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Spaces of Morrey Type and BMO Spaces in Unbounded Domains of R^n (**)

SUMMARY. — In this paper we study the connection between $M^{p,\lambda}$ spaces of Morrey type and BMO spaces of functions with bounded mean oscillation. Some embedding and density results are proved, involving the continuity of the translation operator with respect to product norm of the intersection $BMO \cap M^{1,0}$.

Spazi di tipo Morrey e spazi BMO in aperti non limitati di R^n

SOMMARIO. — In questo lavoro stabiliamo delle relazioni fra gli spazi $M^{p,\lambda}$ di tipo Morrey e gli spazi BMO delle funzioni ad oscillazione media limitata. Proviamo, inoltre, alcuni risultati di immersione e di densità connessi alla continuità dell'operatore di traslazione nello spazio $BMO \cap M^{1,0}$, munito della norma prodotto.

INTRODUCTION

In two recent papers, [TTV₁] and [TTV₂] M. Transirico, M. Troisi and A. Vitolo have studied respectively the $M^{p,\lambda}(\Omega)$ spaces of Morrey type and the $BMO(\Omega)$ spaces of functions with bounded mean oscillation in open subset Ω of R^n , in particular when Ω is unbounded.

We remark that $M^{p,\lambda}(\Omega)$ is a generalization of $M^p(\Omega)$, which was studied by M. Transirico-M. Troisi in [TT₁].

When Ω is a bounded open subset, $M^{p,\lambda}(\Omega)$ agrees with the $L^{p,\lambda}(\Omega)$ space of Morrey (see [C₁] and [KJF]) and with the $\mathcal{L}_\delta^{(p,\lambda)}(\Omega)$ space of Campanato (see [C₂]).

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The results about $M^p(\Omega) \subset M^{p,\lambda}(\Omega)$ have been already applied to the study of second order elliptic equations with discontinuous coefficients in unbounded open subsets of R^n (see e.g. [TT₁], [TT₂], [TTV₁], [CLM₁], [CLM₂], [CDV]).

In this paper we want to examine closely some aspects of the spaces mentioned above.

In Section 1 we recall some subspaces of $M^{p,\lambda}(\Omega)$, that have been introduced in [TTV₁] and denoted by $M_0^{p,\lambda}(\Omega)$, $\tilde{M}^{p,\lambda}(\Omega)$, $VM^{p,\lambda}(\Omega)$, where $VM^{p,\lambda}(\Omega)$ is larger than $\tilde{M}^{p,\lambda}(\Omega)$, which in turn is larger than $M_0^{p,\lambda}(\Omega)$.

The following characterizations hold: $\tilde{M}^{p,\lambda}(\Omega)$ is the closure of $L^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$ and $M_0^{p,\lambda}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$.

The latter result needs the continuity in $M_0^{p,\lambda}(R^n)$ of the *translation operator* τ_b , i.e. that

$$(1) \quad \tau_b g \rightarrow g \text{ in } M^{p,\lambda}(R^n) \text{ as } |b| \rightarrow 0$$

for all $g \in M_0^{p,\lambda}(R^n)$.

In Section 2, by a counterexample (see Example 2.1), we show that in general (1) is not true when $g \in \tilde{M}^{p,\lambda}(R^n)$, whilst the subspace $M^{sp,\lambda}(R^n)$ of $M^{p,\lambda}(R^n)$, in which (1) holds, is smaller than $\tilde{M}^{p,\lambda}(R^n)$, bigger than $M_0^{p,\lambda}(R^n)$, and can be characterized as the closure of $UC \cap L^\infty(R^n)$, where UC is the class of uniformly continuous functions in R^n .

In Section 3 we are concerned with the $BMO(\Omega)$ spaces, already considered in [TTV₂]. As there, we notice that these spaces agree with $\mathcal{L}_{\omega}^{(1,n)}$ spaces of Campanato (see [C₂]), when Ω is a bounded open subset, and with the classical spaces of functions with bounded mean oscillation, when $\Omega = R^n$. The latter spaces have been largely studied by several authors (see e.g. [A], [BDS], [F], [FS], [H], [JN], [Jo], [N], [R], [RR], [S], [Sp], [St], [Va]).

As in [TTV₂], we consider open subsets Ω of R^n endowed with the property (introduced in [C₁] for bounded open subsets):

$$(2) \quad \alpha = \sup \frac{|B|}{|B \cap \Omega|} < +\infty,$$

where the supremum is taken over all open balls B of R^n centered in Ω of radius ≤ 1 , and, if g is a locally integrable function and E is a bounded subset of R^n , we denote by $g_E = |E|^{-1} \int_E g$ the mean value of g over E .

We recall that $BMO(\Omega)$ is defined as the space of the functions $g \in L_{loc}^1(\overline{\Omega})$ such that

$$(3) \quad \|g\|_{BMO(\Omega)} = \sup \frac{1}{|B \cap \Omega|} \int_{B \cap \Omega} |g - g_{B \cap \Omega}| < +\infty,$$

endowed with the seminorm (3), where the supremum is taken over all open balls B of R^n centered in Ω and such that $|B| \leq \alpha |B \cap \Omega|$.

It is worth noticing that in general $BMO(\Omega)$ is smaller than the $BMO_f(\Omega)$ space (introduced by P. W. Jones in [Jo] for connected open subsets) consisting of the functions $g \in L^1_{loc}(\overline{\Omega})$ such that

$$\|g\|_{BMO_f(\Omega)} = \sup \frac{1}{|Q|} \int_Q |g - g_Q| < +\infty,$$

where the supremum is taken over all cubes $Q \subset \Omega$. There exist unbounded connected open subsets Ω of R^n , even of class C^∞ , such that $BMO(\Omega) \subsetneq BMO_f(\Omega)$. In Example 3.1 we exhibit a function $g \in BMO_f(\Omega) \setminus BMO(\Omega)$, with Ω Lipschitzian.

In Section 4 we establish some embedding results, which we also use to prove that $BMO(\Omega) \cap M^{1,0}(\Omega) \subset VM^{1,1}(\Omega)$.

In Section 5 we consider the $VMO(\Omega)$ space (introduced by D. Sarason in [S]) of functions with *vanishing mean oscillation*.

It is known (see [S]) that $g \in VMO(R^n)$ if and only if $g \in BMO(R^n)$ and $\tau_b g \rightarrow g$ in $BMO(R^n)$ as $|b| \rightarrow 0$.

Using the latter result and Lemma 5.4 below, which yields a suitable estimate for the difference between the averages of g on close cubes of R^n , we show that $VMO(R^n) \cap M^{1,0}(R^n) \subset M^{1,1}(R^n)$.

At the end, we characterize $VMO(R^n) \cap M^{1,0}(R^n)$ as the closure of $L^\infty \cap UC$ in $BMO(R^n) \cap M^{1,0}(R^n)$, equipped with the product norm of $BMO(R^n) \times M^{1,0}(R^n)$.

1. - PRELIMINARIES

Let Ω be an open subset of R^n , $n \geq 1$.

For each $x \in \Omega$ and $r \in R_+$, we set

$$\Omega(x, r) = B(x, r) \cap \Omega, \quad \Omega(x) = \Omega(x, 1),$$

where $B(x, r)$ is the open ball of R^n centered at x of radius r .

We put $B_r = B(0, r)$ and denote by ξ_r a $C_0^\infty(R^n)$ function such that

$$0 \leq \xi_r \leq 1, \quad \xi_{r/2} = 1, \quad \text{supp } \xi_r \subset B_{2r}.$$

Let us denote by $\Sigma(\Omega)$ the σ -algebra of Lebesgue measurable subsets of Ω and by $|E|$ the Lebesgue measure of $E \in \Sigma(\Omega)$.

Moreover we set:

$$\|g\|_{p, \Sigma} = \|g\|_{L^p(E)}.$$

As in [TTV₁], for each $p \in [1, +\infty[$, $\lambda \in [0, \pi[$ and $t \in R_+$, we denote by $M^{p, \lambda}(\Omega, t)$

the space of the functions $g \in L^1_{\text{loc}}(\bar{\Omega})$ such that

$$(1.1) \quad \|g\|_{M^{p,\lambda}(\Omega, t)} = \sup_{\substack{x \in \Omega \\ r < \min(t, t_1)}} r^{-\lambda/p} |g|_{p, \Omega}(x, r) < +\infty,$$

endowed with the norm (1.1).

It holds:

$$M^{q, \nu}(\Omega, t) \subset M^{p, \lambda}(\Omega, t), \quad \text{if } p \leq q \text{ and } \frac{\lambda - \nu}{p} \leq \frac{\nu - \nu}{q}.$$

We recall that

$$g \in M^{p, \lambda}(\Omega, t) \Leftrightarrow g \in M^{p, \lambda}(\Omega, t_1) \quad \forall t, t_1 \in \mathbb{R}_+$$

and there exists a constant $c = c(n)$ such that

$$(1.2) \quad \|g\|_{M^{p, \lambda}(\Omega, t_1)} \leq \|g\|_{M^{p, \lambda}(\Omega, t)} \leq c \|g\|_{M^{p, \lambda}(\Omega, t_1)} \left(\frac{t}{t_1} \right)^{(\lambda - \nu)/p}, \quad t > t_1.$$

In fact the left-hand side of (1.2) is evident, while the right one comes out from the following

LEMMA 1.1: *There exists a constant $c = c(n)$ such that, for each $f \in L^1_{\text{loc}}(\Omega)$, we have:*

$$\frac{1}{t''} \int_{Q(s, t)} |f| \leq \frac{c}{t_1''} \sup_{x \in \Omega} \int_{Q(s, t_1)} |f|, \quad \forall t, t_1 \in \mathbb{R}_+, t \geq t_1.$$

PROOF: The assertion follows at once from the proof of Lemma 3.1 in [TTV₂]. ■

For each function g defined in Ω , let us denote by g_0 the zero-extension of g on \mathbb{R}^n .

Since

$$\|g\|_{M^{p, \lambda}(\Omega, t)} \leq \|g_0\|_{M^{p, \lambda}(\mathbb{R}^n, t)} \leq 2^{1/p} \|g\|_{M^{p, \lambda}(\Omega, 2t)},$$

as a consequence of (1.2), we have that $g \in M^{p, \lambda}(\Omega, t)$ if and only if $g_0 \in M^{p, \lambda}(\mathbb{R}^n, t)$.

Now we set:

$$M^{p, \lambda}(\Omega) = M^{p, \lambda}(\Omega, 1), \quad M^{p, \lambda} = M^{p, \lambda}(\mathbb{R}^n), \quad M(\Omega, t) = M^{1, 0}(\Omega, t),$$

$$M(\Omega) = M(\Omega, 1), \quad M = M^{1, 0}.$$

As anticipated in the Introduction, in [TTV₁] (see also [VI]) there have been introduced some subspaces defined as follows:

- $VM^{p,1}(\Omega)$ is the subspace of $M^{p,1}(\Omega)$ of the functions g such that

$$\lim_{\varepsilon \rightarrow 0^+} \|g\|_{M^{p,1}(\Omega, \varepsilon)} = 0;$$

- $\tilde{M}^{p,1}(\Omega)$ is the subspace of $M^{p,1}(\Omega)$ of the functions g such that

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{\substack{E \in \mathcal{E}(\Omega) \\ \sup_{x \in E} |E(x)| < \varepsilon}} \|g|_E\|_{M^{p,1}(E)} = 0;$$

- $M_0^{p,1}(\Omega)$ is the subspace of $M^{p,1}(\Omega)$ of the functions g such that

$$g \in \tilde{M}^{p,1}(\Omega), \quad \lim_{r \rightarrow +\infty} \|(1 - \xi_r)g\|_{M^{p,1}(\Omega)} = 0.$$

We have already given a characterization of $\tilde{M}^{p,1}(\Omega)$ and $M_0^{p,1}(\Omega)$ in the Introduction.

2. - THE $M^{sp,1}$ SPACES

For each function g defined in R^n and $b \in R^n$ we set:

$$\tau_b g(x) = g(x - b).$$

It is trivial that

$$\|\tau_b g\|_{M^{p,1}} = \|g\|_{M^{p,1}} \quad \forall g \in M^{p,1}.$$

Let us denote by $M^{sp,1}$ the subspace of $M^{p,1}$ consisting of the functions g such that

$$\tau_b g \rightarrow g \text{ in } M^{p,1} \quad \text{as } |b| \rightarrow 0.$$

In the following we mean by $(J_k)_{k \in N}$ a sequence of mollifiers in R^n and by UC the class of uniformly continuous functions in R^n .

LEMMA 2.1: For every $M^{sp,1}$ we have:

$$J_k * g \in L^\infty \cap UC \quad \forall k \in N, \quad J_k * g \rightarrow g \text{ in } M^{p,1} \text{ as } k \rightarrow \infty.$$

PROOF: If $g \in M^{sp,1}$, then for each x' , $x'' \in R^n$ and $k \in N$

$$\begin{aligned} |J_k * g(x') - J_k * g(x'')| &\leq \int_{R^n} J_k(y) |g(x' - y) - g(x'' - y)| dy \leq \\ &\leq \left(\int_{R^n} J_k(y) |g(x' - y) - g(x'' - y)|^p dy \right)^{1/p} \leq c_k \left(\int_{B(0,1)} |g(x' - y) - g(x'' - y)|^p dy \right)^{1/p} = \\ &= c_k \|\tau_{(x' - x'')} g - g\|_{M^{p,1}}. \end{aligned}$$

From the last inequalities we deduce that $J_k * g \in UC$. Moreover, for each $x \in R^n$, we have

$$\|J_k * g(x)\| \leq \int_{R^n} |g(x-y)| J_k(y) dy \leq \|g\|_{M^{p,1}} \|J_k\|_{L^p},$$

and then $J_k * g \in L^\infty \cap UC$.

On the other side (see [Z])

$$\begin{aligned} \frac{1}{r^1} \int_{B(x,r)} |J_k * g(x) - g(x)|^p dx &\leq \frac{1}{r^1} \int_{B(0,1/k)} J_k(y) dy \int_{B(x,r)} |g(x-y) - g(x)|^p dx \leq \\ &\leq \sup_{|y| < 1/k} \|\tau_y g - g\|_{M^{p,1}}, \end{aligned}$$

so the desired result is proved. ■

It is plain that

$$L^\infty \cap UC \subset M^{p,1} \cap UC \subset M^{ap,1}.$$

LEMMA 2.2: $M^{ap,1}$ is the closure of $L^\infty \cap UC$ in $M^{p,1}$.

PROOF: From Lemma 2.1 it follows that $\overline{L^\infty \cap UC}^{M^{p,1}} \supset M^{ap,1}$. On the other hand, if $g \in L^\infty \cap UC^{M^{p,1}}$, then there exists $(g_n)_{n \in N}$ such that $g_n \in L^\infty \cap UC$ and

$$(i) \quad g_n \rightarrow g \text{ in } M^{p,1},$$

$$(ii) \quad \|\tau_y g - g\|_{M^{p,1}} \leq 2\|g - g_n\|_{M^{p,1}} + \|\tau_y g_n - g_n\|_{M^{p,1}},$$

whence the result. ■

We note that, since $\tilde{M}^{p,1}$ is the closure of L^∞ in $M^{p,1}$, Lemma 2.2 implies:

$$(2.1) \quad M^{ap,1} \subset \tilde{M}^{p,1}.$$

Let us show by a counterexample that in (2.1) the inclusion is strong.

EXAMPLE 2.1: Let g belong to $L^\infty(R)$ such that

$$g_{e,k} = g_{[k(n+(k-1)/n), n+k/n]} = \begin{cases} 0 & \text{if } k \text{ is even,} \\ 1 & \text{if } k \text{ is odd.} \end{cases}$$

For $n \geq 2$ and $k \leq n-1$, we have

$$\int_{x+(k-1)/n}^{x+k/n} |\tau_{1/n}g - g| = \frac{1}{n},$$

so

$$\|\tau_{1/n}g - g\|_{BMO} \geq \frac{1}{2}$$

and

$$g \in \widehat{M}(R) \setminus M^*(R). \quad \blacksquare$$

Since $M_0^{p,1}$ is the closure of $C_0^\infty(R^n)$ in $M^{p,1}$, then Lemma 2.2 implies that

$$(2.2) \quad M_0^{p,1} \subset M^{sp,1}.$$

In (2.2) the inclusion is strong; in fact it is known (see Remark 2.2 in [TTV₁]) that

$$g \in M_0^{p,1} \Rightarrow \lim_{|x| \rightarrow +\infty} \sup_{z \in B(0,1)} \frac{1}{z^{1/p}} \|g\|_{p, Q(x,z)} = 0,$$

where $Q(x,t) = \{y \in R^n : |y_i - x_i| < t, i = 1, \dots, n\}$, and so $g \in M^{sp,1} \setminus M_0^{p,1}$ for every $g \in L^\infty \cap UC$ such that $\inf |g| > 0$.

3. THE $BMO(\Omega)$ AND $BMO_1(\Omega)$ SPACES

In the following we assume that the open subset Ω satisfies (2).

For each $t \in]0, +\infty]$ we set (as in [TTV₂])

$$\alpha_t = \sup_{B \in \mathcal{B}(\Omega, t)} \frac{|B|}{|B \cap \Omega|},$$

where $\mathcal{B}(\Omega, t) = \{B(x, r) | x \in \Omega, r \leq t\}$.

Clearly $t \in R_+ \rightarrow \alpha_t \in R_+$ is an increasing function and verifies the relations

$$\alpha_t \leq \alpha \quad \forall t \leq 1, \quad \alpha_t \leq \alpha t^* \quad \forall t > 1.$$

We denote by $BMO(\Omega, t)$, $t \in]0, +\infty]$, the space of functions $g \in L^1_{loc}(\overline{\Omega})$ such that

$$(3.1) \quad \|g\|_{BMO(\Omega, t)} = \sup_{B \in \mathcal{B}(\Omega, t)} \frac{1}{|B \cap \Omega|} \int_{B \cap \Omega} |g - g_{B \cap \Omega}| < +\infty,$$

endowed with the seminorm defined by (3.1).

Setting

$$t_0 = \sup \{t \in R_+ | \alpha_t \leq \alpha\},$$

we observe that (see (3) of Introduction)

$$BMO(\Omega) = BMO(\Omega, t_0), \quad \|g\|_{BMO(\Omega)} = \lim_{t \rightarrow t_0^+} \|g\|_{BMO(\Omega, t)}.$$

Moreover we put $BMO = BMO(R^+)$.

It is known that if $\Omega \subset \Omega'$, then $BMO(\Omega', t) \subset BMO(\Omega, t)$, $t \in [0, +\infty]$, and

$$\|g\|_{BMO(\Omega, t)} \leq 2\alpha_t \|g\|_{BMO(\Omega', t)}.$$

We also denote by $BMO_1(\Omega, t)$, $t \in [0, +\infty]$, the space of the functions $g \in BMO(\Omega, t) \cap M(\Omega, t)$ equipped with the norm

$$\|g\|_{BMO_1(\Omega, t)} = \|g\|_{BMO(\Omega, t)} + \|g\|_{M(\Omega, t)},$$

and set

$$BMO_1(\Omega) = BMO_1(\Omega, 1), \quad BMO_1 = BMO_1(R^+).$$

Let us remark that $BMO_1(\Omega)$ is in general smaller than $BMO(\Omega)$: the function $g(x) = \log|x|$ is, indeed, an example of function in $BMO \setminus BMO_1$.

About the connection with the $BMO_f(\Omega)$ space of P. W. Jones (see Introduction) we recall that $BMO(\Omega) \subset BMO_f(\Omega)$.

Here below, we exhibit a function g , defined in a sufficiently regular and connected open subset Ω , that belongs to $BMO_f(\Omega) \setminus BMO(\Omega)$.

EXAMPLE 3.1: Let $f: R_+ \rightarrow R$ the function such that $f(x) = \log x$ if $x > 1$, $f(x) = 0$ if $x \leq 1$. Since f is the zero extension outside $I_1 = [1, +\infty]$ of a function in $BMO(I_1)$, which is bounded on bounded subsets of I_1 , then f belongs to $BMO(R_+, t_0)$ for every $t_0 \in R_+$ (see Lemma 4.1 in [TFV2]).

In particular, choosing $t_0 = 1$, we have:

$$\sup_{\substack{I \subset K_r \\ |I|/2 \leq 1}} \frac{1}{|I|} \int_I |f - f_I| < +\infty.$$

On the other hand, if $I \subset R_+$ is an interval such that $|I|/2 = t > 1$, then we have the following two cases:

- 1) if $I \subset I_1$, it turns out that $|I|^{-1} \int_I |f - f_I| \leq \|\log x\|_{BMO(I_1)}$;
- 2) if $I \cap \partial I_1 \neq \emptyset$, putting $I = [a, a+2t]$, $0 < a < 1$, $b = a+2t$ and $a_I = (b-1)^{-1} \int_a^b \log x dx$, we have

$$\frac{1}{|I|} \int_I |f - f_I| = \frac{1}{|I|} \int_{I \cap I_1} |f - f_I| + \frac{1}{|I|} \int_{I - I_1} |f - f_I| =$$

$$\begin{aligned}
&= \frac{1}{2t} \int_1^b \left| \log x - \frac{1}{(b-1)} \int_1^b \log x dx \right| + \frac{1}{2t} \int_t^1 \frac{1}{(b-1)} \int_1^b \log x dx \leq \\
&\leq [\log x]_{BMO(I_1)} + \frac{(1-t)}{2t(b-1)} \int_1^b \log x dx \leq \\
&\leq [\log x]_{BMO(I_1)} + \frac{1}{2t} \log(1+2t) \leq [\log x]_{BMO(I_1)} + 1.
\end{aligned}$$

From the above considerations it follows that $f \in BMO_f(\mathbb{R}_+)$.

Let us put $A = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$, $C = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y \leq 0\}$,

$$S = \left\{ (x, y) \in A \cup C \mid x \geq 1, -\frac{1}{2} \leq y \leq 0 \right\}.$$

We define $g: A \cup C \rightarrow \mathbb{R}$, setting

$$g(x, y) = \begin{cases} f(x) & \text{if } (x, y) \in A, \\ 0 & \text{if } (x, y) \in C. \end{cases}$$

Then, if $\Omega = (A \cup C) \setminus S$, we have:

$$g \in BMO_f(\Omega), \text{ but } g \notin BMO(\Omega).$$

Moreover

$$g \in BMO\left(\Omega, \frac{1}{4}\right), \text{ but } g \notin BMO(\Omega, 1).$$

In fact, considering a square $Q \subset \Omega$ and setting $D = \{(x, y) \in \mathbb{R}^2 \mid x > 1, y > 0\}$, it turns out that

- a) if $Q \cap D = \emptyset$, then $|Q|^{-1} \int_Q |g - g_Q| = 0$;
- b) if $Q \cap D \neq \emptyset$, then $Q = [x_0 - r, x_0 + r] \times [y_0 - r, y_0 + r] \subset A$ and

$$\frac{1}{|Q|} \int_Q \left| g - \frac{1}{2r} \int_{x_0-r}^{x_0+r} f \right| = \frac{1}{2r} \int_{x_0-r}^{x_0+r} \left| f - \frac{1}{2r} \int_{x_0-r}^{x_0+r} f \right| \leq [f]_{BMO_f(\Omega)}.$$

Therefore $g \in BMO_f(\Omega)$.

If we put $(x_k, y_k) = (k, 1/k)$, $B_k = B((x_k, y_k), 1)$, with $k \in \mathbb{N}$, we get

$$\lim_{k \rightarrow +\infty} \frac{1}{|B_k \cap \Omega|} \int_{B_k \cap \Omega} |g - g_{B_k \cap \Omega}| = +\infty.$$

Hence $g \notin BMO(\Omega, 1)$ and, *a fortiori*, $g \notin BMO(\Omega)$.

Finally, let $0 < r \leq 1/4$, $(x_0, y_0) \in \Omega$ and $B_r = B((x_0, y_0), r)$.

We distinguish the following three cases:

i) if $x_0 \leq 2$, then $|B_r \cap \Omega|^{-1} \int_{B_r \cap \Omega} |g - g_{B_r \cap \Omega}| \leq 2 \|g\|_{B_r \cap \Omega} \leq 2 \log 3$;

ii) if $x_0 > 2$ and $y_0 \leq 0$, then $B_r \subset C$ and so

$$\frac{1}{|B_r \cap \Omega|} \int_{B_r \cap \Omega} |g - g_{B_r \cap \Omega}| = 0;$$

iii) if $x_0 > 2$ and $y_0 > 0$, then

$$\frac{1}{|B_r \cap \Omega|} \int_{B_r \cap \Omega} \left| g - \frac{1}{2r} \int_{y_0-r}^{y_0+r} \log x dx \right| \leq \frac{16}{\pi} [\log x]_{BMO}.$$

Therefore we conclude that $g \in BMO(\Omega, 1/4)$. ■

It is clear that, by suitable modifying this example, one can take Ω to be C^∞ .

4. · SOME EMBEDDING RESULTS.

LEMMA 4.1: *There exists $c = c(n)$ such that*

$$(4.1) \quad [g]_{BMO(\Omega, t)} \leq [g]_{BMO(\Omega, t_0)} + \frac{\alpha_t c}{t_0^n} \|g\|_{M(\Omega, t_0)}$$

$\forall t, t_0, t_1 \in \mathbb{R}_+$ with $t_0 \leq t_1$ and $\forall g \in BMO(\Omega, t_1) \cap M(\Omega, t_0)$.

PROOF: From the proof of Lemma 3.1 in [TTV₂], we can deduce that there exists a constant $c = c(n) \in \mathbb{R}_+$ such that

$$[g]_{BMO(\Omega, t)} \leq [g]_{BMO(\Omega, t_1)} + \frac{\alpha_t c}{t_1^n} \|g\|_{M(\Omega, t_1)}.$$

Then we obtain (4.1) as a consequence of Lemma 1.1. ■

Let $t^* \in]0, +\infty]$ such that $\sup_{t \in]0, t^*[} \alpha_t < +\infty$ and fix $t_1 \in \mathbb{R}_+$. From (4.1) we have:

$$(4.2) \quad BMO_1(\Omega, t_1) \subset BMO(\Omega, t) \quad \forall t \in]0, t^*];$$

in particular

$$(4.3) \quad BMO_1(\Omega) \subset BMO(\Omega).$$

REMARK 4.2: We point out that in general (4.2) is not true with $BMO(\Omega, t_1)$ instead of $BMO_1(\Omega, t_1)$: for example, if g is the function of Example 3.1, then $g \in BMO(\Omega, 1/4)$, whilst $g \notin BMO(\Omega, 1)$.

LEMMA 4.3: For every $\lambda \in [0, n]$, there exists a constant $c = c(n, \lambda)$ such that for each $t \in \mathbb{R}_+$ and $g \in BMO(\Omega, t)$ we have:

$$(4.4) \quad \frac{1}{\tau^{\lambda}} \int_{Q(x, \tau)} |g| \leq ca_t \left(t^{n-\lambda} [g]_{BMO(\Omega, t)} + \frac{1}{t^{\lambda}} \left| \int_{Q(x, t)} g \right| \right) \quad \forall x \in \Omega, \quad \forall \tau \in]0, t],$$

$$(4.5) \quad \|g\|_{M^{1,1}(\Omega, t)} \leq ct^{n-\lambda} \left(\left(1 + \log \frac{t}{r} \right) [g]_{BMO(\Omega, t)} + \frac{1}{t^n} \sup_{x \in \Omega} \left| \int_{Q(x, t)} g \right| \right) \\ \forall r \in]0, te^{-1/(n-\lambda)}],$$

PROOF: From Lemma 3.3 in [TTV₂], there exists a constant $c_1 = c_1(n)$ such that

$$\frac{1}{\tau^{\lambda}} \int_{Q(x, \tau)} |g| \leq c_1 a_t \tau^{n-\lambda} \left(\left(1 + \log \frac{t}{r} \right) [g]_{BMO(\Omega, t)} + \frac{1}{t^n} \left| \int_{Q(x, t)} g \right| \right) \quad \forall \tau \in]0, t].$$

Using the previous inequality, we have the desired result observing that the function

$$f: \tau \in]0, t] \rightarrow \tau^{n-\lambda} \log \frac{t}{\tau}$$

is increasing in $]0, te^{-1/(n-\lambda)}]$ and that

$$\sup_{\tau \in]0, t]} \tau^{n-\lambda} \log \frac{t}{\tau} = \frac{t^{n-\lambda}}{e(n-\lambda)}. \quad \blacksquare$$

Of course (4.4) implies

$$(4.6) \quad \|g\|_{M^{1,1}(\Omega, t)} \leq ca_t \left(t^{n-\lambda} [g]_{BMO(\Omega, t)} + \frac{1}{t^{\lambda}} \|g\|_{M(\Omega, t)} \right) \\ \forall t \in \mathbb{R}_+, \quad \forall g \in BMO_1(\Omega, t).$$

THEOREM 4.4: We have:

$$BMO_1(\Omega) \subset BMO(\Omega) \cap M^{1,1}(\Omega) \quad \forall \lambda \in [0, n],$$

$$[g]_{BMO(\Omega)} + \|g\|_{M^{1,1}(\Omega)} \leq c \|g\|_{BMO_1(\Omega)}, \quad \forall g \in BMO_1(\Omega),$$

where $c = c(n, \lambda, \alpha)$.

PROOF: It is an easy consequence of (4.3) and (4.6). \blacksquare

THEOREM 4.5: For every $\lambda \in [0, n]$, we have:

$$(4.7) \quad BMO_1(\Omega) \subset VMO^{1,\lambda}(\Omega).$$

Moreover, for a fixed $K \in \mathbb{R}_+$, there exists a constant $c = c(K, n, \lambda)$ such that

$$(4.8) \quad \|g\|_{VMO^{1,\lambda}(\Omega)} \leq c \tau^{\pi - \lambda} \log\left(\frac{K}{\tau}\right) \|g\|_{BMO_1(\Omega)} \quad \forall \tau \in]0, K\tau^{-1/(n-\lambda)}], \quad \forall g \in BMO_1(\Omega).$$

PROOF: (4.7) is an obvious consequence of (4.5) while we obtain (4.8) right applying (4.5), (4.1) and (1.2). ■

5. - THE $VMO(\Omega)$ AND $VMO_1(\Omega)$ SPACES

As D. Sarason [S], we denote by $VMO(\Omega)$ the subspace of $BMO(\Omega)$ consisting of the functions g such that

$$\lim_{\tau \rightarrow 0^+} \|g\|_{BMO(\Omega, \tau)} = 0.$$

EXAMPLE 5.1: Let us consider the function

$$(5.1) \quad g: x \in \Omega \rightarrow]0, e^{-1}[-\log |\log x| \in \mathbb{R}_+.$$

Then $g \in VMO(\Omega)$.

In fact g is non-negative, decreasing, convex and differentiable.

So for every $\tau \in]0, e^{-1}[$ the function

$$x \in [0, e^{-1} - \tau] \rightarrow \frac{1}{\tau} \int_x^{e^{-1}} (g(x) - g(x + \tau)) dx \in \mathbb{R}_+,$$

is decreasing too and

$$(5.2) \quad \frac{1}{\tau} \int_x^{e^{-1}} (g(x) - g(x + \tau)) dx \leq \frac{1}{\tau} \int_0^\tau |g(x) - g(\tau)| dx = \sigma(\tau),$$

where

$$\lim_{\tau \rightarrow 0} \sigma(\tau) = 0.$$

On the other side

$$\frac{1}{\tau} \int_x^{e^{-1}} \left| g(x) - \frac{1}{\tau} \int_x^{e^{-1}} g(y) dy \right| dx \leq \frac{2}{\tau} \int_x^{e^{-1}} (g(x) - g(x + \tau)).$$

whence, by (5.1) and (5.2), it follows

$$\lim_{r \rightarrow 0} \|g\|_{BMO(\Omega, r)} \leq 2 \lim_{r \rightarrow 0} \sup_{x \in r} \sigma(x) = 0. \quad \blacksquare$$

Let us set

$$VMO_1(\Omega) = VMO(\Omega) \cap M(\Omega).$$

As consequence of (4.6), we have that $g \in VMO_1(\Omega)$ if and only if $g \in BMO_1(\Omega)$ and

$$\lim_{r \rightarrow 0^+} \|g\|_{BMO_1(\Omega, r)} = 0.$$

We remark that in general $VMO_1(\Omega) \subset VMO(\Omega)$.

In fact the function

$$g: x \in \Omega =]1, +\infty[\rightarrow \log x \in R,$$

belongs to $VMO(\Omega)$ but is not in $M(\Omega)$.

We put

$$VMO = VMO(R^*), \quad VMO_1 = VMO_1(R^*).$$

LEMMA 5.2: *We have*

$$(5.3) \quad UC(\Omega) \cap M(\Omega) \subset VMO_1(\Omega).$$

PROOF: If $g \in UC(\Omega) \cap M(\Omega)$ then for every $\varepsilon \in R_+$, there exists $\delta \in R_+$ such that

$$\|g\|_{BMO(\Omega, r)} \leq \sup_{B \in S(\Omega, r)} \frac{1}{|B \cap \Omega|^2} \int_{B \cap \Omega} \left(\int_{B \cap \Omega} |g(x) - g(y)| dy \right) dx < \varepsilon \quad \forall r \in \left]0, \frac{\delta}{2}\right[.$$

Let $\delta \in R_+$ such that $|g(x') - g(x'')| \leq 1$ for each $x', x'' \in \Omega$, $|x' - x''| < \delta$. Then, for every $B \in S(\Omega, r)$, we have

$$a) \quad \frac{1}{|B|} \int_B \left| g - \frac{1}{|B|} \int_B g \right| \leq 1 \text{ if } 2r \leq \delta,$$

$$b) \quad \frac{1}{|B|} \int_B |g - g_B| \leq \frac{2}{|B|} \int_B |g| \leq \frac{2}{|B_{\delta/2}|} \|g\|_{M(\Omega)} \text{ if } 2r > \delta,$$

whence the result. \blacksquare

The function g of (5.1) belongs to $VMO_1(\Omega)$, but is not uniformly continuous.

THEOREM 5.3: We have

$$VMO_1 \subset M^*.$$

The proof of this theorem is based on two lemmas, which we state in advance. The first one is a generalization of Lemma 2.1 of P. W. Jones (see [Jo]).

LEMMA 5.4: Suppose that $g \in BMO$ and Q_0, Q_1 are two cubes in \mathbb{R}^n . Then

$$|g_{Q_0} - g_{Q_1}| \leq \left(9 + 3n \left| \log \frac{l(Q_1)}{l(Q_0)} \right| + 6n \log \left(1 + \frac{\text{dist}(Q_0, Q_1)}{l(Q_0) + l(Q_1)} \right) \right) \|g\|_{BMO},$$

where $l(Q)$ is the length of the sides of the cube Q .

PROOF: Let us first distinguish the following two cases:

- i) $Q_0 \subset Q_1$,
- ii) $l(Q_0) = l(Q_1) = L$.

Case i). Let us divide the interval $[1, +\infty[$ in subintervals of type $[3^{m-1}, 3^m[$, $m \in N$.

Then there exists $m \in N$ such that $3^{m-1} \leq |Q_1| / |Q_0| < 3^m$.

Proceeding as Jones, for $m = 1$ we have

$$|g_{Q_0} - g_{Q_1}| \leq 3\|g\|_{BMO}$$

For any m , repeating m times the same argument by means of a chain of $m - 1$ intermediate cubes, it turns out that

$$|g_{Q_0} - g_{Q_1}| \leq 3m\|g\|_{BMO}.$$

So in general if $Q_0 \subset Q_1$, we have

$$(5.4) \quad |g_{Q_0} - g_{Q_1}| \leq 3 \left(1 + n \log \frac{l(Q_1)}{l(Q_0)} \right) \|g\|_{BMO}.$$

Case ii). If Q_2 is a cube such that $Q_0, Q_1 \subset Q_2$ and $l(Q_2) = \text{dist}(Q_0, Q_1) + 2L$, applying (5.4) twice, we get

$$(5.5) \quad \begin{aligned} |g_{Q_0} - g_{Q_1}| &\leq |g_{Q_0} - g_{Q_2}| + |g_{Q_2} - g_{Q_1}| \leq 6 \left(1 + n \log \frac{l(Q_2)}{l} \right) \|g\|_{BMO} = \\ &= 6 \left(1 + n \log \left(2 + \frac{\text{dist}(Q_0, Q_1)}{l} \right) \right) \|g\|_{BMO}. \end{aligned}$$

Finally, we consider the general case.

Let us suppose $l(Q_0) \leq l(Q_1)$ and take a cube Q_2 such that $Q_0 \subset Q_2$ and

$l(Q_1) = l(Q_2)$. Observing that

$$|g_{Q_1} - g_{Q_2}| \leq |g_{Q_1} - g_{Q_3}| + |g_{Q_3} - g_{Q_2}|$$

and applying (5.4) and (5.5), we succeed in our purpose. ■

In particular, if $g \in BMO$, then for every cube Q of R^n and each $b \in R^n$ it turns out that

$$(5.6) \quad |g_{Q+b} - g_Q| \leq \left(9 + 6\pi \log \left(1 + \frac{|b|}{l(Q)} \right) \right) \|g\|_{BMO}.$$

The inequality (5.6) can be improved as $|b| / l(Q) \rightarrow 0$.

LEMMA 5.5: Let $g \in BMO$, Q a cube of R^n and $b \in R^n$ such that $\max_{1 \leq i \leq n} |b_i| \leq l(Q)$, then we have

$$(5.7) \quad |g_{Q+b} - g_Q| \leq 15\pi^{3/2} \frac{|b|}{l(Q)} \left(1 + 2 \log \frac{|b|}{l(Q)} \right) \|g\|_{BMO}.$$

PROOF: Let $l(Q) = l$ and $b = \pm ae_1$, $0 < a \leq l$, where e_i , $i = 1, \dots, n$, denotes the i -th vector of the standard basis of R^n . Then

$$\int_{Q+b} g - \int_Q g = \int_{R+le_1} g - \int_R g,$$

where R is a parallelepiped of volume $a \cdot l^{n-1}$.

Since $a/l \leq 1$, then a has the binary representation

$$a = a_1 2^{-1} l + \dots + a_k 2^{-k} l + \dots$$

where a_k may assume the values 0 or 1.

It follows that R can be subdivided in parallelepipeds R_{i_k} of volume $2^{-i_k} l \cdot l^{n-1}$, $k \in N$, where $i_k < i_{k+1}$ and $i_k \in \{i \in N \mid a_i = 1\}$.

In turn R_{i_k} can be expressed (up to subsets of zero measure) as disjoint union of $2^{(n-1)i_k}$ cubes with side $2^{-i_k} l$.

If Q_{i_k} is one of these cubes, from Lemma 5.3, it follows that

$$(5.8) \quad \left| \int_{Q_{i_k}+le_1} g - \int_{Q_{i_k}} g \right| \leq 2^{-ni_k} l^n (9 + 6\pi i_k) \|g\|_{BMO}.$$

In virtue of the decomposition of R , given by the procedure mentioned above, and

from (5.8) we get:

$$\begin{aligned}
 (5.9) \quad & \left| \int_{R+le_1} g - \int_R g \right| \leq \sum_{i=1}^n a_i \left| \int_{R_i+le_1} g - \int_{R_i} g \right| \leq \\
 & \leq \sum_{i=1}^n a_i 2^{(n-1)i} \cdot 2^{-ni} l^n (9 + 6ni) [g]_{BMO} \leq 15 n^{1/2} \left(\sum_{i=1}^n a_i i 2^{-i} \right) [g]_{BMO} = \\
 & = 15 n^{1/2} \frac{\sum_{i=1}^n a_i i 2^{-i}}{\sum_{i=1}^n a_i 2^{-i}} a[g]_{BMO}.
 \end{aligned}$$

Setting $i_1 = \min \{i \in N : a_i = 1\}$, we have

$$\frac{1}{2^{i_1}} = \frac{d}{l} = \sum_{i=i_1+1}^n a_i 2^{-i} \geq \frac{d}{l} = \frac{1}{2^i},$$

and then

$$i_1 \leq 1 + 2 \left| \log \frac{d}{l} \right|,$$

which, inserted in (5.9), by (5.8), gives

$$\left| \int_{Q+b} g - \int_Q g \right| \leq 15 n^{1/2} \left(3 + 2 \left| \log \frac{|b|}{l} \right| \right) a[g]_{BMO},$$

since $b = \pm ae_1$.

Of course, we obtain similar estimates when $b = \pm ae_i$, $i = 1, \dots, n$.

Let now $b = \sum_{i=1}^n b_i e_i$. Setting $Q_0 = Q$ and $Q_i = Q_{i-1} + b_i e_i$, $i = 1, \dots, n$, we have:

$$\left| \int_{Q+b} g - \int_Q g \right| \leq \sum_{i=1}^n \left| \int_{Q_i} g - \int_{Q_{i-1}} g \right|.$$

Hence, using the above-estimates, we get

$$\left| \int_{Q+b} g - \int_Q g \right| \leq 15 n^{3/2} l^{n-1} \left(1 + 2 \left| \log \frac{|b|}{l} \right| \right) |b| [g]_{BMO},$$

which yields the desired result. ■

PROOF OF THEOREM 5.3: Let B an open ball of \mathbb{R}^n of radius 1 and Q a cube of side 2

which contains B . Then, if $b \in R^*$,

$$(5.10) \quad \int_B |\tau_{bg} - g| \leq \int_Q |(\tau_{bg} - g) - (\tau_{bg} - g)_Q| + \left| \int_Q (\tau_{bg} - g) \right| \leq \\ \leq 2^k ([\tau_{bg} - g]_{BMO} + |g_{Q+1} - g_Q|).$$

Since $g \in VMO$, applying (5.7) and (5.10), we deduce that $\tau_{bg} - g \in M^1$ and

$$\lim_{k \rightarrow \infty} \|\tau_{bg} - g\|_{M^1} = 0. \quad \blacksquare$$

LEMMA 5.6: If $g \in VMO_1$, then:

$$J_k * g \in L^\infty \cap UC \quad \forall k \in N, \quad J_k * g \rightarrow g \text{ in } BMO_1 \text{ as } k \rightarrow \infty.$$

PROOF: If $g \in VMO_1$, then, from Lemma 2.1 and Theorem 5.2, we obtain that $J_k * g \in L^\infty \cap UC$ and $J_k * g \rightarrow g$ in M as $k \rightarrow +\infty$.

On the other hand, since $g \in VMO$, then $J_k * g \rightarrow g$ in BMO as $k \rightarrow +\infty$. \blacksquare

THEOREM 5.7: VMO_1 is the closure of $L^\infty \cap UC$ in BMO_1 .

PROOF: That $VMO_1 \subset \overline{L^\infty \cap UC}^{BMO_1}$ is a consequence of Lemma 5.5. To prove the converse, we observe that if $g \in L^\infty \cap \overline{UC}^{BMO_1}$, then there exists $g_k \in L^\infty \cap UC$, $k \in N$ such that $g_k \rightarrow g$ as $k \rightarrow +\infty$. From (5.3) it follows that $g_k \in VMO$ for every $k \in N$.

Then, from the inequality

$$[g]_{BMO(R^*, t)} \leq [g - g_k]_{BMO(R^*, t)} + [g_k]_{BMO(R^*, t)}$$

we deduce that $g \in VMO$, and this completes the proof. \blacksquare

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