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A General Axiomatic Framework for the Foundations of Mathematics, Logic and Computer Science (**)(***)

SUMMARY. — We present a non-reductionist, self-descriptive, open-ended axiomatic framework for the Foundations of Mathematics, Logic and Computer Science developed by the authors, jointly with Ennio De Giorgi, during the year 1996. This framework is based on the primitive «premathematical» notions of quality and relation, as in [8], [9]. We introduce and axiomatize also the notions of operation, collection and correlation, proposition and predicate, natural number and finite sequence, set, system and function. The engrafting of some basic concepts of Real Analysis and of Category Theory is also sketched. The consistency strength of the whole theory is analyzed in the final section.

Un quadro assiomatico generale per i fondamenti di Matematica, Logica e Informatica

RIASSUNTO. — Proponiamo un quadro assiomatico non riduzionista, autodescrittivo ed aperto ad estensioni per i Fondamenti di Matematica, Logica e Informatica, elaborato con Ennio De Giorgi durante l'anno 1996. Questo quadro è fondato sulle nozioni primitive «prematematiche» di qualità e relazione, come in [8], [9]. In questo quadro si introducono e assiomatizzano le nozioni di operazione, collezione e correlazione, proposizione e predicato, numero naturale e sequenza finita, insieme, sistema e funzione. Si accenna anche all'innesto di alcuni concetti fondamentali dell'analisi reale e della teoria delle categorie. Nella sezione conclusiva si esamina la forza di consistenza dell'intera teoria.

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Introduction

This paper arises within the general research programme on the Foundations of Mathematics, Logic and Computer Science, carried out since the early eighties at the the Seminar directed by Ennio De Giorgi at the Scuola Normale Superiore in Pisa (see [21]). The main aim of this programme is not to provide *safe and unquestionable grounds* to scientific activity, but rather to develop *conceptual environments* where this activity can be carried out naturally and without artificial constraints.

The foundational programme of E. De Giorgi was inspired by the following principles, which he used to epitomize quoting Hamlet (see [4], [8], [9]):

«*There are more things in heaven and earth, Horatio, than are dreamt of in your philosophy*» (Hamlet, Act I, Sc. V, vv. 166-167).

– *Non-reductionism*: there are many kinds of «qualitatively different» objects and concepts which are studied by science and humanity. Although the set-theoretic encoding of many mathematical concepts has been fruitful in providing a deep logical analysis of these concepts, nevertheless reducing *natural mathematical notions* to their set-theoretic codings can undermine the conceptual clarity of the notions themselves. E.g. reducing *natural numbers* to *Von Neumann ordinals*, *ordered pairs* to *Kuratowski doubletons*, *binary relations* and *simple operations* to *graphs* makes it difficult, even impossible at times, to formulate appropriate axioms and conjectures. For instance, the intuitive notion of *operation* subsumes the *intensional* concept of computation process, and so operations cannot be simply coded by their graphs. Similarly, conceiving *collections* as *truth-valued operations* forces unnecessary commitments on the definition of collection, and yet it does not make apparent their intrinsic *extensionality*. Taking *natural numbers* as primitives allows for a clearer analysis of the connections between different implementations such as Frege-Russell cardinals, Von Neumann ordinals, Church numerals, etc. Moreover the possibility of using different kinds of objects simplifies the enterprise of engrafting new notions.

– *Open-endedness*: this framework is open to extensions in any conceivable direction. The introduction of qualitatively new notions, from Mathematics, Logic, Computer Science, Physics, Biology, Economics, Theology, Linguistics, etc. is made possible in a natural way. In fact our aim is to provide a foundational framework suitable for accommodating any sufficiently clear concept arising in the different fields of sciences and humanity. For example, the axiomatic framework of [8] has been used in [9] for engrafting metamathematical concepts such as *metaqualities* and *judgements*; in [18] for a detailed treatment of *operations*, *collections* and *sets*; and in [16] for introducing *contingent* qualities and *modal* concepts. Some basic notions from *Biology* are considered in [12] within that framework.

Other important characters of these foundational theories are:

– *Selfdescription*: the most relevant properties, operations and relations considered in this theory are themselves objects of the theory. In particular, various *qualities* are introduced in order to classify the different *species* of objects, including operations, collections and qualities themselves. Also the *assertions* and the *predicates* which arise in developing this theory are objects of the theory. In fact, we introduce in this paper propositions and predicates which speak about the behaviour of *all* qualified kinds of objects, including predicates and propositions themselves. There are also several logical operations, relations and qualities, involving predicates and propositions, including the quality of being a *true proposition*.

– *Semiformal axiomatization*: this theory is developed using the axiomatic method of traditional mathematics. Moreover, in order to keep the metatheoretic requirements to a minimum, it is *finitely* axiomatized. Although many first-order formalizations can be given (see e.g. [20], [23]), none can be satisfactorily taken as definitive. In fact, any formalization misses part of the intended meaning of the original theory. Last but not least, this theory is not conceived for *specialists* in Foundations only, but rather proposed to the analysis and the criticism of all interested scholars. Therefore an exposition in a rigorous yet informal style seems more appropriate.

Earlier proposals of similar foundational theories appear in [3], [5], [6]. Further investigations, along these lines, have been carried out by various mathematicians, logicians and computer scientists since Spring 1994, starting from the «Basic Theories» introduced in [6]. For example, in [10] and [11], metamathematical notions, such as *formula, proposition*, and *interpretation* are engrafted by introducing suitable kinds of objects together with relations and operations involving them. In [7], the concept of *variable* of classical Mathematical Physics and of Economics is engrafted, in the same style. In [17] and [15] a similar engrafting is given for the general notions of *n-ary operation* and respectively of *collection, correlation, set* and *function*.

This paper amounts to a general presentation of the ideas developed by Ennio De Giorgi, together with the authors, during the year 1996. They should constitute the skeleton of a general paper on the foundations, which remained incomplete after the death of Ennio De Giorgi in October 1996. For the benefit of those readers which have not been previously exposed to presentations of the foundational theories *à la* De Giorgi, we think that it is appropriate to give a brief disclaimer of the intentions of the authors, which can help to eliminate some misunderstanding.

– This theory has no *constructivist* tenet. Although we believe that constructive mathematics is one of the highest achievements of this century, still we think that even constructive mathematics itself can benefit by serious research stemming from different philosophical standpoints. Banning certain methods would merely impose useless constraints to scientific activity.

– A significant point of difference between our theory and the existing frameworks is that all of them come as *formal* systems, while our theory is intended as an axiomatization of many different concepts, which *can*, but not necessarily *has to be*, specified in any formal system. We do not intend to propose a *new* formal system, which should have, because of some technical feature, a stronger expressive power than existing logical frameworks such as First or Higher Order Logic, λ-calculus, Type Theory, Set Theory, Category Theory. The expressive power of this theory derives rather from its possibility of *accommodating naturally* all the framework theories above, without having to utilize arbitrary encodings. Hence it seems probably meaningless to compare *formally* the foundational theory that we propose here to existing ones: our goal is that of *engrafting*, and hence of encompassing, rival theories, not that of *superseding* them. However, when a formalization is carried out, say in First Order Logic (as done in section 8 below), it generates very comprehensive formal systems.

– This foundational theory can appear *prima facie* unnecessarily complicated, but this seems unescapable if we want to represent *naturally* within this framework, at least in principle, the multitude of conceptual systems.

The paper is organized as follows. In Section 1, we introduce the first fundamental concepts of the theory, namely *qualities* and *relations*, as in [8], [9], [18]. In Section 2 we introduce the *operations* and we develop a simplified version of [18]. In Section 3 we consider various kinds of *propositions* and *predicates* together with logical operations acting on them. In Section 4 we propose a general theory of *collections* and *correlations*. In Section 5 we introduce the *natural numbers* and we deal with the concept of *finiteness*. In Section 6 we consider *sets*, *systems* and *functions*. In Section 7 we suggest the engrafting of some basic concepts from *Real Analysis* and *Category Theory*. In the final section we consider the consistency strength of a formal version of the whole theory and we sketch the construction of a model within a suitable set theory.

The authors are grateful for many useful discussions to all the participants in the Seminar on the Foundations of Mathematics at the Scuola Normale Superiore in Pisa, and in particular to Ennio De Giorgi, Furio Honsell and Vincenzo M. Tortorelli. In the spirit of all the papers arising from the foundational research originated by Ennio De Giorgi, we hope that also this paper might foster reactions, further contributions or just comments, from researchers in various areas of Science and Humanity.

1. - Fundamental qualities and relations

Mathematics, Logic and Computer Science, as well as most disciplines in Science and Humanity, deal with *several qualitatively different* objects and study *relations* over them. It seems therefore appropriate to propose, as starting point, a general axiomatic framework consisting of a few fundamental *qualities* and *relations*. Additional qualities and relations peculiar to each specific field can then be naturally introduced within this framework.

Following [8], we isolate as *primitive*, i.e. not reducible to preceding concepts, the following notions:

- *the object q is a quality*;
- *the object x has the quality q* (written as $q\ x$);
- *the object r is a binary relation*;
- *the objects x, y are in the binary relation r* (written as $r\ x, y$);
- *the object s is a ternary relation*;
- *the objects x, y, z are in the ternary relation s* (written as $s\ x, y, z$);
- *the object t is a quaternary relation*;
- *the objects x, y, z, w are in the quaternary relation t* (written as $t\ x, y, z, w$).

In accordance to the general principle of *selfdescription* we introduce first four distinguished qualities corresponding to the primitive kinds of objects considered above, i.e. the *classifying qualities*: *Qqual, Qrelb, Qrelt, Qrelq* (*). Hence we postulate:

Axiom 1.1: *Qqual, Qrelb, Qrelt, Qrelq are qualities.*
1) *No object has simultaneously two of the above qualities*;
2) *x is a quality if and only if Qqual x*;
3) *x is a binary relation if and only if Qrelb x*;
4) *x is a ternary relation if and only if Qrelt x*;
5) *x is a quaternary relation if and only if Qrelq x*.

We introduce next the *fundamental relations Rqual, Rrelb, Rrelt*, which describe the behaviour of qualities and relations. Hence we postulate:

Axiom 1.2: *Rqual is a binary relation*:
1) *if Rqual x, y then Qqual x*;
2) *if Qqual q then Rqual $q, x \Leftrightarrow q\ x$*.

Axiom 1.3: *Rrelb is a ternary relation*:
1) *if Rrelb x, y, z then Qrelb x*;
2) *if Qrelb r then Rrelb $r, x, y \Leftrightarrow r\ x, y$*.

Axiom 1.4: *Rrelt is a quaternary relation*:
1) if Rrelt x, y, z, w *then Qrelt x*;
2) *if Qrelt s then Rrelt $s, x, y, z \Leftrightarrow s x, y, z$*

(*) As an aid to recalling the intended meaning of the distinguished objects we use *suggestive names containing fragments of the corresponding* Latin word, e.g. qualities have prefix Q, *relations* R, qualities and relations concerning qualities or relations usually have suffix or infix *qual and* rel respectively. Moreover basic operations or collections have upper case initials and objects involving them have suffix or infix *op or* coll.

Finally we introduce also an *identity* relation *Rid* between objects of arbitrary kind:

AXIOM 1.5: *Rid is a binary relation such that Rid x, y holds if and only if x and y are the same object.*

Since *Rid* is the *identity relation* between arbitrary objects, we shall write $x = y$ in place of *Rid* x, y.

We do not exclude, of course, that there are also *quinary* relations, and in general relations of any larger arity. The reason for stopping here to quaternary relations is two-fold. On the one hand, in this paper we utilize only relations with arity up to four. On the other hand, we could have considered only *binary* relations and used some internal encoding of n-tuples, such as that of section 5. However, it seems more in line with our informing principles not to have to *encode* natural primitive concepts such as simple and binary operations, which are *naturally* described by means of a *ternary* and a *quaternary* relation in the next section. Apparently the action of quaternary relations is left here without *internal description*. However an internal description of *all kinds* of objects, including quaternary relations, is obtained in section 3, by introducing the *atomic propositions* and the notion of *truth*.

2. - OPERATIONS

We present in this section an axiomatization of the primitive notion of *operation*. It builds on the intuition of an operation as an object which *acts* (operates) on one, or two, objects and possibly produces a *result*. This presentation is a simplified version of section 2 of [18]. According to the pattern which will be followed methodically in engrafting mathematical and logical concepts in the framework of Section 1, we introduce suitable qualities which classify the objects under consideration and suitable relations which describe their behaviour. Here we introduce the qualities *Qops* and *Qopb* of being respectively a *simple* (*unary*) and a *binary* operation and the corresponding relations *Rops* and *Ropb*. The *functionality* of operations is expressed by postulating that the relations *Rops* and *Ropb* are «univalent». Hence we postulate:

AXIOM 2.1: *Qops is a quality and Rops is a ternary relation:*
1) *if Rops* x,y,z *then Qops* x;
2) *if Rops* f, x, y *and Rops* f, x, z *then* $y = z$.

AXIOM 2.2: *Qopb is a quality and Ropb is a quaternary relation:*
1) *if Ropb* x, y, z, w *then Qopb* x;
2) *if Ropb* g, x, y, z *and Ropb* g, x, y, w *then* $z = w$.

Given the axioms above, we can adopt unambiguously the standard notations:

– if *Qops* f, then $y = fx$ stands for *Rops* f, x, y;

– if *Qopb* g then $z = gxy$ stands for *Ropb* g, x, y, z.

If there exists an object y such that $y = fx$ we say that «y is the *result* of f applied to x», that «f is *defined* at x», or that «fx exists»; moreover we denote such y by fx. If no such y exists, then we say that «f is *undefined* (or not defined) at x». Similarly, if there is z such that $z = gxy$, we say that «z is the *result* of g applied to x, y», that «g is *defined* at x, y» or that «gxy exists»; moreover we denote such z by gxy. If no such z exists, we say that «g is *undefined* (or not defined) at x, y».

We have *not* postulated *extensionality* for operations, since we do not want to rule out the possibility that there exist operations, acting on the *same* objects and giving the *same* result on each object, which are nonetheless *different*.

The fact that we have considered only *simple* and *binary* operations is somewhat arbitrary. In fact, having at our disposal ternary and quaternary relations, we can easily describe the behaviour of simple and binary operations by means of the relations *Rops* and *Ropb*. Also operations acting «naturally» on more than two arguments arise in some contexts (e.g. the «birapporto» in projective geometry), but they are rather rare. Moreover, in order to deal with the general concept of n-place operation (as done in [6], [11], [17]), one has to ground on the concepts of arbitrary natural number and tuple, which we prefer to postpone (see section 5). We could have done away even with *binary* operations, by encoding them as «iterated» unary operations. We prefer nonetheless to keep both notions since, in the field of Mathematics, Logic and Computer Science, many operations appear naturally as *binary*, (e.g. addition of real numbers, composition of operations, conjunction of formulae). However, we consider in this paper only binary operations which appear to be *natural* and *basic*.

On the other hand, we intend to achieve a significant completeness at the level of simple operations, hence introduce the operation *Cur* (*abstraction à la* Curry), which reduces each binary operation to the iteration of a simple one, the operation K (which generates simple *constant operations*), the operation *Inv* (*inversion* of simple operations), and the binary operations *Comp* (*composition*) and S (*substitution à la* Schönfinkel). We postulate the following axioms:

Axiom 2.3: *Cur is a simple operation. For all binary operations g, Cur g exists and is a simple operation. For all x, $(Cur\ g)x$ exists and is a simple operation such that*

$$\forall xyz \quad (((Cur\, g)x)y = z \Leftrightarrow gxy = z).$$

Axiom 2.4: *K is a simple operation. For all x, Kx exists and is a simple operation such that*

$$\forall xy.\ (Kx)y = x.$$

AXIOM 2.5: *Comp is a binary operation. If f, g are simple operations, then Comp fg exists and is a simple operation such that*

$$\forall xy \ ((Comp\ fg)x = y \Leftrightarrow \exists z \ (gx = z \wedge fz = y)).$$

AXIOM 2.6: *S is a binary operation. If f, g are simple operations, then Sfg exists and is a simple operation such that*

$$\forall xy \ ((Sfg)x = y \Leftrightarrow \exists hz \ (gx = z \wedge fx = h \wedge hz = y)).$$

AXIOM 2.7: *Inv is a simple operation. If f is a simple operation, then Inv f exists and is a simple operation such that:*

1) $(Inv\ f)x = y \Rightarrow fy = x$;
2) $\forall x (\exists y ((Inv\ f)x = y) \Leftrightarrow \exists y (fy = x))$.

Following [18] we introduce also three operations which are basic in dealing with partial operations: the *equality* and *inequality* tests *Eq* and *Neq* and the *binary union Bun*. Hence we postulate:

AXIOM 2.8: *Eq and Neq are binary operations.*

1) *For all x, y, z, $Eq\ xy = z$ if and only if $x = y = z$.*
2) *For all x, y, z, $Neq\ xy = z$ if and only if $x \neq y$ and $y = z$.*

AXIOM 2.9: *Bun is a binary operation. If f, g are simple operations, then Bun fg exists and is a simple operation such that*

1) $\forall xy \ ((Bun\ fg)x = y \Rightarrow fx = y \vee gx = y)$;
2) $\forall x \ (\exists y \ ((Bun\ fg)x = y) \Leftrightarrow \exists y \ (fx = y \vee gx = y))$.

Although the Axioms 2.1-2.9 constitute a system which is essentially weaker than those of [17], [18], nevertheless they yield a rather rich theory of simple operations. One can show, for instance, the existence of all simple *finite operations*, and that simple operations are closed under *composition*, *intersection*, «*permutation of the arguments*», etc. We have not postulated the existence of an *identity* operation since this can be obtained as usual by taking $I = SKK$.

Of particular interest for the next chapter is the notion of «generalized permutator» which we introduce by means of the quality *Qgperm*. Roughly speaking, a generalized permutator τ is an operation which «rearranges the arguments» of a given operation, i.e.

$$(((\tau f)x_1)x_2)\ldots x_n = ((fx_{i_1})x_{i_2})\ldots x_{i_k},$$

where $i_1, \ldots i_k$ are positive integers not exceeding n.

Clearly, K and I have to be counted among the generalized permutators, as well as any operation of «transposition» T verifying the following condition:

If f is a simple operation whose values are simple operations, then Tf exists and is a simple operation whose values are simple operations and such that

$$\forall xyz.\,(((Tf)x)y = z \Leftrightarrow (fy)x = z).$$

We can obtain T by the following formidable expression:

$$S(S(KS)(S(KK)(S(KS))))(K(S(KK)I)).$$

We do not characterize completely the notion of generalized permutator, since there are several more or less restrictive notions deserving this name. We simply postulate here that generalized permutators are «preserved by composition»:

AXIOM 2.10: *Qgperm is a quality.*

1) If *Qgperm* x, *then Qops* x;
2) *if Qgperm* τ *and* $\tau x = y$, *then Qops* x *and Qops* y;
3) *Qgperm* K *and Qgperm* T;
4) *if Qgperm* τ *then Qgperm* (*Cur Comp*) τ.

3. - PROPOSITIONS AND PREDICATIVE OPERATIONS

In this section we deal with some very general ideas about *propositions* and *predicates* and we develop a sort of propositional and predicative calculus. We state the general axioms in a rather weak form; in particular the axioms about true and false propositions will concern mostly «classical» propositions and predicative operations, so that a wide freedom is left for possible «engraftings» of non-classical logics.

We begin by introducing the «most general» notion of proposition, by means of the quality *Qgprop*; so we will write *Qgprop* x to mean that x is a *proposition*, without further specifications. We single out also some propositions which obey the usual rules of classical propositional calculus, by means of the quality *Qclprop*; hence *Qclprop* x means that x is a *classical proposition*.

AXIOM 3.1: *Qgprop, Qclprop are qualities. If Qclprop* x *then Qgprop* x.

We let the usual logical operations act on all propositions; namely we introduce the operations *Et* (*conjunction*), *Vel* (*disjunction*), *Non* (*negation*), and state the following axioms:

AXIOM 3.2: *Et and Vel are binary operations. Non is a simple operation.*

1) *If* p, q *are propositions, then Et* pq, *Vel* pq *and Non* p *exist and are propositions.*
2) *If* p, q *are classical propositions, then also Et* pq *and Vel* pq *are classical.*
3) *Non* p *is a classical proposition if and only if* p *is a classical proposition.*

Following the usual notation we write $p \wedge q$, $p \vee q$, $\neg p$ instead of *Et pq*, *Vel pq*, *Non p*.

A characteristic feature of this theory is the presence of a quality *Qtver* corresponding to «*absolute and total truth*», differently from [9], [10], [11]; therefore we write *Qtver* x to mean that x is a *(absolutely) true proposition*. Since we intend the quality *Qgprop* as classifying all kinds of propositions (including imperative, interrogative, dubitative propositions, etc.) we cannot postulate the principle of the excluded middle (*tertium non datur*) for general propositions. We give in the following axiom a «principle of *non contradiction*» for propositions of general kind, and we state the classical truth rules for *classical* propositions, including the *tertium non datur*:

AXIOM 3.3: *Qtver is a quality.*

1) *If Qgprop p, then at most one of Qtver p, Qtver* $\neg p$ *holds.*
2) *If* p, q *are classical propositions then:*
   2.1) *Qtver p holds if and only if Qtver*$(\neg p)$ *does not hold.*
   2.2) *Qtver*$(p \wedge q)$ *holds if and only if both Qtver p and Qtver q hold;*
   2.3) *Qtver*$(p \vee q)$ *holds if and only if at least one of Qtver p, Qtver q holds.*

Having introduced the propositions as above, we can consider the classical concept of predicate *à la* Frege, namely as an *operation whose values are propositions* (see [19]). Many introductions of this concept are possible (see for instance [10], [22], [23]). In this exposition we prefer to avoid the use of tuples, hence we consider *simple total operations* which generate propositions either *directly*, or *after some iterations*. These operations are called *predicative operations* (shortly *predicates*): as we did in the case of propositions, we consider *predicative operations* of *general* type (characterized by the quality *Qgpred*) and *classical predicative operations* (characterized by the quality *Qclpred*). In this context, it seems appropriate to consider also propositions as «self-reproducing» predicative operations.

AXIOM 3.4: *Qgpred, Qclpred are qualities.*

1) *If Qclpred x, then Qgpred x.*

2) *If p is a predicate, then p is a simple total operation and px is a predicate for every x.*

3) *Let* τ *be a general permutator. Then, for every predicate p, there exists a predicate* p' *such that* $p'x = (\tau p)x$ *for all x.*

4) *If Qgprop p, then Qgpred p and* $Kp = p$. *If p is a predicate and the values of p are propositions, then* $Tp = Kp$.

5) *If p enjoys Qclpred, then, for every object x, px enjoys Qclpred.*

6) *Let* τ *be a general permutator. Then for every classical predicate p there exists a classical predicate* p' *such that* $p'x = (\tau p)x$ *for all x.*

7) *Qclprop p holds if and only if Qclpred p and Qgprop p.*

The clauses 3) and 6) in the axiom above allow to «permutate variables» in iterated (classical) predicates; moreover, taken together with clause 2), it provides (classical) predicates corresponding to «adding dummy variables» inside a given predicate.

We extend the action of the operations of conjunction, disjunction and negation to all the predicative operations, according to the following commutation rules:

AXIOM 3.5: *Let* p, q *be predicative operations. Then*:

1) *Non* p *exists and is a predicative operation such that* $\forall x.(Non\ p)x = Non(px)$.
2) *Et* pq *exists and is a predicative operation such that* $\forall x.(Et\ pq)x = Et(px)(qx)$.
3) *Vel* pq *exists and is a predicative operation such that* $\forall x.(Vel\ pq)x = Vel(px)(qx)$.
4) *Non* p *is classical if and only if* p *is classical.*
5) *If both* p, q *are classical then also Et* pq, *Vel* pq *are classical.*

Also in the case of predicative operations we adopt the usual notations $p \wedge q$, $p \vee q$, $\neg p$ instead of *Et* pq, *Vel* pq, *Non* p respectively.

In order to enhance the *self-descriptive power* of the theory, we ensure the existence of propositions which describe the action of any kind of objects, such as qualities, relations and operations, including predicates and propositions themselves. These are given by the operation *Gopr*, generating *atomic predicative operations*. The operation *Gopr* satisfies the following axioms:

AXIOM 3.6: *Gopr is a simple operation. If* t *is a quaternary relation, then*:

1) *Gopr* t *is a predicative operation*;
2) $((((Gopr\ t)x)y)z)w$ *is a proposition for any* x, y, z, w;
3) $\forall xyzw.(Qtver\ ((((Gopr\ t)x)y)z)w \Leftrightarrow t\,x, y, z, w)$

(*i.e.* $((((Gopr\ t)x)y)z)w$ *is true if and only if* x, y, z, w *are in the relation* t).

The points 2) and 3) of the axiom above justify the «quoted notation»

$$«t\,x, y, z, w» = ((((Gopr\ t)x)y)z)w\,.$$

Since quaternary relations describe the behaviours of all other kinds of objects, we can give the action of the operation *Gopr* on the remaining kinds of objects considered in the previous sections in the natural way, namely:

AXIOM 3.7:

1) *If* s *is a ternary relation, then* $Gopr\ s = (Gopr\ Rrelt)s$.
2) *If* r *is a binary relation, then* $Gopr\ r = (Gopr\ Rrelb)r$.
3) *If* q *is a quality, then* $Gopr\ q = (Gopr\ Rqual)q$.
4) *If* f *is a simple operation, then* $Gopr\ f = (Gopr\ Rops)f$.
5) *If* g *is a binary operation, then* $Gopr\ g = (Gopr\ Ropb)g$.

The «atomic» propositions generated by the predicates obtained by the operation

Gopr are assertions concerning the qualities or relations to which *Gopr* is applied; hence we extend the quoted notation in the natural way, namely:

- if q is a quality, then «$q\,x$» stands for $(Gopr\ q)x$;
- if r is a binary relation, then «$r\,x, y$» stands for $((Gopr\ r)x)y$;
- if s is a ternary relation, then «$s\,x, y, z$» stands for $(((Gopr\ s)x)y)z$;
- «$x = y$» stands for $((Gopr\ Rid)x)y$, and «$x \neq y$» stands for $\neg(((Gopr\ Rid)x)y)$.

Comparing the point 3) of the axiom 3.6 with the axioms of the previous sections one recognizes an interesting phenomenon of *mutual reference*. On the one hand, the quaternary relations *Rrelt* and *Ropb* describe *directly* the behaviour of ternary relations and binary operations and, by means of *Rrelb*, *Rqual*, *Rops*, also that of binary relations, qualities and simple operations. On the other hand, the simple operation *Gopr* generates predicates producing *atomic propositions* which in turn, by means of the quality *Qtver*, describe the behaviour of objects of *any kind*, including quaternary relations. This circularity justifies the choice of stopping to quaternary relations, since it provides an *internal description* of the action all objects.

Before introducing the *existential* and the *universal quantifiers* as operations *Exist*, *Univ* which act on predicative operations, some remarks are useful. The intended meaning of universal quantification, say, is that for any object x, the meaning of $(Univ\ p)x$ involves all values $(py)x$ for every object y. Hence we have to consider besides p also the «transposed predicate» Tp such that $((Tp)x)y = (py)x$. Actually, we postulated the clauses 2) and 3) of axiom 3.4 in order to make possible the «quantification with respect to arbitrary variables». Therefore we can formulate the axiom on quantifiers as follows:

AXIOM 3.8: *Univ and Exist are simple operations. Let p be a predicate; then*:

1) *Both Exist p and Univ p exist, and are predicates such that*:

$$\forall x.\,(Univ\ p)x = Univ\ (Tp)x \quad \text{and} \quad \forall x.\,(Exist\ p)x = Exist\ (Tp)x.$$

2) *If p is classical then both Univ p and Exist p are classical.*
3) *If p is classical and its values are propositions, then*:
   3.1) *Univ p is true if and only if the proposition px is true for all x*;
   3.2) *Exist p is true if and only if the proposition px is true for some x.*

We will often use the notations $\forall p$, $\exists p$ instead of *Univ p*, *Exist p*. When the values of p are propositions, we use also the more standard notations $\forall x.\,px$, $\forall y.\,py$, etc. instead of *Univ p* and $\exists x.\,px$, $\exists y.\,py$, etc. instead of *Exist p*. Of course this notation does not imply any reference to specific objects x, y, etc.

After introducing the propositions, the predicative operations, the logical operations *Non*, *Et*, *Vel*, *Exist*, *Univ* and the «atomic» predicates defined by means of *Gopr*, we face the problem of finding «classicality axioms» which do not lead to contradiction.

Some negative results, essentially inspired to the Liar Antinomy (or to Tarski's theorem, see [1]), have been proved in analogous situations, see [9], [10], [11]. In particular one cannot consistently postulate that all atomic predicates are classical. In fact one can prove:

THEOREM 1: *At least one of the predicates*

$$Gopr\ Qtver,\ Gopr\ Rid,\ Gopr\ Rops$$

is not classical. Hence Gopr Rrelt is not classical.

PROOF: Assume the contrary. Then by Axiom 3.4 there exists a classical predicate R such that $((Rx)y)z =$ «$Rops\ y, z, x$». Put

$$P = Non(Gopr\ Qtver) \wedge K(Gopr\ Rid) \wedge R\,;$$

$$Q = Gopr\ Qtver \wedge K(Gopr\ Rid) \wedge R\,.$$

Then $((Px)y)z$ means that «y is an operation, $x = yz$, $y = z$ and x does not enjoy $Qtver$», whereas $((Qx)y)z$ means that «y is an operation, $x = yz$, $y = z$ and x enjoys $Qtver$».

Now put $L_1 = \exists\exists P$ and $L_2 = \forall\forall\neg Q$. Then $L_1 z$ means that «z is an operation which, applied to itself, gives a result which is not a true proposition», whereas $L_2 z$ means that «if z is an operation which applies to itself, then the result is not a true proposition».

Clearly the propositions $L_1 L_1$ and $L_2 L_2$ are statements of Liar's type, since both are true (enjoy $Qtver$) if and only if they are not true (do not enjoy $Qtver$). Hence both propositions $L_1 L_1$ and $L_2 L_2$ cannot be classical and the thesis follows. Q.E.D.

The search for the strongest axioms of classicality which do not lead to contradiction is an interesting problem which has not yet received a definitive answer.

In order to deal classically with ordinary mathematics, the following axioms could be postulated:

AXIOM Cl: *The following predicates are classical:*

$$Gopr\ Rid,\ Gopr\ Rcoll,\ Gopr\ Rcorr,\ Gopr\ Rops \text{ and } Gopr\ Ropb.$$

In order to have an environment suitable for freely using predicates in the axiomatizations of collections, sets and natural numbers, we introduce the intermediate notion of *semiclassical* predicative operation. Roughly speaking, this notion is intended to be sufficiently wide to encompass all atomic predicates generated by *Gopr* and to be closed under the logical operations. On the other hand, it should be narrow enough to allow truth rules as close as possible to the classical ones. We introduce semiclassical predicates by means of the quality *Qscpred*, and state that all classical and all atomic predicates are semiclassical and that semiclassical predicates are closed under the logical operations.

Axiom 3.9: *Qscpred and Qscprop are qualities.*

1) $Qclpred\ x \Rightarrow Qscpred\ x \Rightarrow Qypred\ x.$
2) $Qscprop\ x \Leftrightarrow Qscpred\ x \wedge Qyprop\ x.$
3) $Qrelq\ t \Rightarrow Qscpred(Gopr\ t).$
4) *If p, q are semiclassical predicates then $p \wedge q$, $p \vee q$, $\neg p$, $\forall p$, $\exists p$, and px for all x are semiclassical.*
5) *If τ is a general permutator and p is a semiclassical predicate, then there exists a semiclassical predicate p' such that $p'x = (\tau p)x$ for any x.*

We cannot put on semiclassical predicates the classical truth rules, since now we can prove that both predicates P and Q of Theorem 1 above are semiclassical.

Therefore we are led to isolate truth rules which are weaker than the classical ones but still preserve most of the common intuitive meaning of truth. Taking into account that, as far as the atomic predicates generated by *Gopr* are concerned, the universe «determines» the truth of all boolean combinations, we propose an axiom which maintains the usual classical boolean structure to the propositional connectives and gives a more «constructive» content to the quantifiers.

Axiom Scl: *If p, q are semiclassical propositions then:*

1) $Qtver(\neg p) \Leftrightarrow \neg(Qtver\ p);$
2) $Qtver(p \wedge q) \Leftrightarrow Qtver\ p \wedge Qtver\ q;$
3) $Qtver(p \vee q) \Leftrightarrow Qtver\ p \vee Qtver\ q.$

If p is a semiclassical predicate and px is a proposition, then:

4) $Qtver(px) \Rightarrow Qtver(\exists p);$
5) $Qtver(\forall p) \Rightarrow Qtver(px).$

Notice that assuming the axiom Scl, the intuitively equivalent Liar's statements L_1L_1 and L_2L_2 behave differently. In fact L_1L_1 becomes true whereas L_2L_2 becomes false, i.e. $\neg(L_2L_2)$ is true!

4. - Collections and correlations

We present in this section an axiomatization of the primitive notions of *collection* and *correlation* (or correspondence). Our notion of collection is intended to capture the most general concept of aggregate underlying the notions of *class*, as conceived by Frege and Russell, see [2]. In ordinary set theory, *graphs* (i.e. sets of pairs) play the role of relations and operations. In our non-reductionist setting, we prefer to single out *correlations* (or correspondences) as an independent primitive kind of objects, a sort of «binary analogue» of collections, which will be suitably connected, but not identified, with that of *graph*.

We introduce the quality *Qcoll* of being a *collection*, together with the relations *Rcoll* of *membership* and *Rincl* of *inclusion* and the quality *Qcorr* of being a correlation together with the ternary relation *Rcorr*. These notions are «extensional» in nature, in the sense that collections which have the same members are *identical*, and correlations which have «the same graph» are also identical. Hence we postulate *extensionality* of collections and correlations.

AXIOM 4.1: *Qcoll is a quality, Rcoll and Rincl are binary relations.*

1) *If* $Rcoll\ x, y$ *then* $Qcoll\ x$.
2) *If* C, D *are collections then* $Rincl\ C, D \Leftrightarrow \forall x (Rcoll\ \ D, x \Rightarrow Rcoll\ \ C, x)$.
3) *If* C, D *are collections then* $Rincl\ C, D \wedge Rincl\ D, C \Rightarrow C = D$.

AXIOM 4.2: *Qcorr is a quality. Rcorr is a ternary relation.*

1) *If* $Rcorr\ x, y, z$ *then* $Qcorr\ x$.
2) *If* F, G *are correlations, then* $Rincl\ F, G \Leftrightarrow \forall x\ y (Rcorr\ G, x, y \Rightarrow Rcorr\ F, x, y)$.
3) *If* F, G *are correlations then* $Rincl\ F, G \wedge Rincl\ G, F \Rightarrow F = G$.

When C, D are collections and x is any object, we use the standard notation $x \in C$ for $Rcoll\ C, x$, and $C \supseteq D$, $D \subseteq C$ for $Rincl\ C, D$.

When G is a correlation, we adopt the same notation of binary relations and write $G\, x, y$ for $Rcorr\ G, x, y$.

We postulate the existence of some general collections and correlations, namely: the collections *Coll* of all collections, *Corr* of all correlations, *Corfun* of all the *univalent* or *functional correlations* and the correspondence *Graph* between correlations and collections.

AXIOM 4.3: *Coll, Corr and Corfun are collections. Graph is a correlation.*

1) $x \in Coll$ *if and only if* $Qcoll\ x$.
2) $x \in Corr$ *if and only if* $Qcorr\ x$.
3) $F \in Corfun$ *if and only if* $F \in Corr$ *and* $\forall xyz (F\, x, y\ \wedge\ F\, x, z \Rightarrow y = z)$.

4.1) *If* $Graph\ x, y$ *then* $x \in Corr$ *and* $y \in Coll$.
4.2) *For all* $x \in Corr$ *there exists a unique* $y \in Coll$ *such that* $Graph\ x, y$.

The third clause of Axiom 4.3 allows the use of the standard functional notation for functional correlations: if $F \in Corfun$, we write $F(x) = y$ instead of $F\, x, y$.

We could now introduce the usual class operations à la Von Neumann-Gödel (union, intersection, complement, domain, etc.), and let them act on collections and correlations. We could then derive an external «Bernays-like» *Comprehension Principle* for collections, as done in [18]. We prefer not to deal with *external* languages, but rather with the internal notion of predicate defined in § 3. Hence we introduce the collection *Compr* of the *comprehensible predicates* and we formulate the corresponding «Comprehension Principle». To this end we give the definition of *logically closed* collection of predicates,which will be useful in the sequel.

DEFINITION 4.1: A collection C of predicates is called *logically closed* if the following conditions are satisfied:

1) If $p \in C$ then $px \in C$ for any x.

2) If $p, q \in C$, then $p \wedge q$, $p \vee q$, $\neg p$, $\exists p$ and $\forall p$ belong to C.

3) If $p \in C$ and τ is a generalized permutator, then there is a predicate $p' \in C$ such that $p'x = (\tau p)x$ for any x.

AXIOM 4.4: *Compr is a logically closed collection of classical predicates.*

1) *If $p \in Compr$ and px is a proposition for all x, then there exists a collection $C = \{x|px\}$ such that $x \in C \Leftrightarrow Qtver\ px$.*

2) *If $p \in Compr$, and $(px)y$ is a proposition for all x, y, then there exists a correlation G such that $G\,x, y \Leftrightarrow Qtver\ (px)y$.*

The strength of the axiom above depends on the predicates that one puts *directly* into *Compr*. Notice that putting a predicate into *Compr* implies that it is classical. As a starting point we state the following uncompromised axiom, which corresponds to the principle underlying Gödel-Bernays comprehension. We leave it up to interested researchers in the field to propose or suggest stronger axioms.

AXIOM 4.5: 1) *If C is a collection then $Gopr\ C = (Gopr\ Rcoll)\ C \in Compr$.*
2) *If G is a correlation then $Gopr\ G = (Gopr\ Rcorr)\ G \in Compr$.*
3) *$Gopr\ Rid \in Compr$.*

Axioms 4.4-4.5 imply the existence of many collections and correlations:

i) There exist the *universal collection* V of all objects, the *empty collection* $\emptyset$, and the *identity correlation* Id, connecting each object to itself.

ii) Collections and correlations are closed under *binary union*, *binary intersection* and *difference* (relative complement); moreover correlations are closed under *composition* and *inversion*.

iii) If G is a correlation, then there exist collections $D = Dom\ G$ and $C = Cod\ G$, called the *domain* and the *codomain* of G respectively, such that $x \in D$ if and only if there is y such that $G\,x, y$, and $y \in C$ if and only if there is x such that $G\,x, y$.

Moreover, if C and D are collections, then there exists a correlation $D \times C$ called the *cartesian product* of D and C such that $(D \times C)x, y$ holds if and only if $x \in D$ and $y \in C$. Finally, given a collection C, there exists also the collection $\widehat{G}(C)$, the *image of C under G*, defined by $x \in \widehat{G}(C)$ if and only if there is $y \in C$ such that $G\,y, x$.

iv) If x, y are arbitrary objects, there exists the collection $\{x, y\}$ whose elements are exactly x and y, called the *doubleton* (or unordered pair) of x and y. If $x = y$ it is called the *singleton* of x and denoted by $\{x\}$.

v) If x, y are arbitrary objects there exists the correlation $\binom{x}{y}$, called the *singular correlation* of x,y, whose domain is the singleton of x and whose codomain is the singleton of y. Notice that, if G is a correlation, then $\binom{x}{y} \subseteq G \Leftrightarrow G\, x, y$.

In ordinary class and set theories, *functions, relations, operations*, etc. are *identified* with their «graphs». As already remarked, an unpleasant consequence of this «ultra-reductionist» attitude is that many fundamental relations and operations cannot be objects of the theory. We have chosen instead the maximal selfdescriptive capability, and we have introduced from the very beginning many descriptive relations, such as *Rqual, Rcoll, Rincl*. Therefore we cannot consistently postulate that all atomic predicates generated by *Gopr* are comprehensible. Russell's argument yields in fact:

THEOREM 2: (see [18]):

$$Gopr\ Rcoll,\ Gopr\ Rcorr,\ Gopr\ Rincl,\ Gopr\ Rrelb,\ Gopr\ Rrelt \notin Compr.$$

Theorem 2 yields that we cannot use the general membership relation in order to «produce» collections. In fact, we cannot perform on arbitrary collections some of the manipulations which are normally carried out in ordinary mathematical practice. We have the possibility of applying the basic Von Neumann-Gödel operations on classes, as seen in remarks i)-v) above. On the other hand we cannot prove even that collections are stable under *unary union* or *unary intersection*, or that the collections $\mathcal{SC}(X)$ (of all *subcollections* of the collection X) and X^Y (of all functional correlations with domain Y and taking values in X) always exist. *A fortiori* we cannot prove the existence of «cartesian products» of collection-valued correlations, and of «transpositions» of correlation-valued correlations. Since these constructions play an important rôle in many areas of Mathematics, Logic and Semantics, it is interesting to postulate the existence of operations *Un, Int, Subcoll, Cart, Trans* that carry out the intended tasks. Hence we follow [18] and give the axiom:

AXIOM 4.6: *Un, Int, Subcoll, Cart and Trans are simple operations.*

1) (Union) *If X is a collection of collections and $Un\ X = \bigcup X$ exists, then it is a collection such that*

$$t \in \bigcup X \Leftrightarrow \exists x \in X (t \in x).$$

2) (Intersection) *If X is a collection of collections and $Int\ X = \bigcap X$ exists, then it is a collection such that*

$$t \in \bigcap X \Leftrightarrow \forall x \in X (t \in x).$$

3) (Subcollections) *If X is a collection and Subcoll $X = \mathcal{SC}(X)$ exists, then it is a collection such that*

$$Y \in \mathcal{SC}(X) \Leftrightarrow Qcoll\, Y \wedge Y \subseteq X .$$

4) (Cartesian Product) *If F is a collection-valued functional correlation and Cart $F = \prod F$ exists, then it is a collection of functional correlations such that*

$$G \in \prod F \Leftrightarrow (Dom\, G = Dom\, F) \wedge \forall x \in Dom\, G.\ G(x) \in F(x)) .$$

5) (Transposition) *If F is a correlation-valued functional correlation and Trans $F = {}^t F$ exists, then it is a functional correlation taking on non-empty correlations as values, such that*

$$\forall x \in Dom\ F.\ \ \forall yz.\ F(x)\, y, z \Leftrightarrow {}^t F(y)\, x, z .$$

Notice that the *power* of collections can be obtained from the cartesian product by putting $X^Y = Cart\ F$, where $F = Y \times \{X\}$.

In order to grasp the action of the *transposition* operation *Trans*, it is convenient to visualize a special case which makes use of the «functional» notion of n-tuple, which we will introduce in the next section. Taking the n-tuple $(x_1, \ldots, x_n)$ to be the functional correlation assigning x_i to i, and viewing a matrix $m \times n$ as an n-tuple of m-tuples, then its *transposition* is an m-tuple of n-tuples, corresponding exactly to the *transposed matrix*. Noteworthy is the transposition of *n-tuples of univalent correlations with the same domain*, which gives a sort of *fibred product*, namely: ${}^t(f_1, \ldots, f_n)x = (f_1x, \ldots, f_nx)$.

The *non-emptyness* clause is given in order to characterize *uniquely* the transposition. It is particularly useful in view of the limitation of size principle, since it implies that transpositions of correlations of «small size» are small. In particular the empty correlation has empty transpose.

The axiom 4.6 is formulated «prudentially». It asserts only that, *if the operations involved are defined*, then they behave according to their intendend meaning. It is not enforced that the operations are defined anywhere. In order to make an effective use of arbitrary collections, the following «problematic» axiom could be appropriate:

Axiom Coll (see [18]): *The operations Un and Int are defined for every collection of collections, Subcoll is defined for every collection, Cart is defined for every collection-valued functional correlation, and Trans is defined for every correlation-valued functional correlation.*

The axiom Coll is a very powerful tool for using arbitrary collections in developing Mathematics and Logic. However, its consistency relative to traditional foundational theories is still unknown. Therefore we do not state it as part of our axiomatic framework, but we only propose it as an interesting topic for further investigation. In order to engraft the ordinary mathematical practice we choose instead a more traditional approach and introduce in section 6 the notions of *set* and *function*.

We conclude this section with some remarks on the operation *Graph*. Up to now, we have not specified which concept of *ordered pair* is used in the correspondence *Graph* between a correlation and the corresponding «graph». Obviously, once a suitable notion of ordered pair (x, y) is chosen, the axiom on *Graph* should state:

AXIOM 4.7: *Let G be a correlation. Then:*

$$\forall z(z \in Graph(G) \Leftrightarrow \exists xy(z = (x, y) \wedge G\, x, y)).$$

Following the usual set-theoretical approach, one could identify the ordered pairs with the *Kuratowski doubletons*: $(x, y)_K = \{\{x\}, \{x, y\}\}$. Also natural in this context should be the choice of [8] and [18], where *singular correlations* are taken as ordered pairs: $(x, y)_S = \binom{x}{y}$. These two approaches are equivalent, provided one has at disposal the one-to-one correspondence *Kur*, the Kuratowski's *encoding of pairs* which associates to any singular correlation $\binom{x}{y}$ the *Kuratowski pair* $\{\{x, y\}, \{x\}\}$. E.g. the cartesian product *à la Kuratowski* of two collections C and D can be obtained by putting $C \times_K D = \overline{Kur}(C \times D)$. The use of singular correlations has some technical advantages, in particular when dealing with *iterated pairs* (which are used as n-tuples in ordinary set theory). However we prefer not to fix at this stage a particular implementation of pairs. In the next section, a natural notion of *n-tuple* will be introduced, and then it will be natural to identify ordered pairs with *2-tuples*.

5. - Natural numbers and finite sequences

As we remarked in the Introduction, it would be a violation of the non-reductionist attitude of this theory to rely on artificial codings (like Von Neumann Ordinals or Church Numerals) in defining such important concepts as *natural numbers* and *finite sequences* (or lists). As suggested in [8], [9], [18], we introduce here the quality *Qnat* of being a *true (standard) natural number*, together with the operations *Nadd* and *Nmult* (*addition* and *multiplication* of natural numbers) and the binary relation *Rnord* (the *ordering* between natural numbers), subject to axioms inspired by those of Peano's Arithmetic. We also introduce the natural numbers 0 and 1.

AXIOM 5.1: *Qnat is a quality; Nadd, Nmult are binary operations; Rnord is a binary relation.*

1) $Qnat\, x \wedge Qnat\, y \Rightarrow \exists z(Qnat\, z \wedge z = Nadd\, xy)$;
2) $Qnat\, x \wedge Qnat\, y \Rightarrow \exists z(Qnat\, z \wedge z = Nmult\, xy)$;
3) $Rnord\, x, y \Leftrightarrow \exists z(Nadd\, xz = y)$;
4) $Qnat\ 0$ *and* $Qnat\ 1$.

We adopt the standard notation $x + y = Nadd\, xy$, $xy = Nmult\, xy$, $x \leqslant y$ for $Rnord\, x, y$ and $x < y$ for $x \leqslant y \wedge x \neq y$.

AXIOM 5.2: *Let x, y, z be natural numbers. Then:*
1) $x + y = y + x$ *and* $xy = yx$;
2) $(x + y) + z = x + (y + z)$ *and* $(xy)z = x(yz)$;
3) $x(y + z) = xy + xz$;
4) $x + 0 = x$, $x0 = 0$, $x1 = x$;
5) $x \neq 0 \Rightarrow x \geq 1$.

The *Induction Principle* can be formulated in several substantially different ways in theories where many different kinds of objects are present. E.g. the mere assumption:

«non-empty collections of natural numbers have a least element»

seems rather weak. In fact it is much weaker than the traditional version:

«if a proposition-valued predicate is true for some natural number, then it is true for a least natural number».

Since in our theory there are general predicative operations of any kind, such a statement could look somewhat «awkward». Therefore it seems more appropriate to isolate a suitable collection *Ind* of «inductive» predicates and postulate:

AXIOM 5.3: *Ind is a logically closed collection of semiclassical predicates.*

1) *Compr* $\subseteq$ *Ind.*

2) *If p* $\in$ *Ind and there is a natural number n such that pn is a true proposition, then there is a least natural number m for which pm is true.*

If *Compr* = *Ind*, the axiom above amounts essentially to assuming that any collection to which some natural number belongs contains a least natural number. Since we introduced the quality *Qnat* in order to qualify the «true natural numbers», it seems natural to postulate a stronger form, obtained by including more predicates in the collection *Ind*, so as to allow for a free use of operations, collections and correlations in forming inductive predicates.

AXIOM 5.4: *Gopr Rops*, *Gopr Ropb*, *Gopr Rcoll*, *Gopr Rcorr* $\in$ *Ind.*

An ultimate version could be the «problematic» axiom:

AXIOM Ind: $x \in$ *Ind* $\Leftrightarrow$ *Qscpred x.*

Having at our disposal the natural numbers, we can deal with the notion of *finiteness* in a natural way. We introduce the quality *Qfseq* of being a *finite sequence*, and the related qualities *Qfcorr* and *Qfcoll* of being respectively a *finite correlation* and a *finite collection*. We state the axiom:

AXIOM 5.5: *Qfseq, Qfcorr and Qfcoll are qualities.*

1) *Qfseq s if and only if s is a functional correlation and there is a natural number n such that*

$$x \in Dom\, s \Leftrightarrow 1 \leq x \leq n .$$

2) *Qfcoll x if and only if there exists a finite sequence s such that* $Cod\, s = x$.
3) *Qfcorr x if and only if there exist two finite sequences* s, t *such that* $x = s \circ t^{-1}$.

If s is a finite sequence and $i \in Dom\, s$, we often write s_i instead of $s(i)$.
Many interesting operations involving finite sequences and natural numbers could be introduced in this framework. We only give the example of the operations *Lseq* (*length* of finite sequences) and *Conc* (*concatenation* of finite sequences):

AXIOM 5.6: *Lseq is a simple operation, Conc is a binary operation.*

1) *If s is a finite sequence, then Lseq s exists and is the natural number n such that* $x \in Dom\, s$ *if and only if* $1 \leq x \leq n$.
2) *If s and t are finite sequences, then Conc st* $= s \frown t$ *exists and is a finite sequence such that:*

$$Lseq(s \frown t) = Lseq\, s + Lseq\, t$$

and

$$(s \frown t)_i = \begin{cases} s_i & \text{if } 1 \leq i \leq Lseq\, s; \\ t_{i - Lseq\, s} & \text{if } Lseq\, s < i \leq Lseq\, s + Lseq\, t . \end{cases}$$

We say that s is an *n-tuple* if *Qfseq s* and $Lseq\, s = n$.
Many finite correlations and collections can be built up «by hand». We give also a general «principle of finite comprehension» for inductive predicates:

AXIOM 5.7: *Let* $p \in Ind$ *and let n be a natural number such that, for* $1 \leq i \leq n$, *there is exactly one object x such that Qtver* $(pi)x$. *Then there exists an n-tuple s such that Qtver* $(pi)s_i$ *holds for* $1 \leq i \leq n$.

We conclude this section by introducing an operation *Pot* which provides the collection of all n-tuples whose components belong to a given collection.

AXIOM 5.8: *Pot is a binary operation. If Qnat n and Qcoll C, then* $Pot\, nC = C^n$ *exists and is a collection such that*

$$\forall s (s \in C^n \Leftrightarrow Qfseq\, s \wedge Lseq\, s = n \wedge Cod\, s \subseteq C) .$$

6. - SETS, SYSTEMS AND FUNCTIONS

In this section we deal with the important notions of set and function. In our view, the *naïve* concept of *set* mediates two contrasting concepts: that of «extension of an arbitrary property» and that of «range of a finite list». The former is captured in our theory by the notion of *collection*, while the latter has been dealt with in the previous section, after introducing the concept of *natural number*. The classical Theory of Sets, as conceived by Cantor and axiomatized by Zermelo, is an attempt of characterizing as *sets* those collections which *can be freely manipulated* as mathematical objects, the *Limitation of Size Principle* being the basic criterion of «sethood» (see [2]). Here we introduce *sets* as «small, well-controlled» collections, which can be freely handled by all mathematical and logical operations. Just as sets are singled out from general collections, similarly we single out from general correlations the (indexed) *systems*, which correspond to «graphs» which are sets, and the *functions*, which are the *univalent* systems.

Sets are particular collections and constitute the collection *Ins* (*), whereas systems constitute the collection *Sys* and functions constitute the subcollection *Fun*.

Axiom 6.1: *Ins, Sys and Fun are collections.*

1) $Ins \subsetneq Coll$;
2) $S \in Sys \Leftrightarrow S \in Corr \wedge Dom\ S \in Ins \wedge Cod\ S \in Ins$;
3) $Fun = Corfun \cap Sys$.

Although the notion of finiteness has not been completely developed in the previous section, nevertheless we take as guidelines to the axiomatization of *sets, systems* and *functions* the main intuitive properties of finite lists.

A first axiom expressing the *manipulability* of sets and systems is obtained by postulating that the fundamental relations *Rcoll* and *Rcorr*, when restricted to sets and systems respectively, give rise to comprehensible predicates:

Axiom 6.2: *The following predicates belong to Compr:*

$$Gopr\ Rcoll \wedge Gopr\ Ins \quad \text{and} \quad Gopr\ Rcorr \wedge Gopr\ Sys.$$

Axiom 6.2 yields the existence of collections giving the *union* and the *intersection* of every *collection of sets*, and of the collection of all subsets of any collection. More generally we obtain a suitable restriction to sets and functions of the problematic axiom Coll:

Theorem 3: *The operations Un and Int are defined for every collection of sets. Subcoll is defined for every set, Cart is defined for every set-valued functional correlation, and Trans is defined for every functional correlation taking on systems as values.*

We state as an axiom the «dual» restriction to sets of the axiom Coll:

(*) The word *Ins* has been chosen in accordance with the Latin *insimul* (meaning *simultaneously*) which is the root of the word for *sets* in some Neo-Latin languages, e.g. *insieme* (Italian), *ensemble* (French).

Axiom 6.3: *The operations Un and Int are defined for every set of collections, Cart is defined for every collection-valued function, and Trans is defined for every correlation-valued function. Moreover the cartesian product of a set-valued function is a set.*

The «Limitation of Size Principle» is embodied in the *Axiom of Replacement*, which we formulate relatively to a collection *Repl* of «replaceable predicates», which is much more extended than the collection *Compr*, and allows for a free use of operations, in addition to collections and correlations.

Axiom 6.4: (Replacement-union):
Repl is a logically closed collection of semiclassical predicates.

1) *Compr* $\subseteq$ *Repl*;
2) *Gopr Rcoll, Gopr Rcorr, Gopr Rops and Gopr Ropb are in Repl*;
3) *Let* $p \in Repl$ *be a predicate such that, for all* x, y $(px)y$ *is a proposition. If* X *is a set such that for all* $x \in X$ *the collection* $\{y \mid Qtver\ (px)y\}$ *exists and is a set, then also the collection* $\{y \mid \exists x \in X.\ Qtver\ (px)y\}$ *exists and is a set.*

Axiom 6.4 subsumes the traditional axioms of *union* and *replacement*. Namely, if E is a set of sets, then $\bigcup E$ can be obtained by taking $X = E$ and $p = Gopr\ Rcoll$. The usual formulation of the Replacement Axiom in Zermelo-Fraenkel set theory corresponds to the case when all sets $\{y \mid Qtver\ (px)y\}$ are singletons. Similarly one obtains the axiom of *separation*, in particular that any subcollection of a set is a set. Finally, from axiom 6.3 together with 6.4, one obtains also the usual *Powerset Axiom*.

If the axiom Cl is assumed, then a «cautious» postulate could be that all the replaceable predicates are classical. A more «audacious» attitude leads to grant a manipulability of sets similar to that of finite collections by postulating that all inductive predicates are replaceable:

Axiom Repl: *Repl* = *Ind*.

We have not yet postulated the existence of any set. An obvious interpretation of the Limitation of Size Principle suggests that *finite* collections and correlations are *small*. Hence we postulate:

Axiom 6.5: *If Qfcoll x then* $x \in Ins$. *If Qfcorr s then* $s \in Sys$.

Which collections are sets is still undetermined. We have assumed the «sethood» of all *intuitively finite* collections, while we can deduce from Theorem 2 that there are collections which are not sets. Various alternative choices can be made as to what we want to be a set (see e.g. [15], [18]). In view of the Limitation of Size Principle, the most «liberal» attitude is expressed by Von Neumann's axiom:

AXIOM VN (Von Neumann): *A collection X is not a set if and only if there is a functional correlation F such that* $\bar{F}(X) = V$.

Axiom VN allows to derive also the existence of a one-to-one correspondence between *Ord*, the collection of all *Von Neumann's ordinals*, and the universe V (see [24]). It is interesting to compare the *true natural numbers* (i.e. the objects enjoying the quality *Qnat*) with their usual set-theoretic implementation, namely the elements of Von Neumann's least limit ordinal ω. We could assume the following «axiom of standardness»:

AXIOM St: *There exists a collection* N *such that* $x \in N$ *if and only if* $Qnat\ x$.

Assuming the axiom St, there is a one-to-one functional correlation from ω onto N, but it is still possible that N is *not* a set. In order to preserve the modern approach to classical analysis, the following «axiom of Cantorian Analysis» should be appropriate:

AXIOM CA: *The collection* N *is a set.*

A proper alternative could be an Axiom of Finitism:

AXIOM Fin: $x \in Ins \Leftrightarrow Qfcoll\ x$.

Some of the axioms above can be combined with the following «axiom of Skolem» (see [15]), which introduces an *enumeration* of the universe *Ren*:

AXIOM Sk: *Ren is a binary relation.*

1) $\forall x \in V.\ \exists n.\ Qnat\ n \wedge Ren\ n, x$;
2) $\forall nxy. (Qnat\ n \wedge Ren\ n, x \wedge Ren\ n, y \Rightarrow x = y)$.

7. - FURTHER ENGRAFTINGS

In this section we outline an axiomatization of some basic notions from two important fields of Mathematics, namely *Standard and Nonstandard Analysis* and *Category Theory*.

7.1. *Standard and nonstandard analysis.*

We can engraft the first notions of *standard and nonstandard analysis* in the following way, which is reminiscent of [8], and follows closely the pattern of a short preliminary draft of De Giorgi's on the subject, dated July 1996 and recently rediscovered (*).

We introduce:

– the qualities *Qreal* and *Qgreal* of being a *true (standard) real number* and a *generalized (nonstandard) real number*;

– the binary operations *Gradd* and *Grmult* (*addition* and *multiplication* of generalized reals);

– the binary relation *Rgrord* (the *ordering* of generalized reals);

– the quality *Qgnat* of being a *generalized (nonstandard) natural number*.

We propose here a very weak axiomatization, which should be compatible with any kind of nonstandard real numbers. In fact we admit a great variety of generalized reals, which can appear in several completely different, even incompatible, «universes». In particular, we do not postulate that addition and multiplication can be performed on *every* pair of generalized reals; similarly the ordering relation *Rgrord* is only a *partial* ordering. The ordinary development of standard and nonstandard analysis is intended to take place within the various models of analysis, whose introduction is outlined in the axiom 7.1.3 below.

We state first a purely descriptive axiom:

Axiom 7.1.1: *Qreal and Qgreal are qualities; Gradd and Grmult are binary operations; Rgrord is a binary relation.*

1) $Qreal\, x \Rightarrow Qgreal\, x$;
2) $Gradd\, xy = z \Rightarrow Qgreal\, x \wedge Qgreal\, y \wedge Qgreal\, z$;
3) $Grmult\, xy = z \Rightarrow Qgreal\, x \wedge Qgreal\, y \wedge Qgreal\, z$;
4) $Qreal\, x \wedge Qreal\, y \Rightarrow \exists zw: Qreal\, z \wedge Qreal\, w \wedge z = Gradd\, xy \wedge w = Grmult\, xy$;
5) $Qnat\, x \wedge Qnat\, y \Rightarrow Gradd\, xy = Nadd\, xy \wedge Grmult\, xy = Nmult\, xy$.

We adopt the standard notation $x + y = Gradd\, xy$, $xy = Grmult\, xy$, $x \leq y$ for $Rgrord\, xy$ and $x < y$ for $x \leq y \wedge x \neq y$.

We state next an axiom which describes the algebraic structure of the generalized reals:

Axiom 7.1.2: *Let x, y, z be general real numbers. Then:*

1) $x + y = z \Leftrightarrow y + x = z$ *and* $xy = z \Leftrightarrow yx = z$;

2) *if* $x + y$ *and* $y + z$ *exist, then both* $x + (y + z)$ *and* $(x + y) + z$ *exist and they are equal;*

3) *if* xy *and* yz *exist, then both* $x(yz)$ *and* $(xy)z$ *exist and they are equal;*

(*) E. De Giorgi, *L'analisi matematica standard e nonstandard rivista in una nuova prospettiva scientifica e culturale*, Lecce, 7.6.96 (manuscript). The authors are grateful to A. Leaci for providing copy of the manuscript.

4) *if* xy, xz *and* $y+z$ *exist, then both* $x(y+z)$ *and* $xy+xz$ *exist and they are equal*;

5) $x+0=x$, $x0=0$, $x1=x$;

6) *if* $Qgreal\ x$ *and* $x \neq 0$ *then there is* y *such that* $xy=1$;

7) *if* $Qgreal\ x$ *and* $x \neq 0$, *then* $\exists z(x=z^2) \Leftrightarrow \neg\exists w(x+w^2=0)$;

8) $x \leq y \Leftrightarrow \exists z(x+z^2=y)$;

9) $\forall xy\ \exists z(x^2+y^2=z^2)$.

As pointed out above, we commit the development of mathematical analysis to the study of several different «universes». We cannot take into account here the most general notion of *model of the analysis*. We restrict instead our attention to «realistic models», which are constituted by collections, correlations and general real numbers and are subject to natural axioms (*). Hence we introduce the quality *Qrealmod* of being a *realistic model* and we postulate:

AXIOM 7.1.3: *Qrealmod is a quality.*

If $Qrealmod\ \mathfrak{M}$, *then* $\mathfrak{M}$ *is a pair* $(C^{\mathfrak{M}}, R^{\mathfrak{M}})$ *such that*:

1) $C^{\mathfrak{M}}$ *is a collection of collections, and* $R^{\mathfrak{M}}$ *is a collection of general real numbers.*

2) $C^{\mathfrak{M}} \in C^{\mathfrak{M}}$, $R^{\mathfrak{M}} \in C^{\mathfrak{M}}$ *and there exists* $U^{\mathfrak{M}} \in C^{\mathfrak{M}}$ *(the universe of* $\mathfrak{M}$*) such that*

$$\forall x(x \in U^{\mathfrak{M}} \Leftrightarrow \exists y \in C^{\mathfrak{M}}(x \in y)).$$

3) $R^{\mathfrak{M}}$ *is closed under sum, product, opposite and inverse.*

4) *There exists a set* $N^{\mathfrak{M}} \in C^{\mathfrak{M}}$ *(the natural numbers of* $\mathfrak{M}$*) such that*:

4.1) $N^{\mathfrak{M}} \subseteq R^{\mathfrak{M}}$;

4.2) $0 \in N^{\mathfrak{M}}$ *and* $\forall x \in N^{\mathfrak{M}}(x+1 \in N^{\mathfrak{M}})$;

4.3) *If* $X \in C^{\mathfrak{M}}$ *satisfies* 4.1-2, *then* $N^{\mathfrak{M}} \subseteq X$.

5) *There exists a collection* $I^{\mathfrak{M}} \in C^{\mathfrak{M}}$ *(the sets of the model* $\mathfrak{M}$*) such that*:

$$I^{\mathfrak{M}} \subseteq C^{\mathfrak{M}} \cap Ins,\ R^{\mathfrak{M}}, N^{\mathfrak{M}} \in I^{\mathfrak{M}} \text{ and } I^{\mathfrak{M}} \notin I^{\mathfrak{M}}.$$

6) *There exists a collection* $F^{\mathfrak{M}} \in C^{\mathfrak{M}}$ *(the functions of the model* $\mathfrak{M}$*) such that*:

$$f \in F^{\mathfrak{M}} \Leftrightarrow f \in Fun \wedge Graph\ f \in I^{\mathfrak{M}}.$$

In order to make an effective use of realistic models, one should assume more comprehension and stability properties for the collections $C^{\mathfrak{M}}$, $I^{\mathfrak{M}}$ and $F^{\mathfrak{M}}$. We leave the search for such conditions to the interested readers. It is interesting to notice that one could characterize the *standard realistic models* by the following property of «fullness»:

$$\textit{If } x \in I^{\mathfrak{M}} \textit{ and } y \subseteq x, \textit{ then } y \in I^{\mathfrak{M}}.$$

(*) The attribute «realistic» was justified in De Giorgi's draft by appealing to the *realistic* attitude of the paper [8], which is in some sense mirrored in these models.

We conclude this subsection by dealing with the essential property of the *true* real numbers, namely *order completeness*. As it was to be expected, order completeness can be formulated as an axiom in several different ways, according to the kinds of objects which are considered. The mere assertion that

any bounded, nonempty collection of real numbers has a least upper bound

seems rather weak. More in line with our framework is the introduction of a collection *Compl* of *completable* predicates, which contains those predicates which are suitable for taking least upper bounds. Hence we postulate:

AXIOM Compl: *Compl is a logically closed collection of semiclassical predicates.*

1) *Compr* $\subsetneq$ *Compl*.

2) *Let* $p \in$ *Compl be a predicate such that, for all* x, px *is a proposition. If there are real numbers* x, y *such that* px *is true, while* pz *is false for all real numbers* $z > y$, *then there exists a least real number* w *such that* pz *is false for all real numbers* $z > w$.

The strength of the axiom depends, as usual, on the predicates which are put into *Compl*. A comparison with other basic collections of predicates, such as *Ind* and *Repl*, is left to the interested reader.

7.2. *Categories.*

In this subsection we suggest the engrafting of some categorial notion. We consider only the first fundamental objects, and we state only the descriptive axioms which are needed for dealing with categories in our framework. We believe that this framework is especially suitable for a natural development of Category Theory and we hope that this sketch may foster reactions among interested categorists.

First of all we introduce:

– the quality *Qcat* of being a *category*;
– the quaternary relations *Rhom* and *Rcomp*.

The intended meaning of *Rhom* C, f, a, b and *Rcomp* C, f, g, h is that, in the category C, f is a *(homo)morphism* from the *object* a to the *object* b and, respectively, the *morphism* h is the *composition* of the *morphisms* f, g.

Hence we state the axiom

AXIOM 7.2.1: *Qcat is a quality, Rhom and Rcomp are quaternary relations.*

1) *If Rhom* x, y, z, w *or Rcomp* x, y, z, w *then Qcat* x.

Let C *be a category. Then*:

2) $\forall faba'b'.(Rhom\ C, f, a, b\ \wedge\ Rhom\ C, f, a', b' \Rightarrow a = a'\ \wedge\ b = b')$.
3) $\forall fghh'.(Rcomp\ C, f, g, h\ \wedge\ Rcomp\ C, f, g, h' \Rightarrow h = h')$.
4) $\forall fgabc.(Rhom\ C, f, b, c \wedge Rhom\ C, g, a, b \Rightarrow$
$\Rightarrow \exists h(Rcomp\ C, f, g, h \wedge Rhom\ C, h, a, c))$

5) $\forall fgh.(Rcomp\ C,f,g,h \Rightarrow$
$\Rightarrow \exists abc(Rhom\ C,f,b,c \wedge Rhom\ C,g,a,b \wedge Rhom\ C,h,a,c))$
6) $\forall abf.(Rhom\ C,f,a,b \Rightarrow \exists i\ \forall gc(Rhom\ C,g,a,c \Rightarrow Rcomp\ C,g,i,g))$.
7) $\forall abf.(Rhom\ C,f,a,b \Rightarrow \exists j\ \forall gc(Rhom\ C,g,c,b \Rightarrow Rcomp\ C,j,g,g))$.
8) $\forall fghkl.(Rcomp\ C,f,g,l \wedge Rcomp\ C,g,h,k \Rightarrow$
$\Rightarrow \exists m(Rcomp\ C,l,h,m \wedge Rcomp\ C,f,k,m))$.

Assuming this axiom, one can introduce the usual notations and definitions of Category Theory, such as *object*, *morphism* or *arrow*, *domain*, *codomain*, *identity*, *composition* etc. We omit them for sake of brevity.

In order to develop category theory many more notions have to be introduced. We consider here only the notion of *functor*, which we axiomatize as *morphism* of the category *Cat* of *all categories*. We introduce also the quality *Qfunc* of being a functor together with the ternary relation *Rfunc* which describes the action of a functor on objects and morphisms of the appropriate categories.

Axiom 7.2.2: *Qfunc is a quality, Rfunc is a ternary relation and Cat is a category.*

1) If $Rhom\ Cat,x,y,z$ *then* $Qfunc\ x$ *and* $Qcat\ y$, $Qcat\ z$.
2) *If* $Qfunc\ x$ *then there exist* C, D *such that* $Rhom\ Cat,x,C,D$.
3) *If* $Rfunc\ x,y,z$ *then* $Qfunc\ x$.
4) *If* $Rfunc\ x,y,z$ *and* $Rfunc\ x,y,w$ *then* $z = w$.

Assume $Rhom\ Cat,F,C,D$. *Then:*

5) *If* $Rfunc\ F,x,y$ *then* x, y *are either objects or morphisms of* C, D *respectively.*
6) $\forall fab.(Rhom\ C,f,a,b \Rightarrow$

$$\exists a'b'f'(Rfunc\ F,a,a' \wedge Rfunc\ F,b,b' \wedge Rfunc\ F,f,f' \wedge Rhom\ D,f',a',b')).$$

7) $\forall fgh.(Rcomp\ C,f,g,h \wedge Rcomp\ D,f',g',h' \wedge Rfunc\ F,f,f' \wedge Rfunc\ F,g,g'$
$\Rightarrow Rfunc\ F,h,h')$.

In a similar way one should also introduce the notion of *natural transformation* together with the category *Func* of all functors and several other basic categorial notions. Moreover, in order to make an effective use of Category Theory, several fundamental categories are needed, in addition to *Cat* and *Func*. For instance one could introduce the following categories:

– *Coll*, whose objects are all *collections* and whose morphisms are all the pairs (f, C) where f is a *functional correlation* and C includes the codomain of f;
– *Corr*, whose objects are all *collections* and whose morphisms are all the triples (C, g, D) where g is a *correlation* included in $C \times D$;
– *Set*, the *full* subcategory of *Coll* whose objects are all *sets*.

8. - The consistency problem

We conclude this paper by briefly discussing the consistency strength of the axiomatic theory introduced here. To this aim we have to choose a first order axiomatization of the theory. We fix a first order language $\mathcal{L}$ (with equality) corresponding to the primitive notions of section 1. More precisely we put in $\mathcal{L}$:

– a unary predicate symbol Q, corresponding to qualities;
– a binary predicate symbol R_Q describing the behaviour of qualities;
– a unary predicate symbol Rb corresponding to binary relations;
– a ternary predicate symbol R_{Rb} describing the behaviour of binary relations;
– a unary predicate symbol Rt corresponding to ternary relations;
– a quaternary predicate symbol R_{Rt} describing the behaviour of ternary relations;
– a unary predicate symbol Rq corresponding to quaternary relations;
– a quinary predicate symbol R_{Rq} describing the behaviour of quaternary relations.

We also put in $\mathcal{L}$ a constant symbol for each distinguished object of the theory, which we denote *autonymously*. Then each axiom of the semiformal theory has a formal counterpart in $\mathcal{L}$. E.g. Axiom 1.1 becomes:

$$Q(Qqual) \wedge Q(Qrelb) \wedge Q(Qrelt) \wedge Q(Qrelq).$$

1) $\neg \exists x((R_Q(Qqual,x) \wedge R_Q(Qrelb,x)) \vee (R_Q(Qqual,x) \wedge R_Q(Qrelt,x)) \vee$
$\vee (R_Q(Qqual,x) \wedge R_Q(Qrelq,x)) \vee (R_Q(Qrelb,x) \wedge R_Q(Qrelt,x)) \vee$
$\vee (R_Q(Qrelb,x) \wedge R_Q(Qrelq,x)) \vee (R_Q(Qrelt,x) \wedge R_Q(Qrelq,x)))$;
2) $Q(x) \leftrightarrow R_Q(Qqual,x)$;
3) $Rb(x) \leftrightarrow R_Q(Qrelb,x)$;
4) $Rt(x) \leftrightarrow R_Q(Qrelt,x)$;
5) $Rq(x) \leftrightarrow R_Q(Qrelq,x)$.

We denote by **AF** the formal theory whose axioms are the formal counterparts of axioms 1.1-1.5, 2.1-2.10, 3.1-3.9, 4.1-4.7, 5.1-5.8, 6.1-6.5.

Due to the presence of many nonwellfounded collections as well as of objects which are not collections, a *natural transitive model* cannot be built up within **ZFC**. Therefore we prefer to use a metatheory that allows for *urelements* and incorporates a suitable «Free Construction Principle» (antifoundation axiom, see [13]). Hence we work within the theory $\mathbf{ZF_0CU} + \mathbf{X_1(U)}$ + «κ is an inaccessible cardinal», as done in [18]. The theory $\mathbf{ZF_0CU}$ is obtained from Zermelo-Fraenkel set theory by dropping out the axiom of foundation and weakening the extensionality so as to allow for a set U of urelements. $\mathbf{X_1(U)}$ is the antifoundation axiom called «injective free construction principle with respect to a set U of atoms» in [18], namely:

Axiom $X_i(U)$: *Let* $f\colon A \to \mathcal{P}(A) \cup U$ *be injective. Then there exist a transitive set* T *and a bijective function* $g\colon A \to T$ *such that*:

$$g(x) = \begin{cases} f(x) & \text{if } f(x) \in U; \\ \{g(y) \mid y \in f(x)\} & \text{otherwise.} \end{cases}$$

In order not to be forced to repeat the whole construction of [18], we use as a starting point the set-theoretic part of a model $\mathfrak{M}$ of the theory **GOCI** as defined there.

– The universe M of $\mathfrak{M}$ is a *transitive* set of size κ containing κ urelements;
– $\mathcal{P}_\kappa(M) \subsetneq M$, i.e. *all* subsets of M of cardinal *less than* κ belong to M;
– $M \cap \mathcal{P}(M)$ is closed under all *Gödel operations* and also under *unions, intersections, products* and *transpositions* of length *less than* κ.

Therefore $M \cap \mathcal{P}(M)$ and $\mathcal{P}_\kappa(M)$ are suitable for interpreting *Coll* and *Ins*, respectively.

We have to enrich $\mathfrak{M}$ by adding the interpretations of all kinds of objects which are *not* collections, namely: *correlations, natural numbers* and *finite sequences, operations* (including *propositions* and *predicates*), *qualities* and *relations*. In order to accommodate all these kinds of objects, we fix seven pairwise disjoint collections of urelements, each of size κ:

– U_{qual}, whose elements are interpreted as qualities;
– U_{relb}, whose elements are interpreted as binary relations;
– U_{relt}, whose elements are interpreted as ternary relations;
– U_{relq}, whose elements are interpreted as quaternary relations;
– U_{ops}, whose elements are interpreted as simple operations;
– U_{opb}, whose elements are interpreted as binary operations;
– U_{corr}, whose elements are interpreted as correlations.

We also assume that there are left κ unqualified urelements, so as to leave place for further engraftings.

In correspondence of each distinguished object of the theory which is not a collection, we fix an urelement belonging to the appropriate set, which we denote by u with the name of the object as a subscript. When no ambiguity can arise, we shall use simply the same name of the object. E. g. the interpretation of the quality *Qqual* is an urelement $u_{Qqual} \in U_{qual}$, possibly denoted again by *Qqual*.

We have to assign the action of correlations, operations, relations and qualities according to the axioms of **AF**. As done in [18], we follow a procedure which reflects the «stratified» hierarchy of the different kinds of objects. Sets, systems and natural numbers lie at the bottom level. Collections and correlations constitute the next level up. On a higher level are placed operations and among them predicates and propositions. Finally, at the top level, we have relations and qualities which involve all kinds of objects including relations and qualities themselves.

We divide the definition of the model in five steps.

STEP 1: As stated at the beginning, *sets* are those subsets of M which have cardinal less than κ and *collections* are all subsets of M belonging to M. *Correlations* are the elements of U_{cor} and their behaviour is assigned by fixing a one-to-one correspondence between U_{cor} and all collections of Kuratowski pairs belonging to M. Among all correlations, *systems* are those corresponding to (internal) sets of Kuratowski pairs. The collections of predicates *Compr* and *Repl* will be chosen in step 3 so as to satisfy Axioms 4.4, 4.5 and 6.2. Notice that the Axiom 6.4 is automatically satisfied since *all* subsets of M of cardinal less than κ are in M.

STEP 2: *Natural numbers* are dealt with by taking a set of «fresh» urelements U_{nat} in one-to-one correspondence with the elements of the Von Neumann ordinal ω. The collection *Ind* of all inductive predicates will be chosen in Step 3. This choice will be done so as to satisfy the strong axiom Ind together with Axiom 5.7. *Finite sequences* and *finite sets* are defined in the obvious way, so as to satisfy the Axiom 5.5.

STEP 3: *Predicates* and *propositions* are simple operations in the theory **AF**, hence we have to select κ urelements from U_{opr} to interpret predicates. We choose a subset $U_{pred} \subseteq U_{opr}$ in one-to-one correspondence with the formulae of a suitable first order language $\mathcal{L}'$. The nonlogical symbols of $\mathcal{L}'$ are two quaternary predicate symbols R_1, R_2 (intended to represent the predicates *Gopr Rrelt* and *Gopr Ropb*), and a constant symbol c_m for every $m \in M$. We fix an enumeration $x_0, x_1, \dots, x_n, \dots$ of the variables of $\mathcal{L}'$.

In order to make the following definitions more perspicuous, we point out that, denoting by u_φ the predicate corresponding to the formula φ, then:

– $u_{R_1(x_0,x_1,x_2,x_3)}$ and $u_{R_2(x_0,x_1,x_2,x_3)}$ are intended to interpret the predicates *Gopr Rrelt* and *Gopr Rrelb* respectively;

– $u_{R_1(c_a,c_b,c_c,c_d)}$ and $u_{R_2(c_a,c_b,c_c,c_d)}$ are intended to interpret the propositions «*Rrelt* a, b, c, d» and «*Ropb* a, b, c, d» respectively.

More precisely, we stipulate that the urelement u_φ acts as an operation as follows:

– $u_\varphi m = u_\psi$, where ψ is obtained by substituting, in φ, the constant c_m to every free occurrence of x_0 and the variable x_i to every free occurrence of x_{i+1}.

We define the action of the connectives in the natural way, namely:

– *Non* $u_\varphi = u_{\neg\varphi}$, *Et* $u_\varphi u_\psi = u_{\varphi\wedge\psi}$ and *Vel* $u_\varphi u_\psi = u_{\varphi\vee\psi}$.

The action of quantifiers is given as follows. Let φ be a formula whose free variables are exactly $x_{i_0}, x_{i_1}, \dots, x_{i_n}$ with strictly increasing indices. Let k be greater than any index of variable appearing in φ and let $\psi = \varphi(x_{j_0}, x_{i_1-1}, \dots, x_{i_n-1})$, where $j_0 = k$, if $i_0 = 0$, and $j_0 = i_0 - 1$ otherwise. Then:

– *Exist* $u_\varphi = u_{\exists x_k \psi}$ and *Univ* $u_\varphi = u_{\forall x_k \psi}$.

In order to deal with the action of generalized permutators, we stipulate that they constitute the least set of operations including K and T which is closed under *Cur Comp*. Hence the action of all generalized permutators is uniquely defined by putting:

– $Tu_\varphi = u_\chi$, where χ is obtained from φ by exchanging any free occurrence of the variables x_0 and x_1.

– $Ku_\varphi = u_\theta$, where θ is obtained by increasing by 1 all the indices of free variables occurring in φ.

We stipulate that all predicates are semiclassical and that u_φ is classical if and only if the atomic subformulae of φ have one of the following forms:

– $R_2(i, j, k, l)$;
– $R_1(c_{Rops}, j, k, l)$, $R_1(c_{Rcorr}, j, k, l)$;
– $R_1(c_{Rrelb}, c_{Rid}, k, l)$, $R_1(c_{Rrelb}, c_{Rcoll}, k, l)$, $R_1(c_{Rrelb}, c_{Rincl}, k, l)$,
$R_1(c_{Rrelb}, c_{Rcoord}, k, l)$;

where i, j, k, l are arbitrary variables or constants.

We also stipulate that all predicates belong to both collections *Ind* and *Repl*, which are therefore equal. In order to interpret the collection *Compr*, we take the least logically closed collection of predicates containing all predicates u_φ where φ is any of the following formulae:

– $R_1(c_{Rrelb}, c_{Rid}, k, l)$, $R_1(c_{Rrow}, c, k, l)$, $R_1(c_{Rrelb}, c_{Rcoll}, c, l)$;
– $R_1(c_{Rrelb}, c_{Rcoll}, k, l) \wedge R_1(c_{Rrelb}, c_{Rcoll}, c_{Ins}, k)$;
– $R_1(c_{Rcorr}, j, k, l) \wedge R_1(c_{Rrelb}, c_{Rcoll}, c_{Sys}, j)$;

where c is an arbitrary constant and j, k, l are arbitrary variables or constants.

Step 4: In defining the action of all *operations*, we follow an inductive procedure similar to that of [18]. Besides the distinguished operations and the predicates, we have to single out also:

– an element $u_{Cur,g} \in U_{ops}$ for each $g \in U_{oph}$;
– an element $u_{g,x} \in U_{ops}$ for each $g \in U_{oph}$ and each $x \in M$;
– an element $u_{K,x} \in U_{ops}$ for each $x \in M \backslash U_{pred}$.

The intended meaning of $u_{Cur,g}$ is the interpretation of *Cur* applied to g, that of $u_{g,x}$ is the interpretation of *Cur* g applied to x, and that of $u_{K,x}$ is K applied to x (the action of K on U_{pred} has been defined in Step 3). In order to avoid undesired clashings, we stipulate that these elements are all different from each other and from all previously fixed operations.

We pick three (external) injective mappings $\varphi_{Comp}, \varphi_S, \varphi_{Bas}: U^2_{ops} \to U_{ops}$ and an injective mapping $\varphi_{Ins}: U_{ops} \to U_{ops}$, taking care that their ranges are pairwise disjoint and disjoint from the previously considered operations. The intended meaning of $\varphi_{Comp}(f, g)$ is the value of *Comp* on f and g, and similarly for φ_S, φ_{Bun}, φ_{Inv}.

At this point the behaviour of all binary operations should be clear from their natural meaning and the stipulations above. The elements of U_{qub} which have not been «qualified» are taken to be empty.

Also all simple operations which are not of the kind $\varphi_{Comp}(f, g)$, $\varphi_S(f, g)$, $\varphi_{Bun}(f, g)$ and $\varphi_{Inv}(f)$ have a natural action according to the stipulations above. We define the actions of these operations by an inductive procedure. At level 0 we decide that the operations above are all empty and we put

$$F_0 = \{(f, x, y) \mid f \in U_{ops}, fx = y \textit{ at level } 0\}.$$

In order to deal with the *nondeterministic* operations *Inv* and *Bun*, we fix a wellordering of M. Then at level $\alpha + 1$ we put

$$\begin{aligned} F_{\alpha+1} = F_\alpha \cup \\ \cup \{(\varphi_{Comp}(f, g), x, y) \mid \exists z.(g, x, z), (f, z, y) \in F_\alpha\} \cup \\ \cup \{(\varphi_S(f, g), x, y) \mid \exists zw.(f, x, w), (g, x, z), (w, z, y) \in F_\alpha\} \cup \\ \cup \{(\varphi_{Inv}(f), x, y) \mid \forall z((\varphi_{Inv}(f), x, z) \notin F_\alpha) \wedge y = \min\{w \mid (f, w, x) \in F_\alpha\}\} \cup \\ \cup \{(\varphi_{Bun}(f, g), x, y) \mid \forall z.(\varphi_{Bun}(f, g), x, z) \notin F_\alpha \wedge y = \min\{w \mid (f, x, w) \in F_\alpha \vee (g, x, w) \in F_\alpha\}\} \end{aligned}$$

At limit λ we put

$$F_\lambda = \bigcup_{\alpha < \lambda} F_\alpha.$$

Finally for all $f \in U_{ops}$ we put:

– $fx = y$ if and only if $(f, x, y) \in F_\nu$, where ν is the least ordinal such that $F_\nu = F_{\nu+1}$.

Step 5: We are left with the behaviour of *relations* and *qualities*. All qualities are purely descriptive and their intended extension can be easily fixed, with the important exception of the quality *Qtver*. Although many propositions can already be seen to be true or false, nevertheless all propositions involving the fundamental relations *Rrelb*, *Rrelt*, *Rqual* can appear undecidable at this stage. Similarly, the «external graphs» of all relations other than *Rrelb*, *Rrelt*, *Rqual* can be assigned in the natural way, since they depend only on the behaviour of collections, correlations, operations, and natural numbers.

However, once the *extension* $E = \{x \in M \mid Qtver\ x\}$ of the quality *Qtver* is fixed, there is one and only one way to define the «external graphs» of the relations *Rrelt*, *Rrelb*, *Rqual* so as to satisfy the Axioms 1.2, 1.3, 1.4. Let us call $\mathfrak{M}(E)$ the structure so obtained. We have to determine E so as to obtain a model of **AF**. In particular, the crucial Axioms 3.3, 3.6 and 3.8 have to be satisfied.

To this aim we start as in [23] by letting *Qtver* to be empty, and we define inductively sets E_α, F_α of propositions which are «true, and respectively false, at level α». Put:

- $F_\alpha = \{p \mid \neg p \in E_\alpha\}$;
- $E_0 = \emptyset$;
- $E_{\alpha+1} = \{u_\sigma \mid \forall E(E_\alpha \subseteq E \wedge F_\alpha \cap E = \emptyset \Rightarrow \mathfrak{M}(E) \models \sigma)\}$

(where $\mathfrak{M}(E)$ is intended as a structure for the language $\mathcal{L}'$ in the obvious way);

- $E_\lambda = \bigcup_{\alpha<\lambda} E_\alpha$.

The sets E_α, F_α are *disjoint* and closed under *logical equivalence*. The sequence E_α is increasing, hence there is a least ordinal ν such that $E_\nu = E_{\nu+1}$.

Put $E = E_\nu$. Then:

i) $p \in E \Rightarrow \neg p \notin E$;
ii) $p \in E \Leftrightarrow$ «*Qtver* p»$\in E$;
iii) $p \wedge q \in E \Leftrightarrow p, q \in E$;
iv) $p \in E \Rightarrow p \vee q \in E$;
v) $pm \in E \Rightarrow \exists p \in E$;
vi) $\forall p \in E \Rightarrow pm \in E$ for all $m \in M$.

The above properties are crucial for our purposes, hence we give the following

Definition: The set of propositions E is a *good truth set* if it includes E_ν, is closed under logical equivalence and satisfies properties i)-vi).

Moreover, E is *complete* if the converse implication of i) also holds, i.e. $p \in E \Leftrightarrow \neg p \notin E$.

Any good truth set provides a model of a very strong theory, namely:

Theorem 4: *Let E be a good truth set. Then*

$$\mathfrak{M}(E) \models \mathbf{AF} + \{\mathbf{Cl}, \mathbf{Ind}, \mathbf{Repl}, \mathbf{VN}, \mathbf{St}, \mathbf{CA}\}.$$

Moreover $\mathfrak{M}(E) \models$ **Scl** *if and only if E is complete.*

Proof: the principle of non contradiction holds by condition i). Moreover, one can prove that all classical propositions are classically decided already in $\mathfrak{M}(E_1)$. Hence Axioms 3.3 and 3.8 hold in $\mathfrak{M}(E)$. On the other hand, Axiom 3.6 holds by condition ii). The remaining axioms of **AF** hold by definition, as well as the Axioms **Cl**, **Ind**, **Repl**, **VN**, **St**, and **CA**.

The completeness condition on E corresponds exactly to clause 1) of axiom **Scl**. The clauses 2), 4) and 5) of Axiom **Scl** hold in $\mathfrak{M}(E)$ by conditions iii), v) and vi) respectively. Finally clause 3) follows from the fact that E is closed under logical equivalence, provided clause 1) is verified. Q.E.D.

It is easily seen that Zorn's lemma applies to good truth sets. Hence, in looking for a *complete* good set of propositions E, one can assume E *maximal*. However, the fact that a maximal good set is complete is not true, in general. Actually it depends on the choice of the correspondence between formulae of $\mathcal{L}'$ and urelements in U_{prod}. For instance, one could have chosen an «epimenidean» correspondence, where an urelement $u \in U_{prod}$ is chosen to correspond to $\neg R_1(c_{Rrelb}, c_{Requal}, c_{Qtver}, c_u)$. Clearly neither u nor $\neg u$ can be in E because of conditions i) and ii), since $u = \neg$«*Qtver u*».

We conjecture, however, that one can find a correspondence based on some «rank function» on formulae, such that the «rank» of a formula φ is larger than that of any formula which appears as an index of a constant appearing in φ. This could allow for finding, in correspondence to any non-complete good truth set E, some existential proposition which can be added to E (together with its consequences), still preserving goodness. This would imply that maximal truth sets are complete. Also in this case, however, one should face with the somehow unpleasant phenomenon of an *existential proposition* which is *true* while *all its instances* are *false*.

Notice that we can take $\kappa = \omega$ in the whole construction. The model $\mathfrak{M}(E)$ thus obtained satisfies all the axioms considered in Theorem 4, with the exception of **CA** which has to be replaced by **Fin**. In this case the model $\mathfrak{M}(E)$ satisfies also the axiom **Sk** of Skolem. It is interesting to remark that, even in case $\kappa > \omega$, one can obtain a model $\mathfrak{M}(E)$ which satisfies **Sk** by making an effective use of the *non-reductionist* character of the theory **AF**. In fact one can perform the construction within a *countable* model of the metatheory. Then one can pick an urelement $u_{Ren} \in U_{Relb}$ and stipulate that its «graph» is an *external* bijection between ω and M. Clearly, the predicate *Gopr Ren* cannot be replaceable, and so are *a fortiori Gopr Rrelb* and *Gopr Rrelt*. Hence, in such a model, the collection *Repl* cannot contain all semiclassical predicates. It follows that at least one of the axioms **Ind**, **Repl** necessarily fails.

Since the theory $\mathbf{ZF_0CU} + \mathbf{X_1(U)}$ is equiconsistent with **ZFC**, we can state the following consistency results:

THEOREM 5:

$$\mathrm{Con}(\mathbf{ZF}) \Rightarrow \mathrm{Con}(\mathbf{AF} + \{\mathbf{Cl}, \mathbf{Ind}, \mathbf{Repl}, \mathbf{VN}, \mathbf{St}, \mathbf{Sk}\})$$

and

Con (**ZFC** + «there exists an inaccessible cardinal») $\Rightarrow$

$$\mathrm{Con}(\mathbf{AF} + \{\mathbf{Cl}, \mathbf{VN}, \mathbf{St}, \mathbf{CA}\} + \textit{any two among } \mathbf{Repl}, \mathbf{Ind}, \mathbf{Sk}).$$

The use of an inaccessible cardinal in the construction of the model $\mathfrak{M}$ is only instrumental, so as not to be forced to use proper classes within inductive definitions. In fact, we conjecture that one can perform all the required inductions working with proper classes in a metatheory as strong as Kelley-Morse class theory. One could thus obtain a class model which allows for substituting **KM** instead of **ZFC** + *«there exists an inaccessible cardinal»* in the second item of Theorem 5.

REFERENCES

[1] J. Barwise - J. Etchemendy, *The Liar*, Oxford (1987).

[2] Y. Bar Hillel - A. A. Fraenkel - A. Levy, *Foundations of Set Theory*, North-Holland, Amsterdam (1973).

[3] M. Clavelli - E. De Giorgi - M. Forti - V. M. Tortorelli, *A selfreference oriented theory for the Foundations of Mathematics*, in *Analyse mathématique et applications – Contributions en l'honneur de Jacques-Louis Lions*, Gauthier-Villars, Parigi (1988), pp. 67-115.

[4] E. De Giorgi, *Riflessioni su Matematica e Sapienza*, Quad. Acc. Pontaniana, 18, Napoli (1996).

[5] E. De Giorgi - M. Forti, *Una teoria quadro per i fondamenti della Matematica*, Atti Acc. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. (8), 79 (1985), 55-67.

[6] E. De Giorgi - M. Forti - G. Lenzi, *Una proposta di teorie base dei Fondamenti della Matematica*, Rend. Mat. Acc. Lincei (9), 5 (1994), 11-22.

[7] E. De Giorgi - M. Forti - G. Lenzi, *Introduzione delle variabili nel quadro delle Teorie Base dei Fondamenti della Matematica*, Rend. Mat. Acc. Lincei (9), 5 (1994), 117-128.

[8] E. De Giorgi - M. Forti - G. Lenzi, *Verso i sistemi assiomatici del 2000 in Matematica, Logica e Informatica*, Preprint di Matematica n. 26, Scuola Norm. Sup. Pisa (1996).

[9] E. De Giorgi - M. Forti - G. Lenzi, *Verità e giudizi in una nuova prospettiva assiomatica*, in *Il fare della Scienza: i fondamenti e le palafitte* (F. Barone et al., eds), Con-tratto (1996), 233-252.

[10] E. De Giorgi - M. Forti - G. Lenzi - V. M. Tortorelli, *Calcolo dei predicati e concetti metateorici in una teoria base dei Fondamenti della Matematica*, Rend. Mat. Acc. Lincei (9), 6 (1995), 79-92.

[11] E. De Giorgi - G. Lenzi, *La teoria '95. Una proposta di teoria aperta e non riduzionistica dei fondamenti della Matematica*, Atti Accad. Naz. Sci. XL, Mem. Mat. Appl., 20 (1996), 7-34.

[12] M. Forti - L. Galleni, *Un'assiomatizzazione di alcuni concetti della Biologia in una teoria generale dei Fondamenti*, Quad. Dip. Mat. Appl. «U. Dini», 8, Pisa (1997).

[13] M. Forti - F. Honsell, *Set theory with free construction principles*, Ann. Scuola Norm. Sup. Pisa (4), 10 (1983), 493-522.

[14] M. Forti - F. Honsell, *Models of self-descriptive set theories*, in *Partial Differential Equations and the Calculus of Variations – Essays in Honor of Ennio De Giorgi* (F. Colombini et al., eds), Birkhäuser, Boston (1989), 473-518.

[15] M. Forti - F. Honsell, *Collections and Sets within the Basic Theories for the Foundations of Mathematics*, Quad. Ist. Mat. Appl. «U. Dini», Univ. Pisa, 11, Pisa (1994).

[16] M. Forti - F. Honsell, *Positive qualities and the ontological argument*, Quad. Dip. Mat. Appl. «U. Dini», 5, Pisa (1997).

[17] M. Forti - F. Honsell - M. Lenisa, *An axiomatization of partial n-place operations*, Math. Struct. Computer Sci., 7 (1997), 283-302.

[18] M. Forti - F. Honsell - M. Lenisa, *Operations, collections and sets within a general axiomatic framework*, in *Logic in Florence, LMPS '95: Selected Contributed Papers* (A. Cantini et al., eds), Kluwer, Amsterdam (1997), to appear.

[19] G. Frege, *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet*, vol. 1, Jena 1893, vol. 2, Pohle, Jena (1903) (reprinted Olms, Hildesheim (1962)).

[20] M. Grassi - G. Lenzi - V. M. Tortorelli, *A formalization of a basic theory for the foundations of Mathematics*, Rend. Acc. Naz. Sci. XL, Mem. Mat. Appl.; 19 (1995) 129-157.

[21] G. Lenzi (curat.), *Seminario sui Fondamenti della Matematica di E. De Giorgi - Scuola Normale Superiore di Pisa - aa.aa. 1990/91 e segg.*, Pisa 1990/1996 (typewritten lecture notes).

[22] G. Lenzi - V. M. Tortorelli, *Introducing predicates into a basic theory for the foundations of Mathematics*, preprint di Matematica n. 51, Scuola Norm. Sup. Pisa (1989).

[23] G. Lenzi - V. M. Tortorelli, *A basic theory with predicates*, Bull. Soc. Math. Belg. (1998) to appear.

[24] J. von Neumann, *Eine Axiomatisierung der Mengenlehre*, J. f. reine und angew. Math., 154 (1925), 219-240.