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Existence for a Nonlinear Problem Relative to Dirichlet Forms (**)(***)

ABSTRACT. — We prove the existence of a solution for a nonlinear problem relative to a strongly local regular Dirichlet form.

Esistenza per un problema non lineare relativo a forme di Dirichlet

SUNTO. — Dimostriamo l'esistenza di una soluzione di un problema non lineare relativo ad una forma di Dirichlet regolare e fortemente locale.

1. - INTRODUCTION

Starting from the works of Visik and Leray-Lions, there exists a very rich literature on the existence of solutions for nonlinear elliptic systems and equations, at least in the case of controlled growth conditions (see [15], [16], [17], [20], [24] and the references therein, expecially in [15]). Of course an exposition (even a simplified one) of the different results, techniques and kind of nonlinearities considered is widely beyond the purpose of this introduction.

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In the following we will be concerned with an extension of a result originally due to Boccardo, Murat and Puel. Namely in [7] the authors consider the quasilinear equation

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + f(x, u, \nabla u) = b \quad \text{in } \Omega ,$$

where the coefficients a_{ij} are bounded and measurable and satisfy an ellipticity condition, whereas $b \in H^{-1}(\Omega)$. Under proper assumptions on the nonlinear term $f(x, u, \nabla u)$, they prove the existence of a solution $u \in H_0^1(\Omega)$, which cannot be expected to belong also to $L^\infty(\Omega, m)$.

Many are the technical tools used in the proof (Fatou Lemma, suitable test functions, Murat compact imbedding result ([23])) but one of the most important is the compact imbedding of $H_0^1(\Omega)$ in $L^2(\Omega, m)$.

Recently, Biroli and Tchou (see [6]) have proved an extension of such an imbedding to the framework of Dirichlet forms. Therefore the natural question arises, if the previous result by Boccardo, Murat and Puel (and also many more that rely on the same compact inclusion) can be proved in the more general setting of Dirichlet forms. This is of interest since the forms provide a functional framework that takes into account a large class of elliptic operators, both degenerate and non degenerate (for general references, see [12], [13], [19], [25], while the applications to general elliptic operators are discussed, for example, in [2] or [6]).

Obviously some assumptions are to be done in order to deal with the nonlinear term, being the Dirichlet forms theory essentially linear, but this can be done with great generality.

In Chapter 2 we present the theoretical framework, the main assumptions on Dirichlet forms, the hypotheses on our equation and the existence result. Chapter 3 is devoted to the proof, while in Chapter 4 we generalize a property of Sobolev spaces due to Brezis and Browder to the spaces defined by our forms.

2. - General framework and main result

2.1. *Hypotheses on the Dirichlet form*

We consider a locally compact, connected Hausdorff space X and a positive Radon measure m defined on X such that supp $[m] = X$. We suppose we are given a *strongly local, regular, symmetric Dirichlet form* $a(\cdot, \cdot)$ in the Hilbert space $L^2(X, m)$ (see [12] and [25] for more details on a general definition of Dirichlet forms) and we denote its domain with $D[a]$. It is well known that for every $u, v \in D[a]$ this kind of forms admits the representation

$$a(u, v) = \int_X d\alpha(u, v)$$

where $\alpha(u, v)$ is a signed Radon measure on X, uniquely associated to u and v. It is normally referred to as the *energy density of the form*. The locality of the form ensures that for any open $\Omega \subset X$, the restriction of $\alpha(u, v)$ to Ω depends only on the restrictions of u and v to Ω.

If we denote by $C(\Omega)$ the space of continuous functions defined on Ω and by $C_0(\Omega)$ its subspace of functions with compact support contained in Ω, the *strong locality* allows us to introduce the domain of the form restricted to Ω which we denote by $D_0[a, \Omega]$ and which is the closure of $D[a] \cap C_0(\Omega)$ in $D[a]$, endowed with the norm $\|u\| = (a(u, u) + \|u\|_{L^2(X, m)})^{1/2}$.

Furthermore, we can extend unambigously the definition of the measure $\alpha(u, v)$ in X to all m-measurable functions u and v in X, that coincide m-a.e. with some function of $D[a]$ on every compact subset of X (the space of these functions will be denoted by $D_{loc}[a]$) or on every compact subset of Ω (the space of these functions will instead be denoted by $D_{loc}[a, \Omega]$).

The Dirichlet forms have nice properties with respect to Leibniz, chain and truncation rules: for all these we refer to [12] and [2].

We assume that our form a has a so-called *separating core* (again see [2]): this allows us to define a distance associated with the form, namely

$$d(x, y) = \sup \{\varphi(x) - \varphi(y), \ \ \forall \varphi \in D[a] \cap C_0(\Omega) \text{ with } \alpha(\varphi, \varphi) \leq m\}$$

and by $B(x, r)$ we denote the metric balls of centre x and radius r given by the distance d associated to the form, that is

$$B(x, r) = \{y \in X: \ d(x, y) < r\}.$$

We can now present the three fundamental hypotheses we assume on the form a

(D) The distance d defines a topology on X equivalent to the initial one; moreover for every $R_0 > 0$ a *duplication property* holds for the balls $B(x, r)$, $r \leq R_0$, that is

$$m(B(x, 2r)) \leq c_0 m(B(x, r)),$$

where c_0 is a constant independent of x and r, but such that $c_0 = c_0(R_0)$.

(P) For every ball $B(x, r)$, $r \leq \overline{R}$, $\overline{R}$ proper, and every $f \in D_{loc}[a]$ we have

$$\int_{B(x, r)} |f - f_{x,r}|^2 dm \leq c_1 r^2 \int_{B(x, kr)} d\alpha(f, f)$$

(*scaled Poincaré inequality*) where c_1 and k are constants that do not depend on x or r, $r \leq 2\overline{R}$ and $f_{x,r}$ is the average of f on $B(x, r)$. Moreover, assuming

that $B(x, r) \subset B(x, 2r) \subsetneq X$, we get

$$\int_{B(x,r)} |f|^2 dm \leq c_1 r^2 \int_{B(x,r)} da(f, f)$$

for every $f \in D_0[a, B(x, r)]$.

Thanks to (D), the space acqires the structure of homogeneous space (see [10]) and by iteration it can be shown that

$$m(B(x, r)) \geq \frac{1}{c_0}\left(\frac{r}{R}\right)^{\nu} m(B(x, r))$$

$\forall x \in X$, $R \leq (1/2) R_0$ and $\nu = \log_2 c_0$. ν is (the upper bound of) the *intrinsic dimension* of X. Finally, for any $B(x, R) \subset B(x, 2R) \subsetneq X$ and $R \leq \bar{R}$, Sobolev inequalities relative to ν have been proved in [3], [4] and [5]. As a matter of fact there exists a large recent literature about Sobolev and Poincaré inequalities in degenerate setting; without pretending to be exhaustive, let's just mention [11], [18], [21], [22] (for other details relative to our framework, see [2] and also [6]).

(A) We assume

i) the existence of the Radon-Nykodym derivative $\alpha(u, u) = (d\alpha(u, u)) / dm$;

ii) and that there exist n linear operator $L_i : D_0[a, \Omega] \to L^2(\Omega, m)$, $i = 1, \ldots, n$, and two positive constants λ and Λ s.t.

$$\lambda \sum_{i=1}^{n} |L_i(u(x))|^2 \leq \alpha(u, u)(x) \leq \Lambda \sum_{i=1}^{n} |L_i(u(x))|^2 \quad \text{a.e. in } X.$$

Moreover we also assume that the adjoint operators L_i^* restricted to $D[a]$ are bounded linear operators from $D[a]$ into $L^2(X, m)$. The operators L_i are closed from $D_0[a, \Omega]$ to $L^2(X, m)$.

Remark 2.1: (A) was originally given in [6]. ■

Remark 2.2: It could be thought that the complex of the three assumptions (D), (P) and (A) is particularly heavy. On the contrary, there are a lot of applications where they are naturally satisfied; let us just recall classical elliptic operators with bounded and measurable coefficients (the original situation of [7]), sum of squares of Hörmander vector fields, degenerate elliptic operators with a weight in Muckenhoupt A_2 class, ... For a more comprehensive presentation and for general references, see [6]. ■

We end this section stating an important result about test functions, which we will use a lot in the following

LEMMA 2.3: *Let K be a compact set in Ω and set $d_K := d(K, \partial\Omega)$. Then there exists a function $\Phi_K \in D[a] \cap C_0(\Omega)$ s.t.*

$$\Phi_K = 1 \text{ on } K, \qquad \operatorname{supp}[\Phi_K] \subset \left\{x \in \Omega : d(x, K) \leq \frac{d_K}{2}\right\},$$

$$a(\Phi_K, \Phi_K)(x) \leq \frac{C}{d_K^2} \quad \text{a.e. in } \Omega$$

where C is an absolute constant.

PROOF: Lemma 2.1 of [6] or Proposition 3.8 of [14].

2.2. *Functional spaces and functional relations.*

From now on we will always work on Ω, a relatively compact open subset of X such that $\Omega \subsetneq B(0, R) \subsetneq B(0, 2R) \subsetneq X$, with $2R \leq \overline{R}$. Let now $p \in [1, \infty[$. We want to define now the analogous of classical Sobolev spaces in our framework (a natural reference is here [26]).

a) If $p \in [2, \infty[$ we define

$$D^p[a, \Omega] = \left\{u \in D_{loc}[a, \Omega] : \int_\Omega a(u, u)(x)^{p/2}\, dm + \int |u|^p\, dm = \|u\|^p_{D^p[a, \Omega]} < \infty\right\}$$

and instead of $D^2[a, \Omega]$ we will simply write $D[a, \Omega]$.

b) If $p \in [1, 2[$ we define

$$D^p[a, \Omega] = \left\{\text{completion of } D[a, \Omega] \text{ for the norm}\right.$$

$$\left.\|u\|^p_{D^p[a, \Omega]} = \int_\Omega a(u, u)(x)^{p/2}\, dm + \int |u|^p\, dm\right\}.$$

Finally the spaces $D^p_0[a, \Omega]$ will be the closure of $C_0(\Omega) \cap D^p[a, \Omega]$ with respect to the $\|\cdot\|_{D^p[a, \Omega]}$ norm.

The main properties and the relations between these spaces and the operators L_i are studied in the following Lemmas, whose proofs are given in [6], to which we refer.

LEMMA 2.4: *Let $p \in [1, \infty]$ and $p' \in [1, \infty]$ be conjugate exponent, take u, v in $D[a, \Omega]$, suppose $a(u, u)$, $a(v, v) \in L^1_{loc}(\Omega, m)$ and assume that $a(v, v) \in L^p(\Omega, m)$,*

$\alpha(u, u) \in L^{p'}(\Omega, m)$. *Then* $\alpha(u, v) \in L^1(\Omega, m)$ *and*

$$\int_\Omega |\alpha(u, v)| dm \leq \left\{ \int_\Omega \alpha(v, v)^{p/2} dm \right\}^{1/p} \left\{ \int_\Omega \alpha(u, u)^{p'/2} dm \right\}^{1/p'}.$$

Moreover the function $\|\cdot\|_{D^p[a, \Omega]}$ is actually a norm on $D^p[a, \Omega]$ *if* $p \in [2, \infty[$.

Remark 2.5: It follows that $D^p[a, \Omega]$ is a Banach space, both when $p \in [2, \infty[$ and $p \in [1, 2[$. ■

Lemma 2.6: *Let* $p \in [1, 2]$. $D^p[a, \Omega]$ *is continuously imbedded in* $L^p(\Omega, m)$, *the operator* $\alpha(u, u)^{1/2}$ *has a unique extension (which does not depend on* Ω*) to a continuous operator* $\alpha(u, u)^{1/2}: D^p[a, \Omega] \to L^p(\Omega, m)$; *moreover the operator* L_i, $i = 1 \ldots, n$ *has a unique extension to a linear closed operator* $L_i: L^p(\Omega, m) \to L^p(\Omega, m)$ *with domain* $D^p[a, \Omega]$ *(in both cases we use the same notation for the operator and its extension).*

Lemma 2.7: *Let* $p \in]2, \infty[$. $D^p[a, \Omega]$ *is continuously imbedded in* $L^p(\Omega, m)$, *the operator* $\alpha(u, u)^{1/2}$ *is a continuous operator* $\alpha(u, u)^{1/2}: D^p[a, \Omega] \to L^p(\Omega, m)$; *moreover the operator* L_i, $i = 1 \ldots, n$ *is a linear closed operator* $L_i: L^p(\Omega, m) \to L^p(\Omega, m)$ *with domain* $D^p[a, \Omega]$.

Relying on Lemma 2.6 and 2.7 we can conclude that

$$\lambda^* \sum_{i=1}^n |L_i(u(x))|^p \leq \alpha(u, u)(x)^{p/2} \leq \Lambda^* \sum_{i=1}^n |L_i(u(x))|^p \qquad m - \text{a.e. in } \Omega,$$

where the two constants λ^* and Λ^* depend only on λ, Λ, n and p and we can set

$$\|u\|^p_{D[a, \Omega]} = \int_\Omega \alpha(u, u)(x)^{p/2} dm + \int |u|^p dm, \qquad \forall p \in [1, \infty[.$$

It is worth noticing that $\forall p \in]1, \infty[$, thanks to the assumption (A), the spaces $D^p[a, \Omega]$ and $D^p_0[a, \Omega]$ are reflexive Banach spaces (see also Lemma 2.6 of [6]).

In the following we will also need the dual spaces of $D^p_0[a, \Omega]$. More precisely we set $D^{-1}[a, \Omega] := (D_0[a, \Omega])'$ and $D_{p'}^{-1}[a, \Omega] := (D^p_0[a, \Omega])'$, with the usual condition $1/p + 1/p' = 1$. As in the case of Sobolev spaces on bounded domain (see [6]), $D_{p'}^{-1}[a, \Omega] \subset D^{-1}[a, \Omega]$ $\forall p \in]1, 2[$ and $\forall p \in]1, \infty[$ f belongs to $D_{p'}^{-1}[a, \Omega]$ iff there exist $(f_0, f_1, \ldots, f_n) \in (L^{p'}(\Omega, m))^{n+1}$ s.t.

$$\langle f, u \rangle = \int_\Omega f_0 u \, dm + \sum_{i=1}^n \int_\Omega f_i L_i(u) \, dm \qquad \forall u \in D^p_0[a, \Omega],$$

where $\langle \cdot, \cdot \rangle = {}_{D_{p'}^{-1}[a, \Omega]}\langle \cdot, \cdot \rangle_{D^p_0[a, \Omega]}$ denotes the duality pairing between $D_{p'}^{-1}[a, \Omega]$ e $D^p_0[a, \Omega]$ (see Lemma 2.7 of [6]).

We conclude the presentation of our functional framework, denoting by $\mathcal{R}(\Omega)$ the set of Radon measures on Ω.

DEFINITION 2.8: *We say that a sequence $\{\mu_\varepsilon\} \subset \mathcal{R}(\Omega)$ is w^*-bounded if for every compact set $K \subset \Omega$ there exists a constant $C_K > 0$ s.t.*

$$\left| \int_\Omega \Phi \, d\mu_\varepsilon \right| \leq C_K \|\Phi\|_{L^\infty(\Omega, m)} \quad \forall \Phi \in C(\Omega) \text{ with } \operatorname{supp}[\Phi] \subset \Omega .$$

We say that a sequence $\{\mu_\varepsilon\} \subset \mathcal{R}(\Omega)$ w^*-converges to $\mu \in \mathcal{R}(\Omega)$ and write $\mu_\varepsilon \xrightarrow{*} \mu$ if

$$\lim_{\varepsilon \to 0} \int_\Omega \Phi \, d\mu_\varepsilon = \int_\Omega \Phi \, d\mu \quad \forall \Phi \in C(\Omega) \text{ with } \operatorname{supp}[\Phi] \subset \Omega .$$

PROPOSITION 2.9: *Let $\{\mu_\varepsilon\}$ be a sequence in $\mathcal{R}(\Omega)$ satisfying $\sup_\varepsilon \mu_\varepsilon(K) < \infty$ for any compact set $K \subset \Omega$. Then there exist a subsequence $\{\mu_{\varepsilon_i}\}$ and a measure $\mu \in \mathcal{R}(\Omega)$ such that $\mu_{\varepsilon_i} \xrightarrow{*} \mu$.*

PROOF: It is a well known result: see, for example, Teorema 1.2.7 of [1]. ■

A fundamental tool in our work will be the following compact imbedding result

PROPOSITION 2.10: *Consider $B(x, R) \subset X$. Then $D_0[a, B(x, R)]$ is compactly imbedded in $L^2(B(x, R), m)$.*

PROOF: Lemma 2.5 of [6]. ■

2.3. *Our problem.*

REMARK 2.11: In the following the notation *a.e.* is obviously to be understood as *m-a.e.*

DEFINITION 2.12: *We say that $f : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ is a Carathéodory function when*

a) $\forall (s, p) \in \mathbf{R} \times \mathbf{R}^n$ $x \mapsto f(x, s, p)$ is measurable;

b) for a.e. $x \in \Omega$ $(s, p) \mapsto f(x, s, p)$ is continuous;

in such a case we denote by $f(u, L_i(u))$ the function $x \in \Omega \mapsto f(x, u(x), L_i(u(x)))$, where $L_i(u)$ are the operators considered in Hypothesis (A).

DEFINITION 2.13: *Take* $b \in D^{-1}[a, \Omega]$ *and* $g: \Omega \times R \times R^n \to R$ *a bounded Carathéodory function. We say that* $u \in D_0[a, \Omega]$ *is a weak solution of the problem*

$$\begin{cases} Lu + g(u, L_i(u)) = b & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

if

$$\int_\Omega a(u, v)\,dm + \int_\Omega g(u, L_i(u))\,v\,dm = {}_{D^{-1}[a,\Omega]}\langle b, v\rangle_{D_0[a,\Omega]}, \qquad \forall v \in D_0[a, \Omega].$$

REMARK 2.14: The previous definition will be useful in the proof of our theorem in the following chapter.

Let us now state the main assumptions.

I) We take $b \in D^{-1}[a, \Omega]$.

II) We consider a Carathéodory function $g: \Omega \times R \times R^n \to R$, that satisfies the following three hypotheses:

1) GROWTH CONDITION:

given $b: R^+ \to R^+$ continuous and monotone increasing and $C \in L(\Omega, m)$, $C \geq 0$, we have

$$(2.1) \qquad |g(x, s, p)| \leq b(|s|)(C(x) + |p|^2) \qquad \forall a.e.\ x \in \Omega,\ \forall (s, p) \in R \times R^n;$$

2) FIRST ONE-SIDED CONDITION:

given $K \in L^1(\Omega, m)$, $K \geq 0$, and $\beta, \gamma \geq 0$ s.t.

$$(2.2') \qquad \frac{\gamma}{\gamma_1} + \frac{\beta}{\lambda} < 1, \qquad \gamma_1 := \inf_{v \in D_0[a,\Omega]} \frac{\int_\Omega a(v, v)\,dm}{\int_\Omega |v|^2\,dm},$$

we have

$$(2.2) \qquad sg(x, s, p) + K(x) + \gamma s^2 + \beta |p|^2 \geq 0 \qquad \forall a.e.\ x \in \Omega,\ \forall (s, p) \in R \times R^n;$$

where λ is the coercivity constant considered in Hypothesis (A).

3) SECOND ONE-SIDED CONDITION:

given $b_*: R^+ \to R^+$ monotone increasing, $K_* \in L^1(\Omega, m)$, $K_* \geq 0$, $H_* \in L^{2/(2-r)}(\Omega, m)$, $r \in [1, 2[$, $H_* \geq 0$, we have

$$(2.3) \qquad \frac{s}{|s|} g(x, s, p) + b_*(|s|)(K_*(x) + H(x)|p|^r) \geq 0$$

$$\forall\ a.e.\ x \in \Omega,\ \forall (s, p) \in R \times R^n.$$

We have then the following

THEOREM 2.15: *Given b e g as above, there exists at least one function $u \in D_0[a, \Omega]$ s.t.*

$$g(u, L_s(u)) \in L^1(\Omega, m); \tag{2.4}$$

$$ug(u, L_s(u)) \in L^1(\Omega, m); \tag{2.5}$$

$$\int_\Omega a(u, v)\, dm + \int_\Omega g(u, L_s(u))\, v\, dm = {}_{D^{-1}[a,\Omega]}\langle b, v\rangle_{D_0[a,\Omega]} \tag{2.6}$$

$$\forall v \in D[a, \Omega] \cap C_0(\Omega).$$

REMARK 2.16: Due to the low regularity of b, we cannot expect u to be bounded; to be convinced of such a thing, it is enough to see that $g \equiv 0$ satisfies our assumptions.

REMARK 2.17: As already observed in [7] the conditions 2) and 3) are independent and cannot be reduced one to the other one.

3. - PROOF

We will divide the proof in different steps, as in the original paper by Boccardo-Murat-Puel. For the sake of simplicity in the following we will write $\langle \cdot, \cdot \rangle$, without explicitely denoting the spaces involved in the duality.

1) Let us define

$$g_\varepsilon(x, s, p) = \frac{g(x, s, p)}{1 + \varepsilon |g(x, s, p)|}$$

where $\varepsilon \in \{\varepsilon_k\}$ s.t. $\lim_{k \to \infty} \varepsilon_k = 0$ and $\varepsilon_k > 0$. It is easy to verify that

a) $|g_\varepsilon(x, s, p)| \leq b(|s|)(C(x) + |p|^2)$;

b) $sg_\varepsilon(x, s, p) + K(x) + \gamma s^2 + \beta |p|^2 \geq 0$;

c) $(s / |s|) g_\varepsilon(x, s, p) + b_w(|s|)(K_w(x) + H_w(x)|p|^r) \geq 0$;

where all the quantities involved have been introduced in the previous chapter. In other words, g_ε satisfies the same growth and one-sided conditions as g.

Relying on the fact that $|g_\varepsilon(x, s, p)| \leq 1 / \varepsilon$, it is not hard to see that the application $T: D_0[a, \Omega] \to D_0[a, \Omega]$, which associates to $w \in D_0[a, \Omega]$ the function $w_\varepsilon \in D_0[a, \Omega]$, weak solution of

$$\begin{cases} Lw_\varepsilon + g_\varepsilon(w, L_s(w)) = b & \text{in } \Omega, \\ w_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies the hypotheses of Schauder's fixed point theorem (for the definition of weak solution, see the previous chapter). Therefore we can conclude that there exists at least one function $u_\varepsilon \in D_0[a, \Omega]$, weak solution of the problem

$$(3.1) \qquad \begin{cases} Lu_\varepsilon + g_\varepsilon(u_\varepsilon, L_i(u_\varepsilon)) = b & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

II) Since $u_\varepsilon \in D_0[a, \Omega]$ and $g_\varepsilon \in L^\infty(\Omega, m)$, we can use u_ε as test function to obtain

$$\int_\Omega \alpha(u_\varepsilon, u_\varepsilon)\, dm + \int_\Omega u_\varepsilon g_\varepsilon(u_\varepsilon, L_i(u_\varepsilon))\, dm = \langle b, u_\varepsilon \rangle$$

and also

$$(3.2) \qquad \int_\Omega \alpha(u_\varepsilon, u_\varepsilon)\, dm + \int_\Omega \left[u_\varepsilon g_\varepsilon(u_\varepsilon, L_i(u_\varepsilon)) + K(x) + \gamma u_\varepsilon^2 + \beta \sum_{i=1}^n |L_i(u_\varepsilon)|^2 \right] dm =$$

$$= \langle b, u_\varepsilon \rangle + \int_\Omega \left[K(x) + \gamma u_\varepsilon^2 + \beta \sum_{i=1}^n |L_i(u_\varepsilon)|^2 \right] dm.$$

Due to the definition of γ_1, to (2.2') and to Hypothesis (A), we have

$$(3.3) \qquad \int_\Omega \alpha(u_\varepsilon, u_\varepsilon)\, dm \geq \frac{\gamma}{\gamma_1} \gamma_1 \int_\Omega u_\varepsilon^2\, dm + \left(1 - \frac{\gamma}{\gamma_1}\right) \lambda \int_\Omega \sum_{i=1}^n |L_i(u_\varepsilon)|^2\, dm =$$

$$= \gamma \int_\Omega u_\varepsilon^2\, dm + \beta \int_\Omega \sum_{i=1}^n |L_i(u_\varepsilon)|^2\, dm + \lambda \left(1 - \frac{\gamma}{\gamma_1} - \frac{\beta}{\lambda}\right) \int_\Omega \sum_{i=1}^n |L_i(u_\varepsilon)|^2\, dm.$$

If we set

$$j_\varepsilon = u_\varepsilon g_\varepsilon(u_\varepsilon, L_i(u_\varepsilon)) + K + \beta u_\varepsilon^2 + \gamma \sum_{i=1}^n |L_i(u_\varepsilon)|^2,$$

recalling that $j_\varepsilon \geq 0$ thanks to (2.2''), (3.2) and (3.3) allow us to conclude that

$$(3.4) \qquad \|u_\varepsilon\|_{D_0[a,\Omega]} \leq C_1, \qquad \|j_\varepsilon\|_{L^1(\Omega, m)} \leq C_2$$

uniformly with respect to ε, which implies $\|u_\varepsilon g_\varepsilon\|_{L^1(\Omega, m)} \leq C_3$, once more uniformly with respect to ε. As in [7] we can prove that

$$(3.5) \qquad |g_\varepsilon(x, s, p)| \leq b(1)(C(x) + |p|^2) + K(x) + \gamma s^2 + \beta |p|^2 +$$

$$+ s g_\varepsilon(x, s, p) + K(x) + \gamma s^2 + \beta |p|^2.$$

Hence we can conclude that

$$\|g_\varepsilon(u_\varepsilon, L_\varepsilon(u_\varepsilon))\|_{L^1(\Omega, m)} \leq C_4. \tag{3.6}$$

Since $\|u_\varepsilon\|_{D_0[a,\Omega]} \leq C_3$, we obtain that there exists $u \in D_0[a, \Omega]$ s.t.

a) $u_\varepsilon \rightharpoonup u$ in $D_0[a, \Omega]$;

b) $u_\varepsilon \to u$ in $L^2(\Omega, m)$;

c) $u_\varepsilon \to u$ a.e. in Ω,

where the implication $a) \Rightarrow b)$ is a direct consequence of the compact imbedding of $D_0[a, \Omega]$ into $L^2(\Omega, m)$.

III) By (3.6) g_ε is bounded in $L^1(\Omega, m)$. On the other hand, let us remark that the equation itself gives $\|g_\varepsilon\|_{D^{-1}[a,\Omega]} \leq C_5$. Thanks to Proposition 2.9 we have that there exists $g_u \in \mathcal{R}(\Omega)$ such that $g_\varepsilon \overset{*}{\rightharpoonup} g_u$ in $\mathcal{R}(\Omega)$.

This convergence allows us to conclude that actually $u_\varepsilon \to u$ in $D_0^p[a, \Omega]$ with $1 < p < 2$, much in the spirit of a famous Murat's compact imbedding result (see [23]).

In fact take $\psi \in D_{p'}^{-1}[a, \Omega]$ with $\|\psi\|_{D_{p'}^{-1}[a,\Omega]} \leq 1$ ($p' > \nu \vee 2$, $1 < p < 2$, $1/p + 1/p' = 1$), let $w_\varepsilon = u_\varepsilon - u$ and u_ψ be the solution of

$$\begin{cases} u_\psi \in D_0[a, \Omega], \\ a(u, v) = \langle \psi, v \rangle, \qquad \forall v \in D_0[a, \Omega]. \end{cases}$$

Regarding the regularity of u_ψ, we have that $u_\psi \in L^\infty(\Omega, m)$ (Theorem 2.12 of [6]) and is locally Hölder continuous in Ω (Theorem 2.13 of [6]). If we take a compact set $K \subset \subset \Omega$ and $\Phi_K \in D_0[a, \Omega] \cap C(\Omega)$ as in Lemma 2.3, we want to prove that

$$\lim_{\varepsilon \to 0} \Phi_K w_\varepsilon = 0 \quad \text{in } D_0^p[a, \Omega].$$

Now

$$\|\Phi_K w_\varepsilon\|_{D_0^p[a,\Omega]} = \sup_{\psi \in D_{p'}^{-1}[a,\Omega], \|\psi\| \leq 1} a(\Phi_K w_\varepsilon, u_\psi) = \sup_{\psi \in D_{p'}^{-1}[a,\Omega], \|\psi\| \leq 1} a(\Phi_K w_\varepsilon, u_\psi);$$

Therefore we have to prove that $a(\Phi_K w_\varepsilon, u_\psi)$ converges to 0 uniformly with respect to $\psi \in D_{p'}^{-1}[a, \Omega]$, $\|\psi\|_{D_{p'}^{-1}[a,\Omega]} \leq 1$ as $\varepsilon \to 0$. We have

$$a(\Phi_K w_\varepsilon, u_\psi) = \int_\Omega \alpha(\Phi_K w_\varepsilon u_\psi)\, dm =$$

$$= \int_\Omega \alpha(\Phi_K, u_\psi) w_\varepsilon\, dm + \int_\Omega \alpha(w_\varepsilon, u_\psi \Phi_K)\, dm - \int_\Omega \alpha(w_\varepsilon, \Phi_K) u_\psi\, dm.$$

The first term on the right-hand side goes to zero uniformly with respect to $\psi \in D_{p'}^{-1}[a, \Omega]$, $\|\psi\|_{D_{p'}^{-1}[a,\Omega]} \leq 1$ when $\varepsilon \to 0$, as it is easily verified. Moreover $\alpha(w_\varepsilon, \Phi_K)$ converges weakly to zero in $L^2(\Omega, m)$ (Lemma 2.8 of [6]); since u_ψ is in a compact set

of $L^2(\Omega, m)$ we can conclude that also the third term converges to zero uniformly with respect to $\psi \in D_p^{-1}[a, \Omega]$, $\|\psi\|_{D_p^{-1}[a,\Omega]} \leq 1$ when $\varepsilon \to 0$. We are therefore left with the estimate of the second term. We know that

$$\int_\Omega \alpha(u_\varepsilon, v)\, dm + \int_\Omega g_\varepsilon(u_\varepsilon, L_\varepsilon(u_\varepsilon))\, v\, dm = \langle b, v\rangle, \qquad \forall v \in D_0[a, \Omega];$$

Since $u_\varepsilon \rightharpoonup u$ in $D_0[a, \Omega]$ and $u_\varepsilon \to u$ in $L^2(\Omega, m)$ we can conclude that

$$\int_\Omega \alpha(u_\varepsilon, v)\, dm \to \int_\Omega \alpha(u, v)\, dm, \qquad \forall v \in D_0[a, \Omega].$$

On the other hand

$$\int_\Omega \alpha(w_\varepsilon, u_\psi \Phi_K)\, dm = \langle b, u_\psi \Phi_K\rangle - \int_\Omega g_\varepsilon(u_\varepsilon, L_\varepsilon(u_\varepsilon))\, u_\psi \Phi_K\, dm - \int_\Omega \alpha(u, u_\psi \Phi_K)\, dm$$

and $g_\varepsilon \overset{*}{\rightharpoonup} g_*$ in $\mathcal{R}(\Omega)$ with

$$\int_\Omega u_\psi \Phi_K\, dg_* = -\int_\Omega \alpha(u, u_\psi \Phi_K)\, dm + \langle b, u_\psi \Phi_K\rangle$$

since $u_\psi \Phi_K \in C_0(\Omega) \cap D_0[a, \Omega]$ (see what we said above about u_ψ); hence

$$\int_\Omega \alpha(w_\varepsilon, u_\psi \Phi_K)\, dm \to 0.$$

The problem is solved if we show that this last convergence occurrs uniformly with respect to $\psi \in D_p^{-1}[a, \Omega]$, which is the same to say that

$$\int_\Omega u_\psi \Phi_K g_\varepsilon\, dm \to \int_\Omega u_\psi \Phi_K\, dg_*$$

uniformly with respect to $\psi \in D_p^{-1}[a, \Omega]$. But $u_\psi \Phi_K$ belongs to a compact set of $C(\Omega)$ as it easily seen and we can conclude applying Lemma 2.9 of [6]. So we have proved that

$$\lim_{\varepsilon \to 0} \Phi_K w_\varepsilon = 0 \quad \text{in } D^p[a, \Omega]$$

and then

$$\lim_{\varepsilon \to 0} \Phi_K w_\varepsilon = 0 \quad \text{a.e. } in\ \Omega.$$

Taking into account that $\alpha(w_\varepsilon, w_\varepsilon)^{1/2}$ is bounded in $L^2(\Omega, m)$, then $\alpha(w_\varepsilon, w_\varepsilon)^{p/2}$ is equi-integrable in Ω with $1 < p < 2$, so that for every $p \in (1, 2)$

$$\lim_{\varepsilon \to 0} \alpha(w_\varepsilon, w_\varepsilon)^{1/2} = 0 \quad \text{in } L^p(\Omega, m);$$

as $\lim_{\varepsilon \to 0} w_\varepsilon = 0$ in $L^2(\Omega, m)$, we obtain $\lim_{\varepsilon \to 0} w_\varepsilon = 0$ in $D_0^p[a, \Omega]$ and we are done.

IV) Relying on Lemmas 2.6 and 2.7 we have that the L_i are linear and continuous from $D^p[a, \Omega]$ to $L^p(\Omega, m)$, hence also from $D_0^p[a, \Omega]$ to $L^p(\Omega, m)$. Therefore if $u_\varepsilon \to u$ in $D_0^p[a, \Omega]$ with $1 < p < 2$, we have that $L_i(u_\varepsilon) \to L_i(u)$ in $L^p(\Omega, m)$ with the same p and also $L_i(u_\varepsilon) \to L_i(u)$ a.e. in Ω.

Thanks to the properties of g, we can then repeat what done in [7] and conclude that

$$u_\varepsilon g_\varepsilon(u_\varepsilon, L_i(u_\varepsilon)) + K + \gamma u_\varepsilon^2 + \beta \sum_{i=1}^{n} |L_i(u_\varepsilon)|^2 \xrightarrow{a.e.} ug(u, L_i(u)) +$$

$$+ K + \gamma u^2 + \beta \sum_{i=1}^{n} |L_i(u)|^2.$$

Moreover, applying Fatou's Lemma we obtain

$$ug(u, L_i(u)) + K + \gamma u^2 + \beta \sum_{i=1}^{n} |L_i(u)|^2 \in L^1(\Omega, m).$$

As a direct consequence $ug(u, L_i(u)) \in L^1(\Omega, m)$ and thanks to (3.5) we have that $g(u, L_i(u)) \in L^1(\Omega, m)$.

V) We want now to choose the right test function that will allow us to pass to the limit and conclude the proof.

Let us first observe that since $u_\varepsilon \in D_0[a, \Omega]$, due to the markovianity of the form it must be $u_\varepsilon^+ \in D_0[a, \Omega]$, where $v^+ := \max\{0, v\}$.

If we define

$$B(t) = \int_0^t b(s)\,ds$$

where b is the function that comes up in (2.1) and set

$$f(t) = \exp\left(-\frac{B(t)}{\gamma}\right) \qquad t \geq 0,$$

then $f(t) \in C^1(\mathbf{R}_+)$ and f, f' are in $L^\infty(\mathbf{R}_+)$.

Let us now take $H : \mathbf{R} \to \mathbf{R}$, $H \in C^1(\mathbf{R})$ such that $0 \leq H \leq 1$, $H = 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $H = 0$ in $]-\infty, -1] \cup [1, +\infty[$: it is evident that H and H' are in $L^\infty(\mathbf{R})$.

We can then apply the chain rule and conclude that

a) $\exp(-B(u_\varepsilon^+)/\gamma) \in D[a, \Omega] \cap L^\infty(\Omega, m)$;

b) $H((1/n)u_\varepsilon) \in D[a, \Omega] \cap L^\infty(\Omega, m)$.

If we now take $\varphi \in D[a, \Omega] \cap C_0(\Omega)$ such that $a(\varphi, \varphi) \leq C$ a.e. in Ω, it is immediate to verify that

$$v_\varepsilon^n := \varphi \exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right) H\left(\frac{1}{n} u_\varepsilon\right) \in D_0[a, \Omega],$$

thanks to the fact that $D[a, \Omega] \cap L^\infty(\Omega, m)$ and $D_0[a, \Omega] \cap L^\infty(\Omega, m)$ are algebras, and we can use v_ε^n as test function in (3.1).

VI) Since $0 \leq v_\varepsilon^n \leq \varphi$ a.e. in Ω, we have $\|v_\varepsilon^n\|_{L^\infty(\Omega, m)} \leq C$, $\forall \varepsilon > 0$, $\forall n$. Let us now fix n: we want to study the limit as $\varepsilon \to 0$. First of all we need some preliminary remarks. As $u_\varepsilon \to u$ a.e., we have

$$v_\varepsilon^n \to \varphi \exp\left(-\frac{B(u^+)}{\alpha}\right) H\left(\frac{1}{n} u\right) \quad \text{a.e. in } \Omega.$$

Setting in the following $w_\varepsilon^n := \exp(-B(u_\varepsilon^+)/\alpha) H((1/n) u_\varepsilon)$ for the sake of simplicity,

$$\|v_\varepsilon^n\|^2_{L^2(\Omega, m)} = \int_\Omega \varphi^2 (w_\varepsilon^n)^2 \, dm \leq \int_\Omega \varphi^2 \, dm = \|\varphi\|^2_{L^2(\Omega, m)};$$

$$\begin{aligned} a(v_\varepsilon^n, v_\varepsilon^n) = \int_\Omega a(v_\varepsilon^n, v_\varepsilon^n) \, dm &\leq \|w_\varepsilon^n\|^2_{L^\infty(\Omega, m)} a(\varphi, \varphi) + \\ &+ \|\varphi\|^2_{L^\infty(\Omega, m)} a(w_\varepsilon^n, w_\varepsilon^n) \leq \|w_\varepsilon^n\|^2_{L^\infty(\Omega, m)} a(\varphi, \varphi) + \\ &+ \|\varphi\|^2_{L^\infty(\Omega, m)} \left[\left\| \exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right) \right\|^2_{L^\infty(\Omega, m)} a\left(H\left(\frac{1}{n} u_\varepsilon\right), H\left(\frac{1}{n} u_\varepsilon\right)\right) + \right. \\ &\left. + \left\| H\left(\frac{1}{n} u_\varepsilon\right) \right\|^2_{L^\infty(\Omega, m)} a\left(\exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right), \exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right)\right) \right] \end{aligned}$$

and relying on the fact that $\int_\Omega a(u_\varepsilon, u_\varepsilon) \, dm \leq C_1$, $\int_\Omega a(\varphi, \varphi) \, dm \leq C_2$, we can conclude that for n fixed

$$\|v_\varepsilon^n\|_{D_0[a, \Omega]} \leq C_3$$

where the constants C_i do not depend on ε (here we rely on the fact that

$$0 \leq b(u_\varepsilon^+) \exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right) H\left(\frac{1}{n} u_\varepsilon\right) \leq b(n)$$

since b is monotone increasing and $H((1/n) u_\varepsilon) = 0$ when $|u_\varepsilon| \geq n$).

Using the chain rule once again, the approximate equation (3.1) can be written in

this way

$$(3.7)\qquad \int_\Omega \alpha(u_\varepsilon, \varphi)\exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right)H\left(\frac{1}{n}u_\varepsilon\right)dm +$$

$$+\frac{1}{n}\int_\Omega \alpha(u_\varepsilon, u_\varepsilon)\exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right)H'\left(\frac{1}{n}u_\varepsilon\right)\varphi\, dm +$$

$$-\int_\Omega \alpha(u_\varepsilon, u_\varepsilon^+)\frac{b(u_\varepsilon^+)}{\gamma}\exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right)H\left(\frac{1}{n}u_\varepsilon\right)\varphi\, dm +$$

$$+\int_\Omega g_\varepsilon(u_\varepsilon, L_\varepsilon(u_\varepsilon))\exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right)H\left(\frac{1}{n}u_\varepsilon\right)\varphi\, dm - \langle b, v_\varepsilon^n\rangle = 0\,.$$

Let us then go to the limit with $\varepsilon \to 0^+$ keeping n fixed and consider separately the different terms in (3.7).

a) Since $\|v_\varepsilon^n\|_{D_0[\alpha,\Omega]} \leq C$, we have

$$v_\varepsilon^n \xrightarrow{D_0[\alpha,\Omega]} v^n := \varphi\exp\left(-\frac{B(u^+)}{\gamma}\right)H\left(\frac{1}{n}u\right);$$

therefore

$$(3.8)\qquad \langle b, v_\varepsilon^n\rangle \to \left\langle b, \varphi\exp\left(-\frac{B(u^+)}{\gamma}\right)H\left(\frac{1}{n}u\right)\right\rangle.$$

b) Since $u_\varepsilon \xrightarrow{D_0[\alpha,\Omega]} u$, we can rely on Lemma 2.8 of [6] and conclude that

$$\alpha(u_\varepsilon, \varphi) \xrightarrow{L^2(\Omega,m)} \alpha(u, \varphi)\,.$$

Moreover

$$\exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right)H\left(\frac{1}{n}u_\varepsilon\right) \xrightarrow{a.e.} \exp\left(-\frac{B(u^+)}{\gamma}\right)H\left(\frac{1}{n}u\right);$$

$$0 \leq \exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right)H\left(\frac{1}{n}u_\varepsilon\right) \leq 1\,.$$

Since $m(\Omega) < \infty$, we can apply the dominated convergence theorem and conclude that

$$\exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right)H\left(\frac{1}{n}u_\varepsilon\right) \xrightarrow{L^2(\Omega,m)} \exp\left(-\frac{B(u^+)}{\gamma}\right)H\left(\frac{1}{n}u\right).$$

Hence

$$(3.9)\qquad \int_\Omega \alpha(u_\varepsilon, \varphi)\exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right)H\left(\frac{1}{n}u_\varepsilon\right)dm \to$$

$$\to \int_\Omega \alpha(u, \varphi)\exp\left(-\frac{B(u^+)}{\gamma}\right)H\left(\frac{1}{n}u\right)dm\,.$$

It is more complicated to treat the second, third and fourth term.

c) We have

$$\frac{1}{n}\int_\Omega \alpha(u_\varepsilon, u_\varepsilon)\exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right)H'\left(\frac{1}{n}u_\varepsilon\right)\varphi\, dm \leq$$

$$\leq \frac{1}{n}\|H'\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^\infty(\Omega,m)}\|u_\varepsilon\|^2_{D_0[a,\Omega]} \leq \frac{1}{n}C_*\|\varphi\|_{L^\infty(\Omega,m)}.$$

Therefore

$$(3.10)\qquad \limsup_{\varepsilon\to 0^+}\frac{1}{n}\int_\Omega \alpha(u_\varepsilon, u_\varepsilon)\exp\left(-\frac{B(u_\varepsilon^+)}{\gamma}\right)H'\left(\frac{1}{n}u_\varepsilon\right)\varphi\, dm \leq \frac{1}{n}C_*\|\varphi\|_{L^\infty(\Omega,m)}.$$

d) We consider the third and the fourth term together. As $u_\varepsilon \to u$ in $D_0^p[a,\Omega]$ with $p<2$, we have $L_i(u_\varepsilon)\to L_i(u)$ in $L^p(\Omega, m)$, with the same p and also $L_i(u_\varepsilon)\to$ $\to L_i(u)$ a.e. in Ω, as already remarked. Due to the continuity of g and g_ε we have

$$g_\varepsilon(u_\varepsilon, L_i(u_\varepsilon)) \to g(u, L_i(u)) \qquad \text{a.e. } in\ \Omega\,.$$

Moreover, due to the strong locality property of Dirichlet forms, given a function v in the proper space, it holds

$$\int_\Omega \alpha(u_\varepsilon, u_\varepsilon^+)v\, dm = \int_\Omega \alpha(u_\varepsilon^+, u_\varepsilon^+)v1\,|_{\{\tilde{u}_\varepsilon>0\}}\, dm\,;$$

$$\int_\Omega \alpha(u, u^+)v\, dm = \int_\Omega \alpha(u^+, u^+)v1\,|_{\{\tilde{u}>0\}}\, dm\,;$$

where $\tilde{u}_\varepsilon$ and $\tilde{u}$ stand for the quasi-continuous representatives of u_ε and u. In the following we will omit the $\tilde{}$ to simplify the notation. We now have the following

Lemma 3.1: *Let us consider* $\{u_\varepsilon\}\subseteq D_0[a,\Omega]$ *and* $u\in D_0[a,\Omega]$. *If* $u_\varepsilon\to u$ *in* $D_0^p[a,\Omega]$ *with* $1<p<2$, *then* $u_\varepsilon^+\to u^+$ *in* $D_0^p[a,\Omega]$.

Proof: Due to the markovianity of the form, $\alpha(u_\varepsilon^+, u_\varepsilon^+)\leq\alpha(u_\varepsilon, u_\varepsilon)$. Therefore we immediately conclude that $u_\varepsilon\in D_0[a,\Omega]\Rightarrow u_\varepsilon^+\in D_0[a,\Omega]\Rightarrow u_\varepsilon^+\in D_0^p[a,\Omega]$.

Moreover, if $u_\varepsilon \to u$ in $D_0^p[a, \Omega]$, it is evident that $u_\varepsilon^+ \to u^+$ in $L^p(\Omega, m)$. Therefore we must show that

$$\lim_{\varepsilon \to 0^+} \int_\Omega |\alpha(u_\varepsilon^+ - u^+, u_\varepsilon^+ - u^+)|^{p/2} dm = 0 .$$

Relying on the strong localization property of the form we have

$$\int_\Omega |\alpha(u_\varepsilon^+ - u^+, u_\varepsilon^+ - u^+)|^{p/2} dm = \int_{\{u_\varepsilon > 0, u < 0\}} |\alpha(u_\varepsilon, u_\varepsilon)|^{p/2} dm +$$

$$+ \int_{\{u_\varepsilon > 0, u > 0\}} |\alpha(u_\varepsilon - u, u_\varepsilon - u)|^{p/2} dm + \int_{\{u_\varepsilon < 0, u > 0\}} |\alpha(u, u)|^{p/2} dm .$$

Moreover, taking into account that

$$\alpha((u_\varepsilon - u) + u, (u_\varepsilon - u) + u) \leq 2\alpha(u_\varepsilon - u, u_\varepsilon - u) + 2\alpha(u, u) ,$$

we can further estimate in this way

$$\int_{\{u_\varepsilon > 0, u < 0\}} |\alpha(u_\varepsilon, u_\varepsilon)|^{p/2} dm = \int_{\{u_\varepsilon > 0, u < 0\}} |\alpha((u_\varepsilon - u) + u, (u_\varepsilon - u) + u)|^{p/2} dm \leq$$

$$\leq c(p) \left[\int_{\{u_\varepsilon > 0, u < 0\}} |\alpha(u_\varepsilon - u, u_\varepsilon - u)|^{p/2} dm + \int_{\{u_\varepsilon > 0, u < 0\}} |\alpha(u, u)|^{p/2} dm \right].$$

We can put everything together to obtain

$$\int_\Omega |\alpha(u_\varepsilon^+ - u^+, u_\varepsilon^+ - u^+)|^{p/2} dm \leq C \left[\int_\Omega |\alpha(u_\varepsilon - u, u_\varepsilon - u)|^{p/2} dm + \right.$$

$$\left. + \int_{\{u < 0, u_\varepsilon > 0\} \cup \{u > 0, u_\varepsilon < 0\}} |\alpha(u, u)|^{p/2} dm \right].$$

As $u_\varepsilon \to u$ in $D_0^p[a, \Omega]$, the first term goes to zero. What is left is to check that also the second one does the same. Let us deal with $\int_{\{u > 0, u_\varepsilon < 0\}} |\alpha(u, u)|^{p/2} dm$; analogously we can work on the other integral.

If we fix $\sigma > 0$ we have

$$\int_{\{u > 0, u_\varepsilon < 0\}} |\alpha(u, u)|^{p/2} dm = \int_{\{0 < u < \sigma, u_\varepsilon < 0\}} |\alpha(u, u)|^{p/2} dm +$$

$$+ \int_{\{u > \sigma, u_\varepsilon < 0\}} |\alpha(u, u)|^{p/2} dm .$$

We have

$$m\{x \in \Omega : u > \sigma, u_\varepsilon < 0\} \leq m\{x \in \Omega : |u - u_\varepsilon| > \sigma\} \leq \frac{1}{\sigma^r} \int_\Omega |u_\varepsilon - u|^r dm$$

and the last integral goes to zero since $u_\varepsilon \to u$ in $D_0^r[a, \Omega]$. Hence

$$\limsup_{\varepsilon \to 0} \int_{\{u > 0, u_\varepsilon < 0\}} |\alpha(u, u)|^{p/2} dm \leq \int_{\{0 < u < \sigma\}} |\alpha(u, u)|^{p/2} dm \quad \forall \sigma > 0$$

and this term goes to zero as $\sigma \to 0$, since $m\{x \in \Omega : 0 < u < \sigma\} = O(\sigma)$. ■

As a direct consequence of $u_\varepsilon^+ \to u$ in $D_0^r[a, \Omega]$, we have $u_\varepsilon^+ \to u$ a.e. in Ω. Since $m(\Omega) < +\infty$, due to the dominated convergence theorem it is easy to see that

$$1|_{\{u_\varepsilon > 0\}} \xrightarrow{L^p(\Omega, m)} 1|_{\{u > 0\}} \quad \forall p \in [1, \infty[;$$

thanks to the previous lemma, we immediately obtain $\alpha(u_\varepsilon^+, u_\varepsilon^+) \xrightarrow{L^p(\Omega, m)} \alpha(u^+, u^+)$ with $1 < p < 2$.

We can then conclude that

$$\left.\begin{array}{l} \alpha(u_\varepsilon^+, u_\varepsilon^+) \xrightarrow{L^p(\Omega, m)} \alpha(u^+, u^+) \\ 1|_{\{u_\varepsilon > 0\}} \xrightarrow{L^{p'}(\Omega, m)} 1|_{\{u > 0\}} \end{array}\right\} \Rightarrow \alpha(u_\varepsilon^+, u_\varepsilon^+) 1|_{\{u_\varepsilon > 0\}} \xrightarrow{L^1(\Omega, m)} \alpha(u^+, u^+) 1|_{\{u > 0\}}$$

and therefore, up to subsequences

$$\alpha(u_\varepsilon^+, u_\varepsilon^+) 1|_{\{u_\varepsilon > 0\}} \xrightarrow{\text{a.e. in } \Omega} \alpha(u^+, u^+) 1|_{\{u > 0\}}.$$

If we set

$$l_\varepsilon := \left[g_\varepsilon(u_\varepsilon, L_i(u_\varepsilon)) - \alpha(u_\varepsilon, u_\varepsilon^+) \frac{b(u_\varepsilon^+)}{\gamma} \right] \exp\left(- \frac{B(u_\varepsilon^+)}{\gamma} \right) H\left(\frac{1}{n} u_\varepsilon \right) \varphi ,$$

we get

$$(3.11) \quad l_\varepsilon \xrightarrow{\text{a.e.}} l := \left[g(u, L_i(u)) - \alpha(u, u^+) \frac{b(u^+)}{\gamma} \right] \exp\left(- \frac{B(u^+)}{\gamma} \right) H\left(\frac{1}{n} u \right) \varphi . +$$

Exactly as in [7], we can prove that

$$(3.12) \quad \begin{cases} \forall v \in D^1[a, \Omega], \quad \text{a.e. in } \Omega , \\ \left[g_\varepsilon(v, L_i(v)) - \dfrac{b(v^+)}{\gamma} \alpha(v, v^+) \right](x) \leq \\ \qquad \leq b(|v(x)|) C(x) + b_n(|v(x)|) \left[K_n(x) + H_n(x) \sum_{i=1}^{n} |L_i(v(x))|^r \right]. \end{cases}$$

Once more exactly as in [7], starting from (3.12) and relying on the properties of v_ε^n and $H((1/n)u_\varepsilon)$, we obtain

$$(3.13)\qquad l_\varepsilon \leq \left[b(n)C(x) + b_*(n)\left(K_*(x) + H_*(x)\sum_{i=1}^{n}|L_i(u_\varepsilon)|^r\right)\right]\varphi(x) \quad \text{a.e. in } \Omega;$$

Thanks to (3.11) and (3.13) we can apply Fatou's lemma to conclude that

$$\limsup_{\varepsilon\to 0}\int_\Omega l_\varepsilon\,dm \leq \int_\Omega l\,dm\,.$$

Remark 3.2: As a matter of fact, Fatou's Lemma is used to conclude about $u_\varepsilon g_\varepsilon$, since the term could be treated directly.

If we now put everything together, we end up with

$$(3.14)\qquad \int_\Omega a(u,\varphi)\exp\left(-\frac{B(u^+)}{\gamma}\right)H\left(\frac{1}{n}u\right)dm + \frac{1}{n}C_*\|\varphi\|_{L^\infty(\Omega,m)} +$$

$$+\int_\Omega\left[g(u,L_i(u)) - a(u,u^+)\frac{b(u^+)}{\gamma}\right]\exp\left(-\frac{B(u^+)}{\gamma}\right)H\left(\frac{1}{n}u\right)\varphi\,dm +$$

$$-\left\langle b,\varphi\exp\left(-\frac{B(u^+)}{\gamma}\right)H\left(\frac{1}{n}u\right)\right\rangle \geq 0\,.$$

We can now add and subtract the term

$$T_n = \frac{1}{n}\int_\Omega a(u,u)\exp\left(-\frac{B(u^+)}{\gamma}\right)H'\left(\frac{1}{n}u\right)\varphi\,,$$

use the estimate

$$-T_n \leq \frac{1}{n}\|\varphi\|_{L^\infty(\Omega,m)}$$

and rely on the fact that $g \in L^1(\Omega,m)$ and $b(u^+)H((1/n)u) \in L^\infty(\Omega,m)$ to conclude that $\forall n$ and $\forall\varphi \in D[a,\Omega]\cap L^\infty(\Omega,m)$ with $\varphi \geq 0$

$$(3.15)\qquad \int_\Omega a\left(u,\exp\left(-\frac{B(u^+)}{\gamma}\right)H\left(\frac{1}{n}u\right)\varphi\right)dm +$$

$$+\int_\Omega g(u,L_i(u))\exp\left(-\frac{B(u^+)}{\gamma}\right)H\left(\frac{1}{n}u\right)\varphi\,dm +$$

$$+\frac{2}{n}C_*\|\varphi\|_{L^\infty(\Omega,m)} \geq \left\langle b,\exp\left(-\frac{B(u^+)}{\gamma}\right)H\left(\frac{1}{n}u\right)\varphi\right\rangle.$$

VII) We can now let $n \to \infty$. Let us then consider

$$\varphi = \varphi_n = \Phi \exp\left(\frac{B(u^+)}{\gamma}\right) H\left(\frac{1}{p(n)} u\right), \tag{3.16}$$

where

1) $\Phi \in \mathrm{D}[a, \Omega] \cap C_0(\Omega)$, $\Phi \geq 0$;

2) B is as above;

3) H is as above;

4) $p(n) \in \boldsymbol{R}_+$ and is defined by the equation $B(p(n)) = \gamma \ln \sqrt{n}$.

It is clear that $p(n) \to \infty$ as $n \to \infty$. Thanks to the chain rule we have that

$$\exp\left(\frac{B(u^+)}{\gamma}\right) H\left(\frac{1}{p(n)} u\right) =$$

$$= \exp\left(\frac{B(u \vee 0 \wedge p(n))}{\gamma}\right) H\left(\frac{1}{p(n)} u\right) \in L^\infty(\Omega, m) \cap \mathrm{D}[a, \Omega].$$

Therefore

$$\varphi_n \in \mathrm{D}_0[a, \Omega] \cap L^\infty(\Omega, m) \quad \text{and} \quad \|\varphi_n\|_{L^\infty(\Omega, m)} \leq \sqrt{n} \|\Phi\|_{L^\infty(\Omega, m)}.$$

If we then substitute in (3.15), we obtain

$$\int_\Omega a\left(u, H\left(\frac{1}{n} u\right) H\left(\frac{1}{p(n)} u\right) \Phi\right) dm + \int_\Omega g(u, L_i(u)) H\left(\frac{1}{n} u\right) H\left(\frac{1}{p(n)} u\right) \Phi \, dm + \tag{3.17}$$

$$+ \frac{2}{\sqrt{n}} C_* \|\Phi\|_{L^\infty(\Omega, m)} \geq \left\langle b, H\left(\frac{1}{n} u\right) H\left(\frac{1}{p(n)} u\right) \Phi \right\rangle.$$

It is evident that $(2/\sqrt{n}) C_* \|\Phi\|_{L^\infty(\Omega, m)} \to 0$ when $n \to \infty$. Moreover

$$g(u, L_i(u)) H\left(\frac{1}{n} u\right) H\left(\frac{1}{p(n)} u\right) \Phi \xrightarrow{a.e.} g(u, L_i(u)) \Phi,$$

$$\left| g(u, L_i(u)) H\left(\frac{1}{n} u\right) H\left(\frac{1}{p(n)} u\right) \Phi \right| \leq |g(u, L_i(u))| \Phi$$

and $|g(u, L_i(u))| \Phi \in L^1(\Omega, m)$ thanks to what we proved above. We can then apply the dominated convergence theorem and conclude that

$$\int_\Omega g(u, L_i(u)) H\left(\frac{1}{n} u\right) H\left(\frac{1}{p(n)} u\right) \Phi \, dm \to \int_\Omega g(u, L_i(u)) \Phi \, dm.$$

Moreover

$$H\left(\frac{1}{n}u\right)H\left(\frac{1}{p(n)}u\right)\Phi \xrightarrow{L^2(\Omega,m)} \Phi .$$

Since $\left\|H\left(\frac{1}{n}u\right)H\left(\frac{1}{p(n)}u\right)\Phi\right\|_{D_0[a,\Omega]} \leq C$, we have $H\left(\frac{1}{n}u\right)H\left(\frac{1}{p(n)}u\right)\Phi \xrightarrow{D_0[a,\Omega]} \Phi$. Hence

$$\int_\Omega \alpha\left(u, H\left(\frac{1}{n}u\right)H\left(\frac{1}{p(n)}u\right)\Phi\right)dm \to \int_\Omega \alpha(u, \Phi)\,dm ,$$

$$\left\langle b, H\left(\frac{1}{n}u\right)H\left(\frac{1}{p(n)}u\right)\Phi\right\rangle \to \langle b, \Phi\rangle$$

and we can conclude that $\forall \Phi \in D[a, \Omega] \cap C_0(\Omega)$ with $\Phi \geq 0$

$$\int_\Omega \alpha(u, \Phi)\,dm + \int_\Omega g(u, L_i(u))\,\Phi\,dm \geq \langle b, \Phi\rangle . \tag{3.18}$$

VIII) To obtain the opposite implication, it is enough to repeat the same calculations as in steps VI) and VII) using as test function

$$v_\varepsilon^n := \varphi \exp\left(-\frac{B(u_\varepsilon^-)}{\gamma}\right)H\left(\frac{1}{n}u_\varepsilon\right), \qquad \varphi \geq 0 .$$

We then let $\varepsilon \to 0$ keeping n fixed. Finally we use

$$\varphi = \varphi_n = \Phi \exp\left(\frac{B(u^-)}{\gamma}\right)H\left(\frac{1}{p(n)}u\right), \qquad \Phi \geq 0 ,$$

with $\Phi \in D[a, \Omega] \cap C_0(\Omega)$ as before. We obtain

$$\int_\Omega \alpha(u, \Phi)\,dm + \int_\Omega g(u, L_i(u))\,\Phi\,dm \leq \langle b, \Phi\rangle \tag{3.19}$$

that is exactly the opposite inequality with respect to (3.18). Actually we can conclude that $\forall \Phi \in D[a, \Omega] \cap C_0(\Omega)$ with $\Phi \geq 0$ we have

$$\int_\Omega \alpha(u, \Phi)\,dm + \int_\Omega g(u, L_i(u))\,\Phi\,dm = \langle b, \Phi\rangle$$

and from here we easily deduce that u is weak solution of our initial equation.

4. - Further remarks

As it was evident in the previous chapter, the proof of the Theorem 2.15 is constructive, in the sense that the function u satisfying (2.4), (2.5), (2.6) is properly built.

On the other hand, the operator L, autoadjoint in $L^2(\Omega, m)$ and defined by

$$_{D^{-1}[a,\Omega]}\langle Lu, v\rangle_{D_0[a,\Omega]} := \int_\Omega \alpha(u, v)\, dm \qquad \forall u, v \in D_0[a, \Omega],$$

estabilishes a correspondence between $D_0[a, \Omega]$ and $D^{-1}[a, \Omega]$; since $b \in D^{-1}[a, \Omega]$ too, it would be natural to read (2.6) also in the sense of $D^{-1}[a, \Omega]$, namely

$$\text{(4.1)} \qquad \begin{cases} Lu + g(u, L_i(u)) = b & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore a natural question arises, actually if

$$\int_\Omega g(u, L_i(u))\, v\, dm = {}_{D^{-1}[a,\Omega]}\langle g(u, L_i(u)), v\rangle_{D_0[a,\Omega]}.$$

In the context of classical Sobolev spaces, a result like this has been proved by Brezis and Browder (see [8] and [9]); the lucky fact is that it remains true also in our framework. In fact we have the following

Proposition 4.1: *Let us consider a relatively compact open subset $\Omega \subsetneq X$ s.t. $\Omega \subset B(0, R) \subset X$ with R proper. Let $T \in D^{-1}[a, \Omega] \cap L^1_{loc}(\Omega, m)$ and $u \in D_0[a, \Omega]$ s.t.*

$$Tu \geq f \quad a.e. \text{ in } \Omega$$

with $f \in L^1(\Omega, m)$. Then $Tu \in L^1(\Omega, m)$ and

$$_{D^{-1}[a,\Omega]}\langle T, u\rangle_{D_0[a,\Omega]} = \int_\Omega Tu\, dm.$$

It is immediate to see that this Proposition applies to our case due to (2.4), (2.5) and the unilateral growth condition given by (2.2).

The proof of Proposition 4.1 is based on the following lemma

Lemma 4.2: *Let $\beta \in R_+$, $u \in D_0[a, \Omega]$, $v \in D_0[a, \Omega] \cap L^\infty(\Omega, m)$. If we set*

$$w := \frac{u}{\sqrt{u^2 + \beta^2}} \min\left\{\sqrt{u^2 + \beta^2} - \beta, \sqrt{v^2 + \beta^2} - \beta\right\}$$

then $w \in D_0[a, \Omega] \cap L^\infty(\Omega, m)$ and

$$\alpha(w, w) \leq K \max\{\alpha(u, u), \alpha(v, v)\}.$$

PROOF: Let us consider $\varphi(t) = t/\sqrt{t^2+\beta^2}$. It is evident that

i) $\varphi(t) \in C^1(\mathbf{R})$;

ii) $\varphi(0) = 0$;

iii) $|\varphi(t)| < 1$;

iv) $|\varphi'(t)| = |(\beta^2/\sqrt{(t^2+\beta^2)^3})| \leq C$.

Therefore it is easy to verify that $\varphi(u) \in L^\infty(\Omega, m) \cap D_0[a, \Omega]$. Moreover

$$\alpha(\varphi(u), \varphi(u)) = \left(\frac{\beta^2}{\sqrt{(u^2+\beta^2)^3}}\right)^2 \alpha(u, u) = \frac{\beta^4}{(u^2+\beta^2)^3}\alpha(u, u).$$

Let us now consider

$$\zeta(t) = \sqrt{t^2+\beta^2} - \beta.$$

Since $\zeta(0) = 0$ and $|\zeta'(t)| \leq 1$, it is easy to see that $\forall z \in D_0[a, \Omega]$, $\zeta(z) \in D_0[a, \Omega]$. Finally, taken p and q in $D_0[a, \Omega]$, if we set $\psi = \min\{p, q\}$, using regularization and truncation results, it can be shown that $\psi \in D_0[a, \Omega]$. Therefore

$$\psi = \min\{\zeta(u), \zeta(v)\} \in D_0[a, \Omega] \cap L^\infty(\Omega, m),$$

the last inclusion due to the fact that $|\psi| \leq |v|$.

Since $D_0[a, \Omega] \cap L^\infty(\Omega, m)$ is an algebra, we can conclude that

$$w = \varphi(u) \min\{\zeta(u), \zeta(v)\} \in D_0[a, \Omega] \cap L^\infty(\Omega, m).$$

Moreover

$$\alpha(w,w) = \alpha(\varphi(u)\psi, \varphi(u)\psi) \leq \|\varphi(u)\|^2_{L^\infty(\Omega,m)}\alpha(\psi,\psi) + \|\psi\|^2_{L^\infty(\Omega,m)}\alpha(\varphi(u),\varphi(u)) \leq$$

$$\leq \alpha(\psi, \psi) + \|\psi\|^2_{L^\infty(\Omega,m)} \frac{\beta^4}{(u^2+\beta^2)^3}\alpha(u, u) \leq \alpha(\psi, \psi) + \max\{1, \beta^4\}\alpha(u, u).$$

Moreover

$$\alpha(\psi, \psi) = \alpha(\zeta(u), \zeta(u))\, 1|_{\{\zeta(u) < \zeta(v)\}} + \alpha(\zeta(v), \zeta(v))\, 1|_{\{\zeta(v) < \zeta(u)\}},$$

$$\alpha(\zeta(u), \zeta(u)) = |\zeta'(u)|^2\alpha(u, u), \qquad \alpha(\zeta(v), \zeta(v)) = |\zeta'(v)|^2\alpha(v, v).$$

Therefore

$$\alpha(\psi, \psi) \leq \alpha(u, u)\, 1|_{\{\zeta(u) < \zeta(v)\}} + \alpha(v, v)\, 1|_{\{\zeta(v) < \zeta(u)\}} \leq 2 \max\{\alpha(u, u), \alpha(v, v)\}.$$

Hence

$$\alpha(w, w) \leq 2 \max\{\alpha(u, u), \alpha(v, v)\} + \max\{1, \beta^4\}\, \alpha(u, u) \leq$$

$$\leq \max\{3, 2 + \beta^4\} \max\{\alpha(u, u), \alpha(v, v)\}.$$

If we set $K = \max\{3, 2 + \beta^4\}$, we are done. ■

Proof of proposition 4.1: If $u \in D_0[a, \Omega] \cap L^\infty(\Omega, m)$ is such that $\operatorname{supp} u \subseteq K \subset\subset \Omega$, relying on the density of $D[a, \Omega] \cap C_0(\Omega)$ in $D_0[a, \Omega]$ the result is immediate.

Let us now consider the general case. Thanks to the above recalled density, we can take a sequence $\{v_n\} \subseteq D[a, \Omega] \cap C_0(\Omega)$, such that $v_n \to u$ in $D_0[a, \Omega]$ and $v_n \to u$ a.e. in Ω. Let us now set

$$w_n = \frac{u}{\sqrt{u^2 + 1/n^2}} \min\left\{\sqrt{u^2 + \frac{1}{n^2}} - \frac{1}{n}, \sqrt{v^2 + \frac{1}{n^2}} - \frac{1}{n}\right\}.$$

Due to Lemma 4.2 we have that $w_n \in D_0[a, \Omega] \cap L^\infty(\Omega, m)$. Moreover

i) $|w_n| \leq |u|$ a.e. in Ω;

ii) $\alpha(w_n, w_n) \leq 3 \max\{\alpha(u, u), \alpha(v_n, v_n)\}$;

iii) $w_n \to u$ a.e. in Ω.

We can then conclude that

a) $\|w_n\|_{D_0[a, \Omega]} \leq C$, $\forall n$;

b) $w_n \rightharpoonup \chi$ in $D_0[a, \Omega]$;

c) thanks to iii) it must be $\chi = u$.

Therefore

$$w_n \xrightarrow{D_0[a, \Omega]} u \Rightarrow {}_{D^{-1}[a, \Omega]}\langle T, w_n\rangle_{D_0[a, \Omega]} \to {}_{D^{-1}[a, \Omega]}\langle T, u\rangle_{D_0[a, \Omega]}.$$

If we set

$$\lambda_n = \frac{1}{\sqrt{u^2 + 1/n^2}} \min\left\{\sqrt{u^2 + \frac{1}{n^2}} - \frac{1}{n}, \sqrt{v_n^2 + \frac{1}{n^2}} - \frac{1}{n}\right\},$$

it is clear that $0 \leq \lambda_n \leq 1$ and $w_n = \lambda_n u$. Therefore

$$Tw_n = T\lambda_n u = \lambda_n Tu \geq \lambda_n f \geq -|f|.$$

Since $w_n \in D_0[s, \Omega] \cap L^\infty(\Omega, m)$ and $\operatorname{supp} w_n \subseteq K \subset\subset \Omega$, we obtain

$$D^{-1}[s, \Omega]\langle T, w_n \rangle_{D_0[s, \Omega]} = \int_\Omega T w_n \, dm .$$

Applying Fatou's Lemma we have

$$\int_\Omega Tu \, dm \leq \lim_{n \to \infty} \int_\Omega T w_n \, dm = \lim_{n \to \infty} D^{-1}[s, \Omega] \langle T, w_n \rangle_{D_0[s, \Omega]} = D^{-1}[s, \Omega]\langle T, u \rangle_{D_0[s, \Omega]} .$$

Hence $Tu \in L^1(\Omega, m)$. Moreover $|Tw_n| \leq |Tu|$ and applying the dominated convergence theorem we obtain

$$D^{-1}[s, \Omega]\langle T, u \rangle_{D_0[s, \Omega]} = \lim_{n \to \infty} D^{-1}[s, \Omega]\langle T, w_n \rangle_{D_0[s, \Omega]} = \lim_{n \to \infty} \int_\Omega T w_n \, dm = \int_\Omega Tu \, dm ,$$

that is

$$D^{-1}[s, \Omega]\langle T, u \rangle_{D_0[s, \Omega]} = \int_\Omega Tu \, dm . \quad \blacksquare$$

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