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Differential Inclusions with Nonconvex Right Hand Side and Applications to Implicit Integral and Differential Equations (**)

ABSTRACT. — In this work an existence theorem for an ordinary differential inclusion, where the usual conditions of lower or upper semicontinuity on the multifunction are not assumed, is established. As a consequence, some results on implicit integral and differential equations are obtained. Moreover, a similar problem for elliptic differential inclusions and equations is considered.

Inclusioni differenziali con secondo membro non convesso ed applicazioni alle equazioni integrali e differenziali in forma implicita

RIASSUNTO. — In questo lavoro viene stabilito un teorema di esistenza per un'inclusione differenziale ordinaria dove non vengono assunte le usuali condizioni di semicontinuità inferiore o superiore sulla multifunzione. Come conseguenza si ottengono alcuni risultati sulle equazioni differenziali ed integrali in forma implicita. Viene, inoltre, considerato l'analogo problema per le inclusioni ed equazioni differenziali di tipo ellittico.

INTRODUCTION

Let E be a separable real Banach space, $x_0 \in E$, $a \in]0, +\infty[$. Given a multifunction $F: [0, a] \times E \to 2^E$, consider the Cauchy problem

$$\begin{cases} x' \in F(t, x), \\ x(0) = x_0. \end{cases} \tag{1}$$

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In dealing with the existence of solutions to this problem, one usually assumes some kind of continuity of F with respect to x and, accordingly, some suitable properties of the values of F. For a thorough account on the subject, we refer to [1] and [4].

In this paper, on the contrary, we deal with a class of multifunctions F which are able to satisfy none of the most usual continuity conditions, namely lower semicontinuity and upper semicontinuity. At the same time, the values of our F's can be non-convex, non-closed and unbounded. But, the best way of introducing our existence result is to state here two of its consequences which cannot be directly obtained from the other known existence theorems for problem (1).

THEOREM A: *Let V be a compact, connected, locally connected metric space, $f: V \to E$ a continuous function, and $g: V \to \mathbf{R}$ a continuous function such that* $\operatorname{int}(g^{-1}(r)) = \emptyset$ *for every* $r \in \operatorname{int}(g(V))$.

Then, for every Carathéodory function $\varphi: [0, a] \times E \to \mathbf{R}$ *such that* $\varphi([0, a] \times E) \subseteq g(V)$ *and every* $x_0 \in E$, *there exists a measurable function* $v: [0, a] \to V$ *such that*

$$\varphi\left(t, x_0 + \int_0^t f(v(\tau))\, d\tau\right) = g(v(t))$$

for almost every $t \in [0, a]$.

THEOREM B: *Let $(S, \mathfrak{F}, \mu)$ be a finite non-atomic complete measure space, Φ a linear homeomorphism from E onto $L^1(S)$, and $f: S \times \mathbf{R} \to \mathbf{R}$ a Carathéodory function such that the function* $s \to \inf\{|r| : f(s, r) = 0\}$ *belongs to* $L^1(S)$ *and*

$$\sup\{r \in \mathbf{R} : f(s, r) = 0\} = +\infty, \quad \inf\{r \in \mathbf{R} : f(s, r) = 0\} = -\infty$$

for μ-a.e. $s \in S$.

Then, for every $x_0 \in E$, for every $\Psi \in E^* \setminus \{0\}$, every continuous function $g: \mathbf{R} \to \mathbf{R}$ such that $\operatorname{int}(g^{-1}(r)) = \emptyset$ *for all* $r \in \mathbf{R}$, *and every bounded Carathéodory function* $\varphi: [0, a] \times E \to \mathbf{R}$ *such that* $\overline{\varphi([0, a] \times E)} \subseteq g(\mathbf{R})$, *there exists* $u \in AC([0, a], E)$ *such that*

$$\begin{cases} f(s, \Phi(u'(t))(s)) = 0 & \text{for a.e. } t \in [0, a] \text{ and for } \mu\text{-a.e. } s \in S, \\ g(\Psi(u'(t))) = \varphi(t, u(t)) & \text{for a.e. } t \in [0, a], \\ u(0) = x_0. \end{cases}$$

The present paper is arranged as follows. In Section 1, we give some preliminaries. In Section 2, we establish our existence theorem for Problem (1), that is Theorem 2.1. Section 3 is devoted to proving theorems A and B, deducing them from Theorem 2.1.

Finally, in Section 4, we deal with partial differential inclusions associated to multifunctions of the same kind considered in connection with Problem (1).

1. - Preliminaries

Let X and Z be two nonempty sets. A multifunction Φ from X into Z (briefly, $\Phi: X \to 2^Z$) is a function from X into the family of all subsets of Z. The graph of Φ, denoted by $\mathrm{gr}(\Phi)$, is the set $\{(x, z) \in X \times Z : z \in \Phi(x)\}$. If $A \subseteq X$, $\Omega \subseteq Z$, we put $\Phi(A) = \bigcup_{x \in A} \Phi(x)$ and $\Phi^-(\Omega) = \{x \in X : \Phi(x) \cap \Omega \neq \emptyset\}$. When $(X, \mathfrak{J})$ is a measurable space, Z is a topological space and for every open set $\Omega \subseteq Z$ one has $\Phi^-(\Omega) \in \mathfrak{J}$, we say that the multifunction Φ is measurable. If X and Z are two topological spaces and for every open (resp. closed) set $\Omega \subseteq Z$ the set $\Phi^-(\Omega)$ is open (resp. closed) in X, we say that Φ is lower (resp. upper) semicontinuous. If Z is a metric space and if $\Phi(X)$ is a bounded set in Z, we say that Φ is a bounded multifunction.

Let $I, V \subseteq \mathbf{R}$, with $I \subseteq V$. We denote by $\mathrm{int}_V(I)$ the interior of I in V; if $V \equiv \mathbf{R}$ we denote the interior of I by $\mathrm{int}(I)$. Moreover, if T is a subset of a topological space X we denote the closure of T by $\overline{T}$.

Let $(S, \mathfrak{J}, \mu)$ be a finite non-atomic complete measure space and let $(E, \|\cdot\|_E)$ be a separable real Banach space; we denote by $L^1(S, E)$ the space of all (equivalence classes of) strongly $\mathfrak{J}$-measurable function $v: S \to E$ such that

$$\int_S \|v(s)\|_E \, ds < +\infty .$$

The norm on $L^1(S, E)$ is the usual one:

$$\|v\|_{L^1(S, E)} = \int_S \|v(s)\|_E \, ds .$$

If $v \in L^1(S, E)$, we denote by $\int_S v(s) ds$ the Bochner integral of v. When $E = \mathbf{R}$ and there is no ambiguity, we simply write $L^1(S)$ in place of $L^1(S, \mathbf{R})$. A nonempty set $K \subseteq L^1(S)$ is said to be decomposable if, for every $w_1, w_2 \in K$ and every measurable set $A \subseteq S$, one has $\mathcal{X}_A w_1 + (1 - \mathcal{X}_A) w_2 \in K$, where $\mathcal{X}_A$ is the characteristic function of A.

Now, let $T = [0, a]$ be a compact interval with Lebesgue measure m, we denote by $AC([0, a], E)$ the set of all strongly absolutely continuous functions from $[0, a]$ into E which are almost everywhere strongly differentiable.

We say that a mapping $F: T \times E \to 2^E$ has the weak Scorza Dragoni property if for any $\varepsilon > 0$ and any compact $K \subseteq E$ there exists a compact set $T_\varepsilon \subseteq T$ with $m(T \setminus T_\varepsilon) \leq \varepsilon$ such that the restriction of F to $T_\varepsilon \times K$ is lower semicontinuous. Classes of multifunctions that have the weak Scorza Dragoni property can be obtained, for instance, from Theorem 1 and 2 of [2].

Theorem 1 of [8] plays a fundamental role in the proof of Theorem B. For the reader's convenience, we recall it here.

Theorem 1.1 ([8], Theorem 1 and Remark 2): *Let $(S, \mathfrak{I}, \mu)$ be a finite non-atomic complete measure space; E a separable real Banach space; $\Phi: E \to L^1(S)$ a linear homeomorphism; $f: S \times \mathbf{R} \to \mathbf{R}$ a Carathéodory function such that*

$$\sup\{r \in \mathbf{R}: f(s, r) = 0\} = +\infty; \quad \inf\{r \in \mathbf{R}: f(s, r) = 0\} = -\infty \quad \text{for } \mu\text{-a.e. } s \in S,$$

and the function $s \to \inf\{|r|: f(s, r) = 0\}$ belongs to $L^1(S)$.

Then, the set $Y = \{u \in E: f(s, \Phi(u)(s)) = 0 \text{ for } \mu\text{-a.e. } s \in S\}$ is a retract of E and intersects every closed hyperplane of E.

In the fourth section we use the following notation. Ω is a nonempty, bounded, open, connected subset of $\mathbf{R}^n$, $n \geq 3$, with a boundary $\partial\Omega$ of class $C^{1,1}$ and $W^{k,p}(\Omega)$, with $k \in \{1, 2\}$ and $n < p < \infty$, is the set of all real functions defined on Ω whose weak partial derivatives up to the order k lie in $L^p(\Omega)$. The symbol $W_0^{1,p}(\Omega)$ is used to denote the set of all $u \in W^{1,p}(\Omega)$ such that $u(x) = 0$ for every $x \in \partial\Omega$. Given a positive integer h, we denote $W^{k,p}(\Omega, \mathbf{R}^h)$ the set of all functions $u: \Omega \to \mathbf{R}^h$, $u = (u_1, u_2, \ldots, u_h)$, such that $u_i \in W^{k,p}(\Omega)$ for every $i = 1, 2, \ldots, h$. In similar way one defines the set $W_0^{k,p}(\Omega, \mathbf{R}^h)$. The symbol $\mathcal{L}(\Omega)$ is used for the Lebesgue σ-algebra of Ω, while $\mathcal{B}(\mathbf{R}^h \times \mathbf{R}^{nh})$ denotes the Borel σ-algebra of $\mathbf{R}^h \times \mathbf{R}^{nh}$.

Let

$$\mathcal{L}u = -\sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

where $a_{i,j} \in C^0(\Omega)$, $a_{i,j} = a_{j,i}$ for every $i, j = 1, 2, \ldots, n$, and $\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq \lambda(\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2)$ for some $\lambda > 0$, every $x \in \Omega$ and every $(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{R}^n$; $b_i \in L^\infty(\Omega)$ for all $i = 1, 2, \ldots, n$; $c \in L^\infty(\Omega)$ and $c(x) \geq 0$ almost everywhere in Ω.

Finally, for every $u = (u_1, u_2, \ldots, u_h) \in W^{2,p}(\Omega, \mathbf{R}^h)$, we define

$$Lu = (\mathcal{L}u_1, \mathcal{L}u_2, \ldots, \mathcal{L}u_h)$$

and for every $u = (u_1, u_2, \ldots, u_h) \in W^{1,p}(\Omega, \mathbf{R}^h)$ we set

$$Du = (Du_1, Du_2, \ldots, Du_h),$$

where Du_i is the gradient of the function u_i.

2. - An existence theorem for Problem (1)

The main result of this section is Theorem 2.1 below.

Theorem 2.1: *Let I be a subset of* $\mathbf{R}$*; E a separable real Banach space; $F: I \to 2^E$ a multifunction with nonempty and closed values such that* (*a*) gr(F) *is connected and locally connected;*

(*b*) *for every open set* $\Omega \subseteq E$, *the set* $F^-(\Omega) \cap \mathrm{int}(I)$ *has no isolated points.*

Then, for every bounded multifunction $G: [0, a] \times E \to 2^{\mathbb{R}}$ *with nonempty and closed values having the weak Scorza Dragoni property and such that*

$$\overline{G([0, a] \times E)} \subseteq I,$$

and for every $x_0 \in E$, *there exists a function* $u \in AC([0, a], E)$ *such that*

$$\begin{cases} u'(t) \in F(G(t, u(t))) & \text{for almost every } t \in [0, a], \\ u(0) = x_0. \end{cases}$$

Proof: Thanks to our assumptions, we can apply Theorem 11 of [9] to the multifunction F. Then, there exist two multifunctions $\Phi_1, \Phi_2: I \to 2^E$ such that Φ_1 is lower semicontinuous, Φ_2 is upper semicontinuous with compact values and $\emptyset \neq \Phi_1(r) \subseteq \Phi_2(r) \subseteq F(r)$ for every $r \in I$.

We now put

$$\Gamma(t, x) = \overline{\Phi_1(G(t, x))}$$

for every $(t, x) \in [0, a] \times E$.

It is easily seen that $\Gamma: [0, a] \times E \to 2^E$ has the weak Scorza Dragoni property. Moreover, Γ is a multifunction with relatively compact range. Indeed, from the upper semicontinuity of Φ_2 and the compactness of $\overline{G([0, a] \times E)}$ it follows that $\Phi_2(\overline{G([0, a] \times E)})$ is a compact set in E and

$$\Gamma([0, a] \times E) \subseteq \Phi_2(\overline{G([0, a] \times E)}).$$

At this point, we can apply Theorem 4.1 of [12] to the multifunction Γ. From that result, there exists a function $u \in AC([0, a], E)$ such that

$$\begin{cases} u'(t) \in \overline{\Phi_1(G(t, u(t)))} & \text{for almost every } t \in [0, a], \\ u(0) = x_0. \end{cases}$$

Taking into account that for every $t \in [0, a]$ the set $\Phi_2(G(t, u(t)))$ is closed in E, we have

$$\overline{\Phi_1(G(t, u(t)))} \subseteq \Phi_2(G(t, u(t))) \subseteq F(G(t, u(t))),$$

for every $t \in [0, a]$. Hence, the function u satisfies our conclusion. ■

Remark 2.1: From [11] (see the example and Theorem 11) it is not difficult to obtain an example of a differential inclusion so that the conclusion of Theorem 2.1 is no longer true without assuming (*b*).

Clearly, from classical examples (see, for instance, Example 5.1 of [4]) it is also possible to show that the conclusion of Theorem 2.1 is no longer true without assuming (a).

3. - Proofs of Theorems A and B

In this section we prove Theorems A and B; moreover, we give a variant of Theorem A and present a further consequence of Theorem 2.1.

Proof of Theorem A: We put

$$I = g(V) \quad \text{and} \quad F(r) = f(g^{-1}(r))$$

for every $r \in I$. We claim that the graph of multifunction $F: I \to 2^E$ is a connected and locally connected set. Let us prove this. Let $b: V \to I \times E$ be a function defined $b(v) = (g(v), f(v))$ for every $v \in V$. Obviously, b is a continuous function and $\mathrm{gr}(F) = b(V)$. Taking into account that V is compact, connected and locally connected then, the set $b(V)$ is so too (see, for instance, Theorem 5 of [6] p. 257). Hence, our claim is proved.

Now, let $r_0 \in \mathrm{int}(I)$ and Ω an open set in E such that $f(g^{-1}(r_0)) \cap \Omega \neq \emptyset$. Taking into account that f and g are continuous functions, since $\mathrm{int}(g^{-1}(r_0)) = \emptyset$, for every $\varrho > 0$ there exists $\bar{r} \in \mathrm{int}(I)$, with $0 < |\bar{r} - r_0| < \varrho$, such that $g^{-1}(\bar{r}) \cap f^{-1}(\Omega) \neq \emptyset$; that is $f(g^{-1}(\bar{r})) \cap \Omega \neq \emptyset$. Hence, the set $F^{-}(\Omega) \cap \mathrm{int}(I)$ has no isolated points. Clearly, F is a multifunction with closed values. At this point, we can apply Theorem 2.1. Then, there exists a function $u \in AC([0, a], E)$ such that

$$(2) \qquad \begin{cases} u'(t) \in f(g^{-1}(\varphi(t, u(t)))) & \text{for almost every } t \in [0, a], \\ u(0) = x_0. \end{cases}$$

We now put

$$H(t) = f^{-1}(u'(t)) \cap g^{-1}(\varphi(t, u(t)))$$

for every $t \in [0, a]$.

From (2), we have $H(t) \neq \emptyset$. Moreover, H is a measurable multifunction with closed values. Hence, by the classical Kuratowski and Ryll-Nardzewski theorem, there exists a measurable function $v: [0, a] \to V$ such that $v(t) \in H(t)$ for almost every $t \in [0, a]$. To finish the proof, it is enough to observe that $f(v(t)) = u'(t)$ and $g(v(t)) = \varphi(t, u(t))$ for almost every $t \in [0, a]$, namely

$$\varphi\left(t, x_0 + \int_0^t f(v(\tau))\, d\tau\right) = g(v(t))$$

for almost every $t \in [0, a]$. ■

Now we point out a variant of Theorem A, when V is not compact.

THEOREM 3.1: *Let V be a connected and locally connected metric space; E a separable real Banach space. Further, let f from V onto a subset of E a homeomorphism and $g: V \to \mathbf{R}$ a continuous function such that $\operatorname{int}(g^{-1}(r)) = \emptyset$ for every $r \in \operatorname{int}(g(V))$.*

Then, for every bounded Carathéodory function $\varphi: [0, a] \times E \to \mathbf{R}$ such that $\overline{\varphi([0, a] \times E)} \subseteq g(V)$ and, for every $x_0 \in E$, there exists a measurable function $v: [0, a] \to V$ such that

$$\varphi\left(t, x_0 + \int_0^t f(v(\tau))\, d\tau\right) = g(v(t))$$

for almost every $t \in [0, a]$.

PROOF: Keeping the same notation as that the proof of Theorem A, we just point out that the function h is a homeomorphism and so, taking into account that V is a connected and locally connected metric space, $h(V) = \operatorname{gr}(F)$ is a connected and locally connected set. At this point the proof is similar to the previous one and so we omit it. ■

Before giving the proof of Theorem B, we point out the following consequence of Theorem 2.1.

THEOREM 3.2: *Let E be a separable real Banach space; Y a subset of E and $h: Y \to \mathbf{R}$ a function such that*

(*a*) *$\operatorname{gr}(h)$ is connected and locally connected;*
(*b*) *$\operatorname{int}_Y(h^{-1}(r)) = \emptyset$ for every $r \in \operatorname{int}(h(Y))$,*
(*c*) *for every $r \in h(Y)$ one has $h^{-1}(r)$ is closed in E.*

Then, for every bounded Carathéodory function $\varphi: [0, a] \times E \to \mathbf{R}$ such that $\overline{\varphi([0, a] \times E)} \subseteq h(Y)$ and, for every $x_0 \in E$, there exists a function $u \in AC([0, a], E)$ such that

$$\begin{cases} u'(t) \in Y & \text{for almost every } t \in [0, a], \\ h(u'(t)) = \varphi(t, u(t)) & \text{for almost every } t \in [0, a], \\ u(0) = x_0. \end{cases}$$

PROOF: We put

$$I = h(Y) \quad \text{and} \quad F(r) = h^{-1}(r)$$

for every $r \in I$.

Clearly, the multifunction $F: I \to 2^E$ verifies all the assumptions of Theorem 2.1. Then, there exists a function $u \in AC([0, a], E)$ such that

$$\begin{cases} u'(t) \in b^{-1}(\varphi(t, u(t))) & \text{for almost every } t \in [0, a], \\ u(0) = x_0. \end{cases}$$

Hence, $b(u'(t)) = \varphi(t, u(t))$ and $u'(t) \in Y$ for almost every $t \in [0, a]$. The function u is our solution. ∎

Remark 3.1: If Y is a closed, connected and locally connected subset of E and b is a continuous function such that $\operatorname{int}_Y(b^{-1}(r)) = \emptyset$ for every $r \in \operatorname{int}(b(Y))$, then the conditions ($a$), ($b$) and ($c$) are verified. So, Theorem 3.2 extends to infinite-dimensional separable Banach spaces Theorem 2.2 of [10].

Remark 3.2: We point out that, as the example in [9, p. 227] shows, there are discontinuous functions b satisfying the hypotheses of Theorem 3.2.

Finally, we give the proof of Theorem B.

Proof of Theorem B: Put

$$Y = \{u \in E : f(s, \Phi(u)(s)) = 0 \text{ for } \mu\text{-a.e. } s \in S\}.$$

By Theorem 1.1, Y is a retract of the Banach space E. It follows that Y is a closed, connected and locally connected set (see, for instance, [3, p.19]). Now, let g and Ψ be as in the statement and put $b = g \circ (\Psi_{|Y}): Y \to \mathbf{R}$. Let us prove that the function b satisfies all the assumptions of Theorem 3.2 (see also Remark 3.1). Clearly, b is a continuous function. Moreover, we state that b is such that $\operatorname{int}_Y(b^{-1}(r)) = \emptyset$ for every $r \in \operatorname{int}(b(Y))$. To prove this, we show that $\Psi_{|Y}$ is a surjective and open function. Clearly, $\Psi_{|Y}$ is a surjective function, since Y intersects every closed hyperplane of E, owing to Theorem 1.1.

Since Y is locally connected, to prove the openness of $\Psi_{|Y}$ it is enough to show that it has no local extrema. To this end, we put

$$F(s) = \{r \in \mathbf{R} : f(s, r) = 0\}$$

for every $s \in S$ and

$$S_F = \{u \in L^1(S) : u(s) \in F(s) \text{ for every } s \in S\}.$$

Clearly, $\Phi(Y) = S_F$ and S_F is a decomposable set. Moreover, owing to the classical Riesz Theorem, there exists $v \in L^\infty(S)$ such that

$$(\Psi \circ \Phi^{-1})(u) = \int_S v(s) u(s) \, d\mu$$

for every $u \in L^1(S)$. At this point it is easy to see that all the hypotheses of the Theorem of [5] are satisfied. Hence, the functional $(\Psi \circ \Phi^{-1})_{|S}$ has no local extrema since its range is R. So, since Φ is a homeomorphism, $\Psi_{|Y}$ has no local extrema. From the openness of $\Psi_{|Y}$, taking into account the hypothesis on g, we have that $\mathrm{int}_Y(b^{-1}(r)) = \emptyset$ for every $r \in \mathrm{int}(b(Y))$.

Now, we apply Theorem 3.2. Then, there exists a function $u \in AC([0, a], E)$ such that

$$\begin{cases} u'(t) \in Y & \text{for a.e. } t \in [0, a], \\ b(u'(t)) = \varphi(t, u(t)) & \text{for a.e. } t \in [0, a], \\ u(0) = x_0, \end{cases}$$

hence

$$\begin{cases} f(s, \Phi(u'(t))(s)) = 0 & \text{for a.e. } t \in [0, a] \text{ and for } \mu\text{-a.e. } s \in S, \\ g(\Psi(u'(t))) = \varphi(t, u(t)) & \text{for a.e. } t \in [0, a], \\ u(0) = x_0. \quad \blacksquare \end{cases}$$

4. - Elliptic differential inclusions and applications

In this section we present a theorem on elliptic differential inclusions and, as an application a result on implicit elliptic equations.

Theorem 4.1: *Let I be a subset of* R; *$F: I \to 2^{\mathbb{R}^b}$ a multifunction with nonempty and closed values such that*

(*a*) *$\mathrm{gr}(F)$ is connected and locally connected;*

(*b*) *for every open set $A \subseteq \mathbb{R}^b$, the set $F^-(A) \cap \mathrm{int}(I)$ has no isolated points.*

Then, for every bounded multifunction $G: \Omega \times \mathbb{R}^b \times \mathbb{R}^{nb} \to 2^{\mathbb{R}}$ with nonempty and closed values such that

(i) *the multifunction G is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^b \times \mathbb{R}^{nb})$-measurable;*

(ii) *for almost every $x \in \Omega$, the multifunction $(z, w) \to G(x, z, w)$ is lower semicontinuous;*

(iii) *one has*

$$\overline{G(\Omega \times \mathbb{R}^b \times \mathbb{R}^{nb})} \subseteq I.$$

Then, there exists a function $u \in W^{2,p}(\Omega, \mathbb{R}^b) \cap W_0^{1,p}(\Omega, \mathbb{R}^b)$ such that

$$Lu(x) \in F(G(x, u(x), Du(x))) \quad \text{for a.e. } x \in \Omega.$$

Proof: Thanks to our assumptions, we can apply Theorem 11 of [9] to the multifunction F. Then, there exist two multifunctions $\Phi_1, \Phi_2 : I \to 2^{\mathbb{R}^h}$ such that Φ_1 is lower semicontinuous, Φ_2 is upper semicontinuous with compact values and $\emptyset \neq \Phi_1(r) \subseteq \Phi_2(r) \subseteq F(r)$ for every $r \in I$.

We now put

$$\Gamma(x, z, w) = \overline{\Phi_1(G(x, z, w))}$$

for every $(x, z, w) \in \Omega \times \mathbb{R}^h \times \mathbb{R}^{nh}$. It is easy to see that for almost every $x \in \Omega$, the multifunction $(z, w) \to \Gamma(x, z, w)$ is lower semicontinuous and the multifunction $(x, z, w) \to \Gamma(x, z, w)$ is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^h \times \mathbb{R}^{nh})$-measurable. Moreover, from the upper semicontinuity of Φ_2 and the compactness of $\overline{G(\Omega \times \mathbb{R}^h \times \mathbb{R}^{nh})}$ we obtain that Γ is a bounded multifunction. At this point, we can apply Theorem 3.1 of [7]. Then, there exists a function $u \in W^{2,p}(\Omega, \mathbb{R}^h) \cap W_0^{1,p}(\Omega, \mathbb{R}^h)$ such that $Lu(x) \in \overline{\Phi_1(G(x, u(x), Du(x)))}$ for almost every $x \in \Omega$.

Taking into account that for every $x \in \Omega$ we have

$$\overline{\Phi_1(G(x, u(x), Du(x)))} \subseteq F(G(x, u(x), Du(x))),$$

the function u satisfies our conclusion. ∎

An immediate consequence of Theorem 4.1 is the following

Theorem 4.2: *Let Y a subset of $\mathbb{R}^h$ and $h: Y \to \mathbb{R}$ a function such that*

(*a*) *$\mathrm{gr}(h)$ is connected and locally connected;*

(*b*) *$\mathrm{int}_Y(h^{-1}(r)) = \emptyset$ for every $r \in \mathrm{int}(h(Y))$;*

(*c*) *for every $t \in h(Y)$ one has $h^{-1}(t)$ is closed in $\mathbb{R}^h$.*

Then, for every bounded Carathéodory function $\varphi: \Omega \times \mathbb{R}^h \times \mathbb{R}^{nh} \to \mathbb{R}$ such that $\varphi(\Omega \times \mathbb{R}^h \times \mathbb{R}^{nh}) \subseteq h(Y)$; there exists a function $u \in W^{2,p}(\Omega, \mathbb{R}^h) \cap W_0^{1,p}(\Omega, \mathbb{R}^h)$ such that $Lu(x) \in Y$ and $h(Lu(x)) = \varphi(x, u(x), Du(x))$ for a.e. $x \in \Omega$.

Proof: It follows immediately from Theorem 4.1 putting $I = h(Y)$ and $F(r) = h^{-1}(r)$ for every $r \in I$. ∎

Remark 4.1: If Y is a closed, connected and locally connected subset of $\mathbb{R}^h$ and h is a continuous function such that $\mathrm{int}_Y(h^{-1}(r)) = \emptyset$ for every $r \in \mathrm{int}(h(Y))$, then the conditions (*a*), (*b*) and (*c*) are verified. On the contrary, if h satisfies all the hypotheses of Theorem 4.2 it may be non continuous on Y (see [9, p. 227]), contrary to that required in Theorem 3.4 of [7].

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