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Paul Lévy Type Inequalities for Symmetric Random Variables

SUMMARY. — We prove some inequalities for a jointly symmetric system of n random variables with values in a measurable group. These inequalities include, as a particular case, the classical inequalities of Paul Lévy.

Diseguaglianze del tipo di Paul Lévy per variabili aleatorie simmetriche

SUNTO. — Si dimostrano alcune diseguaglianze per una n-upla globalmente simmetrica di variabili aleatorie a valori in un gruppo misurabile. Queste diseguaglianze contengono come caso particolare le diseguaglianze classiche di Paul Lévy.

1. - THE CASE OF MEASURABLE GROUP

Let G be a measurable abelian group, that is an abelian group (for which we shall use the additive notation) endowed with a σ-field $\mathcal{G}$ with respect to which the operation $(x, y) \mapsto y - x$ is measurable as a mapping from the measurable space $(G \times G, \mathcal{G} \otimes \mathcal{G})$ into the measurable space $(G, \mathcal{G})$.

Let $(X_j)_{1 \leq j \leq n}$ be a finite sequence of random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ and taking values in G. We shall say that $(X_j)_{1 \leq j \leq n}$ is *jointly symmet-*

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ric if, for each sequence $(\varepsilon_j)_{1 \leqslant j \leqslant n}$ of elements of $\{-1, 1\}$, the two random vectors

$$[X_j]_{1 \leqslant j \leqslant n}, \qquad [\varepsilon_j X_j]_{1 \leqslant j \leqslant n}$$

have the same distribution. (In particular, this condition holds when the random variables X_j are independent and symmetric.)

A random variable T defined on the probability space $(\Omega, \mathcal{A}, P)$ and taking its values in $\{1, \ldots, n, \infty\}$, is called an *optional time* with respect to the natural filtration associated with $(X_j)_{1 \leqslant j \leqslant n}$ if, for $1 \leqslant j \leqslant n$, we have

$$I_{\{T=j\}} = I_{A_j}(X_1, \ldots, X_j), \tag{1.1}$$

where A_j is a suitable measurable set. If, moreover, it is possible to take the sets A_j such that (for each j)

$$I_{\{T=j\}} = I_{A_j}(X_1, \ldots, X_j) = I_{A_j}(-X_1, \ldots, -X_{j-1}, X_j), \tag{1.2}$$

then we shall say that the optional time T is *symmetric*.

We observe that, if X, Y are two random variables with values in G, then it is the same also for $X + Y$.

Theorem 1.1: *Let $(X_j)_{1 \leqslant j \leqslant n}$ be a jointly symmetric finite sequence of random variables on a probability space $(\Omega, \mathcal{A}, P)$ with values in G, and T be an optional time with respect to the natural filtration associated with $(X_j)_{1 \leqslant j \leqslant n}$.*

For $1 \leqslant j \leqslant n$, let $S_j = X_1 + \ldots + X_j$. Further let $S_T = \sum_{j=1}^{n} I_{\{T=j\}} S_j$. Finally let f be any positive (or bounded) measurable function defined on $(G, \mathcal{G})$ such that

$$f(x) \leqslant f(x+y) + f(x-y) \tag{1.3}$$

for each pair x, y of elements in G. Then

$$\int_{\{T<\infty\}} f(S_T)\, \mathrm{d}P \leqslant 2 \int_{\{T<\infty\}} f(S_n)\, \mathrm{d}P .$$

Proof: For $1 \leqslant j \leqslant n$, let

$$Z_j = S_n - S_j = X_{j+1} + \ldots + X_n .$$

Then the two random vectors $[I_{\{T=j\}}, S_j, Z_j]$, $[I_{\{T=j\}}, S_j, -Z_j]$ have the same distribution. (This follows from the hypothesis of joint symmetry of $(X_j)_{1 \leqslant j \leqslant n}$ by the property (1.1).) Therefore we have

$$\int_{\{T=j\}} f(S_n)\,\mathrm{d}P = \int_{\{T=j\}} f(S_j+Z_j)\,\mathrm{d}P = \int_{\{T=j\}} f(S_j-Z_j)\,\mathrm{d}P\,.$$

Let now $Z_T = \sum_{j=1}^{n} I_{\{T=j\}} Z_j$. Adding the above equalities, we find

$$\int_{\{T<\infty\}} f(S_n)\,\mathrm{d}P = \int_{\{T<\infty\}} f(S_T+Z_T)\,\mathrm{d}P = \int_{\{T<\infty\}} f(S_T-Z_T)\,\mathrm{d}P\,.$$

Hence, using the property (1.3), we finally get

$$\int_{\{T<\infty\}} f(S_T)\,\mathrm{d}P \leqslant \int_{\{T<\infty\}} f(S_T+Z_T)\,\mathrm{d}P + \int_{\{T<\infty\}} f(S_T-Z_T)\,\mathrm{d}P = 2\int_{\{T<\infty\}} f(S_n)\,\mathrm{d}P\,.$$

In the same way we can prove the following theorem:

THEOREM 1.2: *Under the same hypotheses as that of the preceding theorem, let us suppose that the optional time T is symmetric, and let* $X_T = \sum_{j=1}^{n} I_{\{T=j\}} X_j$. *Then*

$$\int_{\{T<\infty\}} f(X_T)\,\mathrm{d}P \leqslant 2\int_{\{T<\infty\}} f(S_n)\,\mathrm{d}P\,.$$

PROOF: For $1 \leqslant j \leqslant n$, let

$$Y_j = S_n - X_j = \sum_{1\leqslant i\leqslant n,\, i\neq j} X_i\,.$$

Then the two random vectors $[I_{\{T=j\}}, X_j, Y_j]$, $[I_{\{T=j\}}, X_j, -Y_j]$ have the same distribution. (This follows from the hypothesis of joint symmetry of $(X_j)_{1\leqslant j\leqslant n}$ by the property (1.2).) Therefore we have

$$\int_{\{T=j\}} f(S_n)\,\mathrm{d}P = \int_{\{T=j\}} f(X_j+Y_j)\,\mathrm{d}P = \int_{\{T=j\}} f(X_j-Y_j)\,\mathrm{d}P\,.$$

Let now $Y_T = \sum_{j=1}^{n} I_{\{T=j\}} Y_j$. Adding the above equalities, we find

$$\int_{\{T<\infty\}} f(S_n)\,\mathrm{d}P = \int_{\{T<\infty\}} f(X_T+Y_T)\,\mathrm{d}P = \int_{\{T<\infty\}} f(X_T-Y_T)\,\mathrm{d}P\,.$$

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2. - The case of a normed space

Now we shall specialize the hypotheses of the previous section supposing, in addition, that there exists a positive real function $x \mapsto |x|$, measurable on $(G, \mathcal{G})$, such that

$$(2.1) \qquad |-x| = |x|, \qquad |x+y| \leq |x| + |y|, \qquad |x| \leq |x+y| \vee |x-y|$$

for each pair x, y of elements in G.

These particular hypotheses are clearly satisfied when G is the underlying additive group of a separable normed space and $\mathcal{G}$ coincides with the Borel σ-field of G.

Remark 2.1: More generally, the hypotheses of the present section are satisfied when G is the underlying additive group of a (not necessarily separable) normed space for which there exists a countable set D of continuous linear forms generating the σ-field $\mathcal{G}$ and such that the norm of any element x of G coincides with $\sup_{l \in D} l(x)$. This situation includes, for instance, the important case of the space l^{∞}.

Let us remark that the last of the properties (2.1) implies that every function f such that $f(x) = g(|x|)$, with g positive and increasing on $\mathbb{R}_+$, has the property (1.3).

If X is a random variable taking its values in G, we shall denote by $|X|$ the real random variable $\omega \mapsto |X(\omega)|$.

Theorems (1.1), (1.2) imply the following corollary:

Corollary 2.2: *Let $(X_j)_{1 \leq j \leq n}$ be a jointly symmetric finite sequence of random variables on a probability space $(\Omega, \mathcal{A}, P)$ with values in G.*

Let $S_j = X_1 + \ldots + X_j$. Then, for each positive real number s, we have:

$$(2.2) \qquad P\{\sup_{1 \leq j \leq n} |S_j| > s\} \leq 2P\{|S_n| > s\},$$

$$(2.3) \qquad P\{\sup_{1 \leq j \leq n} |X_j| > s\} \leq 2P\{|S_n| > s\}.$$

Proof: In order to prove inequality (2.2), it is sufficient to apply Theorem (1.1) with

$$T(\omega) = \inf\{j\colon |S_j(\omega)| > s\} \quad \text{and} \quad f(x) = I_{]s,\infty[}(|x|).$$

In the same way, in order to prove inequality (2.3), it is sufficient to apply Theorem (1.2) with

$$T(\omega) = \inf\{j\colon |X_j(\omega)| > s\} \quad \text{and} \quad f(x) = I_{]s,\infty[}(|x|).$$

Inequalities (2.2), (2.3) are known as inequalities of Paul Lévy.

Now we shall prove the following theorem:

THEOREM 2.3: *Let $(X_j)_{1 \leqslant j \leqslant n}$ be a finite sequence of independent symmetric random variables on a probability space $(\Omega, \mathcal{A}, P)$ with values in G, and let T be an optional time with respect to the natural filtration associated with $(X_j)_{1 \leqslant j \leqslant n}$.*

For $1 \leqslant j \leqslant n$, let $S_j = X_1 + \ldots + X_j$. Further let $S_T = \sum_{j=1}^{n} I_{\{T=j\}} S_j$, $s_T = \sup_{\omega \in \Omega} |S_T(\omega)|$. Then:

(*a*) *For each positive increasing function g defined on $[0, +\infty]$, we have*

$$\int_{\{T<\infty\}} g(|S_n|)\, dP \leqslant 2P\{T<\infty\} \int g(s_T + |S_n|)\, dP. \tag{2.4}$$

(*b*) *For each positive real number s, we have*

$$P\{|S_n| > s,\ T<\infty\} \leqslant 2P\{T<\infty\}\, P\{|S_n| > s - s_T\}.$$

PROOF: Let $Z_j = S_n - S_j = X_{j+1} + \ldots + X_n$ and g be a positive increasing function defined on $[0, +\infty]$. Then

$$\int_{\{T=j\}} g(|S_n|)\, dP \leqslant \int_{\{T=j\}} g(|S_j| + |Z_j|)\, dP \leqslant$$

$$\leqslant \int_{\{T=j\}} g(s_T + |Z_j|)\, dP = P\{T=j\} \int g(s_T + |Z_j|)\, dP.$$

(The final equality follows from the fact that Z_j and $I_{\{T=j\}}$ are independent.) Hence, adding these relations, we get

$$\int_{\{T<\infty\}} g(|S_n|)\, dP \leqslant \sum_{j=1}^{n} P\{T=j\} \int g(s_T + |Z_j|)\, dP. \tag{2.5}$$

Let us now define the function f by setting $f(x) = g(s_T + |x|)$. Applying to this function Theorem (1.1) (in which we replace the sequence $(X_1, \ldots, X_n)$ by the sequence $(X_n, \ldots, X_1)$ and take for optional time the constant $n-j$), we find:

$$\int g(s_T + |Z_j|)\, dP \leqslant 2 \int g(s_T + |S_n|)\, dP.$$

Introducing this inequality into (2.5), we get finally the relation (2.4) and thus statement (*a*) is proved.

Statement (*b*) is a particular case of (*a*), obtained by taking for g the indicator function $I_{]s,\infty]}$.

Using the theorem just proved, we can easily deduce the following corollary:

COROLLARY 2.4: *Under the same hypotheses as that of the preceding theorem, let us suppose also that there exists a constant c such that* $|X_j| \leqslant c$, *for each* j. *Then we have*

$$(1 - 2P\{\sup_{1 \leqslant j \leqslant n} |S_j| > s\}\, e^{\lambda(s+c)}) \int \exp(\lambda |S_n|)\, dP \leqslant P\{\sup_{1 \leqslant j \leqslant n} |S_j| \leqslant s\}\, e^{\lambda s}$$

for each pair s, λ *of positive real numbers.*

The proof follows on applying Theorem (2.3) with

$$T(\omega) = \inf\{j\colon |S_j(\omega)| > s\} \quad \text{and} \quad g(x) = e^{\lambda x}.$$

REMARK 2.5: Let $(X_j)_{j \geqslant 1}$ be an infinite sequence of independent symmetric random variables on a probability space $(\Omega, \mathcal{A}, P)$ with values in G, such that $|X_j| \leqslant c$ for each j.

Let $S_n = X_1 + \ldots + X_n$ and let us suppose that $(|S_n|)_{n \geqslant 1}$ converges almost surely to a positive real random variable Z.

Then, following an argument similar to that of Ledoux-Talagrand in [3], we find, as a consequence of corollary (2.4), that $\int \exp(\lambda Z)\, dP < \infty$ for some $\lambda > 0$. Indeed, if in corollary (2.4) we choose s such that $P\{\sup_n |S_n| > s\} \leqslant (4e)^{-1}$, and let $\lambda = (s + c)^{-1}$, we get, for each strictly positive integer n,

$$\int \exp(\lambda |S_n|)\, dP \leqslant 2e^{\lambda s},$$

and hence, by Fatou's Lemma, we obtain

$$\int \exp(\lambda Z)\, dP \leqslant 2e^{\lambda s}.$$

Further, from the above inequality, we can easily deduce that Z has finite moments of any order.

Applying once more Theorem (2.3), we can also deduce the following result, due to J. Hoffmann-Jørgensen [1,2]:

THEOREM 2.6: *Under the same hypotheses as that of the preceding theorem, we have*

$$P\{|S_n| > 2s + t\} \leqslant 4(P\{|S_n| > s\})^2 + P\{\sup_{1 \leqslant j \leqslant n} |X_j| > t\}$$

for each pair s, t *of positive real numbers.*

PROOF: Let us consider the two optional times U, V defined by:

$$U(\omega) = \inf\{j \in \mathbb{N}: 1 \leqslant j \leqslant n,\ |S_j(\omega)| > s\},$$

$$V(\omega) = \inf\{j \in \mathbb{N}: 1 \leqslant j \leqslant n,\ |X_j(\omega)| > t\}.$$

Then $\{V < \infty\} = \{\sup_{1 \leqslant j \leqslant n} |X_j| > t\}$. Moreover, the obvious inclusion

$$\{|S_n| > 2s + t\} \subset \{U < \infty\}$$

implies

$$P\{|S_n| > 2s + t\} \leqslant P\{|S_n| > 2s + t,\ U < V\} + P\{V < \infty\}.$$

Therefore, it is enough to prove

$$P\{|S_n| > 2s + t,\ U < V\} \leqslant 4(P\{|S_n| > s\})^2. \tag{2.6}$$

To this end, let us denote by T the optional time which coincides with U on $\{U < V\}$ and with $+\infty$ on $\{U \geqslant V\}$. Let ω be an element in $\{T < \infty\}$ and $j = T(\omega) = U(\omega)$. Then $j < V(\omega)$ implies $|X_j(\omega)| \leqslant t$ and consequently we have

$$|S_j(\omega)| \leqslant |S_j(\omega) - X_j(\omega)| + |X_j(\omega)| \leqslant s + t.$$

Hence, with the notations of Theorem (2.3), we have $s_T \leqslant s + t$. Using this inequality and applying statement (*b*) of Theorem (2.3), we get:

$$\begin{aligned}(2.7)\quad P\{|S_n| > 2s + t,\ U < V\} &= P\{|S_n| > 2s + t,\ T < \infty\} \leqslant \\ &\leqslant 2P\{T < \infty\}\, P\{|S_n| > 2s + t - s_T\} \leqslant 2P\{T < \infty\}\, P\{|S_n| > s\}.\end{aligned}$$

On the other hand, applying (2.2), we find

$$P\{T < \infty\} \leqslant P\{U < \infty\} = P\{\sup_{1 \leqslant j \leqslant n} |S_j| > s\} \leqslant 2P\{|S_n| > s\}.$$

Therefore, in order to prove (2.6), it is sufficient to insert the last inequality into (2.7).

3. - THE REAL CASE

Let us now consider the particular case in which G is the additive group $(\mathbb{R}, +)$ of real numbers and $\mathcal{G}$ the Borel σ-field of $\mathbb{R}$. In this situation, the elementary inequality $x \leqslant (x + y) \vee (x - y)$, which holds for each pair x, y of real numbers, shows that any positive increasing function f defined on $\mathbb{R}$ has the property (1.3). Therefore, if in Theorem (1.1) we choose

$$T(\omega) = \inf\{j: S_j(\omega) > s\} \qquad \text{and} \qquad f(x) = I_{]s,\,\infty[}(x),$$

we get the following corollary:

COROLLARY 3.1: *Let* $(X_j)_{1 \leq j \leq n}$ *be a jointly symmetric finite sequence of real random variables on a probability space* $(\Omega, \mathcal{A}, P)$.

Let $S_j = X_1 + \ldots + X_j$. *Then, for each real number s, we have*

$$P\{\sup_{1 \leq j \leq n} S_j > s\} \leq 2P\{S_n > s\}. \tag{3.1}$$

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