Stieltjes integral of H with respect to X. Then, going further, any representative of the equiva-

lence class will be called the Stieltjes integral of H with respect to X,

$$\int H(\omega, t, x)\, X(\omega, dt, dx) = \int H(\omega, dt, dx)\, X'(\omega, dt, dx), \quad \text{a.s.}$$

We remark that the Stieltjes integral $\int H(\omega, t, x)\, X(\omega, dt, dx)$ is defined only almost surely, and that it is not a genuine Stieltjes integral, but just a notation for the equivalence class of the meaningfull Stieltjes integrals with respect to X' and X''.

We define the expectation of the equivalence class to be the common value of the expectation of its representatives:

$$E\left(\int H(\omega, t, x)\, X(\omega, dt, dx)\right) = E\left(\int H(\omega, t, x)\, X'(\omega, dt, dx)\right).$$

If $F\colon \Omega \times \mathbb{R}_+ \times L \to E$ is a right continuous two parameter process with p-integrable semivariation $\tilde{F}_{F,G}$, associated to X by Theorem 2.5, and satisfying

$$F(\omega, t, x) = X_t(\omega, t, (-\infty, x]), \quad \text{a.s.}$$

then we have

$$\int H(\omega, t, x)\, X'(\omega, dt, dx) = \int H(\omega, t, x)\, F(\omega, dt, dx), \quad \text{a.s.}$$

therefore

$$\int H(\omega, t, x)\, X(\omega, dt, dx) = \int H(\omega, t, x)\, F(\omega, dt, dx), \quad \text{a.s.}$$

Then, for the expectations we deduce

$$E\left(\int H(\omega, t, x)\, X(\omega, dt, dx)\right) = E\left(\int H(\omega, t, x)\, F(\omega, dt, dx)\right).$$

2.5. P-measures associated with process measures with integrable semivariation.

Assume $c_0 \not\subset E$ and that E and G are separable. As in the case of process measures with integrable variation ([D5], Theorem 2.6), we shall associate to a process measure X with integrable semivariation $\tilde{X}_{F,G}$ a P-measure μ_X, by means of the Stieltjes integral defined in Section 2.4. In case X is pathwise σ-additive in E, we use the Stieltjes integral defined in Section 2.3, and in this case the restriction for E and G to be separable, can be removed. We have to distinguish between the P-measure μ_X and the measures $m_{X(\omega)}$ considered in Sections 2.3 and 2.4.

THEOREM 2.7: *Assume $c_0 \not\subset E$ and let $X\colon \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ be a right continuous, adapted process measure with integrable semivariation $\tilde{X}_{F,G}$. If X is not pathwise σ-additive in E, assume further that E and G are separable.*

There exists a P-measure $\mu_X\colon \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{L} \to E$ with finite semivariation $(\tilde{\mu}_X)_{F,G}$, satisfying the following condition:

Let X': $\Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ be any right continuous, adapted process measure, pathwise σ-additive in E, with integrable semivariation $\tilde{X}'_{F,G}$ and a modification of X. If H: $\Omega \times \mathbb{R}_+ \times L \to F$ is measurable and if $(\tilde{m}_{X'(\omega)})_{F,G}(H(\omega)) < \infty$, a.s., then $H \in L^1_{F,G}(\mu_X)$ and

$$\int H d\mu_X = E\left(\int H(\omega, t, x)\, X(\omega, dt, dx)\right),$$

and $(\tilde{\mu}_X)_{F,G}(H) \leq E(\tilde{X}'_{F,G}(H(\omega)))$.

Proof: Let F: $\Omega \times \mathbb{R}_+ \times L \to E$ be a right continuous two parameter process with integrable semivariation $\tilde{F}_{F,G}$ associated to X by Theorem 2.5 and satisfying

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{a.s.}$$

For the rest of the proof, all semivariations are relative to (F, G). If X is pathwise σ-additive in E, F is associated to X by Theorem 2.4 and satisfies

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \qquad \text{everywhere}.$$

By ([D3], Theorem 3.3), there is a P-measure μ_X: $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{L} \to E$ with finite semivariation $(\tilde{\mu}_F)_{F,G}$ such that if H: $\Omega \times \mathbb{R}_+ \times L \to F$ is measurable, and if

$$E(\tilde{F}(H(\omega))) = E(\tilde{m}_{F(\omega)}(H(\omega))) < \infty,$$

then $H \in \mathcal{F}_{F,G}(\mu_X)$, the following integrals are defined and are equal

$$\int H d\mu_X = E\left(\int H(\omega, t, x)\, F(\omega, dt, dx)\right),$$

and we have also

$$\tilde{\mu}_X(H) \leq E(\tilde{F}(H(\omega))) = E(\tilde{m}_{F(\omega)}(H(\omega))).$$

For the modification X' of X we have

$$F(\omega, t, x) = X'_t(\omega, (-\infty, x]), \quad \text{a.s.},$$

the negligible set being independent of t and x, and

$$\tilde{m}_{F(\omega)}(H(\omega)) = \tilde{m}_{X'(\omega)}(H(\omega)), \quad \text{a.s.},$$

therefore, if $\tilde{m}_{X'(\omega)}(H(\omega)) < \infty$, then also $\tilde{m}_{F(\omega)}(H(\omega)) < \infty$, therefore $H(\omega) \in \mathcal{F}_{F,G}(m_{X'(\omega)}) = \mathcal{F}_{F,G}(m_{F(\omega)})$ a.s. and

$$\int H(\omega, t, x)\, m_{X'(\omega)}(dt, dx) = \int H(\omega, t, x)\, m_{F(\omega)}(dt, dx), \quad \text{a.s.},$$

therefore, in terms of Stieltjes integrals,

$$\int H(\omega, t, x)\, X'(\omega, dt, dx) = \int H(\omega, t, x)\, F(\omega, dt, dx), \quad \text{a.s.},$$

consequently

$$\int H d\mu_F = E\left(\int H(\omega, t, x)\, X'(t, dt, dx)\right).$$

We take $\mu_{X'} := \mu_F$ and then

$$\int H d\mu_{X'} = E\left(\int H(\omega, t, x)\, X'(\omega), dt, dx\right).$$

This proves the theorem for pathwise σ-additive processes X'.

If X is not pathwise σ-additive, we define further $\mu_X := \mu_{X'} = \mu_F$, and using the definition in Section 2.4 of the Stieltjes integral with respect to X, as a notation for the Stieltjes integral with respect to X', we have, finally

$$\int H d\mu_X = E\left(\int H(\omega, t, x)\, X(\omega, dt, dx)\right).$$

We notice that if X' and X'' are two modifications of X, then $\mu_{X'} = \mu_{X''}$, therefore the measure μ_X depends on X only. To prove the last inequality, let $z \in G^*$ and let $H' : \Omega \times \mathbb{R}_+ \times L \to F$ be a measurable step process such that $|H'| \leq |H|$. Then

$$\left| \int H' d\mu_X \right| \leq E\left(\left| \int H'(\omega, t, x)\, dm_{X'(\omega)}(dt, dx) \right| \right) \leq$$

$$\leq E(\tilde{m}_{X'(\omega)}(H(\omega))) = E(\tilde{X}'(H(\omega))),$$

therefore

$$\tilde{\mu}_X(H(\omega)) \leq E(\tilde{X}'(H(\omega))).$$

This prove the theorem completely.

3. - The Stochastic integral for process measures with integrable semivariation

The summable process measures and the Stochastic integral for summable process measures, have been defined in [D5], Sections 3.1 and 3.2. The stochastic integral $H \cdot X$ is a right continuous, adapted, p-process measure.

In this paragraph we shall prove that under the assumption that $c_0 \not\subset E$ and E, G are separable, the right continuous process measures with integrable semivariation are summable, and their Stochastic integral can be computed pathwise, as a Stieltjes integral, in the sense of Section 2.4.

3.1. *Summability of process measures with integrable semivariation.*

Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration satisfying the usual conditions and assume $c_0 \not\subset E$.

THEOREM 3.1: *Let X: $\Omega \times \mathbb{R}_+ \times L \to E \subset L(F, G)$ be a right continuous, adapted p-process measure with p-integrable semivariation $\tilde{X}_{F,G}$, ie. $\tilde{X}_{F,G}(\cdot, \infty, L) \in L^p$. In case X is not pathwise σ-additive in E on $\mathcal{L}$, assume that E and G are separable. Then X is p-summable relative to (F, G).*

PROOF: Let F: $\Omega \times \mathbb{R}_+ \times L \to E$ be the right continuous, two parameter process with p-integrable semivariation $\tilde{F}_{F,G}$, associated to X by Theorem 2.5 and satisfying

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad \text{a.s.}$$

If X is pathwise σ-additive in E on $\mathcal{L}$, we can apply Theorem 2.4, to obtain a processs F satisfying

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \qquad \text{everywhere}.$$

The two parameter process F is adapted to the filtration $(\mathcal{F}_{tx})_{t\in\mathbb{R}, x\in L}$, with $\mathcal{F}_{tx} = \mathcal{F}_t$ for every t and x. Consider the L^p_E-valued additive measures defined on the ring $\mathcal{T} = r(\mathcal{R} \times \mathcal{S})$, by

$$I_F(A \times \{0\} \times (x, y]) = 1_A F(\cdot, 0, y) - F(\cdot, 0, x), \qquad \text{for } A \in \mathcal{F}_0,$$

$$I_F(A \times (s, t] \times (x, y]) = 1_A \Delta_{(s,t]\times(x,y]}(F) =$$

$$= 1_A(F(s, x) + F(t, y) - F(s, y) - F(t, x)), \qquad \text{for } A \in \mathcal{F}_s,$$

$$I_X(A \times \{0\} \times (x, y]) = 1_A X_0(\cdot, (x, y]) \qquad \text{for } A \in \mathcal{F}_0,$$

and

$$I_X(A \times (s, t](x, y]) = 1_A(X_t(\cdot, (x, y]) - X_s(\cdot, (x, y])), \qquad \text{for } A \in \mathcal{F}_s.$$

Then $I_X = I_F$ in L^p_E on $\mathcal{T}$.

By ([D3], Theorem 4.9), F is p-summable relative to (F, G). It follows that X is also p-summable relative to (F, G) and we have

$$I_F = I_X \quad \text{in } L^p_E \text{ on } \mathcal{P} \otimes \mathcal{L},$$

therefore $\int H dI_F = \int H dI_X$, for $H \in \mathcal{F}_{F,G}(I_X)$.

The relationship between the stochastic integrals $H \cdot F$ and $H \cdot X$ is similar to that between F and X.

THEOREM 3.2: *Let X: $\Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ be a p-summable, right continuous, adapted p-process measure with p-integrable semivariations $\tilde{X}_{F,G}$, and let F: $\Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$*

be a two parameter, right continuous, adapted process with p-integrable semivariation $\tilde{F}_{F,G}$ *satisfying*

$$F(\omega, t, x) = X_t(\omega, (-\infty, x]), \quad a.s.$$

In case X is not pathwise σ-additive in E, assume E and G are separable. Then F is p-summable, and for $H \in L^1_{F,G}(F) \cap L^1_{F,G}(X)$ *we have*

$$(H \cdot F)(\omega, t, x) = (H \cdot X)(\omega, t, (-\infty, x]), \quad a.s.,$$

for $t \geqslant 0$ *and* $x \in L$.

PROOF: The p-summability of F follows from the proof of Theorem 3.1, as well as the equality

$$I_F = I_X \text{ on } \mathcal{P} \otimes \mathcal{L},$$

therefore $\mathcal{F}_{F,G}(I_F) = \mathcal{F}_{F,G}(I_X)$ and

$$\int H dI_F = \int H dI_X, \qquad \text{for } H \in \mathcal{F}_{F,G}(I_X).$$

Then, for $H \in L^1_{F,H}(F) \cap L^1_{F,G}(X)$ we have, a.s.,

$$(H \cdot F)(\omega, t, x) = \int_{[0,t] \times [0,x]} H dI_F = \int_{(-\infty, x] \times (-\infty, x]} H dI_X = (H \cdot X)(\omega, t, (-\infty, x]).$$

for $t \geqslant 0$ and $x \in L$.

The next theorem gives the reationship between the P-measures μ_F and μ_X.

THEOREM 3.3: *Let X and F be as in the statement of Theorem 3.2. Then for every* $\mathcal{C} \in \mathcal{P} \otimes \mathcal{L}$ *we have*

$$\mu_X(C) = \mu_F(C) = E(I_X(C)) = E(I_F(C))$$

and

$$(\tilde{\mu}_X)_{F,G}(C) = (\tilde{\mu}_F)_{F,G}(C) \leqslant (\tilde{I}_X)_{F,G}(C) = (\tilde{I}_F)_{F,G}(C).$$

PROOF: Let $C = A \times (s, t] \times (x, y]$ with $A \in \mathcal{F}_s$, with $s \leqslant t$ in $\mathbb{R}_+$ and $x \leqslant y$ in L. Then

$$I_X(C)(\omega) = 1_A(\omega)[X_t(\omega, (x, y]) - X_s(\omega, (x, y])] =$$

$$\int 1_C(\omega, t, x)\, m_{X(\omega)}(dt, dx) = \int 1_C(\omega, t, x)\, X(\omega, dt, dx),$$

(even though $m_{X(\omega)}$ might not be σ-additive), therefore

$$\mu_X(C) = E\left(\int 1_C(\omega, t, x)\, X(\omega, dt, dx)\right) = E(I_X(C)) .$$

Since both μ_X and $E(I_X)$ are σ-additive on $\mathcal{P}\otimes\mathcal{L}$ and equal on $\mathcal{T}$, it follows that

$$\mu_X(C) = E(I_X(C)), \qquad \text{for } C \in \mathcal{P}\otimes\mathcal{L} .$$

Similarly,

$$\mu_F(C) = E(I_F(C)), \qquad \text{for } C \in \mathcal{P}\otimes\mathcal{L} .$$

From $I_F = I_X$ on $\mathcal{P}\otimes\mathcal{L}$, it follows that

$$E(I_F(C)) = E(I_X(C)), \qquad \text{for } C \in \mathcal{P}\otimes\mathcal{L} ,$$

therefore $\mu_F(C) = \mu_X(C)$, for $C \in \mathcal{P}\otimes\mathcal{L}$. From the last equality we deduce that

$$(\tilde{\mu}_F)_{F,G}(C) = (\tilde{\mu}_X)_{F,G}(C), \qquad \text{for } C \in \mathcal{P}\otimes\mathcal{L} ,$$

and the theorem is proved.

3.2. *The Stochastic Integral as a Stieltjes integral.*

We can prove now one of the main theorems of the paper, that expresses the Stochastic integral as a pathwise Stieltjes integral.

THEOREM 3.4: *Assume $c_0 \not\subset E$. Let $X: \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ be a right continuous, adapted p-process measure with p-integrable semivariation $\tilde{X}_{F,G}$. If X is not pathwise σ-additive in E, assume that E and G are separable.*

Let $X': \Omega \times \mathbb{R}_+ \times \mathcal{L} \to E$ be a right continuous, adapted p-process measure with p-integrable semivariation $\tilde{X}'_{F,G}$, pathwise σ-additive in E and a modification of X.

Let $H: \Omega \times \mathbb{R}_+ \times L \to F$ be a separably valued, predictable process such that $\tilde{X}'_{F,G}(H(\omega)) = (\tilde{m}_{X'(\omega)})_{F,G}(H(\omega))$ belongs to L^p. Then

1) *$H \in \mathcal{F}_{F,G}(I_X)$ and $\tilde{I}_X(H) \leq \|\tilde{X}(H(\omega))\|_p$, and the integrals $\int H dI_X$ and $\int H(\omega, x)\, X(\omega, dt, dx)$ are defined a.s.*

2) *We have $\int_{[0,t]\times B} H dI_X \in L^p_G$ for every $t \in \mathbb{R}_+$ and $B \in \mathcal{L}$ iff $\int_{[0,t]\times B} H(\omega, t, x)\, X(\omega, dt, dx) \in G$ a.s., for every $t \in \mathbb{R}_+$ and $B \in \mathcal{L}$, and in this case we have*

$$\left(\int_{[0,t]\times B} H dI_X\right)(\omega) = \int_{[0,t]\times B} H(\omega, t, x)\, X(\omega, dt, dx), \quad a.s.$$

3) *We have* $H \in L^1_{F,G}(X)$ *iff* $\int_{[0,t]\times B} H(\omega, t, x)\, X(\omega, dt, dx) \in G$, *a.s. and the process set function* $\left[\int_{[0,t]\times B} H(\omega, t, x)\, X(\omega, dt, dx)\right]_{t\in\mathbb{R}_+,\, B\in\mathcal{L}}$ *has a cadlag modification. In this case, for each* $t \in \mathbb{R}_+$ *and* $B \in \mathcal{L}$ *we have*

$$(H\cdot X)(\omega, t, B) = \int_{[0,t]\times B} H(\omega, t, x)\, X(\omega, dt, dx), \quad a.s.$$

Proof: Assume first that X is pathwise σ-additive in E and $X' = X$. We shall divide the proof into several steps.

a) $I_X(M)(\omega) = m_{X(\omega)}(M(\omega))$, a.s., for every $M \in \mathcal{P}\otimes\mathcal{L}$.

Assume first $M = A \times (s, t] \times (x, y]$ with $A \in \mathcal{F}_s$. Then

$$I_X(M)(\omega) = 1_A(\omega)\left(X_t(\omega, (x, y]) - X_s(\omega, (x, y])\right)$$

and

$$m_{X(\omega)}((s, t] \times (x, y]) = X_t(\omega, (x, y] - X_s(\omega, x, y]) ;$$

if we notice that the section $M(\omega)$ satisfies $M(\omega) = (s, t] \times (x, y]$ if $\omega \in A$ and $M(\omega) = \phi$ if $\omega \notin A$, then

$$1_A(\omega)\, m_{X(\omega)}((s, t] \times (x, y]) = m_{X(\omega)}((s, t] \times (x, y]) = m_{X(\omega)}(M(\omega)) ,$$

therefore

$$I_X(M)(\omega) = m_{X(\omega)}(M(\omega)), \qquad \text{everywhere} .$$

The same is true if $M = A \times \{0\} \times (x, y]$ with $A \in \mathcal{F}_0$. Then this equality remains true for M in the ring $r(\mathcal{R} \times \mathcal{S})$ generated by the above sets.

Let now $\mathcal{M}_0$ be the class of sets $M \in \mathcal{P}\otimes\mathcal{L}$ such that

$$I_X(M)(\omega) = m_{X(\omega)}(M(\omega)), \quad \text{a.s.}$$

We notice that I_X is defined for every $M \in \mathcal{P}\otimes\mathcal{L}$ and is σ-additive in L^p_E, since X is p-summable. We remark also that for each $\omega \in \Omega$, the function measure $X(\omega)$: $\mathbb{R}_+ \times \mathcal{L} \to E$ is right continuous, σ-additive in E and has bounded semivariation $(\tilde{X}(\omega))_{F,G}$ a.s., since $\tilde{X}(\omega)_{F,G} = \tilde{X}_{F,G}(\omega)$, and by hypothesis, $\tilde{X}_{F,G}(\omega)$ belongs to L^p, therefore it is finite a.s. It follows then that $m_{X(\omega)}$ is defined and σ-additive on $B(\mathbb{R})\otimes\mathcal{L}$, with values in E. We shall prove that $\mathcal{M}_0$ is a monotone class. Since it contains the ring $r(\mathcal{R} \times \mathcal{S})$, it will follows that $\mathcal{M}_0 = \mathcal{P}\otimes\mathcal{L}$.

Let (M_n) be a monotone sequence from $\mathcal{M}_0$ with limit M and prove that $M \in \mathcal{M}_0$.

Since I_X is σ-additive in L_E^p we have

$$I_X(M) = \lim I_X(M_n), \qquad \text{in } L_E^p.$$

The sequence of sections $(M_n(\omega))$ is monotone in $B(\mathbb{R}) \otimes \mathcal{L}$, with limit $M(\omega)$. Since $m_{X(\omega)}$ is σ-additive, we have

$$m_{X(\omega)}(M(\omega)) = \lim m_{X(\omega)}(M_n(\omega)), \qquad \text{in } E, \qquad \text{for each } \omega \in \Omega .$$

For each n we have

$$I_X(M_n)(\omega) = m_{X(\omega)}(M_n(\omega)), \quad \text{a.s.}$$

therefore

$$I_X(M)(\omega) = \lim m_{X(\omega)}(M_n(\omega)), \quad \text{a.s.}$$

It follows that that two limits of $I_X(M_n)$ are equal a.s.:

$$I_X(M)(\omega) = m_{X(\omega)}(M(\omega)), \quad \text{a.s.},$$

therefore $M \in \mathcal{M}_0$, hence $\mathcal{M}_0 = \mathcal{P} \otimes \mathcal{L}$.

b) $(\tilde{I}_X)_{F,\, L_G^p}(M) = \|X_{F,\, G}(M(\omega))\|_{L^p} \leq \|(\tilde{X}_{F,\, G})_\infty\|_p$, for $M \in \mathcal{P} \otimes \mathcal{L}$.

In fact, let $M \in \mathcal{P} \otimes \mathcal{L}$, let $(M_i)_{i \in I}$ be a finite family of disjoint sets from $\mathcal{P} \otimes \mathcal{L}$ contained in M and $(x_i)_{i \in I}$ a family of elements from F with $|x_i| \leq 1$. Then

$$\|\sum I_X(M_i)\, x_i\|_{L_G^p}^p = E\left(\left|\sum I_X(M_i)\, x_i\right|^p\right) =$$

$$= E\left(\left|\sum m_{X(\omega)}(M_i(\omega))\, x_i\right|^p\right) \leq E\left((\tilde{m}_{X(\omega)}(M(\omega)))^p\right) =$$

$$= \|\tilde{m}_{X(\omega)}(M(\omega))\|_{L^p}^p = \|\tilde{X}_{F,\, G}(M(\omega))\|_{L^p}^p,$$

where $\tilde{m}_{X(\omega)}$ is the semivariation of $m_{X(\omega)}$ relative to (F, G). Then

$$(\tilde{I}_X)_{F,\, L_G^p}(M) \leq \|\tilde{X}_{F,\, G}(M(\omega))\|_{L^p} \leq \|(\tilde{X}_{F,\, G})_\infty\|_p.$$

We can now start the proof of the theorem.

c) We assume first $H = 1_M x$, with $M \in \mathcal{P} \otimes \mathcal{L}$ and $x \in F$. Then

$$(\tilde{I}_X)_{F,\, L_G^p}(H) = (\tilde{I}_X)_{F,\, L_G^p}(1_M\, |x|) = (\tilde{I}_X)_{F,\, L_G^p}(M)\, |x| \leq$$

$$\leq \|\tilde{X}_{F,\, G}(M(\omega))\|_p\, |x| = \|\tilde{X}_{F,\, G}(1_{M(\omega)} x)\|_p = \|\tilde{X}_{F,\, G}(H)\|_p < \infty, \quad \text{by hypothesis},$$

hence $H \in \mathcal{F}_{F,\, G}(I_X)$, and $(\tilde{I}_X)_{F,\, G}(H) \leq \|\tilde{X}_{F,\, G}(H(\omega))\|_p$, and the first assertion of the theorem is proved. To prove assertion 2), let $t \in \mathbb{R}_+$ and $B \in \mathcal{L}$. Then $1_{[0,\, t] \times B} H = 1_{(\Omega \times [0,\, t] \times B) \cap M}$ and $(\Omega \times [0, t] \times B) \cap M \in \mathcal{P} \times \mathcal{L}$, therefore $\int\limits_{[0,\, t] \times B} H dI_X \in L_G^p$. Also

$\int\limits_{[0,\,t]\times B} H(\omega,\,t,\,x)\,X(\omega,\,dt,\,dx) \in G$, a.s., and we have

$$\int\limits_{[0,\,t]\times B} HdI_X = \int 1_M 1_{[0,\,t]\times B}\,x dI_X =$$

$$= I_X(M \cap (\Omega \times [0,\,t] \times B))x = m_{X(\omega)}(M(\omega) \cap ([0,\,t] \times B))\,x =$$

$$= \int\limits_{[0,\,t]\times B} 1_M(\omega,\,t,\,x)\,xX(\omega,\,dt,\,dx) = \int\limits_{[0,\,t]\times B} H(\omega,\,t,\,x)\,X(\omega,\,dt,\,dx),$$

a.s., which proves assertion 2).

To prove assertion 3), let $t_n \downarrow t$ in $\mathbb{R}_+$ and let $B \in \mathcal{L}$. Then

$$\lim \int\limits_{[0,\,t_n]\times B} H(\omega,\,t,\,x)\,X(\omega,\,dt,\,dx) =$$

$$= \lim m_{X(\omega)}((\Omega \times [0,\,t_n] \times B) \cap M(\omega))\,x = m_{X(\omega)}((\Omega \times [0,\,t] \times B) \cap M(\omega))\,x =$$

$$= \int\limits_{[0,\,t]\times B} H(\omega,\,t,\,x)\,X(\omega,\,dt,\,dx),$$

hence the mapping $t \mapsto \int\limits_{[0,\,t]\times B} H(\omega,\,t,\,x)\,X(\omega,\,dt,\,dx)$ is right continuous. It follows that $\left(\int\limits_{[0,\,t]} HdI_X\right)$ has a right continuous modification, therefore $H \in L^1_{F,\,G}(X)$ and the equality in assertion 3) can be written

$$(H \cdot X)(\omega,\,t,\,B) = \int\limits_{[0,\,t]\times B} H(\omega,\,t,\,x)\,X(\omega,\,dt,\,dx), \quad \text{a.s.}$$

This proves the theorem for $H = 1_M x$. Then the theorem remains valid for $\mathcal{P} \otimes \mathcal{L}$-step functions H.

d) Assume now H is as in the hypothesis of the theorem.

Then $(\tilde{m}_{X(\omega)})_{F,\,G}(H(\omega)) < \infty$, a.s., hence $H(\omega) \in \mathcal{F}_{F,\,G}(m_{X(\omega)})$ a.s. and the following Stieltjes integral is defined a.s.

$$\int H(\omega,\,t,\,x)\,X(\omega,\,dt,\,dx) = \int H(\omega,\,t,\,x)\,m_{X(\omega)}(dt,\,dx)$$

and belongs to G^{**}, in general. Then

$$\left| \int H(\omega, t, x)\, X(\omega, dt, dx) \right| = \left| \int H(\omega, t, x)\, m_{X(\omega)}(dt, dx) \right| \leqslant$$

$$\leqslant (\tilde{m}_{X(\omega)})_{F,G}(H(\omega)) = \tilde{X}_{F,G}(H(\omega))$$

and the last function belongs to L^p, by hypothesis. It follows that $\int H(\omega, t, x)\, X(\omega, dt, dx) \in L^p_{G^{**}}$.

Let $H' : \Omega \times \mathbb{R}_+ \times L \to F$ be a $\mathcal{P} \otimes \mathcal{L}$-simple process with $|H'| \leqslant |H|$. Then

$$\left\| \int H'\, dI_X \right\|_{L^p_G} = \left\| \int H'(\omega, t, x)\, X(\omega, dt, dx) \right\|_{L^p} \leqslant$$

$$\leqslant \|\tilde{X}_{F,G}(H'(\omega))\|_p \leqslant \|\tilde{X}_{F,G}(H(\omega))\|_p .$$

Taking the supremum for all such H' we obtian

$$(\tilde{I}_X)_{F, L^p_G}(H) \leqslant \|\tilde{X}_{F,G}(H(\omega))\|_p < \infty ,$$

therefore $H \in \mathcal{F}_{F,G}(I_X)$ and the integral $\int H dI_X$ is defined, with values in $(L^q_{G^*})^*$, and assertion 1) is proved. To prove assertion 2), let (H^n) be a sequence of $\mathcal{P} \otimes \mathcal{L}$-simple F-valued processes, such that $H^n \to H$ everywhere and $|H^n| \leqslant |H|$ for every n. Let $\omega \in \Omega$ and $z \in L^q_{G^*}$, with $1/p + 1/q = 1$. Then $y = z(\omega) \in G^*$. For each $(t, x) \in \mathbb{R}_+ \times L$ we have $H^n(\omega, t, x) \to H(\omega, t, x)$. Since the function $\omega \mapsto \tilde{m}_{X(\omega)}(H(\omega))$ belongs to L^p, we deduce that $\tilde{m}_{X(\omega)}(H(\omega)) < \infty$, a.s., hence $H(\omega) \in \mathcal{F}_{F,G}(m_{X(\omega)})$, a.s., consequently $H(\omega) \in L^1_F(m_{X(\omega)})_y)$, a.s. We can apply Lebesgue's theorem and deduce that $H(\omega) = \lim H^n(\omega)$ in $L^1_F((m_{X(\omega)})_y)$, therefore

$$\int H(\omega)\, d(m_{X(\omega)})_y = \lim \int H^n(\omega)\, d(m_{X(\omega)})_y ,$$

that is

$$\left\langle \int H(\omega, t, x)\, X(\omega, dt, dx), z(\omega) \right\rangle = \lim \left\langle \int H^n(t, \omega, x)\, X(\omega, dt, dx), z(\omega) \right\rangle .$$

Since $\left| \left\langle \int_{[0,t] \times B} H(\omega, t, x)\, X(\omega, dt, dx), z(\omega) \right\rangle \right| \leqslant \tilde{X}_{F,G}(H(\omega))|z(\omega)|$ and the function $\omega \mapsto \tilde{X}_{F,G}(H(\omega))|z(\omega)|$ belongs to L^1, we can apply Lebesgue's theorem in L^1

and deduce that

$$\left\langle \int_{[0,t]\times B} H(\omega, t, x)\, X(\omega, dt, dx), z(\omega) \right\rangle =$$

$$= \lim \left\langle \int_{[0,t]\times B} H^n(\omega, t, x)\, X(\omega, dt, dx, z(\omega)) \right\rangle, \quad \text{in } L^1.$$

We can apply Lebesgue's theorem in the space $L^1_F((I_X)_z)$ and deduce that

$$\int_{[0,t]\times B} H d(I_X)_z = \lim \int_{[0,t]\times B} H^n d(I_X)_z,$$

that is

$$\left\langle \int_{[0,t]\times B} H dI_X, z \right\rangle = \lim \left\langle \int_{[0,t]\times B} H^n d(I_X), z \right\rangle,$$

where the brackets express the duality between the spaces $L^p_{G^{**}}$ and $L^q_{G^*}$. For each n we have

$$\left(\int_{[0,t]\times B} H^n dI_X \right)(\omega) = \int_{[0,t]\times B} H^n(\omega, t, x)\, X(\omega, dt, dx);$$

therefore

$$\left\langle \int_{[0,t]\times B} H^n dI_X, z \right\rangle = E\left[\left\langle \int_{[0,t]\times B} H^n(\omega, t, x)\, X(\omega, dt, dx), z(\omega) \right\rangle \right].$$

Passing to the limit we obtain

$$\left\langle \int_{[0,t]\times B} H dI_X, z \right\rangle = E\left[\left\langle \int_{[0,t]\times B} H(\omega, t, x)\, X(\omega, dt, dx), z(\omega) \right\rangle \right].$$

Since $L^q_{G^*}$ is norming for $L^p_{G^{**}}$, it follows that

$$\int_{[0,t]\times B} H dI_X = \int_{[0,t]\times B} H(\omega, t, x)\, X(\omega, dt, dx), \quad \text{in } L^p_{G^{**}}.$$

If $\int_{[0,t]\times B} H dI_X \in L^p_G$, this equality is written

$$\left(\int_{[0,t]\times B} H dI_X \right)(\omega) = \int_{[0,t]\times B} H(\omega, t, x)\, X(\omega, dt, dx), \quad \text{a.s.},$$

and from this we deduce that $\int_{[0,t]\times B} H(\omega, t, x)\, X(\omega, dt, dx) \in G$ a.s. Conversely, if

$\int_{[0,t]\times B} H(\omega, t, x)\, X(\omega, t, dx) \in G$, then the function

$$\omega \mapsto \int_{[0,t]\times B} H(\omega, t, x)\, X(\omega, dt, dx)$$

belongs to L^p_G, hence $\int_{[0,t]\times B} HdI_X \in L^p_G$, and assertion 2) is proved.

From the equality

$$\left(\int_{[0,t]\times B} HdI_X\right)(\omega) = \int_{[0,t]\times B} H(\omega, t, x)\, X(\omega, dt, dx), \quad \text{a.s.},$$

it follows that the right hand side has a cadlag modification iff the left hand side has a cadlag modification; therefore $H \in L^1_{F,G}(X)$ iff the process measure $\left[\int_{[0,t]\times B} H(\omega, t, x)\, X(\omega, dt, dx\right]_{t\in\mathbf{R}_+,\, B\in\mathcal{E}}$ has values in G a.s. and has a cadlag modification. In this case, the above equality is written

$$(H\cdot X)(\omega, t, x) = \int_{[0,t]\times B} H(t, \omega, x)\, X(\omega, dt, dx), \quad \text{a.s.}$$

and the theorem is proved, in case $X' = X$.

If X is not pathwise σ-additive in E, we have

$$I_X = I_{X'}, \qquad \tilde{I}_X = \tilde{I}_{X'}, \qquad \mathcal{F}_{F,G}(X) = \mathcal{F}_{F,G}(X'),$$

and $\int HdI_X = \int HdI_{X'}$ for $H \in \mathcal{F}_{F,G}(X)$, as well as $L^1_{F,G}(X) = L^1_{F,G}(X')$ and $H\cdot X = H\cdot X'$ for $H \in L^1_{F,G}(X)$. Finally, according to Section 3.4, we have, as a notation

$$\int H(\omega, t, x)\, X(\omega, dt, dx) = \int H(\omega, t, x)\, X'(\omega, dt, dx),$$

so the theorem remains true in this case too.

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