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Bounded Solutions Relative to a Vibrating and not Tensioned String (**) (***)

SUMMARY. — We examine the problem of the motion of a vibrating and not tensioned string, when some friction is present. In particular, using the model already studied in a recent paper, we investigate the asymptotic behaviour of the solutions of the Cauchy-Dirichlet problem. We prove both the existence of a bounded solution in R, and the boundedness of all solutions in R^+.

Soluzioni limitate relative a una corda vibrante non completamente tesa

RIASSUNTO. — Si considera il problema della corda non completamente tesa, studiando il comportamento asintotico delle soluzioni del problema di Cauchy-Dirichlet, in presenza di attrito. Utilizzando un modello introdotto in un precedente lavoro, si ottengono teoremi di limitatezza per $t \to +\infty$ per tutte le soluzioni, nonché un teorema di esistenza di una soluzione limitata in tutto R.

1. - THE PROBLEM AND THE MODEL

In a recent paper [1] we studied a possible mathematical model of the motion of a vibrating and not tensioned string with fixed ends (for different models concerning other problems see, for instance, [2], [3], [4], [5], [6], [7], [8]). Following a classical procedure, we substituted the string with a system of n suitable elements, connected by hinges at the points $P_j^{(n)}$ sliding trasversally and without friction. We started studying a system of n ordinary differential equations, suggested by this problem. Taking then $n \to \infty$, the solution of the discrete model approaches, in a suitable sense, a solution of a partial differential equation which we assumed

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as the model of the problem considered above. This procedure allowed us to state an existence theorem for the Cauchy-Dirichlet problem.

The purpose of the present paper is to investigate the asymptotic behaviour of the solutions, when some friction is present.

Taking $t \to +\infty$, the first aim is to have appropriate assumptions on the external force, so that all solutions are bounded, and, moreover, to find a bound good for all solutions.

Later, considering $t \in \mathrm{R}$, we look for a bounded solution.

Supposing that the string, with proper length A, has fixed ends at the points $A(0, 0)$ and $B(L, 0)$, with $L < A$, and that some elastic reaction arises when its length reaches A, we introduce the following notations, where $n \in \mathrm{N}$, $i = -2, -1, \ldots, n+1, n+2$,

$$b^{(n)} = L/n\,, \qquad n \in \mathrm{N}\,;$$

$$\Delta = \{x\colon 0 \leqslant x \leqslant L\}\,;$$

$$\Delta_i^{(n)} = \{x\colon (i - 1/2)b^{(n)} \leqslant x \leqslant (i + 1/2)b^{(n)}\}\,;$$

$$\bar{\Delta}_i^{(n)} = \{x\colon ib^{(n)} \leqslant x \leqslant (i+1)b^{(n)}\}\,;$$

$$T_n = \{t\colon -n \leqslant t \leqslant n+1\}\,;$$

$$\mathrm{R}_{(\tau)}^{+} = \{t\colon \tau \leqslant t < +\infty\}\,;$$

$$Q_t = \{(x, t, \eta)\colon x \in \Delta, \eta \in T_0\}\,, \qquad t \in \mathrm{R}\,;$$

$$Q = \{(x, t)\colon x \in \Delta, t \in \mathrm{R}\}\,;$$

$$\{f_i^{(n)}\} = \{f_{n-2}^{(n)}, f_{n-1}^{(n)}, \ldots, f_{n+1}^{(n)}, f_{n+2}^{(n)}\}\,;$$

$$\delta f_i^{(n)} = f_{i+1}^{(n)} - f_i^{(n)}\,;$$

$$S = \{f_i^{(n)}\}\colon f_{-2}^{(n)} = -f_2^{(n)}, f_{-1}^{(n)} = -f_1^{(n)}, f_{n-2}^{(n)} = -f_{n+2}^{(n)}, f_{n-1}^{(n)} = -f_{n+1}^{(n)}, f_0^{(n)} = f_n^{(n)} = 0\}\,;$$

$$L_{\mathrm{loc}}^{p,k}(T) = \left\{f(t)\colon \operatorname*{Sup}_{t \in T}\left\{\int_{\Delta} |f(t+\eta)|^p\, d\eta\right\}^{1/p} < +\infty\right\};$$

obviously, for the sake of simplicity, we shall dropp out the (n) when no confusion will be possible; moreover, if necessary, we shall extend functions defined on Δ or Q as follows:

$$f(-x, t) = -f(x, t)\,, \qquad f(2L - x, t) = -f(x, t)\,, \qquad (x, t) \in \bigcup_{i=1}^{n} \Delta_i^{(n)} \times \mathrm{R}\,.$$

The function which defines the stress–strain relationship obtained when considering

the discrete model introduced above is

$$(1.1)\qquad \varphi(\alpha)=\begin{cases} L\left\{\sqrt{1+\alpha^2}-\dfrac{\Lambda}{L}\right\}\dfrac{\alpha}{\sqrt{1+\alpha^2}}, & |\alpha|\geq\sqrt{(\Lambda/L)^2-1}=\bar{\alpha}, \\ 0, & |\alpha|<\sqrt{(\Lambda/L)^2-1}=\bar{\alpha}, \end{cases}$$

in the classical case, when the proper length of the string is $\Lambda < L$, setting $\varphi(\alpha)=\alpha$, we have the Hooke's law.

Moreover the physical problem suggest to examine two families of discrete Cauchy–Dirichlet problems (depending on h): the first one for $t\geq 0$, $0<\mu<2$,

$$(1.2)\qquad \begin{cases} z_i'' - \dfrac{\delta}{b}\varphi\left(\dfrac{\delta}{b}z_i\right) + b^{\mu}\dfrac{\delta^4}{b^4}z_{i-2} + z_i' - f_i(t) = 0, \quad i=1,2,\dots n-1, \\ z_0(t)=z_n(t)=0, \quad z_{-1}(t)=-z_1(t), \quad z_{-2}(t)=-z_2(t), \\ z_{n-1}(t)=-z_{n+1}(t), \quad z_{n-2}(t)=-z_{n+2}(t), \end{cases}$$

$$(1.3)\qquad z_i(0)=\bar{z}_i, \quad z_i'(0)=\bar{z}_i';$$

this problem allows us to study the behaviour of the solutions for $t\to+\infty$.

In fact, for suitable $\{\bar{z}_i^{(n)}\}$ and $\{\bar{z}_i'^{(n)}\}$, taking $n\to+\infty$, the poligonal $\delta^{(n)}(x,t)$, connecting the points $P_i^{(n)}(t)=\{ih^{(n)}, z_i^{(n)}(t)\}$, approaches a solution of the following variational problem

$$(1.4)\qquad \int_0^t \{(y'(\eta),u'(\eta))_{L^2}-(\varphi(Dy(\eta)),Du(\eta))_{L^2}-(y'(\eta),u(\eta))_{L^2}+$$

$$+(f(\eta),u(\eta))_{L^2}\}\,d\eta+(\bar{y}',u(0))_{L^2}-(y'(t),u(t))_{L^2}=0, \qquad t\in\mathbb{R}^+ \text{ a.e.},$$

$$(1.5)\qquad y(0)=\bar{y}.$$

In this case, studying the asymptotic behaviour for $t\to+\infty$, we want a family of solutions with similar properties, therefore we examine the family of solutions of (1.4) (1.5) which we can approach with a sequence of poligonals $\mathbb{Z}^{(n)}(x,t)$. Then, bearing in mind the boundary conditions, we shall say that $y(x,t)$ is a solution of the problem above considered if

i) $y(t)\in L^2_{loc}(\mathbb{R}^+;H_0^1)\cap H^1_{loc}(\mathbb{R}^+;L^2)$;

ii) (1.4) holds $\forall u\in L^2_{loc}(\mathbb{R}^2;H_0^1)\cap H^1_{loc}(\mathbb{R}^+;L^2)$.

iii) $\forall T\in\mathbb{R}^+$ there exists a sequence of poligonals $\delta^{(n)}(x,t)$ approaching $y(x,t)$ weakly in $L^2(0,T;H_0^1)\cap H^1(0,T;L^2)$.

The second discrete problem, where we set $t \geq -n$ changing initial conditions (1.3) into

$$z_i(-n) = 0\,, \qquad z_i'(-n) = 0\,, \tag{1.6}$$

allows us to study the behaviour of solutions on R. In fact, taking $n \to +\infty$, the functions $\xi^{(n)}(x,t)$ approach a solution of the variational problem

$$\begin{aligned} \int_{t'}^{t''} \{ & (y'(\eta), u'(\eta))_{L^2} - (\varphi(Dy(\eta)), Du(\eta))_{L^2} + \\ & - (y'(\eta), u(\eta))_{L^2} + (f(\eta), u(\eta))_{L^2} \} \, d\eta + \\ & + (y'(t'), u(t'))_{L^2} - (y'(t''), u(t''))_{L^2} = 0\,, \qquad t', t'' \in \mathrm{R}^+ \ \text{ a.e.}\,, \end{aligned} \tag{1.7}$$

Now, since we are looking for a bounded solution on R, we say that $y(x,t)$ is a solution of the problem if

i) $y(t) \in L^2_{loc}(\mathrm{R}; H^1_0) \cap H^1_{loc}(\mathrm{R}; L^2)$;

ii) (1.7) holds $\forall u \in L^2_{loc}(\mathrm{R}; H^1_0) \cap H^1_{loc}(\mathrm{R}; L^2)$.

The equations (1.4) and (1.7), similar to the equation studied in [1], have a linear dissipative term which takes into account the friction.

Finally setting

$$\begin{cases} \Phi(\alpha) = \displaystyle\int_0^\alpha \varphi(\beta)\, d\beta = \begin{cases} \dfrac{L}{2}\left[\sqrt{1+\alpha^2} - \dfrac{A}{L}\right]^2, & |\alpha| \geq \bar{\alpha}\,, \\ 0\,, & |\alpha| \leq \bar{\alpha}\,, \end{cases} \\ E^{(n)}(z_i(t)) = \displaystyle\sum_{i=1}^{n} \left\{ \frac{1}{2} z_i'^2(t) + \Phi\left(\frac{\delta}{h} z_{i-1}(t)\right) + h^n \frac{\delta^2}{h^2} z_{i-2} \right\}, \\ E(z_i(t)) = \displaystyle\sum_{i=1}^{n} \left\{ \frac{1}{2} z_i'^2(t) + \Phi\left(\frac{\delta}{h} z_{i-1}(t)\right) \right\}, \\ E(y(t)) = \dfrac{1}{2} \|y'(t)\|^2_{L^2} + \|\Phi(Dy)\|_{L^1}\,, \end{cases} \tag{1.8}$$

we observe that $E(z_i(t))$ and $E(y(t))$ have un obvious physical meaning.

In fact, in the classical case, where $\varphi(\alpha) = \alpha$, the quantities

$$\begin{cases} E(z_i(t)) = \dfrac{1}{2} \displaystyle\sum_{i=1}^{n} \left\{ z_i'^2(t) + \left(\frac{\delta}{h} z_i(t)\right)^2 \right\}, \\ E(y(t)) = \dfrac{1}{2} \{ \|y'(t)\|^2_{L^2} + \|Dy(t)\|^2_{L^2} \}\,, \end{cases} \tag{1.9}$$

give the «energy» of the system or of the string respectively, the unique rest position

corresponds to the null solution, and we can assume (1.9) as a square of the norm in the «energy» space.

By analogy with the classical case we shall say that the functions $E^{(n)}(z_i)$ and $E(y)$ give the energy of the solutions of (1.2), (1.7) respectively. Nevertheless neither $E^{(n)}(z_i)$, nor $E(y)$ are any norms, in fact we have infinite rest positions corresponding to solutions not equal to zero.

In § 2 we give the statements on the continuous problem; in § 3 we give the statements on the discrete problem useful for the study of the continuous problem; in § 4 and § 5 we give the proofs of the statements of § 2 and § 3 respectively.

Finally we recall in § 6 some classical results and usefull formulas.

2. - The continuous problem

Supposing $f(t) \in L^{2,h}_{loc}(\mathrm{R}^+, L^2)$, we set

$$\|f\|_{L^{2,h}_{loc}(\mathrm{R}^+, L^2)} = \operatorname*{Sup}_{t \in \mathrm{R}^+} \|f(t+\eta)\|_{L^2(Q_1)} = \operatorname*{Sup}_{t \in \mathrm{R}^+} \left\{ \int_{I_0} d\eta \int_{\Omega} f^2(x, t+\eta)\, dx \right\}^{1/2} < \infty .$$

Now we can state the following theorems.

Theorem 2.1 (Boundedness of solutions): *Let*

$$\|f\|_{L^{2,h}_{loc}(\mathrm{R}^+, L^2)} = K < \infty , \tag{2.1}$$

$$\bar{y} \in H^1_0 , \quad \bar{y}' \in L^2 . \tag{2.2}$$

Then there exists at least a solution $y = y(t)$ in the sense indicated above (see § 1) of the Cauchy-Dirichlet problem (1.4), (1.5).

Moreover there exists a positive constant M (depending on $K, \bar{y}, \bar{y}'$) such that

$$E(y(t)) \leq M , \quad \|Dy(t)\|^2_{L^2} \leq M , \quad \|y'(t)\|^2_{L^2} \leq M , \quad t \in \mathrm{R}^+ \text{ a.e.} . \tag{2.3}$$

Theorem 2.2 (Asymptotic behaviour): *Let* (2.2) *hold and there exists $\beta > 1/2$ such that $t^\beta f(t) \in L^2(\mathrm{R}^+, L^2)$, with*

$$\|t^\beta f(t)\|_{L^2(\mathrm{R}^+, L^2)} = K . \tag{2.4}$$

Then

$$E(y(t)) \in L^1(\mathrm{R}^+, L^2) . \tag{2.5}$$

(2.5) *implies that $\forall \varepsilon > 0$ there exists a positive constant t_ε depending on $\varepsilon, \bar{y}, \bar{y}'$, such that*

$$\|E(y(t))\|_{L^{2,h}_{loc}(R_{t_\varepsilon})} \leq \varepsilon , \quad \|y'(t)\|^2_{L^{2,h}_{loc}(R_{t_\varepsilon}; L^2)} \leq \varepsilon , \tag{2.6}$$

$$\|Dy(t)\|^2_{L^{2,h}_{loc}(R_{t_\varepsilon}; L^2)} \leq \varepsilon + \bar{a}^2 . \tag{2.7}$$

THEOREM 2.3 (Boundeness in R): *Let*

$$\|f\|_{L^2_{\mathbb{R}}L^2(\mathbb{R},L^2)} = \operatorname*{Sup}_{t\in\mathbb{R}}\left\{\int_{T_0} d\eta \int_{\Omega} f^2(x,t+\eta)\,dx\right\}^{1/2} = K < +\infty\,, \tag{2.8}$$

Then there exists at least a solution of (1.7) *bounded in* R (*in the sense indicated in* § 1).

3. - THE DISCRETE PROBLEM

The first result that we state is the following lemma concerning a Cauchy-Dirichlet problem for the system (1.2)

$$z_i'' - \frac{\delta}{b}\varphi\left(\frac{\delta}{b}z_{i-1}\right) + b^{\mu}\frac{\delta^4}{b^4}z_{i-2} + z_i' - f_i(t) = 0\,,\quad i = 1, 2, \ldots, n-1\,,\quad t \leq \tau\,, \tag{3.1}$$

$$z_i(\tau) = \bar{z}_i\,,\qquad z_i'(\tau) = \bar{z}_i'\,. \tag{3.2}$$

LEMMA 3.1: *Assume that there exists a constant* $C > 0$ *such that*

$$\sum_{i=1}^{n} \|f_i(t)\|_{L^2_{\mathbb{R}}L^2(\mathbb{R}_t^+)} = \operatorname*{Sup}_{t\in\mathbb{R}^+_{(1)}}\left\{\sum_{i=1}^{n}\int_{T_0}\{f_i(t+\eta)\}^2\,d\eta\right\} \leq \frac{C}{b}\,. \tag{3.3}$$

Then there exists a constant $C_1 > 0$ *such that*

$$E^{(n)}(z_i^{(n)}(\bar{t}+1)) \leq \operatorname{Sup}\left\{E^{(n)}(z_i^{(n)}(\bar{t})),\ \frac{C_1}{b}\right\}. \tag{3.4}$$

We can then state the following Theorems.

THEOREM 3.2: *Assume that* (3.3) *holds and moreover*

$$\sum_{i=1}^{n}\bar{z}_i^2 \leq \frac{C}{b}\,,\quad \sum_{i=1}^{n}\bar{z}_i'^2 \leq \frac{C}{b}\,,\quad \sum_{i=1}^{n}\left(\frac{\delta}{b}\bar{z}_i\right)^2 \leq \frac{C}{b}\,,\quad \sum_{i=1}^{n}\left(\frac{\delta^2}{b^2}\bar{z}_i\right)^2 \leq \frac{C}{b}b^{-\mu/2}\,. \tag{3.5}$$

Then the solution $\{z_i^{(n)}(t)\} \in S$ *of* (3.1), (3.2), *which is unique, is bounded in* $\mathbb{R}^+_{(1)}$. *Precisely there exists a constant* C_1, *depending on* C, *such that*

$$\operatorname*{Sup}_{t\in\mathbb{R}^+_{(1)}} E^{(n)}(z_i^{(n)}(t)) \leq \frac{C_1}{b}\,. \tag{3.6}$$

Obviously we have

$$\sum_{i=1}^{n} z_i'^2(t) \leq \frac{C_1}{b}\,,\qquad \sum_{i=1}^{n} b^{\mu}\left(\frac{\delta^2}{b^2}z_{i-2}(t)\right)^2 \leq \frac{C_1}{b}\,; \tag{3.7}$$

and, by (6.6), *there exists a constant* $\mathcal{C}_2$ *such that*

$$\sum_{i=1}^{n} z_i^2(t) \leq \frac{\mathcal{C}_2}{b}, \quad \sum_{i=1}^{n}\left(\frac{\delta}{b} z_{i-1}(t)\right)^2 \leq \frac{\mathcal{C}_2}{b}. \tag{3.8}$$

THEOREM 3.3: *Assume that* (3.5) *holds and there exists* $\beta > 1/2$ *such that*

$$\sum_{i=1}^{n} \int_{\tau}^{\infty} \eta^{2\beta} f_i^2(\eta)\, d\eta \leq \frac{\mathcal{C}}{b}. \tag{3.9}$$

Then the «Energy» of the solution of (3.1), (3.2) $E^{(n)}(z_i^{(n)}(t))$ *vanishes for* $t \to +\infty$. *Moreover there exists a constant* $\mathcal{C}_3$, *depending on* $\mathcal{C}$, *such that*

$$\int_{\tau}^{+\infty} E^{(n)}(z_i^{(n)}(t))\, dt \leq \frac{\mathcal{C}_3}{b}. \tag{3.10}$$

Obviously (3.10) *implies that*

$$\int_{\tau}^{+\infty} \left\{\sum_{i=1}^{n} z_i'^2(t)\right\} dt \leq \frac{\mathcal{C}_3}{b};$$

moreover, $\forall \varepsilon > 0$ *there exists a constant* t_ε *(depending only on* ε*) such that*

$$\operatorname*{Sup}_{t > t_\varepsilon} \int_{t}^{t+1} \left\{\sum_{i=1}^{n}\left(\frac{\delta}{b} z_{i-1}(\eta)\right)^2\right\} d\eta \leq \frac{1}{b}(\overline{\alpha}^2 + \varepsilon).$$

COROLLARY 3.4: *Let* (3.3) *hold and*

$$\bar{z}_i = \bar{z}_i' = 0; \qquad f_i(t) \equiv 0, \qquad \text{for } t < \tau.$$

Then the solution of (3.1), (3.2) *is bounded in* $\mathbb{R}$. *Precisely there exists a constant* $\mathcal{C}_4$ *depending only on* $\mathcal{C}$, *such that*

$$\operatorname*{Sup}_{t \in \mathbb{R}} E^{(n)}(z_i^{(n)}(t)) \leq \frac{\mathcal{C}_4}{b}.$$

4. - PROOF OF THE STATEMENTS OF § 2

We can prove the theorems of § 2 using the results concerning the discrete problem considered in § 1. In order to do it, we fix the following connections between the continuos and the discrete problems.

When we consider the problem (1.2), (1.3) firstly we choose a sequence $\{\alpha^{(p)}(x)\}$ with $\alpha^{(p)} \in H^2 \cap H_0^1$, such that

$$\lim \|\alpha^{(p)} - \bar{y}\|_{H_0^1} = 0. \tag{4.1}$$

Then, if

$$\|\alpha^{(p)}\|_{H^2 \cap H_0^1} = a^{(p)},$$

and $\{n_p\} \subseteq \mathrm{N}$ is a sequence such that $[a^{(p)}]^{4/3} \leq n_p$, we construct a new sequence $\bar{z}^{(n)}(x)$ setting successively, for example

$$\bar{z}^{(1)}(x) = \bar{z}^{(2)}(x) = \ldots = \bar{z}^{(n_1 - 1)}(x) = \frac{1}{a^{(1)}}\alpha^{(1)}(x),$$

$$\bar{z}^{(n_1)}(x) = \bar{z}^{(n_1 + 1)}(x) = \ldots = \bar{z}^{(n_2 - 1)}(x) = \alpha^{(1)}(x),$$

$$\bar{z}^{(n_p)}(x) = \bar{z}^{(n_p + 1)}(x) = \ldots = \bar{z}^{(n_{p+1} - 1)}(x) = \alpha^{(p)}(x), \quad p \in \mathrm{N}.$$

In such a way we have, $\forall n_p + 1 \leq n \leq n_{p+1}$

$$\|\bar{z}^{(n)}\|_{H^2 \cap H_0^1} \leq \|\alpha^{(p)}\|_{H^2 \cap H_0^1} \leq (n_p)^{3/4} \leq (n)^{3/4} = L^{3/4} b^{-3/4}. \tag{4.2}$$

Finally we set in (1.2)

$$f_i^{(n)}(t) = \frac{1}{b^{(n)}} \int_{\Delta_i^{(n)}} f(x, t)\, dx, \quad t \in \mathrm{R}^+, \tag{4.3}$$

and in (1.3)

$$\bar{z}_i^{(n)} = \frac{1}{b^{(n)}} \int_{\Delta_i^{(n)}} \bar{z}^{(n)}(x)\, dx, \quad \bar{z}_i^{\prime(n)} = \bar{y}_i^{\prime(n)} = \frac{1}{b^{(n)}} \int_{\Delta_i^{(n)}} \bar{y}^{\prime(n)}(x)\, dx. \tag{4.4}$$

Considering the second problem we set

$$f_i(t) = \begin{cases} \dfrac{1}{b^{(n)}} \displaystyle\int_{\Delta_i^{(n)}} f(x, t)\, dx, & t \geq -n, \\ 0, & t < -n. \end{cases} \tag{4.5}$$

Proof of Theorem 2.1: As in [1] we divide the proof in the following steps:

a) Setting

$$\delta^{(n)}(x, t) = z_i(t) + \frac{\delta}{b} z_i(t)(x - ib), \quad x \in \bar{\Delta}_i^{(n)}, \tag{4.6}$$

it is possible to select a subsequence $\{\delta^{n_i - 1}(t)\}$ *such that*

$$\lim_{n \to \infty} \delta^{(n_i - 1)}(t) = y_1(t) \tag{4.7}$$

strongly in $L^2(T_1; L^2)$, *weakly* in* $L^\infty(T_1; H_0^1)$,

$$\lim_{n\to\infty} \bar{y}^{\prime(n,1)}(t) = y_1'(t) \tag{4.8}$$

weakly in* $L^\infty(T_1; L^2)$.

We have (4.7) and (4.8) directly from (4.6) and Theorem 3.2.

In fact (2.1) implies (3.3); (2.2), (4.2) and (4.4) imply (3.5), then by Theorem 3.2 (that is (3.7) and (3.8)) there exists a constant K_1, depending only on $\mathcal{C}$ (that is on K, $\bar{y}$, $\bar{y}'$), such that

$$\|\bar{y}^{(n)}(t)\|_{L^\infty(\mathbb{R}^+; H_0^1)} \leq K_1 , \quad \|\bar{y}^{\prime(n)}(t)\|_{L^\infty(\mathbb{R}^+; L^2)} \leq K_1 , \tag{4.9}$$

b) We can extend $y_1(t)$ *to* $\mathbb{R}^+$.

In fact we can consider the intervals $(0, k)$, $\forall k \in \mathbb{N}$, and choose the successive subsequences $\{\bar{y}^{(n,k)}(t)\} \subseteq \{\bar{y}^{(n,k-1)}(t)\}$ such that

$$\lim_{n\to\infty} \bar{y}^{(n,k)}(t) = y_k(t) \tag{4.10}$$

strongly in $L^2(0, k; L^2)$, weakly* in $L^\infty(0, k; H_0^1)$,

$$\lim_{n\to\infty} \bar{y}^{\prime(n,k)}(t) = y_k'(t) \tag{4.11}$$

weakly* in $L^\infty(0, k; L^2)$. Moreover $y_k(t) \equiv y_{k-1}(t)$ for $0 \leq t \leq k-1$, and we can set

$$y(t) = y_k(t), \quad t \in (0, k).$$

c) The function $y(t)$ *satisfies* (2.3).

In fact, $\forall t \in \mathbb{R}^+$, we can choose $k > t$ and considering the subsequence $\{\bar{y}^{(n,k)}(t)\}$ we have (2.3) by (4.9), (4.10) and (4.11).

d) $y(t)$ *is a solution of* (1.4), (1.5) *in the sense indicated in* § 1.

Dropping out the k for the sake of symplicity, (1.5) holds by (4.1) and (4.4), since

$$\bar{y}^{(n)}(0) - \bar{y} = \bar{y}^{(n)}(0) - \bar{z}^{(n)} + \bar{z}^{(n)} - \bar{y}.$$

Now we recall how to verify (1.4), for the reader's convenience.

Assuming at first that $u(t) \in H^{1,\infty}(\mathbb{R}^+; L^2) \cap L^\infty(\mathbb{R}^+; H^2 \cap H_0^1)$, we calculate

$$\int_0^t \{(\mathfrak{z}^{\prime(n)}(\eta), u'(\eta))_{L^2} - (\varphi(D\mathfrak{z}^{(n)}(\eta)), Du(\eta))_{L^2} - (\mathfrak{z}^{\prime(n)}(\eta), u(\eta))_{L^2} +$$

$$+ (f(\eta), u(\eta))_{L^2}\} d\eta - (\mathfrak{z}^{\prime(n)}(t), u(t))_{L^2} + (\mathfrak{z}^{\prime(n)}(0), u(0))_{L^2}$$

and pass to the limit fot $t \to +\infty$.

Setting

$$g_i = \frac{\delta}{b}\left\{\frac{1}{h}\int_{\bar{\Delta}_i} dx \int_0^x u(\xi, t)\, d\xi\right\} \in H^1(0, k),$$

and

$$\mathfrak{A}^{(n)}(t) = \frac{\delta}{b} g_i(t), \quad \mathfrak{B}^{(n)}(t) = g_i(t), \quad x \in \bar{\Delta}_i,$$

we have very easily

$$\lim_{n \to \infty} \|\mathfrak{A}^{(n)}(t) - Du(t)\|_{L^\infty(0,k;L^2)} = 0,$$

$$\lim_{n \to \infty} \|\mathfrak{B}^{(n)}(t) - u(t)\|_{L^\infty(0,k;L^2)} = 0,$$

and there exists a constant $G > 0$ (depending only on u) such that

$$\sum_{i=1}^{n} \left(\frac{\delta^2}{b^2} g_i\right)^2 \leq \frac{G}{b}. \tag{4.12}$$

Moreover

$$\int_0^t (\mathfrak{z}^{\prime(n)}(\eta), u'(\eta))_{L^2}\, d\eta = b \int_0^t \sum_{i=1}^{n} z_i'(\eta) g_i'(\eta)\, d\eta,$$

and then, by (6.3),

$$\int_0^t (\mathfrak{z}^{\prime(n)}(\eta), u'(\eta))_{L^2}\, d\eta = -b \int_0^t \left\{\sum_{i=1}^{n} \varphi\left(\frac{\delta}{b} z_{i-1}(\eta)\right) \frac{\delta}{b} g_{i-1}(\eta) + \right.$$

$$\left. + b^\mu \frac{\delta^2}{b^2} z_{i-2}(\eta) \frac{\delta^2}{b^2} g_{i-2}(\eta) + z_i'(\eta) g_i(\eta) + -f_i(\eta) g_i(\eta)\right\} d\eta +$$

$$+ b \sum_{i=0}^{n} z_i'(t) g_i(t) - b \sum_{i=0}^{n} z_i'(0) g_i(0),$$

where, by (3.7) and (4.12)

$$(4.13)\qquad b^{1+\mu}\int_0^t \sum_{i=1}^n \frac{\delta^2}{b^2} z_{i-2}(\eta)\frac{\delta^2}{b^2} g_{i-2}(\eta)\,d\eta \leq$$

$$\leq b^{1+\mu/2}\left\{\int_0^t \sum_{i=1}^n b^{\mu}\left(\frac{\delta^2}{b^2} z_{i-2}(\eta)\right)^2 d\eta\right\}^{1/2}\left\{\int_0^t \sum_{i=1}^n \left(\frac{\delta^2}{b^2} g_{i-2}(\eta)\right)^2 d\eta\right\}^{1/2} \leq kb^{\mu/2}G\mathcal{C}$$

and moreover

$$(4.14)\qquad b\int_0^t \sum_{i=1}^n f_i(\eta)g_i(\eta)\,d\eta \to \int_0^t (f(\eta), u(\eta))_{L^2}\,d\eta\,,$$

$$(4.15)\qquad b\sum_{i=1}^n z_i'(t)g_i(t) - b\sum_{i=1}^n z_i'(0)g_i(0) \to (y'(t), u(t))_{L^2} - (\bar{y}', u(0))_{L^2}\,.$$

Finally, in order to calculate

$$\lim_{b\to 0} b\int_0^t \left\{\sum_{i=1}^n \varphi\left(\frac{\delta}{b} z_{i-1}(\eta)\right)\frac{\delta}{b} g_{i-1}(\eta)\right\} d\eta\,,$$

we consider a new sequence $\{Z^{(n)}\}$ defined as follows

$$\eta^{(n)}(x) = \frac{\delta}{b} z_{i-1}^{(n)} + \frac{\delta^2}{b^2} z_{i-1}^{(n)}\left\{x - \left(i - \frac{1}{2}\right)b\right\}, \quad x \in \Delta_i^{(n)}\,,$$

$$Z^{(n)}(x) = \int_{-b/2}^{x} \eta^{(n)}(\xi)\,d\xi - \frac{L-x}{L}\int_{-b/2}^{0} \eta(\xi)\,d\xi - \frac{x}{L}\int_{-b/2}^{L} \eta(\xi)\,d\xi =$$

$$= \frac{1}{2}(z_i + z_{i-1}) + \frac{\delta}{b} z_{i-1}\left\{x - \left(i - \frac{1}{2}\right)b\right\} + \frac{\delta^2}{b^2} z_{i-1}\left\{x - \left(i - \frac{1}{2}\right)b\right\}^2, \quad x \in \Delta_i^{(n)}\,,$$

which is more regular than $\{g^{(n)}\}$.

We can verify that

$$\|Z'^{(n)}(t)\|_{L^2} \leq \mathcal{C}_8\,, \qquad \|Z'^{(n)}(t)\|_{H^1_0} \leq \mathcal{C}_8\,, \qquad t \in [0, k]\,,$$

$$\|Z'^{(n)}\|_{W^1(0,k;(H^2\cap H_0^1)')} \leq \mathcal{C}_8\,, \qquad \|Z^{(n)}(t)\|_{H^2\cap H_0^1} \leq b^{\mu/2}\mathcal{C}_9\,, \qquad t \in [0, k]\,,$$

and moreover

$$(4.16)\qquad \|DZ^{(n)}(t) - Dg^{(n)}(t)\|_{L^2} \leq b^{2-\mu}\mathcal{C}\,.$$

Finally proving as in [1] that

$$\lim_{n\to\infty} \varphi(D\mathcal{Z}^{(n)}) = \varphi(Dy)$$

weakly in $L^2(0,k;L^2)$, by (4.16) we have

$$\lim_{n\to\infty} \varphi(D\mathfrak{z}^{(n)}) = \varphi(Dy), \tag{4.17}$$

weakly in $L^2(0,k;L^2)$. Now bearing in mind (4.13), (4.14), (4.15), we complete the proof, by (4.17).

Proof of Theorem 2.2: Let $k \in \mathbb{N}$, $\sigma > 0$ and $y \in H_0^1(0,k;L^2) \cap L^2(0,k;H_0^1)$; we set

$$\|y(t)\|_{\mathbb{E}_\sigma} = E(y(t)) + \sigma\|Dy(t)\|_{L^2}^2 .$$

By (4.10) and (4.11) (dropping out the k), we have, $\forall n \in \mathbb{N}$,

$$\int_0^k \|y(t)\|_{\mathbb{E}_\sigma} dt = \int_0^k \{E(y(t)) + \sigma\|Dy(t)\|_{L^2}^2\} dt \leq$$

$$\leq \int_0^k \min \lim_{n\to\infty} \|z^{(n)}(t)\|_{\mathbb{E}_\sigma} dt \leq \int_0^k \min \lim_{n\to\infty} \{E(z^{(n)}(t)) + \sigma\|Dz^{(n)}(t)\|_{L^2}^2\} dt \leq$$

$$\leq \mathcal{C}_{10} b \int_0^k \min \lim_{n\to\infty} E^{(n)}(z_0^{(n)}(t)) dt + \sigma k \min \lim_{n\to\infty} \|Dz^{(n)}(t)\|^2_{L^2(0,k;L^2)} .$$

Now (2.4) implies (3.5), then (3.10) and Theorem 3.3 give

$$\int_0^k \|y(t)\|_{\mathbb{E}_\sigma} dt \leq \mathcal{C}_{11} + \sigma\mathcal{C}_{12}$$

and therefore

$$\int_0^k E(y(t))\, dt = \int_0^k \|y(t)\|_{\mathbb{E}} dt - \int_0^k \sigma\|y(t)\|_{H_0^1} dt \leq \mathcal{C}_{11} + \sigma\mathcal{C}_{12} , \quad \forall \sigma > 0 . \tag{4.18}$$

Since (4.18) holds $\forall \sigma > 0$ we have

$$\int_0^k E(y(t))\, dt \leq \mathcal{C}_{11} ,$$

where $\mathcal{C}_{11}$ does not depend on k, and $E(y(t)) \geq 0$, that is (2.5), and obviously (2.6). In

order to have (2.7), we can choose, by (2.6), t_ε so that

$$\|E(y(t))\|_{L^2_{loc}(R_{t_\varepsilon}^+)} < \frac{L^2 \mathcal{C}^2}{16(A^2 + L^2 \varepsilon)} \varepsilon$$

and then by (6.6)

$$\|Dy(t)\|^2_{L^2_{loc}(R_{t_\varepsilon}^+)} < \overline{\alpha}^2 + \frac{\varepsilon}{2} + \frac{A^2 + L_\varepsilon^2}{2L^2 \varepsilon} \|\Phi(Dy(t))\|_{L^2_{loc}(R_{t_\varepsilon}^+)} \leq \overline{\alpha}^2 + \varepsilon .$$

Proof of Theorem 2.3: Consider the sequence (4.6), where now $\{z_i(t)\}$ is the solution of (1.2), (1.6).

a) It is possible to select a subsequence $\{\delta^{(n,1)}(t)\}$ such that

$$\lim_{n \to \infty} \delta^{(n,1)}(t) = y_1(t) \tag{4.19}$$

strongly in $L^2(T_1; L^2)$, weakly in $L^\infty(T_1, H_0^1)$,*

$$\lim_{n \to \infty} \delta'^{(n,1)}(t) = y_1'(t) \tag{4.20}$$

weakly in $L^\infty(T_1; L^2)$.*

We have (4.19) and (4.20) directly from (4.6) and Theorem 3.2.

In fact by (3.7) and (3.8) there exists a constant K_1, depending only on $\mathcal{C}_1$, such that

$$\|\delta^{(n)}(t)\|_{L^\infty(R, H_0^1)} \leq K_1 , \quad \|\delta'^{(n)}(t)\|_{L^\infty(R, L^2)} \leq K_1 .$$

b) We can extend $y_1(t)$ to $\mathbb{R}$, as in the step b) of Theorem 2.2, setting

$$\lim_{n \to \infty} \delta^{(n,k)}(t) = y_k(t)$$

strongly in $L^2(T_k; L^2)$, weakly* in $L^\infty(T_k; H_0^1)$,

$$\lim_{n \to \infty} \delta'^{(n,k)}(t) = y_k'(t)$$

weakly* in $L^\infty(T_k; L^2)$, and finally

$$y(t) = y_k(t), \quad t \in T .$$

c) The function $y(t)$ satisfies (2.3).

In fact we observe that $\forall \bar{t} \in \mathbb{R}$ we can choose $k \in \mathbb{N}$ such that $\bar{t} + 1 \in T_k$ and we substitute T_k to the interval $(0, k)$ in the step c) of the proof of Theorem 3.1.

d) $y(t)$ is a solution of (1.7) in the sense indicated in § 1.

We reach this purpose as in the proof of Theorem 3.1.

Precisely, we observe first that (1.7) holds $\forall u \in H^{1,\infty}(\mathbb{R}; L^2) \cap L^\infty(\mathbb{R}; H^2 \cap \cap H_0^1)$ with compact support in $\mathbb{R}$.

In fact for such test function, k large enough, we have

$$\int_{\mathbb{R}} \{(\mathfrak{z}^{\prime(n,k)}(\eta), u'(\eta))_{L^2} - (\varphi(D\mathfrak{z}^{(n,k)}(\eta)), Du(\eta))_{L^2} +$$

$$-(\mathfrak{z}^{\prime(n,k)}(\eta), u(\eta))_{L^2} + (f(\eta), u(\eta))_{L^2}\} d\eta =$$

$$= \int_{T_k} \{(\mathfrak{z}^{\prime(n,k)}(\eta), u'(\eta))_{L^2} - (\varphi(D\mathfrak{z}^{(n,k)}(\eta)), Du(\eta))_{L^2} +$$

$$-(\mathfrak{z}^{\prime(n,k)}(\eta), u(\eta))_{L^2} + (f(\eta), u(\eta))_{L^2}\} d\eta ,$$

$$\int_{\mathbb{R}} \{(y'(\eta), u'(\eta))_{L^2} - (\varphi(Dy(\eta)), Du(\eta))_{L^2} - (y'(\eta), u(\eta))_{L^2} + (f(\eta), u(\eta))_{L^2}\} d\eta =$$

$$= \int_{T_k} \{(y'(\eta), u'(\eta))_{L^2} - (\varphi(Dy(\eta)), Du(\eta))_{L^2} - (y'(\eta), u(\eta))_{L^2} + (f(\eta), u(\eta))_{L^2}\} d\eta ,$$

therefore we can pass to the limit for $n \to \infty$, exactly as in the proof of Theorem 2.2

5. - Proof of the statements of § 3

Proof of Lemma 3.1: In what follows, we shall drop out the notation (n) for the sake of symplicity.

If

$$\sum_{i=1}^{n} \int_{\bar{t}}^{\bar{t}+1} z_i'^2(\eta) d\eta \geq \frac{c}{b} ,$$

we have (3.4) by Shwartz inequality, setting in (6.4) $t'' = \bar{t} + 1$, $t' = \bar{t}$; in fact,

$$0 \leq E(z_i(\bar{t}+1)) = E(z_i(\bar{t})) + \sum_{i=1}^{n} \int_{\bar{t}}^{\bar{t}+1} [f_i(\eta) z_i'(\eta) - z_i'^2(\eta)] d\eta .$$

Let's now assume that

$$\sum_{i=1}^{n} \int_{\bar{t}}^{\bar{t}+1} z_i'^2(\eta) d\eta < \frac{c}{b} . \tag{5.1}$$

Then there exist $t' \in [\bar{t}, \bar{t}+1/4]$, $t'' \in [\bar{t}+3/4, \bar{t}+1]$ such that

$$\sum_1^n z_i'^2(t') < 4\,\frac{\mathcal{C}}{b}, \quad \sum_1^n z_i'^2(t'') < 4\,\frac{\mathcal{C}}{b},$$

moreover, $\forall t \in [\bar{t}, \bar{t}+1]$, we have successively

$$\sum_{i=1}^n z_i^2(t) \leq 2\sum_{i=1}^n z_i^2(\bar{t}) + 2\sum_{i=1}^n \left\{ \int_{\bar{t}}^{t} z_i'(\eta)\,d\eta \right\} \leq$$

$$\leq 2\left\{ \sum_{i=1}^n z_i^2(\bar{t}) + \sum_{i=1}^n \int_{\bar{t}}^{t} z_i'^2(\eta)\,d\eta \right\} \leq 2\left\{ \sum_{i=1}^n z_i^2(\bar{t}) + \frac{\mathcal{C}}{b} \right\};$$

$$\left\{ \sum_{i=1}^n z_i^2(t) \right\}^{1/2} \leq \sqrt{2}\left\{ \sum_{i=1}^n z_i^2(\bar{t}) + \frac{\mathcal{C}}{b} \right\}^{1/2}. \tag{5.2}$$

Setting now

$$g^{(n)}(z_i) = \sum_{i=1}^n \left\{ \frac{1}{2} b^n \left(\frac{\delta^2}{b^2} z_{i-1}(t) \right)^2 + \varphi\left(\frac{\delta}{b} z_{i-1}(t) \right) \frac{\delta}{b} z_{i-1}(t) \right\}$$

and $u_i = z_i$ in (6.3), we have by (3.3), (5.1) and (5.2)

$$\int_{t'}^{t''} g^{(n)}(z_i(\eta))\,d\eta = \sum_{i=1}^n \int_{t'}^{t''} \{ z_i'^2(\eta) + f_i(\eta) z_i(\eta) + -z_i'(\eta) z_i(\eta) \}\,d\eta - \tag{5.3}$$

$$- \sum_{i=1}^n z_i'(t'') z_i(t'') + \sum_{i=1}^n z_i'(t') z_i(t') \leq \frac{\mathcal{C}}{b} + \left\{ \frac{\mathcal{C}}{b} \right\}^{1/2} \left\{ \frac{\mathcal{C}}{b} \right\}^{1/2} +$$

$$+ 12 \left\{ \frac{\mathcal{C}}{b} \right\}^{1/2} \sqrt{2} \left\{ \sum_{i=1}^n z_i^2(\bar{t}) + \frac{\mathcal{C}}{b} \right\}^{1/2} \leq \mathcal{C}_6 \left[\frac{1}{b} + \left\{ \frac{1}{b} \right\}^{1/2} \left\{ \sum_{i=1}^n z_i^2(\bar{t}) \right\}^{1/2} \right],$$

and also

$$\int_{t'}^{t''} \{ g^{(n)}(z_i(\eta)) + z_i'^2(\eta) \}\,d\eta \leq \mathcal{C}_9 \left[\frac{1}{b} + \left\{ \frac{1}{b} \right\}^{1/2} \left\{ \sum_{i=1}^n z_i^2(\bar{t}) \right\}^{1/2} \right] \leq$$

$$\leq \mathcal{C}_8 \left[\frac{1}{b} + \left\{ \frac{1}{b} \right\}^{1/2} \left\{ E^{(n)}(z_i(\bar{t})) + \frac{\mathcal{C}_{10}}{b} \right\}^{1/2} \right].$$

Then there exists $t^* \in [t', t'']$ such that

$$g^{(n)}(z_i(t^*)) + \frac{1}{2}\sum_{i=1}^n z_i'^2(t^*) \leq \mathcal{C}_9 \left[\frac{1}{b} + \left\{ \frac{1}{b} \right\}^{1/2} \left\{ E^{(n)}(z_i(\bar{t})) + \frac{\mathcal{C}_{10}}{b} \right\}^{1/2} \right].$$

and, by (6.5),

$$E^{(n)}(z_i(t^*)) \leq \mathcal{C}_{10}\left[\frac{1}{b} + \left\{\frac{1}{b}\right\}^{1/2}\{E^{(n)}(z_i(\bar{t}))\}^{1/2}\right]. \tag{5.4}$$

Setting now $u_i = z_i$, $t' = t^*$, $t'' = t \in [\bar{t}, \bar{t}+1]$, (6.4) gives

$$E^{(n)}(z_i(t)) = E^{(n)}(z_i(t^*)) + \sum_{i=1}^{n} \int_{t^*}^{t} [f_i(\eta) z_i'(\eta) - z_i'^2(\eta)] d\eta ,$$

and, by (3.3), (5.1), (5.4),

$$E^{(n)}(z_i(t)) \leq \mathcal{C}_{11}\left[\frac{1}{b} + \left\{\frac{1}{b}\right\}^{1/2}\{E^{(n)}(z_i(\bar{t})\}^{1/2}\right]. \tag{5.5}$$

Now, if

$$E^{(n)}(z_i(\bar{t}+1)) \leq E^{(n)}(z_i(\bar{t})) ,$$

we have (3.4); if

$$E^{(n)}(z_i(\bar{t}+1)) \geq E^{(n)}(z_i(\bar{t})) ,$$

setting in (5.5) $t = \bar{t}+1$, we have

$$E^{(n)}(z_i(\bar{t}+1)) \leq \frac{\mathcal{C}_{12}}{b} ,$$

and then (3.4) again.

Proof of Theorem 3.2: Setting in (6.4) $t' = \tau$, $t'' = t \in (\tau, \tau+1)$, (3.3) gives

$$E^{(n)}(z_i(t)) = E^{(n)}(z_i(\tau)) +$$

$$+ \sum_{i=1}^{n} \int_{\tau}^{t} \left\{\frac{1}{4} f_i(\eta) - \left[\frac{1}{2} f_i(\eta) - z_i'(\eta)\right]^2\right\} d\eta \leq E^{(n)}(z_i(\tau)) + \frac{\mathcal{C}}{b} .$$

Then, by (3.5) there exists a constant $\mathcal{C}_{13}$ depending on $\mathcal{C}$, $\bar{z}_i$, $\bar{z}_i'$, such that

$$E^{(n)}(z_i(t)) \leq \frac{\mathcal{C}_{13}}{b} , \quad \tau \leq t \leq \tau+1 .$$

By (3.4)

$$E^{(n)}(z_i(t)) \leq \operatorname{Sup}\left\{\frac{\mathcal{C}_{13}}{b}, \frac{\mathcal{C}_{14}}{b}\right\},$$

for suitable $\mathcal{C}_{14}$ and then there exists a constant $\mathcal{C}_1$ depending on $\mathcal{C}$, $\bar{z}_i$, $\bar{z}_i'$, such that (3.6) holds.

Proof of Theorem 3.3: Setting in (6.4) $t' = \tau$, $t'' = t > \tau$, we have successively

$$E^{(n)}(z_i(t)) + \sum_{i=1}^{n} \int_{\tau}^{t} z_i'^2(\eta)\,d\eta = E^{(n)}(z_i(\tau)) + \sum_{i=1}^{n} \int_{\tau}^{t} f_i(\eta) z_i'(\eta)\,d\eta\,,$$

$$\sum_{i=1}^{n} \int_{\tau}^{t} z_i'^2(\eta)\,d\eta \leq E^{(n)}(z_i(\tau)) + \sum_{i=1}^{n} \int_{\tau}^{t} f_i(\eta) z_i'(\eta)\,d\eta\,,$$

and, by (3.5), (3.7), (3.9)

$$\sum_{i=1}^{n} \int_{\tau}^{t} z_i'^2(\eta)\,d\eta \leq \frac{c}{b} + \left\{ \sum_{i=1}^{n} \int_{\tau}^{t} f_i^2(\eta)\,d\eta \right\}^{1/2} \left\{ \sum_{i=1}^{n} \int_{\tau}^{t} z_i'^2(\eta)\,d\eta \right\}^{1/2},$$

$$\left\{ \sum_{i=1}^{n} \int_{\tau}^{t} z_i'^2(\eta)\,d\eta \right\}^{1/2} \leq \sqrt{\frac{c}{b}} + \sqrt{5\,\frac{c}{b}}\,,$$

$$\sum_{i=1}^{n} \int_{\tau}^{t} z_i'^2(\eta)\,d\eta \leq \frac{c_{15}}{b}\,. \tag{5.6}$$

Now (5.3), for $t' = \tau$ and $t'' = t > \tau$, gives

$$\int_{\tau}^{t} g^{(n)}(z_i(\eta))\,d\eta = \sum_{i=1}^{n} \int_{\tau}^{t} (z_i'^2(\eta) + f_i(\eta) z_i(\eta))\,d\eta +$$

$$- \sum_{i=1}^{n} \left\{ \frac{1}{2} z_i^2(t) + z_i'(t) z_i(t) - \frac{1}{2} z_i^2(\tau) - z_i'(\tau) z_i(\tau) \right\},$$

and by (3.7), (3.8), (3.9) (5.6) and Theorem 3.2,

$$\int_{\tau}^{t} E^{(n)}(z_i(\eta))\,d\eta \leq \frac{c_{16}}{b} + \sum_{i=1}^{n} \int_{\tau}^{t} \eta^{\beta} f_i(\eta)\, \eta^{-\beta} z_i(\eta)\,d\eta \leq$$

$$\leq \frac{c_{16}}{b} + \frac{c}{b} \sqrt{\frac{1}{1-2\beta} \left\{ \frac{1}{t^{2\beta-1}} - \frac{1}{\tau^{2\beta-1}} \right\}},$$

$$\int_{\tau}^{t} E^{(n)}(z_i(\eta))\,d\eta \leq \frac{c_3}{b}\,,$$

for suitable c_3.

Corollary 3.4: This corollary follows obviously by theorem 3.2, setting in the proof $\tau = -n$.

6. - Some auxiliary results

We recall here some classical results on the discrete problem.

6.1 Existence and uniqueness: *Let $t \in \mathbb{R}^+_{(\tau)}$, and $\{f_i(t)\} \in S$ a.e. in $\mathbb{R}^+_{(\tau)}$, $f_i(t) \in L^2_{loc}(\mathbb{R}^+_{(\tau)})$; then the system* (1.2)-(1.3), *that we write as*

$$(6.1)\quad \begin{cases} z_i(t) = \bar{z}_i + \int_\tau^t v_i(\eta)\,d\eta, \quad t \geq \tau, \\ v_i(t) = \bar{z}_i' + \int_\tau^t \left\{ \frac{\delta}{b}\varphi\left(\frac{\delta}{b} z_{i-1}(\eta)\right) - b^\mu \frac{\delta^4}{b^4} z_{i-2}(\eta) + -v_i(\eta) + f_i(\eta) \right\} d\eta, \\ \qquad\qquad t \geq \tau, \\ z_0 = z_n = 0,\ z_{-1} = -z_1,\ z_{-2} = -z_2,\ z_{n+1} = -z_{n-1},\ z_{n+2} = -z_{n-2} \end{cases}$$

has a unique solution $\in C^0(\mathbb{R}^+_{(\tau)}) \cap H^1_{loc}(\mathbb{R}^+_{(\tau)})$; moreover $z_i(t), z_i'(t) \in S$, $\forall t \in \mathbb{R}^+_{(\tau)}$.

6.2 Variational and «energy» equations: *The solution $\{z_i(t), z_i'(t)\}$ of* (6.1) *satisfies moreover the variational equations*

$$(6.2)\quad \int_{t'}^{t''} \sum_{i=1}^{n} \left\{ z_i'' u_i + \varphi\left(\frac{\delta}{b} z_{i-1}\right) \frac{\delta}{b} u_{i-1} + b^\mu \frac{\delta^2}{b^2} z_{i-2} \frac{\delta^2}{b^2} u_{i-2} + z_i' u_i - f_i u_i \right\} d\eta = 0,$$

$\forall t', t'' \in \mathbb{R}^+_{(\tau)}$, $\forall \{u_i(t)\} \in S$ *a.e. in* $\mathbb{R}^+_{(\tau)}$, *and* $u_i \in L^2_{loc}(\mathbb{R}^+_{(\tau)})$.

$$(6.3)\quad \int_{t'}^{t''} \sum_{i=1}^{n} \left\{ -z_i' u_i' + \varphi\left(\frac{\delta}{b} z_{i-1}\right) \frac{\delta}{b} u_{i-1} + b^\mu \frac{\delta^2}{b^2} z_{i-2} \frac{\delta^2}{b} u_{i-2} + z_i' u_i + -f_i u_i \right\} d\eta +$$
$$+ \sum_{i=1}^{n} z_i'(t'') u_i(t'') - \sum_{i=1}^{n} z_i'(t') u_i(t') = 0,$$

$\forall t', t'' \in \mathbb{R}^+_{(\tau)}$, $\forall \{u_i(t)\} \in S$ *a.e. in* $\mathbb{R}^+_{(\tau)}$, *and* $u_i \in H^1_{loc}(\mathbb{R}^+_{(\tau)})$.

Setting $u_i = z_i'$ in (6.2) and integrating we have the «energy» equation

$$(6.4)\quad E^{(n)}(z_i(t'')) = E^{(n)}(z_i(t')) + \sum_{i=1}^{n} \int_{t'}^{t''} [f_i(\eta) z_i'(\eta) - z_i'^2(\eta)]\,d\eta, \quad \forall t', t'' \in \mathbb{R}^+_{(\tau)}.$$

Finally (1.1) and (1.8) give $\varphi(\alpha)\alpha - 2\Phi(\alpha) = 0$ if $|\alpha| \leq \bar{\alpha}$ and, if $|\alpha| \geq \bar{\alpha}$,

$$\varphi(\alpha)\alpha - 2\Phi(\alpha) = L\left(\left\{\sqrt{1+\alpha^2} - \frac{A}{L}\right\} \frac{\alpha^2}{\sqrt{1+\alpha^2}} - \left\{\sqrt{1+\alpha^2} - \frac{A}{L}\right\}^2\right) =$$
$$= \frac{L}{\sqrt{1+\alpha^2}} \left\{\sqrt{1+\alpha^2} - \frac{A}{L}\right\} \left\{\frac{A}{L}\sqrt{1+\alpha^2} - 1\right\} \geq 0.$$

Consequently we have

$$E^{(n)}(z_i) \leq \frac{1}{2} \sum_{i=1}^{n} z_i'^2 + \delta^{(n)}(z_i). \tag{6.5}$$

Finally (1.8) gives $\forall \omega > 0$ and $\forall \alpha$

$$\alpha^2 \leq \overline{\alpha}^2 + \omega + 4 \frac{A^2 + L^2 \omega}{L^2 \omega} \Phi(\alpha). \tag{6.6}$$

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