Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica
112° (1994), Vol. XVIII, fasc. 1, pagg. 89-101

MAVIRA MANCINO - LUCA PRATELLI (*)

Some Convergence Properties of the Ogawa Integral Relative to a Martingale (**)

SUMMARY. — We consider the Ogawa sequence (S_n) relative to a martingale H and a complete orthonormal system of $L^2([0, 1])$. In $L^p(P; C^{0,*})$ we give an upper bound for $\|(S_{m+j} - S_m)_j\|$ and, in the trigonometric case, we estimate the rate of convergence of $(S_n)_n$ to the Stratonovich integral of H.

Alcune proprietà di convergenza dell'integrale di Ogawa per una martingala

SUNTO. — Si considera la successione di Ogawa (S_n) relativa ad una martingala H e ad un sistema ortonormale di $L^2([0, 1])$. Negli spazi $L^p(P; C^{0,*})$ si ottiene una maggiorazione per le norme $\|(S_{m+j} - S_m)_j\|$ e, nel caso del sistema trigonometrico, si stima l'ordine di convergenza di $(S_n)_n$ verso l'integrale di Stratonovich di H.

0. - INTRODUCTION

At the beginning of the eighties Ogawa resumed Ito-Nisio results on the uniform convergence of a particular random walk to the Wiener process W. By an analogous procedure, he defined a stochastic integral for a class of real processes, not necessarily adapted to the usual «enlargement» of the filtration associated to W.

More precisely, given a complete orthonormal system (e_n) in $L^2([0, 1])$, he introduced in [6] a notion of integrability for a generic process H, defined on a suitable

(*) Indirizzo degli Autori: M. Mancino: Dipartimento di Matematica DEFAS, via Lombroso 6/17, 50134 Firenze; L. Pratelli: Dipartimento di Statistica, via S. Francesco 33, 35121 Padova.
(**) Memoria presentata il 5 settembre 1994 da Giorgio Letta, uno dei XL.

probability space, for which it results that a process H is integrable, with respect to (e_n), if the sequence of partial sums (S_n), given by

$$S_n(t) = \sum_{j=0}^{n} \int_0^1 e_j(s)\, dW_s \int_0^t H_s e_j(s)\, ds$$

converges in probability for every t of $[0, 1]$.

Using this notion of stochastic integral, Ogawa showed in [7] that every quasi-martingale H of the form $A + K \cdot W$, where A is a Stieltjes process and K is a bounded predictable process, is integrable with respect to (e_n) if and only if the sequence (e_n) verifies a certain summability condition (see (2.1)). In this case the integral coincides with the Stratonovich integral of the process H.

Moreover, the sequence of the partial sums converges uniformly in probability.

Then, as an application of Malliavin's calculus, Nualart and Zakai proved in [2] a criterion of «universal» Ogawa integrability (i.e. independent of the orthonormal system (e_n)), but they require some regularity conditions on the Malliavin derivative of the process H. Nevertheless, if H is a martingale of the form $K \cdot W$, these regularity conditions restrict the choice of the predictable process K.

Furthermore they studied in [4] the relations between Stratonovich and Ogawa integrals.

In this paper we consider the random variables (S_n), valued in the Banach space $C^{0,\alpha}([0, 1])$, with α element of $[0, 1/2[$, associated with the partial sums S_n relative to a continuous martingale.

The aim of this note is to show the convergence in $L^p(P)$ of the sequence $(S_n)_n$ to the random variable $S_\cdot$ associated with the Stratonovich integral of the martingale H. For this purpose we obtain some estimates on the absolute moments of order p of the random variables $(S_n)_n$, uniformly with respect to the given process K and the parameter M, which characterizes the summability condition (2.1).

Finally, in the case of the complete trigonometric system, we estimate the rate of convergence of $(S_n)_n$ to $S_\cdot$.

1. - Preliminary results

Here below we suppose given a probability space $(\Omega, \mathcal{A}, P)$, with a filtration $(\mathcal{F}_t)_t$, verifying the usual hypotheses, and a real Brownian motion W adapted to $(\mathcal{F}_t)_t$.

Let v be a real Borel function on $[0, 1] \times [0, 1]$ for which the functions q_1^v, q_2^v, defined on $[0, 1]$ by

$$q_1^v(s) = \int_0^1 v^2(s, s')\, ds', \qquad q_2^v(s') = \int_0^1 v^2(s, s')\, ds$$

are finite and integrable with respect to the Borel-Lebesgue measure λ on $[0, 1]$.

For every real number $p \geq 1$ and $i = 1, 2$, $\|q_i^p\|_p$ denotes the seminorm of q_i^p in $L^p(\lambda)$. Moreover, let K be a predictable bounded process and let Y, Z be two continuous, square integrable martingales such that, for every t of $[0, 1]$, the random variables Y_t, Z_t are respectively versions of

$$\int_0^t dW_s K_s \int_0^s dW_{s'} v(s, s'), \qquad \int_0^t dW_s \int_0^s dW_{s'} K_{s'} v(s, s').$$

We set $Y_1^* = \sup_{t \in [0,1]} |Y_t|$, $Z_1^* = \sup_{t \in [0,1]} |Z_t|$ and, for any real number $p > 1$, $I_{K,p}$ equals $E\left[\left(\int_0^1 ds\, |K|_s^{2p/(p-1)}\right)^{2p-2}\right]^{1/4p}$.

Then it is possible to obtain an estimate of the absolute moments of order p of Y_1^*, Z_1^*, through $I_{K,p}$ and $\|q_i^p\|_p$.

More precisely:

Proposition (1.1): *For every real number $p > 1$, there exists a constant C_p, depending only on p, such that*

$$\|Y_1^*\|_{L^{2p}(P)} \leq C_p\, I_{K,p} \|q_1^p\|_p^{1/2}. \tag{1.2}$$

Proof: We note that Burkholder and Schwarz-Hölder inequalities ([9]) imply

$$E[(Y_1^*)^{2p}] \leq c_p E\left[\left(\int_0^1 ds\, K_s^2 \left(\int_0^s dW_{s'} v(s, s')\right)^2\right)^p\right] \leq$$

$$\leq c_p E\left[\left(\int_0^1 ds\, |K_s|^{2p/(p-1)}\right)^{p-1} \int_0^1 ds \left(\int_0^s dW_{s'} v(s, s')\right)^{2p}\right]$$

where c_p is a proper constant, depending only on p.

Since, for every l, x, y, with $l > 0$,

$$|xy| \leq 2(lx^2 + l^{-1}y^2), \tag{1.3}$$

then we obtain

$$E[(Y_t^*)^{2p}] \leq 2c_p \inf_l \left(II_{K,p}^{4p} + l^{-1} \int_0^t ds\, E\left[\left(\int_0^s dW_{s'} v(s,s') \right)^{4p} \right] \right) \leq$$

$$\leq 2c_p(1+c_p) \inf_l \left(II_{K,p}^{4p} + l^{-1} \int_0^t ds \left(\int_0^s ds' v^2(s,s') \right)^{2p} \right).$$

By putting $C_p = [2c_p(1+c_p)]^{1/2p}$ inequality (1.2) yields.

Proposition (1.4): *For every real number $p \geq 2$, there exists a constant C_p', depending only on p, such that*

$$\|Z_t^*\|_{L^{2p}(P)} \leq C_p' \, I_{K,p} \, \|q_t^\varepsilon\|_p^{1/2}. \tag{1.5}$$

Proof: From Burkholder inequality we deduce

$$E[(Z_t^*)^{2p}] \leq c_p E\left[\left(\int_0^t ds \left(\int_0^s dW_{s'} K_{s'} v(s,s') \right)^2 \right)^p \right].$$

By applying Ito formula to $\left(\int_0^s dW_{s'} K_{s'} v(s,s') \right)^2$, the right term in the previous inequality is bounded above by the expression

$$c_p E\left[\left[\int_0^t ds \int_0^s ds' K_{s'}^2 v^2(s,s') + 2 \int_0^t ds \int_0^s dW_{s'} K_{s'} v(s,s') \int_0^{s'} dW_r K_r v(s,r) \right]^p \right].$$

Apart from the multiplicative constant $c_p 2^{2p-1}$, the previous expression is less or equal to the sum of the following terms:

$$E\left[\left(\int_0^t ds \int_0^s ds' K_{s'}^2 v^2(s,s') \right)^p \right], \tag{1.6}$$

$$E\left[\left(\int_0^t ds \int_0^s dW_{s'} K_{s'} v(s,s') \int_0^{s'} dW_r K_r v(s,r) \right)^p \right]. \tag{1.7}$$

More precisely (1.6) is less or equal to

$$E\left[\left(\int_0^1 ds' K_{s'}^2 \int_0^1 ds\, v^2(s,s')\right)^p\right] = E\left[\left(\int_0^1 ds' K_{s'}^2 q_2^p(s')\right)^p\right] \leq$$

$$\leq E\left[\left(\int_0^1 ds' |K_{s'}|^{2p/(p-1)}\right)^{p-1} \int_0^1 ds' (q_2^p(s'))^p\right] \leq I_{K,p}^{2p} \|q_2^p\|_p^p.$$

Besides, thanks to Fubini's theorem for stochastic integrals, (see [9], Th. 45), by setting the term (1.7) equals to V, it results

$$V = E\left[\left(\int_0^1 dW_{s'} \int_{s'}^1 ds K_{s'} v(s,s') \int_0^{s'} dW_r K_r v(s,r)\right)^p\right] \leq$$

$$\leq c_{p/2} E\left[\left(\int_0^1 ds' K_{s'}^2 \left(\int_{s'}^1 ds\, v(s,s') \int_0^{s'} dW_r K_r v(s,r)\right)^2\right)^{p/2}\right] =$$

$$= c_{p/2} E\left[\left(\int_0^1 ds' K_{s'}^2 \left(\int_0^{s'} dW_r \int_{s'}^1 ds K_r v(s,r) v(s,s')\right)^2\right)^{p/2}\right].$$

From Schwarz-Hölder inequality and (1.3), apart from $c_{p/2}$, it follows

$$V \leq E\left[\left(\int_0^1 ds' |K_{s'}|^{2p/(p-1)}\right)^{(p-1)/2} \left(\int_0^1 ds' \left(\int_0^{s'} dW_r \int_{s'}^1 ds K_r v(s,r) v(s,s')\right)^{2p}\right)^{1/2}\right] \leq$$

$$\leq 2l E\left[\left[\int_0^1 ds' |K_{s'}|^{2p/(p-1)}\right]^{p-1}\right] + 2/l \int_0^1 ds' E\left[\left[\int_0^{s'} dW_r \int_{s'}^1 ds K_r v(s,r) v(s,s')\right]^{2p}\right]$$

for every strictly positive real number l.

Finally we obtain

$$V \leq 2 \inf_t \left[\Pi_{K,p}^{2p} + c_p t^{-1} \int_0^1 ds' E \left[\left(\int_0^{s'} dr K_r^2 \left(\int_r^1 ds v(s,r) v(s,s') \right)^2 \right)^p \right] \right] \leq$$

$$\leq 2(1+c_p) \inf_t \left[\Pi_{K,p}^{2p} + t^{-1} \int_0^1 ds' E \left[\left(\int_0^1 dr K_r^2 q_2^r(s') q_2^r(r) \right)^p \right] \right] \leq$$

$$\leq 2(1+c_p) \inf_t \left[\Pi_{K,p}^{2p} + t^{-1} \int_0^1 ds' (q_2^r(s'))^p E \left[\left(\int_0^1 dr |K_r|^{2p/(p-1)} \right)^{p-1} \int_0^1 dr (q_2^r(r))^p \right] \right] \leq$$

$$\leq 4(1+c_p) \Pi_{K,p}^{2p} \|q_2^r\|_p^p .$$

So the proof has been completed.

Remark (1.8): Actually, through a more shrewd argument as in the proof of (1.4), we can replace $\|q_1^r\|_p^{1/2}$ with $\|q_1^r\|_p^{1/2}$ in the inequality (1.2).

2. - Notations and results on the Ogawa integral

Assume now given a complete orthonormal system (e_n) in $L^2([0,1],\lambda)$. For every integer n, E_n denotes the integral function, defined on $[0,1]$ by $E_n(s) = \int_0^s ds' e_n(s')$ and u_n, u_n^* denote the Borel functions on $[0,1] \times [0,1]$ given by

$$u_n(s,s') = E_n(s) e_n(s'), \qquad u_n^*(s,s') = e_n(s) E_n(s').$$

We set $v_n = \sum_{j=0}^{n} u_j$, $v_n^* = \sum_{j=0}^{n} u_j^*$. Note that the functions v_n, v_n^* coincide on the diagonal of $[0,1] \times [0,1]$.

In the following we suppose, for the complete orthonormal system (e_n), the existence of a constant M^2 such that

$$\sup_n \int_0^1 ds\, v_n^2(s,s) \leq M^2 . \tag{2.1}$$

In particular, the trigonometric orthonormal system $(\tau_n)_n$, determined by the functions $\cos 2\pi n(\cdot)$, $\sin 2\pi n(\cdot)$, verifies the property (2.1) for any M^2 greater or equal to $1/2$.

According with the notations of the first section, for every pair of integer numbers

h, m , we have

$$q_1^{v_{h+m}-v_m} = q_2^{v_{h+m}^*-v_m^*} = \sum_{j=m+1}^{h+m} E_j^2 .$$

Besides, for every pair of real numbers t_0, t_1, with $0 \leq t_0 \leq t_1 \leq 1$, and for every $p \geq 1$, Bessel inequality implies

$$\sum_{j=0}^{\infty} [E_j(t_1) - E_j(t_0)]^2 \leq \int_{t_0}^{t_1} ds = t_1 - t_0 , \tag{2.2}$$

$$\|q_1^{v_h}\|_{2p}^2 = \left(\int_0^1 dt \left(\sum_{j=0}^{\infty} E_j^2(t) \right)^{2p} \right)^{1/2} \leq 1 . \tag{2.3}$$

For the trigonometric system (τ_n), we also have

$$\int_0^1 (v_{h+m} - v_m)^2(s, s)\, ds + \sum_{j=m+1}^{h+m} [E_j(t_1)]^2 = \tag{2.4}$$

$$= \sum_{j=m+1}^{h+m} \left(\int_0^1 u_j^2(s, s)\, ds + [E_j(t_1)]^2 \right) \leq \sum_{j=m+1}^{h+m} j^{-2}$$

because, in this case, $E_j^2(s)$ and $u_j^2(s)$ are less than $1/4j^2$.

Here $(W(e_n))$ is a given sequence of independent and identically distributed random variables with $W(e_n)(P) = N(0, 1)$ such that, for every integer n, the random variable $W(e_n)$ is a version of $\int_0^1 e_n dW$.

Let K be a predictable bounded process and consider a continuous martingale H, version of the Ito integral $\int K\, dW$. By Burkholder inequality and Kolmogorov lemma it is not restrictive to suppose all trajectories of the process H are elements of $C^{0,\alpha}(R_+)$, for every $\alpha \in [0, 1/2[$.

Then, fixed α in $]0, 1/2[$, let $H_\cdot$ denote the random variable valued in $C^{0,\alpha}([0, 1])$, such that $H_\cdot(\omega)$ is the trajectory $H(\omega, .)$ restricted to $[0, 1]$. It is useful to recall that, for every f in $C^{0,\alpha}([0, 1])$, which vanishes in 0, it is verified

$$\|f\|_{C^{0,\alpha}} \leq h_\alpha \sup_n 2^{n\alpha} \sup_{t \in D_n} |f(t + 2^{-n}) - f(t)| \tag{2.5}$$

where D_n is the set of the dyadic numbers $k2^{-n}$, with $k = 0, \ldots, 2^n - 1$, $\|f\|_{C^{0,\alpha}}$ is the usual Hölder norm of f and h_α is a proper positive constant, depending only on α.

Inequality (2.5) is essentially due to Zygmund. (See [1].)

Now we can give a notion of Ogawa integrability of the martingale H.

Definition (2.6): The sequence (S_n) defined on $[0, 1] \times \Omega$ by

$$(S_n)_t = \sum_{i=0}^{n} W(e_i) \int_0^t H_s e_i(s)\, ds \,, \tag{2.7}$$

is called *the Ogawa sequence* relative to the martingale H and the orthonormal system (e_n). Moreover, if there exists a continuous process S on $[0, 1] \times \Omega$ such that, for any t of $[0, 1]$, independently of the orthonormal basis (e_n) verifying (2.1), the sequence $((S_n)_t)$ converges in probability to S_t, then we say that the martingale H is *Ogawa integrable* and a version of this integral is given by the process S.

In this setting it is well known [6] that the martingale H is Ogawa integrable and its integral coincides with the Stratonovich integral $H \cdot W + (1/2)[H, W]$. Moreover the sequence $(S_n)_t$ converges uniformly in probability with respect to t, that is the random variables $(S_n)_\cdot$, associated with the processes S_n and valued in the Banach space $C^0([0, 1])$, converge in probability [7].

Our aim is to strenghten the convergence of the random variables $(S_n)_\cdot$ which are also valued in $C^{0,\alpha}([0, 1])$, with $\alpha \in [0, 1/2[$, and, in the case of the trigonometric system, to give an estimate which includes the properties of the Ogawa integral relative to the martingale H.

We recall an immediate application of inequality (2.5).

Proposition (2.8): *Let Y be a random variable valued in $C^{0,\alpha}([0, 1])$, with $\alpha > 0$, and $Y(0) = 0$. For every real number $p \geq 1$, it results*

$$E[\|Y\|^p_{C^{0,\alpha}}] \leq b^p_\alpha \sum_{n=0}^{\infty} 2^{n\alpha p} \sum_{t \in D_n} E[|Y(t + 2^{-n}) - Y(t)|^p] \tag{2.9}$$

where D_n is the set of all dyadic numbers $k2^{-n}$ in $[0, 1[$.

Proof: It suffices to observe that

$$\|Y\|^p_{C^{0,\alpha}} \leq b^p_\alpha \sum_{n=0}^{\infty} 2^{n\alpha p} \sum_{t \in D_n} |Y(t + 2^{-n}) - Y(t)|^p .$$

As a consequence of the above proposition, we obtain:

Proposition (2.10): *Let α be an element of $]0, 1/2[$. For every $p > 2/(1 - 2\alpha)$, there exists a positive constant $c_{\alpha,p}$, depending only on α, p, such that*

$$\|\|H\|_{C^{0,\alpha}}\|_{L^p(P)} \leq c_{\alpha,p} \left(E\left[\int_0^1 |K_s|^p\, ds \right] \right)^{1/p} . \tag{2.11}$$

PROOF: From Proposition (2.8) and Burkholder inequality, we deduce

$$E[\|H.\|^p_{C^{0,\alpha}}] \leq b^p_\alpha \sum_{n=0}^{\infty} 2^{n\varphi} \sum_{i \in D_n} E[\,|H_{i+2^{-n}} - H_i|^p\,] \leq$$

$$\leq b^p_\alpha c_{p/2} \sum_{n=0}^{\infty} 2^{n\varphi} \sum_{i \in D_n} E\left[\int_i^{i+2^{-n}} K_s^2\, ds\right]^{p/2} \leq b^p_\alpha c_{p/2} \sum_{n=0}^{\infty} 2^{n(\varphi-(p/2)+1)} E\left[\int_0^1 |K_s|^p\, ds\right].$$

By setting $c_{\alpha,p} = b_\alpha (c_{p/2}^{-1}(1 - 2^{(2\varphi-p+2)/2}))^{-1/p}$, the inequality (2.11) is verified.

PROPOSITION (2.12): *Let α be an element of $]0, 1/2[$ and (W_n) the sequence of processes defined by*

$$W_n = \sum_{i=0}^{n} W(e_i)\, E_i.$$

Then the random variables $(W_n)_n$, valued in the Banach space $C^{0,\alpha}([0, 1])$, converge almost surely to the random variable W, associated to the Brownian motion W.

Moreover, for any $p \geq 1$, the sequence $(\|(W_n - W).\|_{C^{0,\alpha}})$ converges in $L^p(P)$ to 0.

PROOF: For Ito-Nisio theorem on random walks, valued in a Banach space, it suffices to prove the convergence of $(\|(W_n - W).\|_{C^{0,\alpha}})$ to 0 in $L^p(P)$. Because, for all t of $[0, 1]$, $\left(\sum_{i=0}^{n} e_i\, E_i(t)\right)$ converges in $L^2(\lambda)$ to $I_{]0,t]}$, it is enough to show that, for every $p \geq 1$, it results

$$\sup_n E[\|(W_n)_.\|^p_{C^{0,\alpha}}] \leq c'_{\alpha,p} \tag{2.13}$$

where $c'_{\alpha,p}$ is a proper constant, depending only on α, p.

Since $(W_n)_{t_1} - (W_n)_{t_0}$ is a gaussian random variable, there exists a positive constant m_p, depending only on p, such that, for every integer n and every pair of real numbers t_0, t_1, with $0 \leq t_0 \leq t_1 \leq 1$, the relation

$$E[\,|(W_n)_{t_1} - (W_n)_{t_0}|^{2p}\,] = m_p \left(\sum_{i=0}^{n} (E_i(t_1) - E_i(t_0))^2\right)^p \leq m_p (t_1 - t_0)^p$$

is verified for all $p \geq 1$. Then, by applying inequality (2.9), it results

$$E[\|(W_n)_.\|^{2p}_{C^{0,\alpha}}] \leq b^{2p}_\alpha m_p \sum_{n=0}^{\infty} 2^{n(2\varphi-p+1)}. \tag{2.14}$$

From (2.14) the proof easily follows.

3. - Convergence results for the Ogawa integral

Now we can prove the two principal results on the Ogawa integral relative to the martingale H. More precisely:

Theorem (3.1): *Let α be an element of $]0,1/2[$. For every real number $p > 5/(2(1-2\alpha))$, there exists a positive constant $C_{\alpha,p}$, depending only on α, p, such that the Ogawa sequence verifies the inequality*

$$\sup_n \left\| \|(S_n)_\cdot\|_{C^{\alpha,\cdot}} \right\|_{L^{2p}(P)} \leq C_{\alpha,p}(1+M^{2p})^{1/2p} \left(E\left[\int_0^1 |K_s|^{4p}\, ds \right] \right)^{1/4p}$$

where M is a constant which satisfies (2.1).

Proof: Using the same notations as the previous section and applying twice the integration by parts formula, (see also [7], proof of Th. 1), we obtain that, for every integer n and for all t of $[0, 1]$, the random variable $(HW_n)_t - (S_n)_t$ is a version of

$$\int_0^t v_n(s,s) K_s\, ds + \int_0^t dW_s K_s \int_0^s dW_{s'} v_n(s,s') + \int_0^1 dW_s \int_0^{s \wedge t} dW_{s'} K_{s'} v_n^*(s,s').$$

Let t_0, t_1 be a pair of real numbers, with $0 \leq t_0 \leq t_1 \leq 1$, and p a real number greater than 2. Thanks to Propositions (1.1),(1.4), there exist two positive constants C_p, C_p', such that

$$(3.2) \qquad E\left[\left| \int_{t_0}^{t_1} dW_s K_s \int_0^s dW_{s'} v_n(s,s') \right|^{2p} \right] \leq (C_p I_{K;[t_0,t_1],p})^{2p} \|q_1^n\|_{2p}^p,$$

$$(3.3) \qquad E\left[\left| \int_0^1 dW_s \int_0^s dW_{s'} I_{[t_0,t_1]}(s') K_{s'} v_n^*(s,s') \right|^{2p} \right] \leq (C_p' I_{K;[t_0,t_1],p})^{2p} \|q_1^{n*}\|_p^p,$$

where $I_{K;[t_0,t_1],p}$ denotes $E\left[\left(\int_{t_0}^{t_1} |K_s|^{2p/(p-1)} ds \right)^{2p-2} \right]^{1/4p}$.

Moreover from Schwarz-Hölder inequality it follows:

$$(3.4) \qquad E\left[\left| \int_{t_0}^{t_1} v_n(s,s) K_s\, ds \right|^{2p} \right] \leq (t_1 - t_0)^{p-1} \left(\int_0^1 v_n^2(s,s)\, ds \right)^p E\left[\int_0^1 |K_s|^{2p}\, ds \right].$$

Because $q_1^{t_1^*} = q_2^{t_2^*}$, from (2.2) and (2.4), we get

$$E[|(S_n - HW_n)_{t_1} - (S_n - HW_n)_{t_0}|^{2p}] \leq d_p(1 + M^{2p})(t_1 - t_0)^{p-1/2} E\left[\int_0^1 |K_s|^{4p} ds\right]^{1/2}$$

where d_p is a proper constant, depending only on p.

Let α be an element of $]0, 1/2[$, and p greater than $5/(2(1-2\alpha))$. For inequality (2.9), it results

$$E[\|(S_n - HW_n)_\cdot\|^{2p}_{C^{0,\alpha}}] \leq b_\alpha^{2p} d_p(1 + M^{2p}) E\left[\int_0^1 |K_s|^{4p} ds\right]^{1/2} \sum_{n \geq 0} 2^{-n(p-5/2-2\alpha)}.$$

Finally, for Proposition (2.10) and inequality (2.13), there exists a positive constant $c'_{\alpha,p}$, such that

$$E[\|(HW_n)_\cdot\|^{2p}_{C^{0,\alpha}}] \leq E[\|(H)_\cdot\|^{2p}_{C^{0,\alpha}} \|(W_n)_\cdot\|^{2p}_{C^{0,\alpha}}] \leq$$

$$\leq E[\|(H)_\cdot\|^{2p}_{C^{0,\alpha}}]^{1/2} E[\|(W_n)_\cdot\|^{2p}_{C^{0,\alpha}}]^{1/2} \leq c'_{\alpha,p} E\left[\int_0^1 |K_s|^{4p} ds\right]^{1/2}.$$

From the two previous inequalities we deduce the proof.

THEOREM (3.5): *Let α be an element of $]0, 1/2[$ and (τ_n) denote the complete trigonometric system. Then there exists a constant $C'_{\alpha,p}$ such that, for every pair of integers h, m and for every real number $p > 5/(2(1-2\alpha))$, the following inequality is verified*

$$(3.6) \quad \|\|(S_{m+h} - S_m - (HW_{m+h} - HW_m))_\cdot\|_{C^{0,\alpha}}\|_{L^{2p}} \leq C'_{\alpha,p}\left(\sum_{j=m+1}^{h+m} \frac{1}{j^2}\right)^{1/2} \|K\|_{4p}$$

where $\|K\|_{4p}$ denotes $\left(E\left[\int_0^1 |K_s|^{4p} ds\right]\right)^{1/4p}$. Moreover, the random variables $(S_n)_\cdot$ converge in every $L^p(P; C^{0,\alpha})$ and if $(n_k)_k$ is an increasing sequence of integers, with

$$\sum_k \left(\frac{n_{k+1} - n_k}{n_{k+1} n_k}\right)^{1/2} < \infty,$$

the random variables $(S_{n_k})_\cdot$ converge almost surely too.

PROOF: The proof runs in the same way of Theorem (3.1). Let h, m, be two positive integers. If we replace the function v_n with the function $v_{h+m} - v_m$ in inequalities (3.2), (3.3), (3.4) and we apply (2.4), then, for every $p \geq 2$ and every pair t_0, t_1, with

$0 \leq t_0 \leq t_1 \leq 1$, we obtain

$$E[|(S_{k,m} - HW_{k,m})_{t_1} - (S_{k,m} - HW_{k,m})_{t_0}|^{2p}] \leq c_p \left(\sum_{j=m+1}^{k+m} \frac{1}{j^2} \right)^p (t_1 - t_0)^{p-3/2} \|K\|_{\varphi}^{2p}$$

where c_p is a proper constant, depending only on p, and $S_{k,m}$, $W_{k,m}$ denote respectively $S_{k+m} - S_m$, $W_{k+m} - W_m$.

Inequality (3.6) follows from inequality (2.9), because, for every $p > \dfrac{5}{2(1-2\alpha)}$,

$$E[\|(S_{k,m} - HW_{k,m})_\cdot\|_{C^{0,\alpha}}^{2p}] \leq b_\alpha^{2p} c_p \left(\sum_{j=m+1}^{k+m} \frac{1}{j^2} \right)^p \|K\|_{\varphi}^{2p} \sum_{\alpha \geq 0} 2^{-\alpha(p-5/2-2\alpha)}.$$

In particular, the random variables $(S_n - HW_n)_\cdot$ converge in $L^p(P; C^{0,\alpha})$, for every $p \geq 1$. Moreover we have

$$\sum_{j=n_k+1}^{n_{k+1}} \frac{1}{j^2} \leq \frac{n_{k+1} - n_k}{n_k n_{k+1}}$$

and then, for (3.6), the random variables $(S_{n_k} - HW_{n_k})_\cdot$ converge almost surely. Thanks to Proposition (2.12) the theorem is proved.

Remark (3.7): We did not use any result from Ogawa theory or Malliavin calculus to obtain these estimates. Only in the following theorem, we use a result concerning Ogawa integrability relative to a continuous martingale H ([7], Th.1).

Actually it suffices to know that the Brownian motion W is Ogawa integrable and its integral is equal to $(1/2)W^2$, proposition easy to prove (see [10]).

Theorem (3.8): *Let S denote a version of the Stratonovich integral $H \circ W$ and α an elements of $]0, 1/2[$. Then H is Ogawa integrable and, for every $p \geq 1$, it results*

$$\lim_n E[\|(S_n - S)_\cdot\|_{C^{0,\alpha}}^p] = 0. \tag{3.9}$$

In particular, if (τ_n) is the complete trigonometric system and $p > 5/(2(1-2\alpha))$, there exists a positive constant $c_{\alpha,p}$, depending only on α, p, such that, for every integer $m \geq 1$, the following inequality is verified

$$E[\|(S - S_m - (HW - HW_m))_\cdot\|_{C^{0,\alpha}}^p] \leq c_{\alpha,p} \left(\frac{1}{m} \right)^{p/2} \left[\int_0^1 |K_s|^{4p} ds \right]^{1/4}. \tag{3.10}$$

Proof: Thanks to Theorems (3.1) and (3.5), it is enough to prove that the random variables $(S_m - S)_\cdot$ converge in probability to the random variable which vanishes everywhere.

But, for every $t \in [0, 1]$, $(S_n)_t$ converge in probability to S_t (see [6]). Besides

$\{(S_{m+h} - S_m)_\cdot(P) : m, h \in N\}$ is a relatively compact set in the narrow topology because, for every $\alpha < \alpha' < 1/2$,

$$\sup_{m,h} (E[\|S_{m+h} - S_m\|^p_{C^{0,\alpha'}}])^{1/p} \leq 2 \sup_n (E[\|S_n\|^p_{C^{0,\alpha'}}])^{1/p} < \infty ,$$

and bounded sets in $C^{0,\alpha'}$ are relatively compact sets in $C^{0,\alpha}$.

These two facts imply the double sequence $((S_{m+h} - S_m)_\cdot)_{m,h}$ converges in distribution to the random variable which vanishes everywhere, or, equivalently, $(S_n)_\cdot$ converges in probability to $S_\cdot$.

Remark (3.11): In the case of the trigonometric system, we can obtain an analogous extimate as (3.10) with respect to $(S - S_m)_\cdot$, provided that the dependence on m gets worse. More precisely, it is possible to give a proper extimate of the absolute moments of order p of the random variable $\|(W - W_m)_\cdot\|_{C^{0,\alpha}}$, such that, by the same techniques used in (3.5), for every element α in $]0, 1/4[$, there exists a suitable constant $c'_{\alpha,p}$, depending only on α, p, for which it results

$$E[\|(S - S_m)_\cdot\|^p_{C^{0,\alpha}}] \leq c'_{\alpha,p} \left(\frac{1}{m}\right)^{p/4} E\left[\int_0^1 |K_s|^{4p} ds\right]^{1/4}$$

where $p > 5/(1 - 2\alpha)$.

REFERENCES

[1] Z. Ciesielski, *On the isomorphism of the space H_α and m*, Bull. Acad. Pol. Sciences, 8 (1960).

[2] D. Nualart - M. Zakai, *Generalized stochastic integrals and the Malliavin Calculus*, Probability theory and Related Fields, 73 (1986).

[3] D. Nualart - E. Pardoux, *Stochastic calculus with anticipating integrands*, Probability Theory and Related Fields, 78 (1988).

[4] D. Nualart - M. Zakai, *On the relation between the Stratonovich and Ogawa integrals*, The Annals of Probability, 17 (1990).

[5] S. Ogawa, *Sur la question d'existence de solutions d'une équation différentielle stochastique du type noncausal*, J. Math. Kyoto Univ., 24-1 (1984).

[6] S. Ogawa, *Quelques propriétés de l'intégrale stochastique du type noncausal*, Japan J. Appl. Math., 1 (1984).

[7] S. Ogawa, *The stochastic integral of noncausal type as an extension of the symmetric integrals*, Japan J. Appl. Math., 2 (1985).

[8] S. Ogawa, *Une remarque sur l'approximation de l'intégrale stochastique du type noncausal par une suite des intégrales de Stieltjes*, Tôhuku Math. J., 36 (1984).

[9] P. Protter, *Stochastic Integration and Differential Equations*, Springer (1990).

[10] J. Rosinski, *On stochastic integration by series of Wiener integrals*, Appl. Math. Optim., 19 (1989).

[11] C. Stricker - M. Yor, *Calcul stochastique dépendant d'un paramètre*, Z. W., 45 (1978).