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Integral Manifolds for Nonautonomous Equations (**) (***)

SUMMARY. — Local finite dimensional integral manifolds with exponential tracking for the nonautonomous equation in a Hilbert space are constructed. Under some conditions the constructed local exponential approximations are combined into a global uniform approximation of solutions. The obtained abstract results are applied to a nonautonomous nonlinear parabolic equation in a bounded domain.

Varietà integrali per equazioni non autonome

SOMMARIO. — Per un'equazione non autonoma in uno spazio di Hilbert si costruiscono varietà integrali con attrazione esponenziale, e localmente di dimensione finita. Sotto certe condizioni, si riesce, combinando opportunamente approssimazioni locali di tipo esponenziale, ad ottenere un'approssimazione uniforme globale della soluzione. I risultati astratti ottenuti vengono applicati a un'equazione parabolica non lineare e non autonoma in un dominio limitato.

INTRODUCTION

We study the nonautonomous evolution equation in a Hilbert space E:

$$\frac{du}{dt} + A_0 u = R_0(u) + \varepsilon R_1(u, t), \qquad u|_{t=\tau} = u_\tau . \tag{0.1}$$

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This equation is considered as a perturbation of the autonomous equation

$$\frac{du}{dt} + A_0 u = R_0(u) \tag{0.2}$$

by the nonautonomous term $\varepsilon R_1(u, t)$.

Under some conditions on the linear operator $A_0 u$ and the nonlinear operators $R_0(u)$, $R_1(u, t)$ the problem (0.1) possesses a unique solution $u(t) \in E$ for all $t \geq \tau$. This solution can be represented $u(t) = U(t, \tau) u_\tau$, where the two-parametric family of operators $\{U(t, \tau) | t \geq \tau, \tau \in R\}$ is called a process, corresponding to the problem (0.1).

In the first part of the paper (§ 1-§ 6) we study the behavior of solutions $u(t)$ of the equation (0.1) in a neighborhood $O_\varrho(z)$ of an equilibrium point z of the autonomous equation (0.2) (i.e. $A_0 z = R_0(z)$). Under some spectral conditions we prove the existence of a finite dimensional integral manifold $M(z)$ for (0.1) (see Definition 2.1) provided that ϱ and ε are sufficiently small. This manifold in $O_\varrho(z)$ is a graph of a Lipschitz continuous function.

In § 3 we prove that the integral manifold $M(z)$ exponentially attracts all the trajectories $u(t) = U(t, \tau) u_\tau$ passing through $O_\varrho(z)$. Precisely if $u(t) \in O_\varrho(z)$ for $t \in (\tau, T)$ then there exists its «trace» $\bar{u}(t)$ on $M(z)$, $\bar{u}(t)$ is a trajectory of the process $U(t, \tau)$ such that

$$\|u(t) - \bar{u}(t)\| \leq C e^{-\lambda(t-\tau)}, \tag{0.3}$$

for $t \in [\tau, T]$; $\lambda > 0$ we can choose arbitrary large by increasing the dimension of the manifold $M(z)$ (C do not depend on $u(t)$ and $\bar{u}(t)$).

In § 4 we study the structure of an integral manifold $M(z)$ in a case when z is a hyperbolic equilibrium point. In § 5 we investigate the dependence of $M(z)$ on ε.

In the second part of the paper (§ 7-§ 8) the constructed local exponential approximations (0.3) are combined into a global uniform approximation lying on the union $\bigcup_{i=1}^{N} M_i^+(z_i)$ of the finite dimensional integral manifolds $M_i^+(z_i)$, where $M_i^+(z_i)$ is an extension of $M(z_i)$ along the trajectories of (0.1). We assume that the limit autonomous equation (0.2) possesses only a finite number of equilibrium points $z_1, \ldots, z_N$ and all these points z_i are hyperbolic. The global approximation $\bar{u}(t)$ of a solution $u(t)$, $t \geq \tau$, of (0.1) is constructed in a class of piecewise continuous trajectories; $(\bar{u}(t), t) \in \bigcup_{i=1}^{N} M_i^+(z_i)$; the number of discontinuity of $\bar{u}(t)$ is not more than N. In this case we have (0.3) for all $t \geq \tau$, for all the solutions $u(t)$ of (0.1) uniformly with respect to the initial data $u_\tau \in B$, B is a bounded set in E. The important condition for such construction is the existence of a global Lyapunov function on an absorbing set for the autonomous equation (0.2).

All the main results of the present paper are formulated and proved for the abstract evolution equation (0.1). We illustrate how these results can be applied to the nonlinear evolution differential equations by the simplest example of a nonlinear parabolic equation in a bounded domain (§ 6 and § 8). The results obtained in the present paper can be applied to more general examples of evolutionary systems that arise in mathematical physics.

1. - Preliminary results

Consider the Cauchy problem for the equation:

$$\frac{du}{dt} + A_0 u = R_0(u) + \varepsilon R_1(u, t), \tag{1.1}$$

$$u\big|_{t=\tau} = u_\tau . \tag{1.2}$$

Here A_0 is a linear usually unbounded positive self-adjoint operator in a Hilbert space E, its domain $\mathcal{D}(A_0)$ is dense in E. We assume that A_0 has compact resolvent, R_0 and $R_1(\cdot, t)$ are nonlinear operators defined on $\mathcal{D}(A_0)$, R_1 is continuous and R_0 is of class C^1 from $\mathcal{D}(A_0)$ into E, ε is a small parameter, $|\varepsilon| \leq \varepsilon_0$.

We assume that for any $u_\tau \in E$, $\tau \in \mathbf{R}$, the problem (1.1)-(1.2) possesses a unique solution $u(t) = U(t, \tau) u_\tau$ in some functional space and $u(t) \in E$ $\forall t \geq \tau$. The two-parametric family of operators $\{U(t, \tau) | t \geq \tau, \tau \in \mathbf{R}\}$, $U(t, \tau): E \to E$, is called a process, corresponding to the problem (1.1)-(1.2) (see [14], [15], [4], [13], [5], [11], [12], [16], [2]).

If $\varepsilon = 0$ the equation (1.1) becomes autonomous:

$$\frac{du}{dt} + A_0 u = R_0(u). \tag{1.3}$$

Let z be a stationary solution of (1.3), i.e. $A_0 z = R_0(z)$. If we denote $v = u - z$ then (1.1)-(1.2) is equivalent to:

$$\frac{dv}{dt} + Av = B_0(v) + \varepsilon B_1(v, t), \tag{1.4}$$

$$v\big|_{t=\tau} = v_\tau , \tag{1.5}$$

where $A = A_0 - DR_0(z)$, $B_0(v) = R_0(z + v) - R_0(z) - DR_0(z)v$, $B_1(v, t) = R_1(z + v, t)$, $v_\tau = u_\tau - z$. Here DR_0 is the Fréchet differential of R_0. Notice that we have

$$B_0(0) = 0, \qquad DB_0(0) = 0. \tag{1.6}$$

For simplicity assume that A is a linear self-adjoint semibounded from below operator with domain $\mathcal{D}(A) = \mathcal{D}(A_0)$ and with a compact resolvent. If we choose $a > 0$ sufficiently large then the operator $A + aI$ is positive. For these a we can define the powers $(A + aI)^\alpha$ for $\alpha \geq 0$. The space $E^\alpha = \mathcal{D}((A + aI)^\alpha)$ is a Hilbert space with the scalar product $(u, v)_\alpha = ((A + aI)^\alpha u, (A + aI)^\alpha v)$, $\|u\|_\alpha = \|(A + aI)^\alpha u\|$, where $(\cdot, \cdot)$ and $\|\cdot\|$

are the scalar product and the norm in E. We suppose that for all $t > \tau$

$$U(t, \tau) u_\tau \in \mathcal{D}(A), \quad \forall u_\tau \in E. \tag{1.7}$$

As for nonlinear operators B_0 and B_1 we assume that they are «dominated» by A. More precisely, B_0 maps E^α into E and B_1 maps $E^\alpha \times R_t$ into E for some α, $0 \leq \alpha < 1$. Besides we suppose that in a neighborhood $O_\varrho = \{v \in E^\alpha \mid \|v\|_\alpha < \varrho\}$:

$$\|B_1(v, t)\| \leq L_1, \qquad \forall v \in O_\varrho, \quad \forall t \in R, \tag{1.8}$$

$$\|B_1(v_1, t) - B_1(v_2, t)\| \leq L_2 \|v_1 - v_2\|_\alpha, \quad \forall v_1, v_2 \in O_\varrho, \quad \forall t \in R, \tag{1.9}$$

$$\|B_0(v_1) - B_0(v_2)\| \leq L_\varrho \|v_1 - v_2\|_\alpha, \quad \forall v_1, v_2 \in O_\varrho, \tag{1.10}$$

and Lipschitz constant L_ϱ in (1.10) we can make arbitrary small if ϱ is sufficiently small:

$$L_\varrho \to 0, \qquad \text{for } \varrho \to 0. \tag{1.11}$$

From (1.6) and (1.10) one obtains

$$\|B_0(v)\| \leq L_\varrho \varrho, \quad \forall v \in O_\varrho. \tag{1.12}$$

We shall study a solution $v(t)$ of the problem (1.4)-(1.5) in a small neighborhood O_ϱ. Outside of this neighborhood we modify nonlinear operators B_0 and B_1 as follows

$$\begin{cases} B_0^*(v) = \begin{cases} B_0(v) & \text{if } \|v\|_\alpha \leq \varrho, \\ B_0\left(\dfrac{v}{\|v\|_\alpha} \varrho\right) & \text{if } \|v\|_\alpha \geq \varrho, \end{cases} \\ B_1^*(v, t) = \begin{cases} B_1(v, t) & \text{if } \|v\|_\alpha \leq \varrho, \\ B_1\left(\dfrac{v}{\|v\|_\alpha} \varrho, t\right) & \text{if } \|v\|_\alpha \geq \varrho. \end{cases} \end{cases} \tag{1.13}$$

Notice that B_0^* satisfies (1.10), (1.12) in the whole space E^α, and B_1^* satisfies (1.8), (1.9) for all $v \in E^\alpha$, $t \in R$. We denote

$$B(v, t) = B_0^*(v) + \varepsilon B_1^*(v, t). \tag{1.14}$$

Instead of (1.4) we shall study the equation

$$\frac{dv}{dt} + Av = B(v, t). \tag{1.15}$$

The equations (1.4) and (1.15) coincide inside O_ϱ. The nonlinear operator $B(v, t)$ satisfies the following inequalities:

(1.16) $\quad \|B(v,t)\| \leqslant L_\varrho \varrho + |\varepsilon| L_1 = L_0, \quad \forall v \in E^\alpha, \ \forall t \in R,$

(1.17) $\quad \|B(v_1,t) - B(v_2,t)\| \leqslant (L_\varrho + |\varepsilon| L_2)\|v_1 - v_2\|_\alpha =$

$$= L\|v_1 - v_2\|_\alpha, \quad \forall v_1, v_2 \in E^\alpha, \ \forall t \in R,$$

where constants L_0 and L are arbitrary small when ε and ϱ are sufficiently small (see (1.11)).

Let $\lambda \geqslant 0$ do not belong to the spectrum of the operator A: $\lambda \notin \sigma(A)$ and thus $[\lambda - \delta, \lambda + \delta] \cap \sigma(A) = \emptyset$ for some $\delta > 0$. Let us denote P — an orthoprojector in E onto the invariant subspace of the operator A corresponding to the spectral set $\sigma_\lambda^-(A) = \{\mu | \mu < \lambda - \delta\}$; $Q = I - P$. The space $P(E)$ is finite dimensional under the above assumptions.

For $\alpha \geqslant 0$ the spectral properties of A yield the following statements:

a) $\forall q \in Q(E^\alpha)$

(1.18) $$\|q\|_\alpha = \|(A + \alpha I)^\alpha q\| \geqslant c_q^\alpha \|q\|,$$

where $c_q = \lambda + \delta + \alpha > 0$;

b) $\forall p \in P(E^\alpha) = P(E)$

(1.19) $$c_p^\alpha \|p\| \leqslant \|p\|_\alpha \leqslant C_p^\alpha \|p\|,$$

where $C_p = \lambda - \delta + \alpha > 0$, $c_p = \lambda_1 + \alpha > 0$, λ_1 is the minimal eigenvalue of A;

c) $\forall p \in P(E)$ $\forall t > 0$

(1.20) $$\|e^{At} p\|_\alpha \leqslant C_\alpha e^{(\lambda - \delta)t} \|p\|;$$

d) $\forall q \in Q(E)$ $\forall t > 0$

(1.21) $$\|e^{-At} q\|_\alpha \leqslant \left(\frac{\alpha}{t} + C_\alpha\right)^\alpha e^{-(\lambda + \delta)t} \|q\|.$$

Proof of the last estimation one can find, for example, in [8] and [3].

2. - Construction of the integral manifold

Below we formulate the existence theorem for integral manifold $M(\lambda)$, corresponding to the split of the spectrum of the operator A onto the parts $\sigma_\lambda^-(A)$ and $\sigma_\lambda^+(A) = \sigma(A) \backslash \sigma_\lambda^-(A)$.

Under the fixed value of λ defined earlier we denote $p(t) = Pv(t)$, $q(t) = Qv(t)$, where P is an orthoprojector, corresponding to $\sigma_\lambda^-(A)$ and Q — to $\sigma_\lambda^+(A)$. The problem (1.15), (1.5) is equivalent to the following system of equations:

(2.1) $$\frac{dp}{dt} + Ap = PB(p+q,t),$$

(2.2) $$\frac{dq}{dt} + Aq = QB(p+q,t),$$

$$p|_{t=\tau} = p_\tau = Pv_\tau; \quad q|_{t=\tau} = q_\tau = Qv_\tau.$$

(2.1) and (2.2) are obtained from (1.15) applying operators P and Q respectively to the both sides of this equation.

Definition 2.1: A set $M(\lambda)$, lying in the extended phase space $E^\alpha \times \mathbf{R}_t$, $M(\lambda) \subset E^\alpha \times \mathbf{R}_t$, is called an integral manifold of the process $\{U(t,\tau) | t \geq \tau\}$, corresponding to the system (2.1)-(2.2) if it satisfies the following two properties:

1) $M(\lambda)$ consists of integral curves $(v(t), t)$ of the equation (1.15). Precisely if $(v_\tau, \tau) \in M(\lambda)$ then $(v(t), t) \in M(\lambda)$ for all $t \geq \tau$, where $v(t) = U(t,\tau)v_\tau$ is a solution of the problem (1.15), (1.5);

2) $M(\lambda)$ is the graph of a function Φ, defined on $P(E^\alpha) \times \mathbf{R}_t$:

$$M(\lambda) = \{(p,q,t) | q = \Phi(p,t),\ p \in P(E^\alpha),\ t \in \mathbf{R},\ q \in Q(E^\alpha)\}.$$

Here we assume that for some b and l:

(2.3) $$\|\Phi(p,t)\|_\alpha \leq b, \quad \forall p \in P(E^\alpha), \ \forall t \in \mathbf{R},$$

(2.4) $$\|\Phi(p_1,t) - \Phi(p_2,t)\|_\alpha \leq l \|p_1 - p_2\|_\alpha, \quad \forall p_1, p_2 \in P(E^\alpha), \ \forall t \in \mathbf{R}.$$

Class of functions Φ: $P(E^\alpha) \times \mathbf{R}_t \to Q(E^\alpha)$ satisfying (2.3)-(2.4) we denote $\mathcal{F}^\alpha_{b,l}$.

Theorem 2.1: *Let linear operator A satisfies (1.20)-(1.21) for some α, $0 \leq \alpha < 1$, and nonlinear operator $B(v,t) = B_0^\alpha(v) + \varepsilon B_1^\alpha(v,t)$ satisfies (1.16)-(1.17). Then for every $b > 0$, $l > 0$ there exist $\varrho > 0$, $\varepsilon_0 > 0$ such that the process $\{U(t,\tau)\}$, corresponding to (1.15) with $|\varepsilon| < \varepsilon_0$ possesses an integral manifold $M(\lambda) = \{(p,q,t) | q = \Phi(p,t)\}$, $\Phi \in \mathcal{F}^\alpha_{b,l}$.*

Construction of an integral manifold is carried out by the Lyapunov-Perron method similarly to construction of an inertial manifold in the papers [8], [9], [7], [3]. In contrast to inertial manifolds that lie in the phase space E^α, an integral manifold depends on t and respectively lies in the extended phase space $E^\alpha \times \mathbf{R}_t$. The gap property supposed in these papers is replaced in the considered case of the nonautonomous equation (1.15) by the fact that the constants L_0 and L in (1.16)-(1.17) are small when ε and ϱ are small and a gap from $\lambda - \delta$ to $\lambda + \delta$ in the spectrum is fixed.

Let us construct the integral manifold $M(\lambda)$.

Assume that an integral manifold M exists. Then for every $\tau \in R$, $p_\tau \in P(E^\alpha)$ the solution of the system (2.1)-(2.2) is the pair $(p(t), q(t)) = (p(t), \Phi(p(t), t))$, where $p(t)$ is a solution of the ordinary differential equation in $P(E^\alpha)$:

$$\partial_t p = -Ap + PB(p + \Phi(p, t), t), \quad p|_{t=\tau} = p_\tau \tag{2.5}$$

and $q(t) = \Phi(p(t), t)$ is a bounded (see (2.3)) solution of the equation:

$$\partial_t q = -Aq + QB(p(t) + \Phi(p(t), t), t), \quad q \in Q(E^\alpha). \tag{2.6}$$

As the operator A is positive on $Q(E^\alpha)$ and the function $QB(p(t) + \Phi(p(t), t), t)$ is bounded in $Q(E^\alpha)$ since (1.16), then the equation (2.6) possesses the unique bounded solution. It is given by

$$q(t) = \Phi(p(t), t) = \int_{-\infty}^{t} e^{-A(t-\xi)} QB(p(\xi) + \Phi(p(\xi), \xi), \xi)\, d\xi = \tag{2.7}$$

$$= \int_0^{+\infty} e^{-A\eta} QB(p(t-\eta) + \Phi(p(t-\eta), t-\eta), t-\eta)\, d\eta .$$

For functions Φ from $\mathcal{F}^0_{b,l}$ we define a mapping J:

$$J[\Phi](p_\tau, \tau) = \int_0^{+\infty} e^{-A\eta} QB(p(\tau-\eta) + \Phi(p(\tau-\eta), \tau-\eta), \tau-\eta)\, d\eta , \tag{2.8}$$

where $p(t)$ is a solution of (2.5) for $t \leq \tau$. If we put $t = \tau$ in (2.7), we obtain $q(\tau) = \Phi(p_\tau, \tau) = J[\Phi](p_\tau, \tau)$, where (p_τ, τ) is any point of $P(E^\alpha) \times R$. Therefore the function $\Phi \in \mathcal{F}^0_{b,l}$ which define an integral manifold M has to be a fixed point of the mapping J defined by (2.8).

It is easy to verify the following fact: if $J[\Phi] = \Phi$, $\Phi \in \mathcal{F}^0_{b,l}$, then the set $M = \{(p + \Phi(p, t), t)\}$ satisfies the invariance property 1) of Definition 2.1 (see [3], [7]-[9] in an autonomous case or [10] in nonautonomous case).

Hence, the problem of construction of an integral manifold is reduced to the study of the properties of the operator J.

Proposition 2.1: *If L_0 in (1.16) and L in (1.17) are sufficiently small (i.e. ϱ and $|\varepsilon|$ are sufficiently small) then:*

a) the operator J maps $\mathcal{F}^0_{b,l}$ into itself,

b) the operator J is strictly contracting, i.e. $d_\alpha(J[\Phi_1], J[\Phi_2]) \leq \theta d_\alpha(\Phi_1, \Phi_2)$, $\theta < 1$, where

$$d_\alpha(\Phi_1, \Phi_2) = \sup_{p \in P(E^\alpha),\, t \in R} \|\Phi_1(p,t) - \Phi_2(p,t)\|_\alpha , \qquad \Phi_1, \Phi_2 \in \mathcal{F}^0_{b,l} .$$

The full proof of Proposition 2.1 is given in [10]. The similar statements for inertial manifolds for autonomous equations are proved in [8], [9], [7], [3]. This proposition implies the existence of a fixed point for the operator J and, consequently, the existence of an integral manifold of the process $U(t,\tau)$ corresponding to (1.15).

Remark 2.1: It follows from the construction of the integral manifold $M(\lambda)$ that $M(\lambda)$ consists of graphs of those and only those solutions of the equation (1.15) that are defined for all $t \in R$ and have bounded projection onto $Q(E^\alpha)$.

3. - Exponential approximation of solutions in a neighborhood of an equilibrium point

In the present section for any solution $v(t)$ of the equation (1.4) that lies in a sufficiently small neighborhood O_ϱ we construct an integral curve, lying on the integral manifold $M(\lambda)$ that approaches to $v(t)$ exponentially.

The integral manifold $M \equiv M(\lambda) = \{(p + \Phi(p,t) | p \in P(E^\alpha), t \in R\}$ constructed in § 2 corresponds to the initial equation (1.4) only in the neighborhood $O_\varrho = \{v \in E^\alpha \,|\, \|v\|_\alpha < \varrho\}$ and, consequently, in the set $V_\varrho = \{v \in E^\alpha \,|\, \|p\|_\alpha = \|Pv\|_\alpha \leq \varrho/2, \|q\|_\alpha = \|Qv\|_\alpha \leq \varrho/2\} \subset O_\varrho$.

Let us show that we can choose ϱ and ε such that $\|Qv\|_\alpha < \varrho/2$ if $(v,t) \in M$, i.e.

$$\sup_{p \in P(E^\alpha), t \in R} \|\Phi(p,t)\|_\alpha < \frac{\varrho}{2}. \tag{3.1}$$

Indeed, as Φ is a fixed point of the operator J defined by (2.7), then

$$\|\Phi(p,t)\|_\alpha = \|J[\Phi](p,t)\|_\alpha \leq \int_0^{+\infty} \|e^{-A\eta} QB(p(\tau-\eta) + \Phi(p(\tau-\eta),\tau-\eta),\tau-\eta)\|_\alpha \, d\eta.$$

From (1.16),(1.21) it follows for $\lambda \geq 0$:

$$\|\Phi(p,t)\|_\alpha \leq \int_0^{+\infty} \left(\frac{\alpha}{\eta} + C_\alpha\right)^\alpha e^{-(\lambda+\delta)\eta} L_0 \, d\eta \leq L_0 k(\alpha),$$

$$k(\alpha) = \int_0^{+\infty} \left(\frac{\alpha}{\eta} + C_\alpha\right)^\alpha e^{-\delta\eta} \, d\eta < +\infty \, (\alpha \in [0,1), \delta > 0). \tag{3.2}$$

If ϱ is small enough to have $L_\varrho < 1/4k(\alpha)$ and ε satisfies $|\varepsilon| < \varrho/4k(\alpha)L_1$, then $L_0 = L_\varrho \varrho + |\varepsilon| L_1 < \varrho/2k(\alpha)$ and we get (3.1).

Theorem 3.1: *Let $0 \leq \alpha \leq 1/2$; ϱ and ε satisfy the conditions above. There exist $\eta > 0$, $C > 0$ and ϱ_1 $(0 < \varrho_1 < \varrho)$ such that for every trajectory $v(t) = U(t,\tau)v(\tau)$, $v(t) \in V_{\varrho_1}$ for $t \in [\tau, T]$, one can find its «trace» $\bar{v}(t)$ on $M(\lambda)$, i.e.*

$\bar{v}(t) = U(t,\tau)\,\bar{v}(\tau)$, $(\bar{v}(t), t) \in M(\lambda)$, *and*

$$\|v(t) - \bar{v}(t)\|_{\alpha} \leqslant Ce^{-(\lambda+\eta)(t-\tau)} \quad \textit{for} \quad t \in [\tau, T]. \tag{3.3}$$

In addition $\bar{v}(t) \in V_0$ *when* $t \in [\tau, T]$.

The proof is based on two simple Lemmas. In these Lemmas we consider the system of ordinary differential inequalities:

$$\begin{cases} \dot{x} \geqslant \gamma x - \theta y \\ \dot{y} \leqslant \theta x - \gamma y, \end{cases} \qquad \gamma > \theta > 0, \ (x, y) \in R^2. \tag{3.4}$$

Notice that eigenvalues of the matrix $G = \begin{pmatrix} \gamma & -\theta \\ \theta & -\gamma \end{pmatrix}$ are $\lambda_1 = \mu = \sqrt{\gamma^2 - \theta^2} > 0$ and $\lambda_2 = -\mu < 0$; corresponding eigenvectors are $e_1 = (1, \beta)$ and $e_2 = (\beta, 1)$, where

$$\beta = \gamma/\theta - \sqrt{\gamma^2/\theta^2 - 1}, \tag{3.5}$$

$0 < \beta < 1$ for $\gamma > \theta > 0$.

LEMMA 3.1: *Let* $x(t), y(t) \geqslant 0$ *satisfy* (3.4) *for* $t \in [0, T]$ *and* $x(T) \leqslant \beta y(T)$, *where* β *is defined by* (3.5). *Then for* $t \in [0, T]$:

$$x(t) \leqslant \beta y(t); \quad y(t) \leqslant \frac{1}{1-\beta^2} y(0) e^{-\mu t}, \qquad \mu = \sqrt{\gamma^2 - \theta^2}.$$

PROOF: Passing to eigenvectors of G we get from (3.4):

$$\dot{z} \geqslant -\mu z, \qquad \text{where } z = x - \beta y, \tag{3.6}$$

$$\dot{w} \leqslant -\mu w, \qquad \text{where } w = y - \beta x. \tag{3.7}$$

As $z(T) \leqslant 0$, then (3.6) implies $z(t) \leqslant 0$ for $t \leqslant T$, i.e. $x(t) \leqslant \beta y(t)$. From (3.7) and $z(t) \leqslant 0$ it follows the estimation for $y(t)$ (see [10]). ■

LEMMA 3.2: *Let* $x(t) \geqslant 0$, $y(t) \geqslant 0$ *satisfy for* $t \in [\tau, T]$ *the following system of inequalities:*

$$\begin{cases} \dot{x} \geqslant (-\lambda + \gamma)x - \theta y, \\ \dot{y} \leqslant (-\lambda - \gamma)y + \theta x, \end{cases} \qquad \gamma > \theta > 0. \tag{3.8}$$

Assume that $x(T) \leqslant \beta y(T)$, *where* β *is the same as in Lemma* 3.1. *Then for* $t \in [\tau, T]$:

$$x(t) \leqslant \beta y(t), \qquad y(t) \leqslant \frac{1}{1-\beta^2} y(\tau) e^{-(\lambda+\mu)(t-\tau)}, \qquad \mu = \sqrt{\gamma^2 - \theta^2}.$$

PROOF: We apply Lemma 3.1 to the functions $x_1(s) = x(s+\tau)e^{\lambda s}$ and $y_1(s) = y(s+\tau)e^{\lambda s}$, where $s = t - \tau$ (see [10]). ∎

PROPOSITION 3.1: *Let $0 \leq \alpha \leq 1/2$, $(p_1(t), q_1(t))$ and $(p_2(t), q_2(t))$ be two solutions of (2.1)-(2.2), $p(t) = p_1(t) - p_2(t)$, $q(t) = q_1(t) - q_2(t)$. If ϱ and ε are sufficiently small then for $t > \tau$:*

$$\frac{d}{dt}\|p\|_\alpha^2 \geq (-2\lambda + \delta)\|p\|_\alpha^2 - \frac{\delta}{2}\|q\|_\alpha^2\,, \tag{3.9}$$

$$\frac{d}{dt}\|q\|_\alpha^2 \leq (-2\lambda - \delta)\|q\|_\alpha^2 + \frac{\delta}{2}\|p\|_\alpha^2\,. \tag{3.10}$$

PROOF: Functions $p(t)$ and $q(t)$ satisfy

$$\partial_t p = -Ap + P[B(p_1 + q_1, t) - B(p_2 + q_2, t)]\,, \tag{3.11}$$

$$\partial_t q = -Aq + Q[B(p_1 + q_1, t) - B(p_2 + q_2, t)]\,. \tag{3.12}$$

Taking the scalar product of (3.11) with $(A + aI)^{2\alpha}p$ and using (1.17), (1.19) we get (3.9) if L in (1.17) is small enough (see [10]). Similarly taking the scalar product of (3.12) with $(A + aI)^{2\alpha}q$ and using (1.17),(1.18) we obtain (3.10) for sufficiently small ϱ and ε.

REMARK 3.1: If (3.9)-(3.10) hold, then Lemma 3.2 is applicable with $x = \|p\|_\alpha^2$, $y = \|q\|_\alpha^2$, $\gamma = \delta$, $\theta = \delta/2$. In this case $\mu = \sqrt{\gamma^2 - \theta^2} = (\sqrt{3}/2)\delta$, $\beta = \gamma/\theta - \sqrt{\gamma^2/\theta^2 - 1} = 2 - \sqrt{3}$.

Thus if $\|p_1(T) - p_2(T)\|_\alpha^2 \leq \beta\|q_1(T) - q_2(T)\|_\alpha^2$, then

$$\|q_1(t) - q_2(t)\|_\alpha^2 \leq \frac{1}{1-\beta^2}\|q_1(\tau) - q_2(\tau)\|_\alpha^2 e^{-(2\lambda + \mu)(t-\tau)} \tag{3.13}$$

$$\|p_1(t) - p_2(t)\|_\alpha^2 \leq \beta\|q_1(t) - q_2(t)\|_\alpha^2 \tag{3.14}$$

for $t \in [\tau, T]$.

PROOF OF THEOREM 3.1: Let the process $\{U(t,\tau), t \geq \tau\}$ correspond to the equation (1.15). This equation is obtained by the stated above modification of the initial equation (1.4) outside the neighborhood O_ϱ. For a given solution $v(t) = p_1(t) + q_1(t)$ of the equation (1.15), or equivalently of the system (2.1)-(2.2), that belongs to sufficiently small neighborhood of zero for $\tau \leq t \leq T$, we shall construct its approximation $\bar{v}(t)$. The integral curve $(\bar{v}(t), t)$ corresponding to this approximation lies on the integral manifold $M \equiv M(\lambda)$: $(\bar{v}(t), t) \in M$. Sometimes below for brevity we shall use «$\bar{v}(t)$ lies on M» instead of «the integral curve $(\bar{v}(t), t)$ corresponding to the solution $\bar{v}(t)$ lies on M».

Then we shall prove that $v(t)$ and $\bar{v}(t)$ do not leave the neighborhood O_ϱ. Therefore $v(t)$ and $\bar{v}(t)$ for $\tau \leqslant t \leqslant T$ are also solutions of the initial equation (1.4).

We put $\varrho_1 = \varrho(1 - \sqrt{\beta}) / (1 + \sqrt{\beta}) < \varrho$. Let $v(t)$ be a solution of (1.15) and $v(t) \in \in V_{\varrho_1} \subset O_\varrho$ for $t \in [\tau, T]$; $v(T) = p_T + q_T$. Consider in E^α the cone $\|p - p_T\|_\alpha^2 \leqslant \beta \|q - - q_T\|_\alpha^2$, where β is chosen in accordance with Remark 3.1. In the intersection of this cone with the set $M \cap \{t = T\}$ we take a point $p_2(T) + q_2(T) = \bar{v}(T)$ and define an integral curve $(\bar{v}(t), t) \in M$, $t \in R$, passing through the point $(\bar{v}(T), T)$. It follows from Remark 2.1 that such a curve exists. We denote $p_2(t) = P\bar{v}(t)$, $q_2(t) = Q\bar{v}(t)$.

Note that $\|q_2(t)\|_\alpha \leqslant \varrho/2$ since (3.1). As $v(t) = p_1(t) + q_1(t) \in V_{\varrho_1}$, then $\|q_1(t)\|_\alpha \leqslant \leqslant \varrho_1/2$. Thus

$$\|q_1(t) - q_2(t)\|_\alpha \leqslant \frac{\varrho + \varrho_1}{2} \quad \text{for } t \in [\tau, T]. \tag{3.15}$$

Proposition 3.1 and Lemma 3.2 imply (3.13),(3.14) if ϱ and ε are small enough. Thus from (3.13)-(3.15) we get:

$$\|v(t) - \bar{v}(t)\|_\alpha^2 = \|p_1(t) - p_2(t)\|_\alpha^2 + \|q_1(t) - q_2(t)\|_\alpha^2 \leqslant (1 + \beta)\|q_1(t) - q_2(t)\|_\alpha^2 \leqslant$$

$$\leqslant \frac{1+\beta}{1-\beta^2}\|q_1(\tau) - q_2(\tau)\|_\alpha^2 e^{-(2\lambda+\mu)(t-\tau)} \leqslant \frac{1}{1-\beta}\frac{(\varrho+\varrho_1)^2}{4}e^{-(2\lambda+\mu)(t-\tau)}.$$

Taking into account that $\varrho_1 < \varrho$ we obtain (3.3) with $C = \varrho/\sqrt{1-\beta}$, $\eta = \mu/2$.

Let us verify that $\bar{v}(t) \in V_\varrho$ when $t \in [\tau, T]$. As it was noticed, $\|q_2(t)\|_\alpha \leqslant \varrho/2$, so we get from (3.14),(3.15):

$$\|p_2(t)\|_\alpha \leqslant \|p_1(t)\|_\alpha + \|p_2(t) - p_1(t)\|_\alpha \leqslant \frac{\varrho_1}{2} + \sqrt{\beta}\|q_2(t) - q_1(t)\|_\alpha \leqslant$$

$$\leqslant \frac{\varrho_1}{2} + \sqrt{\beta}\frac{\varrho + \varrho_1}{2} = \varrho_1\frac{1+\sqrt{\beta}}{2} + \varrho\frac{\sqrt{\beta}}{2} = \varrho\frac{1-\sqrt{\beta}}{1+\sqrt{\beta}}\frac{1+\sqrt{\beta}}{2} + \varrho\frac{\sqrt{\beta}}{2} = \frac{\varrho}{2}.$$

Thus $\bar{v}(t) \in V_\varrho \subset O_\varrho$ when $\tau \leqslant t \leqslant T$. ■

Now we study the case when a solution $v(t) = U(t, \tau)v(\tau)$ of the equation (1.4) (or (1.15)) does not leave V_{ϱ_1} for all $t \geqslant \tau$. It should be noted that in an autonomous case exponential approximations for solutions, tending to a hyperbolic equilibrium point z, are constructed in [1]. These approximations lie on a finite dimensional invariant manifold, passing through z. The analogous result takes place in our non-autonomous case.

Theorem 3.2: *Let the process $U(t, \tau)$ corresponding to the problem (1.4),(1.5) be continuous in E^α; α, ϱ, ϱ_1, ε are the same as in Theorem 3.1. If $v(t) = U(t, \tau)v(\tau) \in$*

$\in V_{\varrho_1}$ *for all* $t \geq \tau$*, then there exists an integral curve* $(\bar{v}(t), t) \in M(\lambda)$ *such that*

$$\|v(t) - \bar{v}(t)\|_\alpha \leq Ce^{-(\lambda+\eta)(t-\tau)} \quad \text{for } t \geq \tau$$

with the same constants η *and* C *as in Theorem* 3.1.

Proof: Theorem 3.1 is applicable to the solution $v(t)$ on the time interval $[\tau, \tau+n]$ for every $n \in N$ because $v(t) \in V_{\varrho_1}$ for all $t \geq \tau$. According to this Theorem we can find an integral curve $(\bar{v}_n(t), t) \in M(\lambda)$, $\bar{v}_n(t) \in V_\varrho$, such that

$$\|v(t) - \bar{v}_n(t)\|_\alpha \leq Ce^{-(\lambda+\eta)(t-\tau)} \quad \text{for } t \in [\tau, \tau+n].$$

The sequence of points $\bar{v}_n(\tau)$ is bounded (since $\bar{v}_n(\tau) \in V_\varrho$) and finite dimensional (since $\bar{v}_n(\tau) \in M \cap \{t = \tau\}$). So we can choose a convergent subsequence $\bar{v}_{n_k}(\tau) \to \bar{v}(\tau) \in M \cap \{t = \tau\}$ and define an integral curve $(\bar{v}(t), t) \in M, t \in R$, passing through the point $(\bar{v}(\tau), \tau)$. Since the process $U(t, \tau)$ is continuous in E^α we have $\bar{v}_{n_k}(t) = U(t, \tau)\bar{v}_{n_k}(\tau) \to \bar{v}(t) = U(t, \tau)\bar{v}(\tau)$ when $t \geq \tau$. Therefore

$$\|v(t) - \bar{v}_{n_k}(t)\|_\alpha \leq Ce^{-(\lambda+\eta)(t-\tau)} \quad \text{when } t \in [\tau, T],$$

if $\tau + n_k \geq T$. Whence

$$\|v(t) - \bar{v}(t)\|_\alpha \leq Ce^{-(\lambda+\eta)(t-\tau)} \quad \text{when } t \in [\tau, T].$$

As $T > \tau$ is arbitrary the last estimation holds for all $t \geq \tau$. ■

Theorem 3.3: *Let constants* l *in* (2.4) *and* β *in* (3.13), (3.14) *satisfy* $l\sqrt{\beta} < 1$; $M(\lambda) = \{(p+q, t) \in E^\alpha \times R, | q = \Phi(p, t)\}$ *is an integral manifold of the process* $U(t, \tau)$ *corresponding to* (1.15)*. Then there exists a constant* $C_0 = C_0(l, \beta)$ *such that if* $v(t) = p_1(t) + q_1(t) \in V_{\varrho_1}$ *for* $t \in [\tau, T]$*, then*

$$\|q_1(t) - \Phi(p_1(t), t)\|_\alpha \leq C_0\|q_1(\tau) - \Phi(p_1(\tau), \tau)\|_\alpha e^{-(\lambda+\eta)(t-\tau)}$$

for $t \in [\tau, T]$*;* $\eta > 0$ *is the same as in Theorem* 3.1; $\lambda \geq 0$.

Proof: Together with $v(t)$ let us consider corresponding trajectory $\bar{v}(t) = p_2(t) + q_2(t)$ defined in Theorem 3.1. As $(\bar{v}(t), t) \in M(\lambda)$, then $q_2(t) = \Phi(p_2(t), t)$. From (2.4) and (3.14) we get

$$(3.16) \quad \|q_1(t) - \Phi(p_1(t), t)\|_\alpha \leq \|q_1(t) - q_2(t)\|_\alpha + \|\Phi(p_1(t), t) - \Phi(p_2(t), t)\|_\alpha \leq$$

$$\leq \|q_1(t) - q_2(t)\|_\alpha + l\|p_1(t) - p_2(t)\|_\alpha \leq (1 + l\sqrt{\beta})\|q_1(t) - q_2(t)\|_\alpha,$$

$$\|q_1(\tau) - q_2(\tau)\|_\alpha \leq \|q_1(\tau) - \Phi(p_1(\tau), \tau)\|_\alpha + \|\Phi(p_1(\tau), \tau) - \Phi(p_2(\tau), \tau)\|_\alpha \leq$$

$$\leq \|q_1(\tau) - \Phi(p_1(\tau), \tau)\|_\alpha + l\sqrt{\beta}\|q_1(\tau) - q_2(\tau)\|_\alpha.$$

Thus for $l\sqrt{\beta} < 1$ we obtain

$$\|q_1(\tau) - q_2(\tau)\|_\alpha \leqslant \frac{1}{1 - l\sqrt{\beta}} \|q_1(\tau) - \Phi(p_1(\tau), \tau)\|_\alpha . \tag{3.17}$$

By substituting (3.16) and (3.17) into (3.13), we get

$$\|q_1(t) - \Phi(p_1(t), t)\|_\alpha \leqslant \frac{1 + l\sqrt{\beta}}{1 - l\sqrt{\beta}} \frac{1}{\sqrt{1 - \beta^2}} \|q_1(\tau) - \Phi(p_1(\tau), \tau)\|_\alpha e^{-(\lambda + \mu/2)(t - \tau)} . \quad \blacksquare$$

REMARK 3.2: Theorem 3.1 states the existence of integral manifolds for arbitrary small $l > 0$, if ϱ and ε are sufficiently small. So $l < 1/\sqrt{\beta}$ holds for small ϱ and ε.

4. - THE STRUCTURE OF $M(0)$

Consider the equation (1.15) in the case $\lambda = 0 \notin \sigma(A)$. Let P and Q be orthoprojectors onto invariant subspaces of the operator A, corresponding to $\sigma_-(A)$ and $\sigma_+(A)$ — positive and negative eigenvalues of A.

In such a case all the constructions above hold. So process $U(t, \tau)$ corresponding to (1.15), (1.5) possesses an integral manifold $M = M(0)$ provided that ϱ and ε are sufficiently small. The aim of the present section is to study the structure of this manifold. We state the existence and the uniqueness of a solution $v = z(t)$, $t \in R$, of the equation (1.15) that is bounded in E^α. Also it is proved that all trajectories lying on $M(0)$ exponentially approach $z(t)$ when $t \to -\infty$.

One can find similar results for differential equations in Banach space with a bounded operator A in [6].

THEOREM 4.1: *Let $s > 0$ be given. If L_0, L in (1.16)-(1.17) are sufficiently small, then there exists the unique solution $v = z(t)$ of the equation (1.15) such that*

$$\|z(t)\|_\alpha \leqslant s , \qquad t \in R . \tag{4.1}$$

PROOF: We reduce the problem of the existence of such solution to the problem of the existence of a fixed point of some operator Ψ. Let $z(t)$ be the required solution. Then $p(t) = Pz(t) \in P(E^\alpha)$ is a bounded for $t \geqslant \tau$ solution of the linear equation

$$\partial_t p = -Ap + PB(z(t), t) .$$

As $-A|_{P(E^\alpha)}$ is a positive operator and $\|PB(z(t), t)\| \leqslant L_0$ due to (1.16), a bounded solution $p(t)$ of this equation is unique and it is defined by the formula:

$$p(t) = -\int_t^{+\infty} e^{A(\xi - t)} PB(z(\xi), \xi)\, d\xi .$$

Similarly the operator $A|_{Q(E^\alpha)}$ is positive, so the equation

$$\partial_t q = -Aq + QB(z(t), t), \qquad q = Qz(t) \in Q(E^\alpha)$$

possesses a unique bounded when $t \to -\infty$ solution

$$q(t) = \int_{-\infty}^{t} e^{-A(t-\xi)} QB(z(\xi), \xi)\, d\xi .$$

Therefore

$$z(t) = p(t) + q(t) = \int_{-\infty}^{t} e^{-A(t-\xi)} QB(z(\xi), \xi)\, d\xi - \int_t^{+\infty} e^{A(\xi - t)} PB(z(\xi), \xi)\, d\xi .$$

If $v = z(t)$ is a bounded for $t \in R$ solution of (1.15), then the function $z(t)$ is a fixed point of the operator Ψ: $\Psi[z](t) = z(t)$, where

$$\Psi[z(t)] = \int_{-\infty}^{t} e^{-A(t-\xi)} QB(z(\xi), \xi)\, d\xi - \int_t^{+\infty} e^{A(\xi - t)} PB(z(\xi), \xi)\, d\xi =$$

$$= \int_0^{+\infty} e^{-A\eta} QB(z(t-\eta), t-\eta)\, d\eta - \int_0^{+\infty} e^{A\eta} PB(z(t+\eta), t+\eta)\, d\eta .$$

It is easy to check the inverse statement: if $\Psi[z] = z$, then $v = z(t)$ is a solution of (1.15).

Consider the operator Ψ on the set Z_s of all functions z: $R \to E^\alpha$, satisfying (4.1). We define the metric on Z_s by the formula

$$\varrho_\alpha(z_1, z_2) = \sup_{t \in R} \|z_1(t) - z_2(t)\|_\alpha . \tag{4.2}$$

Proposition 4.1: *If L_0 in (1.16) is sufficiently small, then*

$$\sup_{t \in R} \|\Psi[z](t)\|_\alpha \le s , \qquad \forall z \in Z_s . \tag{4.3}$$

PROOF: From the definition of the operator Ψ, taking into account (1.16) and (1.20)-(1.21) with $\lambda = 0$ we obtain:

$$\|\Psi[z](t)\|_\alpha \leq \int_0^{+\infty} \left(\frac{\alpha}{\eta} + C_\alpha\right)^\alpha e^{-\delta\eta} L_0 \, d\eta + \int_0^{+\infty} C_\alpha e^{-\delta\eta} L_0 \, d\eta \geq \tag{4.4}$$

$$\leq k(\alpha) L_0 + \frac{C_\alpha}{\delta} L_0 = C^* L_0 ,$$

where $k(\alpha)$ is defined in (3.2). Thus we have (4.3) if $L_0 C^* \leq s$. ■

PROPOSITION 4.2: *The operator Ψ is contracting on Z, with respect to metric* (4.2) *if L in* (1.17) *is sufficiently small.*

PROOF: From (1.17), (1.20)-(1.21) it follows:

$$\|\Psi[z_1](t) - \Psi[z_2](t)\|_\alpha \leq \int_0^{+\infty} \left(\frac{\alpha}{\eta} + C_\alpha\right)^\alpha e^{-\delta\eta} L \|z_1(t-\eta) - z_2(t-\eta)\|_\alpha \, d\eta +$$

$$+ \int_0^{+\infty} C_\alpha e^{-\delta\eta} L \|z_1(t+\eta) - z_2(t+\eta)\|_\alpha \, d\eta \leq L\left(k(\alpha) + \frac{C_\alpha}{\delta}\right) \varrho_\alpha(z_1, z_2).$$

So if $L(k(\alpha) + C_\alpha/\delta) < 1$ then Ψ is a contraction operator on Z_s. ■

To complete the proof of Theorem 4.1 we have to note that the unique fixed point $z \in Z_s$ of the operator Ψ is exactly the required solution $z = z(t)$, $t \in R$, of the equation (1.15). ■

REMARK 4.1: If we choose ϱ to have $L_\varrho \leq 1/(2C^*)$ and put $|\varepsilon| \leq \varrho/(2L_1 C^*)$, then $L_0 = L_\varrho \varrho + |\varepsilon| L_1 \leq \varrho/C^*$. Thus (4.4) implies $\|z(t)\|_\alpha = \|\Psi[z](t)\|_\alpha < \varrho$ for all $t \in R$, i.e. the solution $v = z(t)$ of (1.15) satisfies $z(t) \in O_\varrho$. The equations (1.4) and (1.15) coincide in a small neighborhood of zero O_ϱ, so $z(t)$ is also a solution of the initial equation (1.4).

Thus, provided ϱ and ε are small enough, the equation (1.4) possesses a unique solution $z(t)$, $t \in R$, lying inside O_ϱ for all $t \in R$.

Due to Remark 2.1 the integral curve $(z(t), t)$ corresponding to the solution $z(t)$, bounded for all $t \in R$, lies on the integral manifold M.

Now we shall study the behavior of other solutions lying on $M = M(0)$.

THEOREM 4.2: *For sufficiently small L in* (1.17) *there exist $\mu > 0$, $C_0 > 0$ such that any solutions $v_1(t), v_2(t), t \in R$, of the equation* (1.15), *lying on $M(0)$, satisfy*

$$\|v_1(\tau - t) - v_2(\tau - t)\|_\alpha \leq C_0 \|v_1(\tau) - v_2(\tau)\| e^{-\mu t} \quad \forall t > 0 .$$

In particular, as $(z(t), t) \in M(0)$, any solution $v(t)$ on $M(0)$ approaches $z(t)$ exponentially as $t \to -\infty$.

PROOF: If $v(t)$ is a solution of (1.15), $p(t) = Pv(t)$, $q(t) = Qv(t)$, then

$$\partial_t p = -Ap + PB(v(t), t), \tag{4.5}$$

$$\partial_t q = -Aq + PQ(v(t), t). \tag{4.6}$$

A solution $p(t)$ of (4.5) with the initial data $p(\tau) = p_\tau = Pv(\tau)$ is given by the formula

$$p(t) = e^{-A(t-\tau)}p_\tau + \int_\tau^t e^{-A(t-\xi)}PB(v(\xi), \xi)\, d\xi. \tag{4.7}$$

If the graph $(v(t), t)$ of the solution $v(t)$ lies on the integral manifold $M(0)$, then $q(t)$ is a bounded as $t \to -\infty$ solution of the equation (4.6). The operator $A|_{QE^\alpha}$, is positive, so such solution is unique and

$$q(t) = \int_{-\infty}^t e^{-A(t-\xi)}QB(v(\xi), \xi)\, d\xi. \tag{4.8}$$

From (4.7) and (4.8) we have

$$v(\tau - t) = p(\tau - t) + q(\tau - t) = e^{-At}p_\tau - \int_0^t e^{A(t-\eta)}PB(v(\tau - \eta), \tau - \eta)\, d\eta +$$

$$+ \int_t^{+\infty} e^{-A(\eta - t)}QB(v(\tau - \eta), \tau - \eta)\, d\eta.$$

Let $v_1(t), v_2(t)$ be two solutions of (1.15) lying on the integral manifold $M(0)$, $r(t) = v_1(t) - v_2(t)$. Then taking into account (1.17) and (1.20)-(1.21) with $\lambda = 0$ we obtain for $t \geq 0$:

$$\|r(\tau - t)\|_\alpha \leq e^{-bt}\|p_{1\tau} - p_{2\tau}\|_\alpha + \int_0^t C_\alpha e^{-b(t-\eta)}L\|r(\tau - \eta)\|_\alpha\, d\eta +$$

$$+ \int_t^{+\infty} \left(\frac{\alpha}{\eta - t} + C_\alpha\right)^\alpha e^{-\delta(\eta - t)}L\|r(\tau - \eta)\|_\alpha\, d\eta =$$

$$= e^{-bt}\|p_{1\tau} - p_{2\tau}\|_\alpha + L\int_0^{+\infty} G(t, \eta)\|r(\tau - \eta)\|_\alpha\, d\eta,$$

where $G(t, \eta)$ is a positive function

(4.9)
$$G(t,\eta)=\begin{cases} C_\alpha e^{-\delta(t-\eta)} & \text{for } 0<\eta\leqslant t, \\ \left(\dfrac{\alpha}{\eta-t}+C_\alpha\right)^\alpha e^{\delta(t-\eta)} & \text{for } 0\leqslant t<\eta. \end{cases}$$

Notice that any solution $v(t)$ lying on $M(0)$ is bounded in E^α for $t\to-\infty$. In fact by definition of integral manifold $\|Qv(t)\|_\alpha\leqslant b$ for all $t\in R$. The projection $p(t)=Pv(t)$ is bounded for $t\to-\infty$ as a solution of the linear nonhomogeneous equation (4.5) with the bounded function $PB(v(t),t)$ and the positive linear operator $-A|_{P(E^\alpha)}$. Therefore $\varphi(\eta)\equiv\|r(\tau-\eta)\|_\alpha$ is a positive bounded function defined for $\eta>0$. This function satisfies the inequality:

$$\varphi(t)\leqslant e^{-\delta t}\|p_{1\tau}-p_{2\tau}\|_\alpha+\mathcal{G}\varphi(t),\qquad \mathcal{G}\varphi(t)=L\int_0^{+\infty}G(t,\eta)\,\varphi(\eta)\,d\eta.$$

The integral operator $\mathcal{G}$ is bounded in the space $C_b([0,+\infty[)$:

$$\|\mathcal{G}\|\leqslant L\sup_{t>0}\int_0^{+\infty}G(t,\eta)\,d\eta\leqslant L\left(\frac{C_\alpha}{\delta}+k(\alpha)\right),$$

where $k(\alpha)$ is defined by (3.2).

If L is sufficiently small, then $\|\mathcal{G}\|<1$, and (see [6])

$$\varphi(t)\leqslant\psi(t)\equiv(I-\mathcal{G})^{-1}[e^{-\delta t}\|p_{1\tau}-p_{2\tau}\|_\alpha],$$

where $\psi(t)\in C_b([0,+\infty[)$ is a solution of the integral equation

(4.10)
$$\psi(t)=e^{-\delta t}\|p_{1\tau}-p_{2\tau}\|_\alpha+\mathcal{G}\psi(t).$$

Let us prove that $\psi(t)\leqslant Ce^{-\mu t}$, $0<\mu<\delta$. Multiplying (4.10) by $e^{\mu t}$ and denoting $\xi(t)=\psi(t)e^{\mu t}$ we obtain

(4.11)
$$\xi(t)=e^{-(\delta-\mu)t}\|p_{1\tau}-p_{2\tau}\|_\alpha+\mathcal{G}_1\xi(t),$$

$$\mathcal{G}_1\xi(t)=e^{\mu t}\mathcal{G}[\xi(t)e^{-\mu t}]=L\int_0^{+\infty}G(t,\eta)\,e^{\mu(t-\eta)}\xi(\eta)\,d\eta.$$

$\mathcal{G}_1$ is an integral operator with the kernel $G_1(t,\eta)=G(t,\eta)e^{\mu(t-\eta)}$, $G(t,\eta)$ is defined in (4.9). The norm of the operator $\mathcal{G}_1$ in the space $C_b([0,+\infty))$ is estimated analogously to the norm of $\mathcal{G}$:

$$\|\mathcal{G}_1\|\leqslant\sup_{t>0}L\int_0^{+\infty}|G_1(t,\eta)|\,d\eta\leqslant L\left(\frac{C_\alpha}{\delta-\mu}+k(\alpha)\right).$$

If L is sufficiently small then $\|\mathcal{G}_1\|<1$.

In such a case the equation (4.11) possesses the unique solution

$$\xi(t) = \|p_{1\tau} - p_{2\tau}\|_\alpha (I - \mathfrak{S}_1)^{-1} e^{-(\delta-\mu)t} \leq C_0 \|p_{1\tau} - p_{2\tau}\|_\alpha$$

in $C_b([0, +\infty))$, because the function $e^{-(\delta-\mu)t}$ is bounded for $t > 0$.

Consequently, we get for $t > 0$:

$$\varphi(t) \leq \psi(t) = \xi(t) e^{-\mu t} \leq C_0 \|p_{1\tau} - p_{2\tau}\|_\alpha e^{-\mu t},$$

$$\|v_1(\tau - t) - v_2(\tau - t)\|_\alpha = \varphi(t) \leq C_0 \|v_1(\tau) - v_2(\tau)\|_\alpha e^{-\mu t},$$

where $v_1(t), v_2(t)$ are two solutions of the equation (1.15) lying on $M(0)$. ∎

Remark 4.2: It is easy to show that if ϱ and ε are sufficiently small then any solution $v(t)$ of the equation (1.15) lying on $M(0)$ with sufficiently small initial data does not leave the neighborhood of zero O_ϱ for all $t \leq \tau$. Thus, in such a case $v(t)$ is also a solution of the initial equation (1.4). Therefore the result of Theorem 4.2 holds also for solutions of (1.4) with small norm $\|v(\tau)\|_\alpha$ at the initial moment $t = \tau$. Remind that $v = 0$ is an equilibrium point of the limit (when $\varepsilon = 0$) equation (1.4).

5. - Estimation of distance between M_ε and M_0

In the present section we study how the integral manifold $M_\varepsilon \equiv M_\varepsilon(\lambda)$, $\lambda \geq 0$, depends on ε.

We consider the equation (1.15), replacing $B(v, t)$ by (1.14):

(5.1) $$\frac{dv}{dt} + Av = B_0^*(v) + \varepsilon B_1^*(v, t),$$

(5.2) $$v|_{t=\tau} = v_\tau.$$

The functions $B_0^*(v)$ and $B_1^*(v, t)$, defined in (1.13) satisfy the following properties:

(5.3) $$\|B_0^*(v_1) - B_0^*(v_2)\| \leq L_\varrho \|v_1 - v_2\|_\alpha, \quad \forall v_1, v_2 \in E^\alpha,$$

moreover the constant L_ϱ can be chosen arbitrary small when $\varrho \to +0$;

(5.4) $$\|B_1^*(v, t)\| \leq L_1, \quad \forall v \in E^\alpha, \ \forall t \in \mathbf{R}.$$

Together with (5.1) we consider the autonomous equation

(5.5) $$\frac{dv}{dt} + Av = B_0^*(v).$$

All constructions of the integral manifold $M_\varepsilon(\lambda)$ evidently hold for $\varepsilon = 0$. The integral manifold $M_0 \equiv M_0(\lambda)$, $\lambda \geq 0$, for $\varepsilon = 0$ corresponds to the autonomous equation (5.5). In this case M_0 is invariant with respect to the substitution $t \to t + \tau$ for all $\tau \in \mathbf{R}$, so $M_0 = M \times \mathbf{R}_t$, where $M = \{p + \Phi_0(p) | p \in P(E^\alpha), \Phi_0(p) \in Q(E^\alpha)\} \subset E^\alpha$. The function $\Phi_0(p)$ satisfies the equation:

$$\Phi_0(p_\tau) = J_0[\Phi_0](p_\tau) = \int_0^{+\infty} e^{-A\xi} QB_0^*(p_0(\tau-\xi) + \Phi_0(p_0(\tau-\xi)))\,d\xi, \tag{5.6}$$

where $p_0(t)$ is a solution of the following Cauchy problem:

$$\partial_t p_0 + Ap_0 = PB_0^*(p_0 + \Phi_0(p_0)), \tag{5.7}$$

$$p_0|_{t=\tau} = p_\tau \in P(E^\alpha) \tag{5.8}$$

in the finite dimensional space $P(E^\alpha) = P(E)$. Notice that $J_0[\Phi_0](p_\tau)$ does not depend on τ since the equation (5.7) is autonomous.

As it was proved in § 2 the equation (5.1) under small ε and ϱ possesses the integral manifold

$$M_\varepsilon = \{(p + \Phi_\varepsilon(p,t),t) \mid p \in P(E^\alpha),\ \Phi_\varepsilon(p,t) \in Q(E^\alpha),\ t \in R\}.$$

The function $\Phi = \Phi_\varepsilon$ satisfies (2.3),(2.4) and the equation

$$\Phi_\varepsilon(p_\tau,\tau) = J_\varepsilon[\Phi_\varepsilon](p_\tau,\tau) = \int_0^{+\infty} e^{-A\xi} Q[B_0^*(p_\varepsilon(\tau-\xi) + \Phi_\varepsilon(p_\varepsilon(\tau-\xi),\tau-\xi) + \tag{5.9}$$

$$+\varepsilon B_1^*((p_\varepsilon(\tau-\xi) + \Phi_\varepsilon(p_\varepsilon(\tau-\xi),\tau-\xi),\tau-\xi)]\,d\xi,$$

where $p_\varepsilon(t)$ is a solution of the equation

$$\partial_t p_\varepsilon + Ap_\varepsilon = PB_0^*(p_\varepsilon + \Phi_\varepsilon(p_\varepsilon,t)) + \varepsilon PB_1^*(p_\varepsilon + \Phi_\varepsilon(p_\varepsilon,t),t) \tag{5.10}$$

in the space $P(E^\alpha)$ with the initial data (5.8).

THEOREM 5.1: *The integral manifolds M_ε and M_0 are close in the following sense: there exists a constant $C > 0$ such that*

$$d_\alpha(\Phi_\varepsilon,\Phi_0) \equiv \sup_{p_\tau \in P(E^\alpha),\, \tau \in R} \|\Phi_\varepsilon(p_\tau,\tau) - \Phi_0(p_\tau)\|_\alpha \leqslant C|\varepsilon|.$$

At first let us prove an auxiliary proposition.

Denote $r(t) = p_\varepsilon(t) - p_0(t)$, where $p_\varepsilon(t)$ is a solution of (5.10) and $p_0(t)$ is a solution of (5.7) with the same initial data (5.8).

PROPOSITION 5.1: *If L_φ in (5.3) is sufficiently small, then*

$$\|r(\tau-t)\|_\alpha \leqslant C_\alpha \frac{L_\varphi d_\alpha(\Phi_\varepsilon,\Phi_0) + |\varepsilon| L_1}{\delta - L_\varphi(1+l)C_\alpha} e^{lt}$$

for $t \geqslant 0$, where C_α and l are the same as in (1.20) and (2.4).

PROOF: Subtracting (5.7) from (5.10) we have:

$$\partial_t r = -Ar + PI_1(t) + PI_2(t), \quad r|_{t=t} = 0, \tag{5.11}$$

$$\begin{cases} I_1(t) = B_0^+(p_\varepsilon(t) + \Phi_\varepsilon(p_\varepsilon(t), t)) - B_0^+(p_0(t) + \Phi_0(p_0(t))), \\ I_2(t) = \varepsilon B_1^+(p_\varepsilon(t) + \Phi_\varepsilon(p_\varepsilon(t), t), t). \end{cases} \tag{5.12}$$

From (2.4), (5.3), (5.4) it follows

$$\|I_2(t)\| \leq |\varepsilon| L_1, \tag{5.13}$$

$$\|I_1(t)\| \leq L_\varphi[\|p_\varepsilon(t) - p_0(t)\|_\alpha + \|\Phi_\varepsilon(p_\varepsilon(t), t) - \Phi_\varepsilon(p_0(t), t)\|_\alpha + \tag{5.14}$$

$$+\|\Phi_\varepsilon(p_0(t), t) - \Phi_0(p_0(t))\|_\alpha] \leq L_\varphi(1+l)\|r(t)\|_\alpha + L_\varphi d_\alpha(\Phi_\varepsilon, \Phi_0).$$

Solution of (5.11) is given by the formula:

$$r(t) = e^{-A(t-\tau)} \cdot 0 + \int_\tau^t e^{-A(t-\xi)}(PI_1(\xi) + PI_2(\xi))\, d\xi.$$

Then

$$r(\tau - t) = -\int_0^t e^{A(t-\eta)}(PI_1(\tau - \eta) + PI_2(\tau - \eta))\, d\eta.$$

Substituting the estimates (5.13), (5.14) for I_1 and I_2 and taking into account (1.20), we obtain:

$$\|r(\tau - t)\|_\alpha \leq C_\alpha \int_0^t e^{(\lambda - \delta)(t-\eta)}[L_\varphi(1+l)\|r(\tau - t)\|_\alpha + L_\varphi d_\alpha(\Phi_\varepsilon, \Phi_0) + |\varepsilon| L_1]\, d\eta \leq$$

$$\leq C_\alpha e^{(\lambda-\delta)t}\left[L_\varphi(1+l)\int_0^t e^{-(\lambda-\delta)\eta}\|r(\tau - \eta)\|_\alpha\, d\eta + (L_\varphi d_\alpha(\Phi_\varepsilon, \Phi_0) + |\varepsilon| L_1)\int_0^t e^{\delta\eta}\, d\eta\right].$$

Multiplying the last inequality by $e^{-(\lambda-\delta)t}$ and denoting $g(t) = \|r(\tau - t)\|_\alpha e^{-(\lambda-\delta)t} \geq 0$ we get:

$$g(t) \leq C_\alpha L_\varphi(1+l)\int_0^t g(\eta)\, d\eta + C_\alpha \frac{L_\varphi d_\alpha(\Phi_\varepsilon, \Phi_0) + |\varepsilon| L_1}{\delta}(e^{\delta t} - 1).$$

If L_ϱ is small then we have $C_\alpha L_\varrho(1+l) < \delta$ and by the Gronwall inequality we obtain

$$g(t) \leqslant C_\alpha \frac{L_\varrho d_\alpha(\Phi_\varepsilon, \Phi_0) + |\varepsilon| L_1}{\delta - C_\alpha L_\varrho(1+l)} e^{\delta t}.$$

By substitution $\|r(\tau - t)\|_\alpha = g(t) e^{(\lambda - \delta)t}$ we get the required estimation. ■

Proof of Theorem 5.1 Subtracting (5.6) from (5.9) we obtain

$$\Phi_\varepsilon(p_\tau, \tau) - \Phi_0(p_\tau) = \int_0^{+\infty} e^{-A\xi} Q[I_1(\tau - \xi) + I_2(\tau - \xi)]\, d\xi,$$

where I_1, I_2 are defined by (5.12).
Using (1.21) to estimate $Qe^{-A\xi}$ and also (5.13),(5.14) we get:

$$\|\Phi_\varepsilon(p_\tau, \tau) - \Phi_0(p_\tau)\|_\alpha \leqslant$$

$$\leqslant \int_0^{+\infty} \left(\frac{\alpha}{\xi} + C_\alpha\right)^\alpha e^{-(\lambda+\delta)\xi} [L_\varrho(1+l)\|r(\tau - \xi)\|_\alpha + L_\varrho d_\alpha(\Phi_\varepsilon, \Phi_0) + |\varepsilon| L_1]\, d\xi.$$

Now we estimate $\|r(\tau - t)\|_\alpha$ due to Proposition 5.1 and obtain

$$\|\Phi_\varepsilon(p_\tau, \tau) - \Phi_0(p_\tau)\|_\alpha \leqslant \int_0^{+\infty} \left(\frac{\alpha}{\xi} + C_\alpha\right)^\alpha e^{-\delta\xi}\, d\xi \times$$

$$\times \left[L_\varrho(1+l) C_\alpha \frac{L_\varrho d_\alpha(\Phi_\varepsilon, \Phi_0) + |\varepsilon| L_1}{\delta - L_\varrho(1+l) C_\alpha} + L_\varrho d_\alpha(\Phi_\varepsilon, \Phi_0) + |\varepsilon| L_1 \right] \leqslant$$

$$\leqslant k(\alpha) \frac{\delta}{\delta - L_\varrho(1+l) C_\alpha} (L_\varrho d_\alpha(\Phi_\varepsilon, \Phi_0) + |\varepsilon| L_1),$$

where $k(\alpha)$ is defined in (3.2). So

$$d_\alpha(\Phi_\varepsilon, \Phi_0) \leqslant \frac{\delta L_\varrho k(\alpha)}{\delta - L_\varrho(1+l) C_\alpha} d_\alpha(\Phi_\varepsilon, \Phi_0) + |\varepsilon| \frac{\delta L_1 k(\alpha)}{\delta - L_\varrho(1+l) C_\alpha}.$$

Thus

$$(\delta - L_\varrho(\delta k(\alpha) + (1+l) C_\alpha))\, d_\alpha(\Phi_\varepsilon, \Phi_0) \leqslant \delta L_1 k(\alpha) |\varepsilon|.$$

If ϱ is sufficiently small, then $L_\varrho(\delta k(\alpha) + (1+l) C_\alpha) < \delta$ and

$$d_\alpha(\Phi_\varepsilon, \Phi_0) \leqslant \frac{\delta L_1 k(\alpha)}{\delta - L_\varrho(\delta k(\alpha) + (1+l) C_\alpha)} |\varepsilon| = C|\varepsilon|.$$

6. - INTEGRAL MANIFOLD FOR A NONAUTONOMOUS PARABOLIC EQUATION

In a bounded domain $\Omega \in R^n$ consider the problem:

$$\partial_t u = \Delta u - f(u) + g(x) + \varepsilon(-f_1(u,t) + g_1(x,t)), \quad x \in \Omega, \ t > \tau, \tag{6.1}$$

$$u|_{\partial\Omega} = 0, \tag{6.2}$$

$$u|_{t=\tau} = u_\tau(x), \quad u_\tau \in L_2(\Omega). \tag{6.3}$$

We suppose that $g \in L_2(\Omega)$, $g_1 \in L_\infty(R, L_2(\Omega))$, i.e.

$$\|g\|, \quad \|g_1(\cdot, t)\| \leq C_0, \quad \forall t \in R. \tag{6.4}$$

Let the following inequalities be satisfied for $|\varepsilon| < \varepsilon_0$:

$$\frac{\partial}{\partial u}(f + \varepsilon f_1) \geq -C, \tag{6.5}$$

$$(f(u) + \varepsilon f_1(u,t))u \geq -C, \tag{6.6}$$

$$|f(u)|, |f_1(u)| \leq C_1(1 + |u|^q), \tag{6.7}$$

$$|f'(u)|, \left|\frac{\partial}{\partial u} f_1(u,t)\right| \leq C_2(1 + |u|^{q-1}), \tag{6.8}$$

$u \in R$, $t \in R$. Here $q = n/(n-2)$ when $n \geq 3$ and q is arbitrary for $n = 1, 2$. Assume that the following Hölder condition is satisfied

$$|f'(u_1) - f'(u_2)| \leq C_3(1 + |u_1| + |u_2|)^s |u_1 - u_2|^\gamma, \tag{6.9}$$

where $s + \gamma = q - 1 = 2/(n-2)$ when $n \geq 3$, $s \geq 0$, $\gamma > 0$.

It is well known that under these conditions the problem (6.1)-(6.3) possesses the unique solution $u(t,x) \in L_\infty((\tau, \tau + T), L_2(\Omega)) \cap L_2((\tau, \tau + T), H_0^1(\Omega))$. This fact is proved by the standard method of Galerkin approximations, using a priori estimates that will be given below in Theorem 8.1.

Let us show that the problem (6.1)-(6.3) satisfy the conditions formulated in § 1. The considered problem is a particular case of (1.1)-(1.2): the linear operator $-A_0$ in (1.1) is now the Laplace operator under the Dirichlet boundary conditions ($-A_0 u = \Delta u$, $u|_{\partial\Omega} = 0$); the nonlinear operators $R_0(u) = -f(u) + g$, $R_1(u,t) = -f_1(u,t) + g_1(x,t)$; the phase space $E = L_2(\Omega)$. We denote $\|u\|_0 = \|u\|$ is a norm in E.

Let z be an equilibrium point of (6.1)-(6.2) with $\varepsilon = 0$:

$$\Delta z - f(z) + g(x) = 0, \quad z \in H_0^1(\Omega) \cap H^2(\Omega).$$

We shall study behavior of the problem (6.1)-(6.3) solutions in a neighborhood of the point z. Denoting $v = u - z$ we obtain

$$\partial_t v = (\Delta - f'(z))v - (f(z + v) - f(z) - f'(z)v) + \varepsilon(-f_1(z + v, t) + g_1(x,t)).$$

This equation corresponds (1.4), where $Av = (-\Delta + f'(z))v$, $v|_{\partial\Omega} = 0$ is a linear self-adjoint operator; $B_0(v) = -(f(z+v) - f(z) - f'(z)v)$; $B_1(v,t) = -f_1(z+v,t) + g_1(x,t)$.

If we choose $a > C$ (C is a constant from (6.5)), then $((A + aI)v, v) = \|\nabla v\|^2 + ((f'(z) + a)v, v) \geq \|\nabla v\|^2$. Using (6.8) we also obtain that $((A + aI)v, v) \leq C(a)\|\nabla v\|^2$ for any $v \in H_0^1(\Omega)$. Whence if $a > C$ then the linear operator A possesses a compact resolvent and the degrees $(A + aI)^\alpha > 0$ are well defined. We shall construct an integral manifold in the space $E^\alpha = \mathcal{D}((A + aI)^\alpha)$ for $\alpha = 1/2$. In such a case $E^{1/2} = H_0^1(\Omega)$, $\|u\|_{1/2} = \|\nabla u\|$ is a norm in $H_0^1(\Omega)$.

To apply to the considered problem the theorems of § 1-§ 5 we have to check the conditions (1.6)-(1.11).

THEOREM 6.1: *If the conditions* (6.4), (6.7)-(6.9) *are satisfied then*

1) $\|B_1(v,t)\|_0 \leq L_1$ *for* $\|v\|_{1/2} \leq \varrho$, $t \in R$;

2) $\|B_1(v_1,t) - B_1(v_2,t)\|_0 \leq L_2\|v_1 - v_2\|_{1/2}$ *for* $\|v_1\|_{1/2}, \|v_2\|_{1/2} \leq \varrho$, $t \in R$;

3) $B_0(0) = 0$;

4) $\|B_0(v_1) - B_0(v_2)\|_0 \leq L_\varrho\|v_1 - v_2\|_{1/2}$ *for* $\|v_1\|_{1/2}, \|v_2\|_{1/2} \leq \varrho$; $L_\varrho \to 0$ *as* $\varrho \to 0$.

PROOF: In this proof we often use the fact that according to the Sobolev embedding theorems we have $H_0^1(\Omega) \subset L_{2q}(\Omega)$ when $q = n/(n-2)$ $(n \geq 3)$.

1) $$\|B_1(v,t)\|_0 \leq \|f_1(z+v,t)\|_0 + \|g_1(x,t)\|_0 \leq$$

$$\leq C_1\left[\int_\Omega (1 + |z(x) + v(x)|^q)^2 dx\right]^{\frac{1}{2}} + C_0 \leq C_4(1 + \|z + v\|^q_{L_{2q}(\Omega)}) \leq$$

$$\leq C_5(1 + \|\nabla z\|^q + \|\nabla v\|^q) \leq L_1 \qquad \text{for } \|\nabla v\| \leq \varrho .$$

2) $$\|B_1(v_1,t) - B_1(v_2,t)\|_0^2 = \int_\Omega (f_1(z+v_1,t) - f_1(z+v_2,t))^2 dx$$

$$= \int_\Omega \left[\int_0^1 (f'_{1u}(z + v_1 + \theta(v_2 - v_1), t)\, d\theta\right]^2 (v_1 - v_2)^2 dx \leq$$

$$\leq C_2^2 \int_\Omega (1 + (|z| + |v_1| + |v_2|)^{q-1})^2 (v_1 - v_2)^2 dx .$$

Using the Hölder inequality with the exponents $q_1 = n/2$, $q_2 = q = n/(n-2)$, we obtain

$$\|B_1(v_1, t) - B_1(v_2, t)\|_0^2 \leq$$

$$\leq C_6 \left(\int_\Omega (1 + (|z| + |v_1| + |v_2|)^{2(q-1)q_1} dx \right)^{1/q_1} \left(\int_\Omega |v_1 - v_2|^{2q_2} dx \right)^{1/q_2}.$$

As $2(q-1)q_1 = 2(n/(n-2) - 1)(n/2) = 2n/(n-2) = 2q$ we have

$$\|B_1(v_1, t) - B_1(v_2, t)\|_0^2 \leq C_7 (1 + \|z\|_{L_{2q}} + \|v_1\|_{L_{2q}} + \|v_2\|_{L_{2q}})^{2(q-1)} \|v_1 - v_2\|_{L_{2q}}^2 \leq$$

$$\leq C_8 (1 + \|\nabla z\| + \|\nabla v_1\| + \|\nabla v_2\|)^{2(q-1)} \|\nabla (v_1 - v_2)\|^2 \leq$$

$$\leq C_8 (1 + \|\nabla z\| + 2\varrho)^{2(q-1)} \|\nabla (v_1 - v_2)\|^2 .$$

3) It is evident that $B_0(0) = 0$.

4) $$\|B_0(v_1) - B_0(v_2)\|_0^2 = \int_\Omega [f(z + v_1) - f(z + v_2) - f'(z)(v_1 - v_2)]^2 dx =$$

$$= \int_\Omega \left[\int_0^1 f'(z + (1-\theta) v_1 + \theta v_2) d\theta (v_1 - v_2) - \int_0^1 f'(z) d\theta (v_1 - v_2) \right]^2 dx$$

$$\leq C_3^2 \int_\Omega (v_1 - v_2)^2 (1 + 2|z| + |v_1| + |v_2|)^{2s} (|v_1| + |v_2|)^{2\gamma} dx .$$

Let us use the Hölder inequality with the exponents $q_1 = q = n/(n-2)$, $q_2 = n/s(n-2)$, $q_3 = n/\gamma(n-2)$; then $1/q_1 + 1/q_2 + 1/q_3 = ((n-2)/n)(1 + s + \gamma) = 1$. Consequently

$$\|B_0(v_1) - B_0(v_2)\|_0^2 \leq$$

$$\leq C_9 \left[\int_\Omega |v_1 - v_2|^{2q} dx \right]^{1/q} \left[\int_\Omega (1 + 2|z| + |v_1| + |v_2|)^{2q} dx \right]^{s/q} \left[\int_\Omega (|v_1| + |v_2|)^{2q} dx \right]^{\gamma/q} \leq$$

$$\leq C_{10} (1 + 2\|\nabla z\| + 2\varrho)^{2s} (2\varrho)^{2\gamma} \|\nabla (v_1 - v_2)\|^2$$

when $\|v_1\|_{L_{2q}} \leq c\|\nabla v_1\| \leq c\varrho$, $\|v_2\|_{L_{2q}} \leq c\|\nabla v_2\| \leq c\varrho$.

Thus we obtain the required statement with $L_\varrho = C_{11} \varrho^\gamma$, $L_\varrho \to 0$ for $\varrho \to 0$. ∎

Consequently all theorems proved in the previous sections can be applied to the problem (6.1)-(6.3).

7. - The uniform global exponential approximation

Let $U_\varepsilon(t,\tau)\colon E \to E$ be a process, corresponding to the problem:

$$\frac{du}{dt} + A_0 u = R_0(u) + \varepsilon R_1(u,t), \tag{7.1}$$

$$u|_{t=\tau} = u_\tau. \tag{7.2}$$

$U_\varepsilon(t,\tau)u_\tau = u(t)$, $u(t)$ is a solution of (7.1), (7.2); $u_\tau \in E$, $u(t) \in E$, $\forall t \geq \tau$. Notice that we sometimes write $U(t,\tau)$ instead of $U_\varepsilon(t,\tau)$.

When $\varepsilon = 0$ the equation (7.1) is autonomous:

$$\frac{du}{dt} + A_0 u = R_0(u), \tag{7.3}$$

and the process $U(t,\tau)$ becomes a semigroup $U_0(t,\tau) = S_{t-\tau}$.

We assume that the semigroup $\{S_t\}$, corresponding to (7.3), possesses only a finite number of equilibrium points $\mathfrak{N} = \{z_1, \ldots, z_N\}$, all these points z_i are hyperbolic and the spaces E^α ($\alpha \leq 1/2$) constructed in a neighborhood of every equilibrium point z_i (see § 1) do not depend on z_i. As it was shown in §6 for the parabolic equation (problem (6.1)-(6.3)) these spaces coincide with $H_0^1(\Omega)$.

Moreover we assume that under $|\varepsilon| < \varepsilon_0$ all conditions of the theorems proved in § 1-§ 5 are satisfied in a ϱ_0-neighborhood of every point z_i. Thus when $\lambda_i \geq 0$ does not belong to the spectrum of the operator $A_i = A_0 - DR_0(z_i)$ ($i = 1, \ldots, N$), the results obtained in § 1-§ 2 imply the existence of integral manifolds $M_i(\lambda_i) \equiv M(\varepsilon, z_i, \lambda_i)$. Furthermore there exist local exponential approximations lying on $M_i(\lambda_i)$ of the solutions of the problem (6.1)-(6.2) (see § 3).

Let us prolong $M_i(\lambda_i)$ down by the integral curves as $t \to +\infty$. Precisely we define the sets $M_i^+(\lambda_i) \subset E \times R_t$, $M_i^+(\lambda_i) \supset M_i(\lambda_i)$ in such a way: $(u,t) \in M_i^+(\lambda_i)$, if there exist $\tau \leq t$ and $u_\tau \in E$ such that $u = U(t,\tau)u_\tau$ and $(u_\tau, \tau) \in M_i(\lambda_i)$. The set $M_i^+(\lambda_i)$ is semiinvariant in the following sense: for all $t_1 \leq t_2$ we have $U(t_2,t_1)(M_i^+(\lambda_i) \cap \{t = t_1\}) \subset M_i^+(\lambda_i) \cap \{t = t_2\}$.

Thus if $(u_\tau, \tau) \in M_i^+(\lambda_i)$, then $(u(t), t) = (U(t,\tau)u_\tau, t) \in M_i^+(\lambda_i)$ for all $t \geq \tau$. We denote $\mathfrak{X}_i = \{u(\cdot) \mid (u(t), t) \in M_i^+(\lambda_i), u(t) = U(t,\tau)u_\tau\}$ is a set of the solutions $u(t)$ of (7.1)-(7.2) such that $(u(t), t) \in M_i^+(\lambda_i)$.

In the present section we construct a global exponential approximation $\tilde{u}(t)$ for any solution $u(t)$ of the problem (7.1)-(7.2). It consists of the local approximations constructed in § 3 and their prolongations as t increases. We present the strict definition of $\tilde{u}(t)$ below. Construction of $\tilde{u}(t)$ for autonomous equations one can find in [1].

First of all we shall prove that under some conditions any trajectory of the process $U(t,\tau)$ cannot twice approach to the same point $z = z_i \in \mathfrak{N}$.

For $|\varepsilon| < \varepsilon_0$ consider $M_\varepsilon(0) = M(\varepsilon, z, 0)$—an integral manifold (local), corresponding to the split of the spectrum of $A = A_0 - DR_0(z)$ onto positive ($\sigma_+(A)$) and negative ($\sigma_-(A)$) eigenvalues (remind that $0 \notin \sigma(A)$ since the equilibrium point z is hyperbolic). Due to results above, $M_\varepsilon(0) \subset V_{\varrho_0}(z) \times \boldsymbol{R}_t$, where $V_{\varrho_0}(z) = \{u \in E^\alpha \mid \|P(u-z)\|_\alpha \leq \varrho_0/2, \|Q(u-z)\|_\alpha \leq \varrho/2\}$, $V_{\varrho_0}(z) \subset O_{\varrho_0}(z)$, $O_{\varrho_0}(z) = \{u \in E^\alpha \mid \|u-z\|_\alpha < \varrho_0\}$; P and Q are orthoprojectors onto invariant subspaces of the operator A corresponding to $\sigma_-(A)$ and $\sigma_+(A)$ respectively. As earlier we denote $p = P(u-z)$, $q = Q(u-z)$.

Remind that for $|\varepsilon| < \varepsilon_0$ the integral manifold $M_\varepsilon(0)$ in $V_{\varrho_0}(z)$ is a graph of a function $q = \Phi_\varepsilon(p, t)$, $\|p\|_\alpha \leq \varrho_0/2$, $t \in \boldsymbol{R}$. The functions $\Phi_\varepsilon(p, t)$, $|\varepsilon| < \varepsilon_0$, uniformly with respect to ε satisfy the Lipschitz condition:

$$\|\Phi_\varepsilon(p_1, t) - \Phi_\varepsilon(p_2, t)\| \leq l\|p_1 - p_2\|_\alpha . \tag{7.4}$$

For $\varepsilon = 0$ the manifold $M_0(0)$ does not depend on t and $M_0(0) = M^+(z) \times \boldsymbol{R}$, where $M^+(z) = \{u \mid q = \Phi_0(p), \|p\|_\alpha \leq \varrho_0/2\}$ is the unstable manifold, passing through the hyperbolic equilibrium point z of the semigroup $\{S_t\}$, corresponding to the autonomous equation (7.3). This unstable manifold $M^+(z)$ consists of such solutions of (7.3) that approach z as $t \to -\infty$ (see [1]). Since $z \in M^+(z)$, then $\Phi_0(0) = 0$ and from (7.4) it follows

$$\|\Phi_0(p)\|_\alpha \leq l\|p\|_\alpha . \tag{7.5}$$

Theorem 5.1 states for $\|p\|_\alpha \leq \varrho_0/2$, $t \in \boldsymbol{R}$:

$$\|\Phi_\varepsilon(p, t) - \Phi_0(p)\|_\alpha \leq C|\varepsilon| . \tag{7.6}$$

Assume there exists a bounded, uniformly with respect to $\tau \in \boldsymbol{R}$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ absorbing set $K_0 \subset E^\alpha$ for the process $U(t, \tau) = U_\varepsilon(t, \tau)$. It means that for any bounded set $K \subset E$ there exists $T(K)$ such that $U_\varepsilon(\tau + t, \tau)K \subset K_0$, $\forall t \geq T(K)$, $\tau \in \boldsymbol{R}$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

Every trajectory $u(t) = U_\varepsilon(t, \tau)u_\tau$ of the process $U_\varepsilon(t, \tau)$ lies in K_0 after a certain period of time that depends only on the norm $\|u_\tau\|$ of the initial data. Below we consider only trajectories $u(t) \in K_0$, $\forall t \geq \tau$.

Definition 7.1: A continuous functional F: $E^\alpha \to \boldsymbol{R}$ is called a global Lyapunov function of the semigroup $\{S_t\}$ if

1) $\forall u \in E^\alpha$, $\forall t_1 \geq t_2 \geq 0$, $F(S_{t_1} u) \leq F(S_{t_2} u)$.

2) If $F(S_{t_1} u) = F(S_{t_2} u)$ for any $t_1 \neq t_2$, then $S_t u = u$ for all $t \geq 0$, i.e. u is an equilibrium point of the semigroup $\{S_t\}$.

Definition 7.2: F is called a relative Lyapunov function in the set K_1, $K_1 \subset E^\alpha$, of the process $U(t, \tau)$ if for any $t_2 \geq t_1$ and any $u(t) = U(t, t_1)u(t_1)$ such that $u(t) \in K_1$ for $t \in [t_1, t_2]$ it follows that $F(u(t_2)) \leq F(u(t_1))$.

Definition 7.3: $T > 0$ is called a time of arrival of the process $U(t, \tau)$ (uniformly with respect to $\tau \in \boldsymbol{R}$) from the bounded set K, $K \subset E$, to the set V, $V \subset E^\alpha$, if $\forall \tau \in \boldsymbol{R}$, $\forall u_\tau \in K$ $\exists t \in [\tau, \tau + T]$: $U(t, \tau)u_\tau \in V$.

LEMMA 7.1: *Let V be a set in E^α and K_0, $K_0 \subset E^\alpha$, be a bounded absorbing set for the process $\{U(t,\tau)\}$. Let $F(u)$ be a bounded function on K_0. Suppose there exists $\nu > 0$ such that*

$$F(u(t_2)) - F(u(t_1)) \leqslant -\nu(t_2 - t_1)$$

for any $t_2 \geqslant t_1$ and for any trajectory $u(t) = U(t, t_1)u(t_1)$ in $K_0 \backslash V$, i.e. $u(t) \in K_0 \backslash V$ for $t \in [t_1, t_2]$.

Then the time T of arrival from the set K_0 to V is finite.

PROOF: As K_0 is an absorbing set, there exists $T(K_0)$ such that $U(t,\tau)K_0 \subset K_0$ for all $t \geqslant \tau + T(K_0)$. We denote $T_0 = (1/\nu)(\sup_{u \in K_0} F(u) - \inf_{u \in K_0} F(u) + 1)$ and put $t_1 = \tau + T(K_0)$, $t_2 = t_1 + T_0$. For all $u_\tau \in K_0$ we have: $u(t) = U(t,\tau)u_\tau \in K_0$, $\forall t \in [t_1, t_2]$.

Suppose $u(t) \notin V$ for $t \in [t_1, t_2]$. Then $u(t) = U(t, t_1)u(t_1) \in K_0 \backslash V$ for $t \in [t_1, t_2]$, and due to the hypothesis of the lemma:

$$F(u(t_2)) - F(u(t_1)) \leqslant -\nu T_0 = \inf_{u \in K_0} F(u) - \sup_{u \in K_0} F(u) - 1 .$$

This inequality contradicts $F(u(t_2)) \geqslant \inf_{u \in K_0} F(u)$, $F(u(t_1)) \leqslant \sup_{u \in K_0} F(u)$. So $u(t) \in V$ for some $t \in [t_1, t_2]$ and the time of arrival of the process $U(t,\tau)$ from K_0 to V is not more than $t_2 - \tau = T(K_0) + T_0$. ∎

THEOREM 7.1: *Let the following properties hold:*

1) *the semigroup $\{S_t\}$, corresponding to (7.3) possesses a global Lyapunov function $F(u)$;*

2) *the set $\mathfrak{N}$ of the equilibrium points of the semigroup S_t is finite: $\mathfrak{N} = \{z_1, \ldots, z_N\}$ and all the points z_i are hyperbolic;*

3) *for any $\varrho > 0$ there exists $\varepsilon^*(\varrho) > 0$ such that under $|\varepsilon| < \varepsilon^*$ the function $F(u)$ is a relative Lyapunov function in the set $K_0 \backslash O_\varrho(\mathfrak{N})$ $(O_\varrho(\mathfrak{N}) = \bigcup_{i=1}^{N} O_\varrho(z_i))$ of the process $U_\varepsilon(t,\tau)$, corresponding to (7.1)-(7.2);*

4) *$U_\varepsilon(t,\tau)u_\tau$ is a continuous in E^α with respect to t function for all $u_\tau \in K_0$.*

Then for any fixed and sufficiently small $\varrho_0 > 0$ there exist ϱ_1, $0 < \varrho_1 < \varrho_0$, and ε_1, $0 < \varepsilon_1 < \varepsilon_0$, such that for any $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$ and any trajectory $u(t) = U_\varepsilon(t,\tau)u_\tau \in K_0$ $\forall t \geqslant \tau$ the following statement holds: if $u(t_1) \in V_{\varrho_1}(z) = \{u \,|\, \|P(u-z)\|_\alpha \leqslant \varrho_1/2, \|Q(u-z)\|_\alpha \leqslant \varrho_1/2\}$ and $u(t_2) \notin V_{\varrho_0}(z)$ for $t_2 > t_1 \geqslant \tau$, then $u(t) \notin V_{\varrho_1}(z)$ for all $t > t_2$.

REMARK 7.1: The statement of Theorem 7.1 can be reformulated in such a way: for $\varrho_0 > 0$ there exists $\varrho_1 < \varrho_0$ such that if $u(t_1) \in V_{\varrho_1}(z)$, $u(t_2) \in V_{\varrho_1}(z)$ for some $t_2 > t_1$, then $u(t) \in V_{\varrho_0}(z)$ for all $t \in [t_1, t_2]$.

Proof of Theorem 7.1: For simplicity we suppose $F(z_i) \neq F(z_j)$ for $i \neq j$. Proof of the case $F(z_i) = F(z_j)$ is similar (see [1]). Let $F(z_1) > F(z_2) > \ldots > F(z_N)$. If ϱ_0 is sufficiently small we get $F(V_{\varrho_0}(z_1)) > \ldots > F(V_{\varrho_0}(z_N))$.

Let $|\varepsilon| \leq \varepsilon^*(\varrho_0/2)$, where ε^* is defined above. Then $F(u)$ is a relative Lyapunov function in the set $K_0 \setminus V$, $V = \bigcup_{i=1}^{N} V_{\varrho_0}(z_i)$ for the process $U_\varepsilon(t,\tau)$. If any solution $u(t) = U_\varepsilon(t,\tau)u(\tau) \in K_0$ leaves $V_{\varrho_0}(z_i)$, then later $u(t)$ does not enter closed neighborhoods $V_{\varrho_0}(z_1), \ldots, V_{\varrho_0}(z_{i-1})$.

Now we shall study in more details the behavior of $u(t)$ in $V_{\varrho_0}(z) \equiv V_{\varrho_0}(z_i)$. We shall show that if $u(t)$ passes through the small neighborhood $V_{\varrho_1}(z)$ and later leaves the bigger neighborhood $V_{\varrho_0}(z)$ at a time moment t_2, then $F(u(t_2))$ is less than values of F in $V_{\varrho_1}(z)$.

As $F(u)$ is a global Lyapunov function of the semigroup $\{S_t\}$ and the unstable manifold $M^+(z)$ consists of trajectories of this semigroup that approach z as $t \to -\infty$, then $F(u) < F(z)$ for $u \in \partial M^+ \equiv M^+(z) \cap \{u \| \|P(u-z)\|_\alpha = \varrho_0/2\}$. The set ∂M^+ is finite dimensional, closed and bounded. Thus there exists $\mu > 0$ such that

$$F(u)|_{u \in \partial M^+} < F(z) - 3\mu\,.$$

The function $F(u)$ is continuous, so there is a θ-neighborhood in E^α of the compact set ∂M^+ which we denote by $O_\theta(\partial M^+)$ such that

$$F(u) < F(z) - 2\mu \quad \text{for } u \in O_\theta(\partial M^+)\,. \tag{7.7}$$

Besides there exists $\varrho_1 > 0$ such that

$$F(u) > F(z) - \mu \quad \text{for } u \in V_{\varrho_1}(z)\,. \tag{7.8}$$

If $\varepsilon_1 \leq \varepsilon^*(\varrho_2)$, $\varrho_2 = \varrho_1/2$ then for $|\varepsilon| < \varepsilon_1$ we get that $F(u)$ is a relative Lyapunov function in the set $K_0 \setminus V$, $V = \bigcup_{i=1}^{N} V_{\varrho_1}(z_i) \supset \bigcup_{i=1}^{N} O_{\varrho_2}(z_i)$ of the process $U_\varepsilon(t,\tau)$.

Let $u(t) = U_\varepsilon(t,\tau)u(\tau) \in K_0$, $t \geq \tau$; $u(t_1) \in V_{\varrho_2}(z)$ and t_2 is a time moment when the trajectory $u(t)$ leaves $V_{\varrho_0}(z)$, thus $u(t) \in V_{\varrho_0}(z)$ for $t \in [t_1, t_2]$. From Theorem 3.3, (7.5) and (7.6) we obtain for $t \in [t_1, t_2]$:

$$\|q(t) - \Phi_0(p(t))\|_\alpha \leq \|q(t) - \Phi_\varepsilon(p(t),t)\|_\alpha + \|\Phi_\varepsilon(p(t),t) - \Phi_0(p(t))\|_\alpha \leq$$

$$\leq C_0\|q(t_1) - \Phi_\varepsilon(p(t_1),t_1)\|_\alpha + C|\varepsilon| \leq$$

$$\leq C_0(\|q(t_1)\|_\alpha + \|\Phi_0(p(t_1))\|_\alpha + \|\Phi_\varepsilon(p(t_1),t_1) - \Phi_0(p(t_1))\|_\alpha) + C|\varepsilon| \leq$$

$$\leq C_0(\|q(t_1)\|_\alpha + l\|p(t_1)\|_\alpha + C|\varepsilon|) + C|\varepsilon|\,.$$

As $u(t_1) = z + p(t_1) + q(t_1) \in V_{\varrho_2}(z)$, i.e. $\|p(t_1)\|_\alpha \leq \varrho_1/2$, $\|q(t_1)\|_\alpha \leq \varrho_1/2$, we get

$$\|q(t) - \Phi_0(p(t))\|_\alpha \leq C_1\varrho_1 + C_2|\varepsilon|\,,\ t \in [t_1, t_2]\,. \tag{7.9}$$

In particular if $u(t) \in V_{\varrho_0}(z)$ then

$$\|q(t)\|_\alpha \leq l\|p(t)\|_\alpha + C_1\varrho_1 + C_2|\varepsilon| \leq l\varrho_0/2 + C_1\varrho_1 + C_2|\varepsilon| . \tag{7.10}$$

If the integral manifold M_0 is constructed under $l < 1$ (it is possible if ϱ_0 is small enough) then we choose ϱ_1 and ε_1 to have (7.8) and also

$$C_1\varrho_1 + C_2\varepsilon_1 < \theta , \tag{7.11}$$

$$l\varrho_0/2 + C_1\varrho_1 + C_2\varepsilon_1 < \varrho_0/2 , \tag{7.12}$$

$$\varepsilon_1 < \varepsilon^*(\varrho_1/2). \tag{7.13}$$

Then under $|\varepsilon| < \varepsilon_1$ due to (7.10), (7.12) the solution $u(t) = U_\varepsilon(t, t_1)u(t_1)$ satisfies $\|q(t)\|_\alpha = \|Q(u(t) - z)\|_\alpha < \varrho_0/2$ for $t \in [t_1, t_2]$. So the continuous in E^α trajectory $u(t)$ leaves $V_{\varrho_0}(z)$ through the cylinder $\{u \,|\, \|P(u - z)\|_\alpha = \varrho_0/2\}$, i.e. $\|p(t_2)\|_\alpha = \|P(u(t_2) - z)\|_\alpha = \varrho_0/2$.

From (7.9) and (7.11) we get $\|q(t_2) - \Phi_0(p(t_2))\|_\alpha < \theta$, so $u(t_2) = z + p(t_2) + q(t_2) \in O_\theta(\partial M^+)$ and $F(u(t_2)) < F(z) - 2\mu$.

Therefore the value $F(u(t_2))$ of the relative Lyapunov function F on $u(t)$ in a time moment t_2 (when $u(t)$ leaves $V_{\varrho_0}(z)$) is not more than $F(z) - 2\mu$ since (7.7). From (7.13) it follows that $F(u(t))$ does not increase outside the set $V = \bigcup_{i=1}^{N} V_{\varrho_1}(z_i)$. Consequently $u(t)$ cannot return into $V_{\varrho_1}(z)$ because the value $F(u(t))$ since (7.8) has to be more than $F(z) - \mu$ when $u(t)$ enters $V_{\varrho_1}(z)$.

We denote the sets constructed in Theorem 7.1 by $V_{\varrho_0}(z_i) \equiv V_i$ and $V_{\varrho_1}(z_i) \equiv W_i$. Then $W_i \subset V_i$, $V_i \cap V_j = \emptyset$ for $i \neq j$. As it is proved in this Theorem the following statement takes place:

i) if $u(t) = U(t, \tau)u_\tau \in K_0$ for all $t \geq \tau$ and $u(t_1) \in W_i$, $u(t_2) \notin V_i$ for some $t_2 > t_1 \geq \tau$, then $u(t) \notin W_i$ for all $t > t_2$.

This property means that every solution $u(t)$ of the nonautonomous equation (7.1) can not (in some sense) twice pass near any equilibrium point z_i of the autonomous equation (7.1).

Besides the condition i) we assume also that the following conditions are satisfied:

ii) (finiteness of a time of arrival from K_0 to $\bigcup_{i=1}^{N} W_i$)

$$\exists T > 0 \;\; \forall \tau \in R \;\; \forall u_\tau \in K_0 \;\; \exists\, t \in (0, T)\colon \; U(\tau + t, \tau)u_\tau \in \bigcup_{i=1}^{N} W_i .$$

iii) (exponential rate of divergence of solutions in E^β for some $\beta \in [0, \alpha]$)

$$\exists a > 0, \;\; C > 0\colon \; \|U(t, \tau)u_1 - U(t, \tau)u_2\|_\beta \leq C\|u_1 - u_2\|_\beta e^{a(t-\tau)}$$

for $t \geq \tau$, for all $\tau \in R$, $u_1, u_2 \in E^\alpha$ such that $U(t, \tau)u_i \in K_0$ for all $t \geq \tau$, $i = 1, 2$.

Let us note that some sufficient conditions for i) and ii) are given respectively in Theorem 7.1 and Lemma 7.1. These conditions claim the existence of a relative Lyapunov function F in $K_0 \setminus \bigcup_{i=1}^{N} W_i$.

Finally from Theorem 3.1 and Theorem 3.2 it follows that for any trajectory $u(t)$ passing through the neighborhood V_i we can define its exponential approximation (trace) $\tilde{u}_i(t)$, lying on the integral manifold $M_i^*(\lambda_i)$, i.e. $\tilde{u}(\cdot) \in \mathcal{K}_i$. Precisely

iv) there exist $\gamma_i > \lambda_i \geq 0$, $C_i > 0$ such that if $u(t) \in V_i$ for $t \in (\tau, T)$ $(T \leq +\infty)$ then there exists $\tilde{u}_i(\cdot) \in \mathcal{K}_i$:

$$\|u(t) - \tilde{u}_i(t)\|_\beta \leq C_i e^{-\gamma_i(t-\tau)} \quad \text{for } t \in (\tau, T) \ (\beta \leq \alpha).$$

Definition 7.4: A combined trajectory of the process $U(t, \tau)$, corresponding to the problem (7.1)-(7.2) is a piecewise continuous function $\tilde{u}(t)$, $t \in [\tau, +\infty)$, with values in E if there exist time moments $\tau = t_0^0 < t_1^0 < \ldots < t_{m+1}^0 = +\infty$ such that $\tilde{u}(t) = \tilde{u}_i(t)$ for $t \in [t_i^0, t_{i+1}^0)$, where $\tilde{u}_i(t) = U(t, t_i^0)\tilde{u}_i(t_i^0)$ is a trajectory of $U(t, \tau)$ lying on the integral manifold $M_i^*(\lambda_i)$, i.e. $\tilde{u}_i(\cdot) \in \mathcal{K}_i$ (the set $\mathcal{K}_i = \{\tilde{u}_i(\cdot)\}$ is defined above).

Theorem 7.2: *Let the process $U(t, \tau)u_\tau$ possesses a bounded in E^α absorbing set K_0 and the conditions i)-iv) hold. Then for any solution $u(t) = U(t, \tau)u_\tau \in K_0$ for $t \geq \tau$, there exists a combined trajectory $\tilde{u}(t)$ such that*

$$\|u(t) - \tilde{u}(t)\|_\beta \leq C^i e^{-\eta_i(t-\tau)} \qquad (i = 0, 1, \ldots, m, m \leq N) \tag{7.14}$$

for $t \in [t_i^-, t_{i+1}^-)$, where $\eta_i = \gamma_i \eta_{i-1} / (\gamma_i + \eta_{i-1} + a) > 0$, $t_i^- \in [t_{i-1}^0, t_i^0]$, $t_0^- = \tau$. Moreover $\eta_0 > 0$ can be arbitrary and C^i do not depend on $u(t)$.

The construction of the combined trajectory is described in [1]. This construction is based only on the conditions i)-iv) and holds without any changes for the process $U(t, \tau)$.

Remark 7.2: We can get estimation (7.14) for every solution $u(t) = U(t, \tau)u_\tau$ of (7.1)-(7.2) uniformly with respect to $u_\tau \in K$, where K is any bounded set in E, because $u(t) \in K_0$ for $t \geq \tau_1 = \tau + T(K)$, and $T(K)$ influences only on the values of the constants C^i.

Remark 7.3: In the neighborhood of every point z_i we can construct the integral manifolds $M(\varepsilon, z_i, \lambda_i)$ for arbitrary large $\lambda_i > 0$. If we put λ_i and η_0 large, then γ_i are also large (as $\gamma_i > \lambda_i$) and we can construct a combined trajectory $(\tilde{u}(t), t) \in \bigcup_{i=1}^{N} M^*(\varepsilon, z_i, \lambda_i)$ such that

$$\|u(t) - \tilde{u}(t)\|_\beta \leq C e^{-\eta(t-\tau)}, \qquad t > \tau,$$

where $\eta = \eta_m = \min(\eta_1, \ldots, \eta_m) > 0$ is arbitrarily large. In this case dimensions of the finite dimensional manifolds $M(\varepsilon, z_i, \lambda_i)$ increase.

8. - The uniform global approximation for the parabolic equation

Consider the problem (6.1)-(6.3). We assume (6.4)-(6.9) and also

$$g'_{1t} \in L_\infty(R, L_2(\Omega)) \quad |f'_{1t}(u,t)| \leq C(1 + |u|^q) \tag{8.1}$$

where $q = n/(n-2)$ is the same as in (6.7)-(6.8).

In this section for any solution $u(t)$ of the problem (6.1)-(6.3) we shall construct a combined trajectory $\tilde{u}(t)$, $(\tilde{u}(t), t) \in \bigcup_{i=1}^{N} M_i^u(\lambda_i)$, that exponentially attracts $u(t)$ in the metric of $E^{1/2} = H_0^1(\Omega)$, $\Omega \subset R^n$. It will be done under some additional conditions on f, f_1 in the case when $n \leq 4$. For arbitrary $n \in N$ we shall prove the attraction only in the metric of $E = L_2(\Omega)$.

At first let us state some properties of the process $U(t, \tau)$, corresponding to the problem (6.1)-(6.3).

Theorem 8.1: *Let the conditions (6.4)-(6.9),(8.1) hold. Then the process $U(t,\tau)$, corresponding to (6.1)-(6.3) satisfies the following properties:*

1) *$U(t,\tau)$ is a continuous mapping from $L_2(\Omega)$ to $L_2(\Omega)$.*

2) *$U(t,\tau)$ possesses a uniformly with respect to τ absorbing set K bounded in $L_2(\Omega)$.*

3) *$U(\tau+1,\tau)$ is $(L_2(\Omega), H_0^1(\Omega))$-bounded uniformly with respect to τ.*

4) *$U(\tau+1,\tau)$ is $(H_0^1(\Omega), H^2(\Omega))$-bounded uniformly with respect to τ.*

5) *$U(t,\tau)$ is a continuous mapping from $H_0^1(\Omega)$ to $H_0^1(\Omega)$, $t \geq \tau$.*

6) *$U(t,\tau)$ possesses a uniformly with respect to τ absorbing set K_0 bounded in $H^2(\Omega)$.*

7) *$U(t,\tau)u_\tau$ is a continuous in $H_0^1(\Omega)$ with respect to t, $t > \tau$, function for all $u_\tau \in K_0$.*

Proof: The following formal computations can easily be justified using Galerkin approximations (see [1]).

1) Let u_1 and u_2 be two solutions of (6.1)-(6.2). Then $v = u_1 - u_2$ satisfies the equation

$$\partial_t v = \Delta v - (f(u_1) + \varepsilon f_1(u_1, t)) + (f(u_2) + \varepsilon f_1(u_2, t)). \tag{8.2}$$

Multiplying this equation by v and using (6.5) we obtain

$$1/2\partial_t\|v\|^2 + \|\nabla v\|^2 \leq a\|v\|^2, \qquad \|v(t)\| \leq \|v(\tau)\|e^{a(t-\tau)}. \tag{8.3}$$

This implies the statement of item 1). Let us note that (8.3) implies also the uniqueness of the solution of the problem (6.1)-(6.3).

2) Multiplying (6.1) by u and integrating with respect to t, taking into account (6.4), (6.6), we deduce

$$\|u(t)\|^2 + \gamma\int_\tau^t \|\nabla u(\theta)\|^2 d\theta \leq \|u(\tau)\|^2 + C_1(t-\tau). \tag{8.4}$$

Hence $\|u(t)\|^2 \leq \|u(\tau)\|^2 e^{-\gamma_1(t-\tau)} + C_1/\gamma_1$. Therefore $K = \{\|u\|^2 \leq 2C_1/\gamma_1\}$ is an absorbing set.

3) Multiplying (6.1) by $-(t-\tau)\Delta u$ and integrating with respect to t, from τ to $\tau+1$, using (6.4), (6.5), we obtain

$$\frac{1}{2}\|\nabla u(\tau+1)\|^2 \leq C_2\left(1 + \int_\tau^{\tau+1}\|\nabla u(t)\|^2 dt\right).$$

Estimating the integral by (8.4) we get

$$\|\nabla u(\tau+1)\| \leq C_3(\|u(\tau)\|).$$

Thus $U(\tau+1,\tau)$ is $(L_2(\Omega), H_0^1(\Omega))$-bounded uniformly with respect to τ.

4) Multiplying (6.1) by $-\Delta u$ similarly to the previous item we obtain for $t \in [\tau, \tau+1]$

$$\|\nabla u(t)\|^2 + \int_\tau^t \|\Delta u\|^2 d\theta \leq C_4(\|\nabla u(\tau)\|). \tag{8.5}$$

Differentiating (6.1) in t and denoting $v = \partial_t u$ we have

$$\partial_t v = \Delta v - f'(u)v - \varepsilon f'_{1u}(u,t)v - \varepsilon f'_{1t}(u,t) + \varepsilon g'_{1t}(x,t).$$

Multiplying this equation by $(t-\tau)v$, using (6.5), (8.1) and integrating in t from τ to $\tau+1$ we deduce

$$\frac{1}{2}\|v(\tau+1)\|^2 \leq C_5\int_\tau^{\tau+1}(1 + \|v(\theta)\|^2 + \|\nabla u(\theta)\|^2)\,d\theta. \tag{8.6}$$

It follows from (8.4) that $\int_\tau^{\tau+1}\|\nabla u\|^2 d\theta$ is bounded. To estimate the integral of $\|v\|^2$

we use the initial equation (6.1):

$$\|v\| = \|\partial_t u\| \leq \|\Delta u\| + \|f(u) + \varepsilon f_1(u,t)\| + C_0 .$$

We get from (6.7) and (8.5):

$$\|f(u(t)) + \varepsilon f_1(u(t),t)\| \leq C_6(1 + \|u(t)\|^{r}_{L_{2r}}) \leq C_7(1 + \|\nabla u(t)\|^{r}) \leq C_8(\|\nabla u(\tau)\|) .$$

for $t \in [\tau, \tau+1]$. $\int_{\tau}^{\tau+1} \|\Delta u\|^2 d\theta$ is bounded since (8.5). Using these estimations we conclude from (8.6) that

$$\|v(\tau+1)\|^2 \leq C_9(\|\nabla u(\tau)\|) .$$

Rewriting (6.1) as

$$\Delta u = v + f(u) + \varepsilon f_1(u,t) - g(x) - \varepsilon g(x,t), \qquad v = \partial_t u$$

we note that all the terms in the right-hand side for $t = \tau + 1$ are bounded in $L_2(\Omega)$ if only $u(\tau)$ is bounded in $H_0^1(\Omega)$. Consequently, $\|\Delta u(\tau+1)\| \leq C_8(\|\nabla u(\tau)\|)$, that proves the item 4).

5) Consider the sequence $u_n(\tau) \to u(\tau)$ in $H_0^1(\Omega)$ as $n \to \infty$ and the sequence $u_n(t) = U(t,\tau)u_n(\tau)$. Since $U(t,\tau)$ is continuous in $L_2(\Omega)$ we have $u_n(t) \to u(t) = U(t,\tau)u(\tau)$ in $L_2(\Omega)$.

From the item 4) we deduce that $u_n(t)$ is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$ for $t > \tau$. Then any subsequence of $u_n(t)$ has a convergent in $H_0^1(\Omega)$ subsequence, and its limit coincides with $u(t)$ as an element of $L_2(\Omega)$. It means that $u_n(t) \to u(t)$ in $H_0^1(\Omega)$.

6) In the item 2) the uniformly with respect to τ absorbing set K is constructed. This set is bounded in $L_2(\Omega)$. It follows from the item 3) that the absorbing set $K_1 = \bigcup_{\tau \in R} U(\tau+1,\tau)K$ is bounded in $H_0^1(\Omega)$, and from the item 4) we deduce that the set $K_0 = \bigcup_{\tau \in R} U(\tau+1,\tau)K_1$ is bounded in $H^2(\Omega)$ and also it is an absorbing set.

7) As it is proved above $u(t) \in L_\infty((\tau+s,T); H^2(\Omega) \cap H_0^1(\Omega))$ for $s > 0$ and $\partial_t u \in L_\infty((\tau+s,T); L_2(\Omega))$. Then after changing the values of u on a set of zero measure we have $u \in C((\tau+s,T); H_0^1(\Omega))$. ■

As any solution $u(t)$ of the problem (6.1)-(6.3) enters K_0 in a certain period of time and do not leave it for all $t \geq \tau + T(\|u_\tau\|)$, we shall consider only solutions $u(t)$ that lie in the absorbing set K_0 for all $t \geq \tau$. For these solutions the norm $\|\Delta u(t)\|$ is bounded and from (6.1) we deduce that $\|\partial_t u\|$ is also finite.

To determinate the relative Lyapunov function for the problem (6.1)-(6.3) we consider the limit equation (as $\varepsilon = 0$) for (6.1):

(8.7) $\partial_t u = \Delta u - f(u) + g(x) = \mathfrak{A}(u), \quad u|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0 \in L_2(\Omega).$

This autonomous problem has a Lyapunov function $F(u)$ defined on $H_0^1(\Omega)$:

$$F(u)=\frac{1}{2}\|\nabla u\|^2+(\Phi(u),1)-(g,u)=\int_\Omega\left[\frac{1}{2}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^2+\Phi(u(x))-g(x)u(x)\right]dx,$$

where $\Phi'=f$.

Proposition 8.1:

1) *$F(u)$ is continuous in $H_0^1(\Omega)$.*

2) *$F(u)$ is bounded on K_0.*

3) *$F(u(t))$ decreases in t for any solution $u(t)$ of* (8.7).

Proof: 1) It is evident that $1/2\|\nabla u\|^2$ is continuous in $H_0^1(\Omega)$, and (g,u) is continuous even in $L_2(\Omega)$ since $g\in L_2(\Omega)$. Let us check the continuity of $(\Phi(u),1)$:

$$\left|\int_\Omega(\Phi(u_1(x))-\Phi(u_2(x)))\,dx\right|=\left|\int_\Omega\int_0^1 f(u_1+\theta(u_2-u_1))\,d\theta(u_2-u_1)\,dx\right|\leqslant$$

$$\leqslant C\left[\int_\Omega(1+|u_1|+|u_2|)^{2q}dx\right]^{1/2}\|u_2-u_1\|\leqslant C(1+\|\nabla u_1\|^r+\|\nabla u_2\|^r)\|u_2-u_1\|.$$

2) The continuous function $F(u)$ is bounded on the compact in $H_0^1(\Omega)$ set K_0.

3) Similar to Theorem 8.1 we can show that $u(t)\in H_0^1(\Omega)\cap H^2(\Omega)$, $\partial_t u\in$ $\in L_2(\Omega)$ for $t>0$. Then

$$\frac{d}{dt}F(u(t))=-(\Delta u,\partial_t u)+(f(u),\partial_t u)-(g,\partial_t u)=-\|\mathfrak{A}(u)\|^2,\quad t>0,$$

$$F(u(t_2))-F(u(t_1))=-\int_{t_1}^{t_2}\|\mathfrak{A}(u(t))\|^2dt,\quad\forall t_2>t_1>0.\quad\blacksquare$$

Proposition 8.2: *For any $\varrho>0$ there exists $\delta>0$ such that $\|\mathfrak{A}(u)\|>\delta$ for $u\in K_0\backslash O_\varrho(\mathfrak{N})$. (Remind that $O_\varrho(\mathfrak{N})$ is a ϱ-neighborhood in $H_0^1(\Omega)$ of the set $\mathfrak{N}$ of equilibrium points of the problem* (8.7)).

Proof: Suppose that the statement is wrong. Then there are $\varrho_0>0$ and a sequence $u_n\in K_0\backslash O_\varrho(\mathcal{N})$ that $\|\mathfrak{A}(u_n)\|\to 0$ as $n\to\infty$. Since $\{u_n\}\subset K_0$ and K_0 is bounded in $H^2(\Omega)$ we can choose a subsequence $u_{n'}\to z$ in $H_0^1(\Omega)$ and $u_{n'}\rightharpoonup z$ weakly in $H^2(\Omega)$. Passing to the limit in the equality $\mathfrak{A}(u_{n'})=\Delta u_{n'}-f(u_{n'})+g(x)$ we have: $\mathfrak{A}(u_{n'})\to 0$ in $L_2(\Omega)$, $f(u_{n'})\to f(z)$ in $L_2(\Omega)$ (as $u_{n'}\to z$ in $H_0^1(\Omega)$ and $f(u)$ is a continuous mapping from $H_0^1(\Omega)$ to $L_2(\Omega)$), $g(x)\in L_2(\Omega)$. Thus the limit of $\Delta u_{n'}$ in $L_2(\Omega)$ is defined and equals to Δz (as $\Delta u_{n'}\rightharpoonup\Delta z$ in $L_2(\Omega)$).

Therefore we get $0 = \Delta z - f(z) + g(x)$ and consequently $z \in \mathfrak{N}$. The last statement contradicts $u_n \notin O_\varrho(\mathfrak{N})$ for all n.

Proposition 8.3: *For any $\varrho > 0$ there exist $\varepsilon^*(\varrho) > 0$ and $\nu > 0$ such that under $|\varepsilon| < \varepsilon^*(\varrho)$ and $t_2 > t_1$:*

$$F(u(t_2)) - F(u(t_1)) \leq -\nu(t_2 - t_1) \tag{8.8}$$

for any solution $u(t) \in K_0 \backslash O_\varrho(\mathfrak{N})$, $t \in [t_1, t_2]$. It means that $F(u)$ is a relative Lyapunov function in the set $K_0 \backslash O_\varrho(\mathfrak{N})$ of the process $U_\varepsilon(t, \tau)$ for $|\varepsilon| < \varepsilon^*(\varrho)$.

Proof: If $u(t)$ satisfies (6.1) then

$$\frac{d}{dt} F(u(t)) = -\int_\Omega |\mathfrak{A}(u)|^2 dx + \varepsilon \int_\Omega (f_1(u, t) - g_1)\, \mathfrak{A}(u)\, dx\,,$$

$$F(u(t_2)) - F(u(t_1)) \leq -\frac{1}{2} \int_{t_1}^{t_2} \|\mathfrak{A}(u(t))\|^2 dt + \frac{\varepsilon^2}{2} \int_{t_1}^{t_2} (\|f_1(u, t)\|^2 + \|g_1\|^2)\, dt\,. \tag{8.9}$$

As $u(t) \in K_0 \backslash O_\varrho(\mathfrak{N})$ Proposition 8.2 implies $\|\mathfrak{A}(u)\| > \delta > 0$. It follows from (6.7) and boundedness of $u(t)$ in $H_0^1(\Omega)$ that $f_1(u, t)$ is bounded in $L_2(\Omega)$ uniformly with respect to t. As $g_1 \in L_\infty(\boldsymbol{R}, L_2(\Omega))$ we obtain from (8.9):

$$F(u(t_2)) - F(u(t_1)) \leq$$

$$\leq -\frac{1}{2}\delta^2(t_2 - t_1) + C\varepsilon^2(t_2 - t_1) = -(\delta^2/2 - C\varepsilon^2)(t_2 - t_1) = -\nu(t_2 - t_1)\,.$$

If $\varepsilon^*(\varrho) > 0$ satisfies $(\delta^2/2 - C\varepsilon^*(\varrho)^2) > 0$ then (8.8) holds under $|\varepsilon| < \varepsilon^*(\varrho)$ with $\nu = (\delta^2/2 - C\varepsilon^2) > 0$. ■

Assume that the set $\mathfrak{N}$ of the equilibrium points of (8.7) is finite: $\mathfrak{N} = \{z_1, \ldots, z_N\}$ and all the points z_i are hyperbolic. Let us check that conditions i), ii), and iv) of Theorem 7.2 hold for the considered problem (6.1)-(6.3).

The condition i) takes place due to Theorem 7.1, Propositions 8.1 and 8.3.

The condition ii) asserting that the time of arrival from K_0 to $\bigcup_{i=1}^{N} O_\varrho(z_i)$ is finite is fulfilled according to Lemma 7.1 and Proposition 8.3.

The existence of a local (in a neighborhood of every equilibrium point z_i) exponential approximation in E^α is proved in § 3 with $\alpha = 1/2$. So the condition iv) takes place with $\beta \leq \alpha = 1/2$.

It remains to check the condition iii) of the exponential rate of the divergence of solutions. For $\beta = \alpha = 1/2$ we shall prove it below for the problem (6.1)-(6.3) in $\Omega \subset \boldsymbol{R}^n$, $n \leq 4$, under some supplementary conditions ((8.11)-(8.12)).

Remark 8.1: The condition iii) with $\beta = 0$ (i.e in the metric of the space $L_2(\Omega)$) is obtained in (8.3). Thus Theorem 7.2 and Remark 7.2 with $\beta = 0$ state: for any solution $u(t)$ of the problem (6.1)-(6.3) there exists combined trajectory $\bar{u}(t)$ lying on the union of the finite dimensional integral manifolds $\left(\bigcup_{i=1}^{N} M_i^{*}(\lambda_i)\right)$ such that

$$\|u(t) - \bar{u}(t)\|_0 \leqslant Ce^{-\eta(t-\tau)},$$

where $C = C(\|u(\tau)\|_0)$; $\eta > 0$ does not depend on $u(t)$.

Proposition 8.4: *Suppose instead of the Hölder condition (6.9) we have the Lipschitz condition:*

$$|f'(u_1) - f'(u_2)| \leqslant C_3(1 + |u_1| + |u_2|)^{q-2}|u_1 - u_2| \tag{8.10}$$

with $q - 2 = n/(n-2) - 2 \geqslant 0$. Let the similar condition on f_1 also holds:

$$|f'_{1u}(u_1, t) - f'_{1u}(u_2, t)| \leqslant C_3(1 + |u_1| + |u_2|)^{q-2}|u_1 - u_2| \tag{8.11}$$

for all $t \in R$.

Then

$$\|\nabla v(t)\| \leqslant \|\nabla v(\tau)\| e^{a(t-\tau)} \tag{8.12}$$

for $v(t) = u_1(t) - u_2(t)$, where $u_1(t)$ and $u_2(t)$ are two solutions of (6.1)-(6.3) lying in K_0 for $t \geqslant \tau$.

Remark 8.2: Notice that $q - 2 \geqslant 0$ is possible only in spaces with dimension n not more than 4. Then in (8.10)-(8.11) $q - 2 = 0$ for $n = 4$, $q - 2 = 1$ for $n = 3$, and $q - 2$ is arbitrary for $n = 1, 2$.

Proof of Proposition 8.4: Multiplying the equation (8.2) by $-\Delta v$ and using (8.10)-(8.11) and (6.5) we get

$$\frac{1}{2}\partial_t\|\nabla v\|^2 + \|\Delta v\|^2 =$$

$$= -\int_\Omega \left[\frac{\partial}{\partial u}(f(u_1) + \varepsilon f_1(u_1, t))\nabla u_1 - \frac{\partial}{\partial u}(f(u_2) + \varepsilon f_1(u_2, t))\nabla u_2\right] \cdot \nabla v \, dx =$$

$$= -\int_\Omega \frac{\partial}{\partial u}(f(u_1) + \varepsilon f_1(u_1, t))|\nabla v|^2 dx -$$

$$-\int_\Omega \left[\frac{\partial}{\partial u}(f(u_1) + \varepsilon f_1(u_1, t)) - \frac{\partial}{\partial u}(f(u_2) + \varepsilon f_1(u_2, t))\right]\nabla u_2 \cdot \nabla v \, dx \leqslant$$

$$\leqslant C\|\nabla v\|^2 + C_3(1 + |\varepsilon|)\int_\Omega (1 + |u_1| + |u_2|)^{q-2}|v|\,|\nabla u_2|\,|\nabla v| dx.$$

We estimate the last integral by the Hölder inequality. For brevity let $n = 3$. We get

$$\int_{\Omega}(1 + |u_1| + |u_2|)|v|\,|\nabla u_2|\,|\nabla v|\,dx \leq C_4(1 + \|u_1\|_{L_4} + \|u_2\|_{L_4})\|\nabla u_2\|_{L_4}\|v\|_{L_4}\|\nabla v\| \leq$$

$$\leq C_5(1 + \|\nabla u_1\| + \|\nabla u_2\|)\|\Delta u_2\|\,\|\nabla v\|^2 \leq C_6\|\nabla v\|^2,$$

because $u_i(t) \in K_0$ for $t \geq \tau$, $i = 1, 2$, and K_0 is bounded in $H^2(\Omega)$. Thus we obtain

$$\frac{1}{2}\partial_t\|\nabla v\|^2 \leq a\|\nabla v\|^2$$

that implies (8.12). The cases $n = 1, 2, 4$ are considered similarly. ∎

Proposition 8.4 imply the condition iii) with $\beta = 1/2$. Thus all the conditions of Theorem 7.2 with $\beta = 1/2$ are fulfilled. It follows from this theorem and Remark 7.2 that for every solution $u(t)$ of the problem (6.1)-(6.3) there exists a combined trajectory $\bar{u}(t)$ lying on the union of finite dimensional integral manifolds $\left(\bigcup_{i=1}^{N} M_i^*(\lambda_i)\right)$ such that

$$\|u(t) - \bar{u}(t)\|_{1/2} = \|\nabla(u(t) - \bar{u}(t))\|_0 \leq Ce^{-\eta(t-\tau)},$$

where $C = C(\|u(\tau)\|_0)$; $\eta > 0$ does not depend on $u(t)$.

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