Rendiconti
Accademia Nazionale delle Scienze detta dei XL
Memorie di Matematica
112° (1994), Vol. XVIII, fasc. 1, pagg. 201-225

LAURA GOTUSSO - GIOVANNI PROUSE
ALESSANDRO VENEZIANI (*)

On Two Nonlinear Models of the Vibrating String (**) (***)

SUMMARY. — We propose two new nonlinear models for a vibrating string, starting from a discrete *Greenspan-like* approach and considering nonlinear stress-strain laws. We prove global existence and uniqueness for the Cauchy-Dirichlet problem associated and study the numerical stability of the *leap-frog* formulas. Finally, we present some numerical results.

Su due modelli non lineari della corda vibrante

RIASSUNTO. — Vengono proposti due modelli non lineari della corda vibrante, assumendo come punto di partenza un modello discreto di tipo *Greenspan* ed una legge sforzo deformazione non lineare. Si dimostra un teorema di esistenza e unicità in grande per la soluzione di un problema di Cauchy-Dirichlet associato e si studia la stabilità delle formule *leap-frog* impiegate per la risoluzione numerica. Da ultimo, vengono presentati alcuni risultati numerici.

1. - INTRODUCTION

In previous papers [1] and [2] it has been remarked that in many cases the difference equations involved in the discrete models proposed by Greenspan (see f.i. [3], [4] and [5]) may be considered as a discretization by the finite difference method of well known differential equations. In [2] it has been also pointed out that in the same way it is possible to use the Greenspan technique in order to obtain new differential equations which model physical problems.

(*) Indirizzo degli autori: Dipartimento di Matematica del Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano (Italia).
(**) The research has been supported by MURST 40% and 60% Research Contracts.
(***) Memoria presentata il 6 dicembre 1994 da Luigi Amerio, uno dei XL.

Following this approach, we will obtain here two nonlinear differential equations, modelling the motion of a vibrating string, fixed at the ends, in the case of a nonlinear stress-strain law: namely, we consider a quadratic dependence of the stress on the strain; this means that, if the string is considered as a finite set of particles, or «quasi-molecules», then the physical law linking the interaction between two adjacent particles and their distance is a quadratic law.

In Section 2 we deduce such differential models and make some remarks useful both for the subsequent analytical and numerical study. In Section 3 we study the two differential equations and prove existence and uniqueness of the solution of the associated Cauchy-Dirichlet problems by using techniques similar to the ones adopted in [7], where a class of models for the vibrating string is studied in which the stress grows at most linearly with the strain. In Section 4 we study the stability of the difference schemes obtained by the discrete Greenspan approach («leap frog formulas»). Finally, in Section 5, we give some numerical results calculated in different realistic conditions.

2. - Discrete and continuous models

We consider, as in [1] and [2], a discrete string composed of an ordered finite set of $n+1$ particles or «quasimolecules» (see [5]), $P_0, P_1, \dots, P_n$, with mass m, location of which will be identified with the location of their centres $C_0, C_1, \dots, C_n$ in the xy-plane; let (x_k, y_k) be the coordinates of the centre C_k of P_k. We assume that P_0 and P_n are not in motion (string fixed at both ends), and that P_i $(i = 1, \dots, n-1)$ are free to move only vertically, namely parallel to the y-axis (transversal vibrations). If l is the distance between the fixed particles P_0 and P_n, i.e. the length of the string in the horizontal position, we set $\Delta x = \dfrac{l}{n}$ and $x_i = i\,\Delta x$, $i = 0, 1, \dots, n$. Newton's law for the particle P_i, at the time t_k, along the y-axis, is

$$ma_{i,k} = F_{i,k},$$

where:

$$F_{i,k} = |T^k_{2,i}| \frac{y_{i+1,k} - y_{i,k}}{\sqrt{\Delta x^2 + (y_{i+1,k} - y_{i,k})^2}} - |T^k_{1,i}| \frac{y_{i,k} - y_{i-1,k}}{\sqrt{\Delta x^2 + (y_{i,k} - y_{i-1,k})^2}} + f_{i,k},$$

denoting by $T^k_{2,i}$ the stress between particles P_{i+1} and P_i, $T^k_{1,i}$ the stress between particles P_i and P_{i-1}, and $f_{i,k}$ the external force acting on the particle P_i at time t_k parallel to the y-axis; actually, $T^k_{j,i}$ is the interaction force between two adjacents quasi-molecules. In what follows, $f_{i,k}$ will be given by the weight and the air resistance, i.e. $f_{i,k} = -mg - \rho v_{i,k}$, where g is the gravity acceleration, ρ the viscosity coefficient, $v_{i,k}$ the speed of particle P_i at t_k. However, the theoretical results of Section 3 hold on also for more general forcing terms.

2.1. *G-model.* Firstly, we assume that the interaction between two adjacent particles, as suggested in [3] and [6], is expressed by the law:

$$\text{(1)}\quad \begin{cases} T_{2,i}^{k} = T_0\left[\dfrac{\sqrt{\Delta x^2 + (y_{i+1,k} - y_{i,k})^2}}{\Delta x}(1-\varepsilon) + \varepsilon\left(\dfrac{\sqrt{\Delta x^2 + (y_{i+1,k} - y_{i,k})^2}}{\Delta x}\right)^2\right], \\[2ex] T_{1,i}^{k} = T_0\left[\dfrac{\sqrt{\Delta x^2 + (y_{i,k} - y_{i-1,k})^2}}{\Delta x}(1-\varepsilon) + \varepsilon\left(\dfrac{\sqrt{\Delta x^2 + (y_{i,k} - y_{i-1,k})^2}}{\Delta x}\right)^2\right], \end{cases}$$

with $0 \le \varepsilon < 1$ and where T_0 is the stress when the string is in the horizontal position (i.e. with length l).

By using the well known *leap-frog* formulas:

$$\text{(2)}\quad \begin{cases} v_{i,1/2} = v_{i,0} + \dfrac{\Delta t}{2} a_{i,0}\,, \\ v_{i,k+1/2} = v_{i,k-1/2} + \Delta t a_{i,k}\,, \\ y_{i,k+1} = y_{i,k} + \Delta t v_{i,k+1/2}\,, \end{cases}$$

and working as in [1], we obtain the discrete system:

$$\text{(3)}\quad m\,\frac{y_{i,k+1} - 2y_{i,k} + y_{i,k-1}}{\Delta t^2} = T_0\left\{\frac{y_{i+1,k} - 2y_{i,k} + y_{i-1,k}}{\Delta x}(1-\varepsilon) + \right.$$

$$+\varepsilon\left[\left(\frac{y_{i+1,k} - y_{i,k}}{\Delta x}\right)\sqrt{1 + \left(\frac{y_{i+1,k} - y_{i,k}}{\Delta x}\right)^2} - \left(\frac{y_{i,k} - y_{i-1,k}}{\Delta x}\right)\cdot\right.$$

$$\left.\left.\cdot\sqrt{1 + \left(\frac{y_{i,k} - y_{i-1,k}}{\Delta x}\right)^2}\right]\right\} - mg - \rho v_{i,k}\,.$$

If we set $M = mn = \dfrac{ml}{\Delta x}$ and $\sigma = \dfrac{l\rho}{M\Delta x}$, then equations (3) may be seen as a finite difference scheme of the differential equation:

$$\text{(4)}\quad \frac{M}{l}\frac{\partial^2 y}{\partial t^2} = T_0\left\{(1-\varepsilon)\frac{\partial^2 y}{\partial x^2} + \varepsilon\frac{\partial}{\partial x}\left[\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}\,\frac{\partial y}{\partial x}\right]\right\} - \frac{M}{l}g - \frac{M}{l}\sigma\frac{\partial y}{\partial t}\,,$$

that in the following we shall call *G-model*.

Remark 2.1: *Because of the condition $0 \le \varepsilon < 1$, (4) is certainly an hyperbolic equation; moreover, for $\varepsilon = 0$ we have the classical D'Alembert equation.*

Remark 2.2: *The assumptions (1) mean that we have hypothesized a quadratic stress-strain law, i.e. a law in which the tension in the string grows quadratically with the defor-*

mation: such law is typical, f.i., of rubber material in the case of small deformations (so that breakage phenomena do not occur) (see f.i. [13]).

REMARK 2.3: *The choice of ε in* (1) *is in a certain sense a «degree of freedom» that we can spend in order to make the adopted stress-strain law fit with good approximation the real law of the matter of the string, deduced by physical experiments. Further degrees of freedom can be obtained, considering in* (1) *terms higher than the quadratic one.*

2.2. *M-model.* We want now to assume explicitly, considering the strain, a nonzero value for the proper length of the string l_0 (i.e. for the length at which the stress in the string vanishes—see [9] and [10]). Therefore, the interaction force between two adjacent particles that we assume is:

$$\text{(5)}\quad \begin{cases} T^k_{2,i} = \bar{K} l \left[\left(\dfrac{\sqrt{\Delta x^2 + (y_{i+1,k} - y_{i,k})^2}}{\Delta x} - \dfrac{l_0}{l} \right)(1-\varepsilon) + \right. \\ \qquad\qquad\qquad \left. + \varepsilon \left(\dfrac{\sqrt{\Delta x^2 + (y_{i+1,k} - y_{i,k})^2}}{\Delta x} - \dfrac{l_0}{l} \right)^2 \right], \\ T^k_{1,i} = \bar{K} l \left[\left(\dfrac{\sqrt{\Delta x^2 + (y_{i,k} - y_{i-1,k})^2}}{\Delta x} - \dfrac{l_0}{l} \right)(1-\varepsilon) + \right. \\ \qquad\qquad\qquad \left. + \varepsilon \left(\dfrac{\sqrt{\Delta x^2 + (y_{i,k} - y_{i-1,k})^2}}{\Delta x} - \dfrac{l_0}{l} \right)^2 \right], \end{cases}$$

where $\bar{K} = \dfrac{K}{1 - \varepsilon(l_0/l)}$ and K is the coefficient of the strain if the string follows the linear Hooke's law, i.e. $K(l - l_0) = T_0$: in this way, it is easy to verify that T_0 is still the stress in the string in the horizontal position with length l.

By operating as for the G-model, with the assumptions (5), we are led to the discrete system:

$$\text{(6)}\quad m \frac{y_{i,k+1} - 2y_{i,k} + y_{i,k-1}}{\Delta t^2} =$$

$$= \bar{K} \left\{ [l(1-\varepsilon) - \varepsilon 2 l_0] \frac{y_{i+1,k} - 2y_{i,k} + y_{i-1,k}}{\Delta x} - l_0 \left((1-\varepsilon) - \varepsilon \frac{l_0}{l} \right) \times \right.$$

$$\times \left[\frac{\dfrac{y_{i+1,k} - y_{i,k}}{\Delta x}}{\sqrt{1 + \left(\dfrac{y_{i+1,k} - y_{i,k}}{\Delta x} \right)^2}} - \frac{\dfrac{y_{i,k} - y_{i-1,k}}{\Delta x}}{\sqrt{1 + \left(\dfrac{y_{i,k} - y_{i-1,k}}{\Delta x} \right)^2}} \right] +$$

$$+\varepsilon l\left[\left(\frac{y_{i+1,k}-y_{i,k}}{\Delta x}\right)\sqrt{1+\left(\frac{y_{i+1,k}-y_{i,k}}{\Delta x}\right)^2}+\right.$$

$$\left.-\left(\frac{y_{i,k}-y_{i-1,k}}{\Delta x}\right)\sqrt{1+\left(\frac{y_{i,k}-y_{i-1,k}}{\Delta x}\right)^2}\right]\Bigg\}-mg-\rho v_{i,k},$$

which may be seen as a finite difference scheme of the differential equation:

$$\text{(7)}\quad \frac{M}{l}\frac{\partial^2 y}{\partial t^2}=\tilde{K}\left\{[l(1-\varepsilon)-2\varepsilon l_0]\frac{\partial^2 y}{\partial x^2}-l_0\left((1-\varepsilon)-\varepsilon\frac{l_0}{l}\right)\frac{\partial}{\partial x}\left[\frac{\frac{\partial y}{\partial x}}{\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}}\right]+\right.$$

$$\left.+\varepsilon l\frac{\partial}{\partial x}\left[\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\frac{\partial y}{\partial x}\right]\right\}-\frac{M}{l}g-\frac{M}{l}\sigma\frac{\partial y}{\partial t},$$

to which we will refer as *M-model*.

REMARK 2.4: *If we consider* $l_0=0$ *(i.e.* $T_0=Kl$*) or* $\left(\frac{\partial y}{\partial x}\right)^2\ll 1$*, it is easy to see that equation* (7) *reduces* (4)*: in other words, G-model can be seen as an approximation of M-model when* $l_0\ll l$ *or the deformations are very small. However* (*see* [9]), *it is not possible to know a priori if the deformations will be sufficiently small during the motion, so this hypothesis is not satisfactory; on the other hand, the hypothesis* $l_0\approx 0$*, verifiable a priori, being due only to physical characteristic of the string, is often not verified* (*see* [2]).

REMARK 2.5: *Setting* $\varepsilon=0$ *in* (7), *i.e. supposing that the stress-strain law is linear, we obtain the equation:*

$$\text{(8)}\quad \frac{M}{l}\frac{\partial^2 y}{\partial t^2}=K\left\{l\frac{\partial^2 y}{\partial x^2}-l_0\frac{\partial}{\partial x}\left[\frac{\frac{\partial y}{\partial x}}{\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}}\right]\right\}-\frac{M}{l}g-\frac{M}{l}\sigma\frac{\partial y}{\partial t},$$

for which, therefore, the nonlinearities are only geometrical; equation (8) *has been proposed and studied from the analytical point of view in* [9] *and* [10], *and from the numerical point of view in* [11] *and* [2], *as a nonlinear model for perfectly elastic (i.e. according to Hooke's linear law) transversal string vibrations: we shall refer to equation* (8) *as «A-model».*

Remark 2.6: *As done in* [2], *if we linearize equation* (7) *in the neighborhood of an arbitrary value* z_0 *of* $\frac{\partial y}{\partial x}$, *we obtain, for the coefficient of* $\frac{\partial^2 y}{\partial x^2}$, *the expression:*

$$(9)\quad \bar{K}\left[(l(1-\varepsilon)-2\varepsilon l_0)-l_0\left((1-\varepsilon)-\varepsilon\frac{l_0}{l}\right)\frac{1}{\left(\sqrt{1+z_0^2}\right)^3}+\varepsilon l\frac{1+2z_0^2}{\sqrt{1+z_0^2}}\right].$$

If we want to ensure that (9) *is positive (that means* (7) *to be a hyperbolic equation), it is easy to see that* $0 \leqslant \varepsilon < 1$ *is no longer sufficient condition for every value of* $l_0 \leqslant l$; *namely, we have to suppose:*

$$(10)\qquad \varepsilon < \frac{l}{2l_0},$$

that is always verified provided that $\varepsilon \leqslant \frac{1}{2}$. *In the sequel, we will assume that condition* (10) *holds in M-model, together with the corresponding in G-model, i.e.* $0 \leqslant \varepsilon < 1$, *when proving existence and uniqueness of Cauchy-Dirichlet problems associated to our equations, during the study of stability and in performing numerical experiments.*

Remark 2.7: *By proceeding as done in* [2] *for the case* $\varepsilon = 0$, *it is possible to see that the discretization error of the discrete scheme* (3) *for* (4) *and of* (6) *for* (7) *is* $O(\Delta x^2 + \Delta t^2)$.

3. - Analytic study of G- and M-models

We now want to prove that, under suitable hypotheses, the Cauchy-Dirichlet problem:

$$(11)\qquad \begin{cases} y(0,t) = y(l,t) = 0, \\ y(x,0) = \alpha(x), \\ \frac{\partial y}{\partial t}(x,0) = \beta(x), \end{cases}$$

associated to equations (4) and (7) has a global unique solution. We will refer to the recent papers [7] and [8], setting $\rho = \sigma = 0$, for simplicity(¹): in these works the equation (²):

$$(12)\qquad \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left[b\left(\frac{\partial y}{\partial x}\right)\right] + f(t,x),$$

(¹) It is easy to verify that in the case $\rho \neq 0$ (and > 0), the Theorems here proved continue to hold.

(²) For the sake of simplicity, in this Section we set $M/l = 1$.

is considered with

$$C_1 |\xi|^\gamma - C_2 \leq |b(\xi)| \leq C_3 |\xi|^\gamma + C_4, \qquad 0 < \gamma \leq 1, \tag{13}$$

and

$$b'(\xi) \leq \bar{N} < \infty. \tag{14}$$

It is clear that in our case these assumptions are not verified: however, we will show that existence and uniqueness of the solution are still ensured, extending the Theorems of the quoted references.

We will consider, firstly, the G-model and subsequently extend the results obtained to the M-model in an immediate way.

3.1. *G-model.* According to (12), we have, in the case of G-model:

$$b(\xi) = A\xi + C\left(\sqrt{1+\xi^2}\right)\xi. \tag{15}$$

It is possible to verify with a straightforward calculation that:

$$C_5 |\xi|^2 - C_6 \leq |b(\xi)| \leq C_7 |\xi|^2 + C_8, \tag{16}$$

while

$$b'(\xi) = A + C\,\frac{1+2\xi^2}{\sqrt{1+\xi^2}}, \tag{17}$$

that clearly do not satisfy (14).

Setting $' = \dfrac{\partial}{\partial t}, D = \dfrac{\partial}{\partial x}, Q = (0, T_{\text{fin}}) \times (0, l)$, and assuming that $f(t,x) \in L^2(Q)$, we will say that y is a weak solution in Q of (4),(11) if:

a) $y(t) \in L^\infty(0, T; H_0^{1,3}) \cap H^{1,\infty}(0, T; L^2)$, $y(0) = \alpha$;

b) $y(t)$ satisfies, a.e. on $(0, T)$, the equation

$$\int_0^t \{-(y', b')_{L^2} + \langle b(Dy), Db\rangle - (f, b)_{L^2}\}\, d\eta + \\ + (y'(t), b(t))_{L^2} - (\beta, b(0))_{L^2} = 0, \tag{18}$$

$\forall b(t) \in L^2(0, T; H_0^{1,3}) \cap H^1(0, T; L^2)$. In (18) we denote by $H^{s,p}$ the classical Sobolev space of functions $\in L^p$, together with their derivatives of order $\leq s$, and by $\langle \cdot, \cdot \rangle$ the duality pairing between L^3 and $L^{3/2}$: in fact, by virtue of (16), if $\xi \in L^3$, then $b(\xi) \in L^{3/2}$, so (18), with the notation specified, has perfectly meaning.

In order to prove existence and uniqueness of the solution, as in [7], it is useful to consider, preliminarly, the equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left[b\left(\frac{\partial y}{\partial x}\right)\right] - \varepsilon\frac{\partial^4 y}{\partial x^4} + f(t,x), \qquad \varepsilon > 0, \tag{19}$$

and the associated problem (11) with the further condition

$$\frac{\partial^2 y}{\partial x^2}(t,0) = \frac{\partial^2 y}{\partial x^2}(t,l) = 0 \tag{20}$$

corresponding to a rod hinged at both ends. The weak solution of this problem will be defined in the following, obvious way:

a_4) $y(t) \in L^\infty(0,T; H_0^1 \cap H^2) \cap H^{1,\infty}(0,T;L^2)$, $y(0) = \alpha$;

b_4) $y(t)$ satisfies, a.e. on $(0,T)$, the equation

$$\int_0^t \{-(y',k')_{L^2} + (b(Dy),Dk) + \varepsilon(D^2y, D^2k)_{L^2} - (f,k)_{L^2}\}\,d\tau + \\ + (y'(t),k(t))_{L^2} - (\beta,k(0))_{L^2} = 0 \tag{21}$$

$\forall k(t) \in L^2(0,T; H_0^1 \cap H^2) \cap H^1(0,T;L^2)$.

We are now able to prove the following Theorem.

Theorem 3.1: *If $\alpha \in H_0^1(0,l) \cap H^2(0,l)$, $\beta \in L^2(0,l)$, there exists in Q a unique weak solution of the problem* (19), (11), (20).

Proof: We give the proof of this theorem in a schematic way, focusing the attention on the steps which cannot be taken directly from the corresponding Theorem in [7]. The existence can be proved with a classical Faedo-Galerkin method.

i) We consider an orthogonal basis (³), orthonormal in L^2, denoted by $\{g_i\}$ in $H_0^1 \cap H^2$ and the projections α_n and β_n of α and β on the n-dimensional subspace spanned by $\{g_j\}$ when $j = 1, 2, \dots, n$. Setting:

$$y_n = \sum_{j=1}^n \psi_j(t) g_j, \tag{22}$$

we consider the system of ordinary differential equations (in t), that the coefficients ψ_j have to verify to solve the Cauchy-Dirichlet problem given by (19), the first of (11), and the initial conditions $y_n(0) = \alpha_n$ and $y_n'(0) = \beta_n$.

ii) Multiplying the j^{th} equation obtained at the preceding step by ψ_j and adding for $j = 1, 2, \dots, n$, by a standard use of Gronwall's lemma, we obtain the following.

(³) In the sequel, we set $H^s = H^s(0,l)$, $L^p = L^p(0,l)$, etc.

fundamental upper bounds for the solution y_n.

$$\text{(23)}\qquad \begin{cases} \|y_n'(t)\|_{L^2} \le M_1\,, \\ \|y_n(t)\|_{H^2} \le \dfrac{M_2}{\sqrt{\delta}}\,, \\ \|y_n(t)\|_{H_0^1} \le M_3\,, \end{cases}$$

$\forall t \in (0, T_{\text{fin}}]$, where M_i are constants depending only on the data. By well known embedding theorems, it follows, passing to the limit for $n \to \infty$, that there exists a subsequence of $\{y_n\}$, again denoted by $\{y_n\}$, such that:

$$\text{(24)}\qquad \lim_{n\to\infty} y_n(t) = y(t)$$

in the weak* topology of $H^{1,\infty}(0, T_{\text{fin}}; L^2) \cap L^\infty(0, T_{\text{fin}}; H^2 \cap H_0^1)$ and in the strong topology of $L^\infty(0, T_{\text{fin}}; H_0^1)$.

Moreover, it is possible to show ([7]) that:

$$\text{(25)}\qquad y_n(t) \in H^1(0, T_{\text{fin}}; (H^2 \cap H_0^1)') \cap L^\infty(0, T_{\text{fin}}; L^2)$$

from which, it follows that

$$\text{(26)}\qquad \lim_{n\to\infty} y_n'(t) = y'(t)$$

in the strong topology of $C^0(0, T_{\text{fin}}, H^{-2})$.

iii) We now want to show that:

$$\text{(27)}\qquad \int_0^t\int_0^l \sqrt{1+Dy_n^2}\,Dy_n\,Dk\,dx\,d\eta \to \int_0^t\int_0^l \sqrt{1+Dy^2}\,Dy\,Dk\,dx\,d\eta$$

$\forall k$ belonging to the functional spaces indicated above. Observe that:

$$\text{(28)}\qquad \left|\int_0^t\int_0^l \sqrt{1+Dy_n^2}\,Dy_n\,Dk\,dx\,d\eta - \int_0^t\int_0^l \sqrt{1+Dy^2}\,Dy\,Dk\,dx\,d\eta\right| =$$

$$= \left|\int_0^t\int_0^l \sqrt{1+Dy_n^2}\,[Dy_n - Dy]\,Dk\,dx\,d\eta\right. +$$

$$\left. - \int_0^t\int_0^l [\sqrt{1+Dy^2} - \sqrt{1+Dy_n^2}\,]\,Dy\,Dk\,dx\,d\eta\right| \le$$

$$\le \int_0^t\int_0^l \left|\sqrt{1+Dy_n^2}\,\|[Dy_n - Dy]\|\,Dk\right|\,dx\,d\eta \,+$$

$$+ \int_0^t\int_0^l \left|\sqrt{1+Dy^2} - \sqrt{1+Dy_n^2}\,\|Dy\|\,Dk\right|\,dx\,d\eta\,.$$

Since $y_n \in H^2$, $Dy_n \in H^1$, we have that, for $x \in R$, $Dy_n \in L^\infty(Q)$. Therefore, the first term in the right hand side of (28) has the upper bound:

$$(29) \qquad M_4 \int_0^t \int_0^l |Dy_n - Dy\|Dk|\, dx\, d\eta \leq M_5 \int_0^t \|Dy_n - Dy\|_{L^2} \|Dk\|_{L^2}\, d\eta$$

and the right hand side of (29) vanishes when $n \to \infty$, by virtue of (24).

Moreover, also the second term of the right hand side of (28) tends to 0, since it is easy to verify that $\sqrt{1 + Dy_n^2} \to \sqrt{1 + Dy^2}$.

Therefore, we can say that:

$$(30) \qquad \langle b(Dy_n), Dk\rangle \to \langle b(Dy), Dk\rangle .$$

Hence, passing suitably to the limit, we have proved that y is a solution of our problem.

Moreover, the solution is unique; suppose, in fact, that exist two solutions, u and v, and consider $w = v - u$, where, by virtue of a_4) and b_4), $w \in L^\infty(0, T_{\theta n}; H_0^1 \cap H^2) \cap \cap H^{1,\infty}(0, T_{\theta n}; L^2)$, $w(0) = 0$, $w'(0) = 0$, and satisfies, a.e. on $(0, T_{\theta n})$, the equation:

$$(31) \qquad \int_0^t \{ -(w', k')_{L^2} + (b(Du) - b(Dv), Dk) + \\ + \delta(D^2 w, D^2 k)_{L^2} - (f, k)_{L^2} \}\, d\eta + (w'(t), k(t))_{L^2} = 0$$

$\forall k(t) \in L^2(0, T; H_0^1 \cap H^2) \cap H^1(0, T; L^2)$. Let G be Green's operator with respect to $-D^2$, relative to the homogeneous Dirichlet problem on $(0, l)$. Assuming in (31) Gw' as test function (which is obviously possible), after some calculations we are led to the inequality:

$$(32) \qquad \frac{1}{2}\|w'(t)\|^2_{H^{-1}} + \frac{\delta}{2}\|w(t)\|^2_{H_0^1} + \int_0^t (b(Du) - b(Dv), DGw')_{L^2}\, d\eta \leq 0 .$$

Now, observe that, if $Dy \in L^\infty$, also $\sqrt{1 + (Dy)^2} \in L^\infty$ and that, by Lagrange's Theorem:

$$\left|\sqrt{1 + (Du)^2} - \sqrt{1 + (Dv)^2}\right| \leq |Dw| ,$$

so we can write the following inequalities:

$$(33) \qquad \left|\int_0^t \int_0^l \sqrt{1 + (Du)^2}\, Du G^{1/2} w'\, dx\, d\eta - \int_0^t \int_0^l \sqrt{1 + (Dv)^2}\, Dv\, G^{1/2} w'\, dx\, d\eta\right| \leq$$

$$\leq \int_0^t \int_0^l \left| \sqrt{1+(Du)^2} \| Dw \| G^{1/2} w' \right| dx\, d\eta +$$

$$+ \int_0^t \int_0^l \left| \sqrt{1+(Du)^2} - \sqrt{1+(Dv)^2} \| Du \| G^{1/2} w' \right| dx\, d\eta \leq$$

$$\leq M_6 \int_0^t \| Dw \|_{L^2} \| G^{1/2} w' \|_{L^2} d\eta \leq M_7 \int_0^t \left(\| w \|^2_{H_0^1} + \| w' \|^2_{H^{-1}} \right) d\eta .$$

In this way, by a standard application of Gronwall's lemma, we have $w = 0$, i.e. the solution is unique. □

By virtue of the Remark 1.1, we observe that $b'(\xi) > 0$, i.e. $b(\xi)$ is strictly increasing. This fact allows to us to claim the following existence Theorem, proof of which is immediatly obtained (see [7]).

THEOREM 3.2: *If $\alpha \in H_0^1$, $\beta \in L^2$ and $f \in L^2(Q)$, then there exists at least one solution of the problem* (4), (11). □

Consider, now, the problem of uniqueness: difficulties relevant to this problem are examined in detail in [7].

We start by proving some auxiliary lemmas, analogous to the ones given in [7].

LEMMA 3.1: *If y_δ is the weak solution of* (19), (11), (20) (*which exsits and is unique by virtue of Theorem* 3.1), *satisfying the initial conditions $y_\delta(0) = \alpha_\delta \in H_0^1 \cap H^2$ and $y'_\delta(0) = \beta_\delta \in L^2$, and $\alpha_\delta \to \alpha_{\bar\delta}$ in H_0^1, $\beta_\delta \to \beta_{\bar\delta}$ in H^{-1}, when $\delta \to \bar\delta$, with $\bar\delta > 0$, then*

$$\lim_{\delta \to \bar\delta} y_\delta = y_{\bar\delta}$$

in the strong topology of $H^{1,\infty}(0, T_{fin}; H^{-1}) \cap L^\infty(0, T_{fin}; H_0^1)$.

PROOF: The proof is analogous to the uniqueness part of Theorem 3.1. Set $w = y_\delta - y_{\bar\delta}$, and, with the same procedure adopted in the proof of Theorem 3.1, consider the equation following directly:

$$(34) \qquad \langle w'' + \bar\delta D^4 y_\delta + D(b(y_\delta)) - D(b(y_{\bar\delta})), Gw' \rangle = (\delta - \bar\delta) \langle D^4 y_\delta, Gw' \rangle ,$$

where $w(0) = \alpha_\delta - \alpha_{\bar\delta}$ and $w'(0) = \beta_\delta - \beta_{\bar\delta}$. Hence, by a_4) and (33) (with y_δ and $y_{\bar\delta}$ instead of u and v), we obtain:

$$\frac{1}{2}\|w'(t)\|^2_{H^{-1}} + \frac{\bar\delta}{2}\|w(t)\|^2_{H_0^1} \leq \frac{1}{2}\|w'(0)\|^2_{H^{-1}} + \frac{\bar\delta}{2}\|w(0)\|^2_{H_0^1} + \tag{35}$$

$$+ \left|\int_0^t \left(\langle b(Dy_\delta) - b(Dy_{\bar\delta}), G^{1/2} w'\rangle + (\bar\delta - \delta)\langle D^2 y_\delta, w'\rangle\right) d\eta\right| \leq$$

$$\leq \int_0^t c_1\left(\|w'\|^2_{H^{-1}} + \|w\|^2_{H_0^1}\right) d\eta + c_2|\delta - \bar\delta| + \frac{1}{2}\|w'(0)\|^2_{H^{-1}} + \frac{\bar\delta}{2}\|w(0)\|^2_{H_0^1}$$

a.e. on $(0, T)$.

On the other hand, by (23) and (25), we have, $\forall \delta > 0$:

$$\begin{cases} \|y_\delta\|_{L^\infty(0,T;H_0^1)} \leq M_3, & \|y'_\delta\|_{L^\infty(0,T;L^2)} \leq M_1, \\ \|y'_\delta\|_{H^1(0,T;(H^2\cap H_0^1)')} \leq M_8, & \|y_\delta\|_{L^\infty(0,T;H^2)} \leq \dfrac{M_2}{\sqrt{\delta}}, \\ \|Dy_\delta\|_{H^{\sigma,\infty}(0,T;H^{1-2\sigma})} \leq M_7, & \end{cases} \tag{36}$$

where $0 < \sigma < 1$. It follows that, when $\delta \to \bar\delta$,

$$\lim_{\delta \to \bar\delta} w = z \tag{37}$$

in the weak and weak* topologies corresponding to (36). Letting then $\delta \to \bar\delta$ in (35), we obtain, by the usual compactness theorems, bearing in mind that $w'(0) \to 0$ in H^{-1} and $w(0) \to 0$ in H_0^1,

$$\frac{1}{2}\|z'(t)\|^2_{H^{-1}} + \frac{\bar\delta}{2}\|z(t)\|^2_{H_0^1} \leq \int_0^t c_1\left(\|z'(\eta)\|^2_{H^{-1}} + \|z(\eta)\|^2_{H_0^1}\right) d\eta. \tag{38}$$

Therefore, by Gronwall's lemma, $z = 0$. $\square$

Now, denote by U_f the set of weak solutions of (4), (11) corresponding to the known term f and by $\tilde U_f$ the set of weak solutions of (4), (11), corresponding to f, which are obtained by the solution of the vibrating rod problem (19), (11), (20), when the flexional rigidity δ tends to zero: we shall call such solutions *approximable solutions*. Moreover, we shall denote by $\tilde L^2$ the subspace of $L^2(Q)$ defined in the following way: let $\{g_i\}$ be a basis in $L^2(Q)$ and $\tilde g_k$ the elements of $\{g_i\}$ orthogonal in $L^2(Q)$ to all the elements $u \in U_f$. The remaining elements $\{g_i^*\}$ form a basis which spans $\tilde{L}^2$. Finally, we shall denote by $\mathcal{K}$ the convex set defined as follows:

$$(39) \qquad \mathcal{K} = \left\{ \begin{array}{l} v \in L^\infty(0, T; H_0^{1,1}) \cap H^{1,\infty}(0, T; L^2), \\ v'(t) \in L^2\text{-weakly continuous on } [0, T], \\ v(0) = \alpha, \qquad v'(0) = \beta \end{array} \right\};$$

in [7] it is proved that the solution of (4), (11) belongs to $\mathcal{K}$. Moreover, setting $By = y'' - Db(Dy)$, the existence theorem can be extended assuming that $f \in B\mathcal{K}$, i.e. there exists at least a weak solution $y \in \mathcal{K}$ if $f \in B\mathcal{K}$; on the other hand, if $y \in \mathcal{K}$, there exists a unique $f \in B\mathcal{K}$, such that $y \in U_f$.

The proofs of the follwing two Lemmas can be found in [7].

Lemma 3.2: *Let $\tilde{U}_f$ be the set of approximable solutions corresponding to the known term f. Then, either $\tilde{U}_f$ contains a single element, or $\tilde{U}_f$ is an infinite set which is compact in $L^2(Q)$, weakly* compact in $\mathcal{K}$ and does not contain any isolated points.* □

Lemma 3.3: *Let $\{u_n\}$ be a sequence $\in U_f$, with $f \neq 0$; then $\{u_n\}$ cannot be a basis in $\tilde{L}^2$.* □

We can now prove the following uniqueness Theorem.

Theorem 3.3: *Under the assumptions made in the existence theorem there exists at most one approximable solution of (4), (11), for nearly all known terms $f \in L^2(Q)$.*

Proof: We shall follow the trace of the corresponding Theorem in [7]. To begin with, we remark that since the theorem holds for nearly all f, we may assume that $f \neq 0$: on the other hand, observe that if $\alpha(x) = \beta(x) = f(t, x) = 0$, then the only approximable solution is $y = 0$.

Moreover, we can assume that U_f contains a sequence $\{y_k\}$: in fact, if U_f contains a finite number of elements, the Theorem is proved, by virtue of Lemma 3.2. The elements of $\{y_k\}$ cannot be linearly independent in $\tilde{L}^2$; in fact, if $y_k = \sum_{j=1}^{\infty} \zeta_{kj} g_j^*$ $(k = 1, 2, \dots)$ were linearly independent in $\tilde{L}^2$, they would constitute a basis in $\tilde{L}^2$ (since $\{g_i^*\}$ is, by definition, a basis in $\tilde{L}^2$) and this, by Lemma 3.3, is not possible. Therefore, all solutions are linear combinations of p linearly independent solutions, i.e. U_f belongs to a p-dimensional convex subset $\mathcal{K}_p$ of $\mathcal{K}$, $(p < +\infty)$, which we can assume is spanned by a set of functions $r_1, \dots, r_p \in \mathcal{K}$.

Indeed, if $y \in \mathcal{K}_p$, then y is a weak solution corresponding to the known term $f = By \in B\mathcal{K}$; on the other hand, if $f \in B\mathcal{K}$, there exists at least one $y \in U_f \cap \mathcal{K}_p$.

Now, the basic idea of the proof is the following: if $y \in \tilde{U}_f$, then, by Lemma 3.2, either y is the only approximable solution corresponding to f, or y is a limit point of $\tilde{U}_f$ (and consequently of U_f). The Theorem will then be proved if we show that, for nearly all $f \in L^2(Q)$, there do not exist any solutions that are limit points of U_f (i.e. all solutions are isolated).

Suppose, therefore, that there exist two weak solutions, $u, v \in \mathcal{K}_p \cap U_f$, with u limit point of U_f and $f \in L^2$; we can then set

$$u = \sum_{k=1}^{p} \alpha_k r_k, \qquad v = \sum_{k=1}^{p} \beta_k r_k, \qquad \eta_k = \beta_k - \alpha_k. \tag{40}$$

On the other hand,

$$b(Dv) - b(Du) = (Dv - Du)\, b'(D\psi), \tag{41}$$

with $D\psi(x, t) = \lambda(x, t)\, Du(x, t) + (1 - \lambda(x, t))\, Dv(x, t)$, $(0 \leq \lambda \leq 1)$. Observe that

$$\left| \frac{1 + 2z^2}{\sqrt{1 + z^2}} \right| \leq 1 + 2|z|,$$

so if $z \in L^3$, then $b'(z) \in L^3$ and since $v \to u \in \mathcal{K}$, it follows that $b'(D\psi) \to b'(Du) \in L^\infty(Q)$.

A straightforward calculation shows that the coefficients η_k satisfy, $\forall b \in \mathcal{D}(Q)$, the equation

$$\sum_{k=1}^{p} \eta_k [\langle r_k'', b \rangle + \langle Dr_k b'(D\psi), Db \rangle] = 0, \tag{42}$$

where the second term on the left hand side of (42) is meaningful, because every function in the crochet $\in L^3$. Taking $b = b_1, b_2, \ldots, b_p$ linearly independent functions, (42) reduces to a linear, homogeneous system of p equations in the p unknowns $\eta_1, \ldots, \eta_p$

$$\sum_{k=1}^{p} \eta_k [\langle r_k'', b_i \rangle + \langle Dr_k b'(D\psi), Db_i \rangle] = 0 \qquad (i = 1, \ldots, p) \tag{43}$$

with determinant of order p,

$$G_{u,v} = \det [\langle r_k'', b_i \rangle + \langle Dr_k b'(D\psi), Db_i \rangle]. \tag{44}$$

Since u is a limit point for U_f, it must necessarily be $G_{u,u} = 0$; in fact, if $G_{u,u} \neq 0$, then by continuity, $G_{u,v} \neq 0$ $\forall v$ in a sufficiently small neighbourhood of u, but, in this case, system (43) would admit only the solution $\eta_1 = \ldots = \eta_p = 0$, i.e. $u = v$. The coefficients $\alpha_1, \ldots, \alpha_p$ of solutions $\in \mathcal{K}_p$ which are limit points of sequences of solutions corresponding to the same known term must therefore satisfy the equation

$$G_{u,u} = \det \left[\langle r_k'', b_i \rangle + \left\langle Dr_k\, b'\left(\sum_{j=1}^{p} \alpha_j Dr_j \right), Db_i \right\rangle \right] = 0, \tag{45}$$

Setting

$$(46)\qquad g_{ki}(\alpha_1, \dots, \alpha_p) = \left\langle Dr_k b'\left(\sum_{j=1}^{p} \alpha_j Dr_j\right), Db_i\right\rangle = \int_Q Dr_k b'\left(\sum_{j=1}^{p} \alpha_j Dr_j\right) Db_i \, dQ,$$

observe that, in our case, the functions g_{ki} are analytic in $\alpha_1, \dots, \alpha_p$.

Consider now the set $z_i = r_i'' - b'(0)D^2 r_i$, $(i = 1, \dots, p)$: it is possible to show (see [7]), by absurd, that the functions z_i are linearly independent. Therefore, it is possible to choose $b_1, \dots, b_p$ in such a way that

$$(47)\qquad \langle r_k'' - b'(0)D^2 r_k, b_i\rangle = \delta_{ki}$$

and the known term in (45) becomes $\prod_{i=1}^{p} \langle r_i'' - b'(0)D^2 r_i, b_i\rangle = 1$.

Moreover, in view of the linear independence of z_i and denoting by $\bar{v}(x, t)$ the solution of the homogeneous D'Alembert equation $u'' - b'(0)D^2 u = 0$, with initial and boundary conditions given by (11), it is shown that for all the sets $\mathcal{K}_p$ such that $\bar{v}(x, t) \notin \mathcal{K}_p$, equation (45) cannot reduce to an identity. In the sequel, we shall denote by $\mathcal{K}_p^{(1)}$ such sets, and by $\mathcal{K}_p^{(2)}$ the sets such that $\bar{v}(x, t) \in \mathcal{K}_p$. We may then conclude that the non isolated weak solutions which form part of a set $\mathcal{K}_p^{(1)}$ all belong necessarily to an analytic $(p-1)$-dimensional manifold Γ_{p-1}, of equation (45). On the other hand, the sets $\mathcal{K}_p^{(2)}$ may contain non isolated weak solutions constituting a p-dimensional set (eventually also $\equiv \mathcal{K}_p^{(2)}$).

Considering now the union of all the p-dimensional sets $\mathcal{K}_p$ (which obviously coincides with $\mathcal{K}$) and the union Φ_p of all the corresponding subsets constituted by non isolated weak solutions, we show that Φ_p has measure zero in $\mathcal{K}$.

Let $\{g_i\}$ be a basis in $\mathcal{K}$, with $g_1(t, x) = \bar{v}(t, x)$.

If meas$(\Phi_p) > 0$ in $\mathcal{K}$, then, denoting by Π_p the set of p-dimensional manifolds $\subset \mathcal{K}$ and which do not contain g_1

$$\Pi_p = \left\{ v \in \mathcal{K},\ v = \sum_{i=2}^{p+1} \alpha_i g_i \right\},$$

there exists necessarily one manifold Π_p^* such that the set $\Phi_p \cap \Pi_p^*$ has dimension p. By definition, however, Π_p^* coincides with some set $\mathcal{K}_p^{(1)*}$ and we have shown that $\Phi_p \cap \cap \mathcal{K}_p^{(1)*}$ is always a $(p-1)$-dimensional set. Hence, *meas*(Φ_p) cannot be > 0.

Repeating this procedure for $p = 1, \dots$, we may therefore conclude that the set $\Phi = \bigcup_p \Phi_p$ has measure zero in $\mathcal{K}$.

Consider now the sets $B\Phi$ and $B\mathcal{K}$, constituted by the known terms corresponding respectively to eventual non isolated weak solutions and to all weak solutions. Since to each $f \in B\Phi$ there correspond infinitely many weak solutions, while to each $f \in B\mathcal{K} \setminus B\Phi$ there correspond at most a finite number of weak solutions, it follows, by what has been proved above, that $B\Phi$ has measure zero in $B\mathcal{K}$, and consequently also in $L^2(Q)$, since $B\mathcal{K}\subset L^2(0,T;H^{-1,3/2})+H^{-1}(0,T;L^2)$ and $L^2(Q)\supset(L^2(0,T;H^{-1,3/2})+$ $+H^{-1}(0,T;L^2))'$.

Hence, with the exception of at most a set of measure zero, to the known terms $f\in$ $\in L^2(Q)$ correspond isolated weak solutions; as we observed, by Lemma 3.2, this proves the Theorem. □

REMARK 3.1: *As observed in [7], the proof of Theorem 3.3 can be extended to any solution obtained as suitable limit of an approximate well-posed problem, depending continuously on a real parameter, and not only by the approximate problem of the rod.*

3.2. *M-model.* In the M-model we have, with respect to (12):

$$b(\xi)=A\xi-B\frac{\xi}{\sqrt{1+\xi^2}}+C\left(\sqrt{1+\xi^2}\right)\xi, \tag{48}$$

where, according to Remark 1.4, $b(\xi)$ is still a strictly increasing monotone function, provided that ε satisfies condition (10). In order to extend the Theorems relative to G-model, it is easy to observe that $b(\xi)$ defined by (48) has, with regard to the proof of such theorems, the same properties as (15) (4), so all the existence and uniqueness results obtained at the preceding subsection are immediatly applicable to the M-model.

4. - STABILITY OF THE DIFFERENCE EQUATIONS

4.1. *G-model.* We begin to study stability conditions for the equation (3), setting as usual (see [12], [1] and [2])

$$v_{i,k}=\frac{y_{i,k+1}-y_{i,k-1}}{2\Delta t}.$$

Following what we have done in [2], we linearize the nonlinear term

$$\left(\frac{y_{i+1,k}-y_{i,k}}{\Delta x}\right)\sqrt{1+\left(\frac{y_{i+1,k}-y_{i,k}}{\Delta x}\right)^2}-\left(\frac{y_{i,k}-y_{i-1,k}}{\Delta x}\right)\sqrt{1+\left(\frac{y_{i,k}-y_{i-1,k}}{\Delta x}\right)^2}$$

in (3) by using a Taylor expansion truncated at the first derivative of the function $\varphi(z)=z\sqrt{1+z^2}$, in the neighborhood of a suitable – in a sense that we will point out –

(4) Indeed, it is easy to verify that the further term added in M-model, i.e. $B\dfrac{\xi}{\sqrt{1+\xi^2}}$, satisfies the assumptions (13) and (14).

value z_0 of $\frac{y_{i,k} - y_{i-1,k}}{\Delta x}$, obtaining

(49) $$\frac{M}{l}\frac{y_{i,k+1} - 2y_{i,k} + y_{i,k-1}}{\Delta t^2} =$$

$$= T_0\left[(1-\varepsilon) + \varepsilon\frac{1+2z_0^2}{\sqrt{1+z_0^2}}\right]\frac{y_{i+1,k} - 2y_{i,k} + y_{i-1,k}}{\Delta x^2} - \frac{M}{l}g - \frac{M}{l}\sigma\frac{y_{i,k+1} - y_{i,k-1}}{2\Delta t}.$$

Equation (49) is obtained by writing $\varphi(z_1) - \varphi(z_2) = \varphi'(z_0)(z_1 - z_2)$, with $z_1 = \frac{y_{i+1,k} - y_{i,k}}{\Delta x}$ and $z_2 = \frac{y_{i,k} - y_{i-1,k}}{\Delta x}$.

Let us set:

(50) $$\begin{cases} r = \dfrac{l}{M} T_0 \left\{(1-\varepsilon) + \varepsilon\dfrac{1+2z_0^2}{\sqrt{1+z_0^2}}\right\}\dfrac{\Delta t^2}{\Delta x^2}; & 2p = \dfrac{2(1-r)}{1+\sigma\dfrac{\Delta t}{2}}; \\ q = \dfrac{r}{1+\sigma\dfrac{\Delta t}{2}}; & s = \dfrac{1-\sigma\dfrac{\Delta t}{2}}{1+\sigma\dfrac{\Delta t}{2}} \end{cases}$$

It is easy to see, by using (50) and neglecting for the sake of simplicity the constant term $\frac{Mg}{l}$, that (49) becomes:

(51) $$y_{i,k+1} = 2p\,y_{i,k} + q\,(y_{i+1,k} + y_{i-1,k}) - s\,y_{i,k-1};$$

setting:

$$y_k = [y_{i,k}], \qquad z_k = \begin{bmatrix} y_k \\ y_{k-1} \end{bmatrix}.$$

(51) becomes therefore, in vector form:

(52) $$z_{k+1} = \Pi\, z_k,$$

with:

(53) $$\Pi = \begin{bmatrix} A & -B \\ I & 0 \end{bmatrix}$$

where I is the identity matrix, $B = sI$ and

$$(54) \qquad A = \begin{bmatrix} 2p & q & 0 & \cdots & 0 \\ q & 2p & q & \cdots & \vdots \\ 0 & q & 2p & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & q & 2p \end{bmatrix}.$$

As is well known (see f.i. [14]), the eigenvalues of $\prod$ are given by the equations:

$$(55) \qquad \lambda^2 - \lambda_j \lambda + s = 0 \qquad (j = 1, 2, \ldots, n),$$

where $\lambda_j = 2p + 2q \cos \dfrac{j\pi}{n+1}$ are the eigenvalues of A. In order to choose r in such a way that $|\lambda| \leqslant 1$, according to the matrix stability analysis, we consider preliminarly the case with $\rho = \sigma = 0$, i.e. without damping. In this case, $2p = 2(1-r)$, $q = r$, $s = 1$, and the (55) becomes:

$$(56) \qquad \lambda^2 - 2\left(1 - 2r \sin^2 \frac{j\pi}{2(n+1)}\right)\lambda + 1 = 0.$$

Since the product of the roots of (56) is equal to 1, we can obtain at most weak stability imposing that such roots are immaginary, conjugate, with modulus = 1, i.e. imposing that:

$$\left[1 - 2r \sin^2\left(\frac{j\pi}{2(n+1)}\right)\right]^2 - 1 \leqslant 0,$$

which is always verified provided that:

$$(57) \qquad r \leqslant 1.$$

Expliciting this stability condition, we obtain:

$$(58) \qquad \Delta t \leqslant \frac{\Delta x}{\sqrt{\dfrac{l}{M} T_0 \left\{(1-\varepsilon) + \varepsilon \dfrac{1 + 2z_0^2}{\sqrt{1+z_0^2}}\right\}}}.$$

REMARK 4.1: *Equation* (58) *has surely sense, because of the condition* $\varepsilon \in [0, 1)$, *that ensures, as observed in Remark* 1.3, *the radical to be positive. Moreover, for* $\varepsilon = 0$ *we have* *the well known stability condition for the D'Alembert equation, as was to be expected, while, for* $\varepsilon \neq 0$, (58) *becomes more restrictive, being* $\dfrac{1 + 2z_0^2}{\sqrt{1 + z_0^2}} \geqslant 1$.

REMARK 4.2: *As already observed in* [2] *with reference to* (8), *the denominator of the right hand side of* (58) *increases with* z_0. *Therefore, a simple and, at the same time, «pessimistic» choice for* z_0 *is:*

$$z_0 = \max_i \left(\left| \frac{y_{i+1,k} - y_{i,k}}{\Delta x} \right| \right) = \left\| \frac{y_{i+1,k} - y_{i,k}}{\Delta x} \right\|_\infty . \tag{59}$$

Indeed, if in the case treated in [2] *it was possible to consider a time-independent pessimistic condition, passing to the limit for* $z_0 \to \infty$, *this procedure is not applicable to condition* (58): *in fact, by so doing, we would have* $\Delta t \leqslant 0$.

Finally, it is possible to verify that in the damped case, (58) still ensures stability (at least weak, and strong if σ is large enough) (5).

4.2. *M-model.* In a way similar to the one adopted in the preceding subsection, with a suitable linearization and with direct calculations that, not being of particular interest we shall not give here, we obtain a condition analogous to (58), where the denominator of the right hand side is:

$$\sqrt{\frac{l\bar{K}}{M} \left\{ [l(1-\varepsilon) - 2l_0\varepsilon] - l_0 \left[(1-\varepsilon) - \varepsilon \frac{l_0}{l} \right] \frac{1}{\left(\sqrt{1+z_0^2}\right)^3} + l\varepsilon \frac{1+2z_0^2}{\sqrt{1+z_0^2}} \right\}}, \tag{60}$$

for which all the Remarks made regarding the corresponding (58) still hold (i.e. positivity of the radical quantity, choice of z_0, stabilty in the damped case).

REMARK 4.3: *In* [2] *it has been observed that, because of the nonlinear nature of the problems considered, we find a time-dependent stability condition. This means that, in using the leap frog formulas, we need to make some interpolations, in order to take into account the variability of* Δt, *so that* Δt *is really a* Δt_k. *If we repeat the above calculations, by considering this variability, we are led to the condition:*

$$\Delta t_{k+1} \leqslant \min \left(\Delta t_k, \frac{\Delta x^2}{\Delta t_k \gamma} \right), \tag{61}$$

(5) In this connection, see f.i. [16].

where:

$$\gamma = \frac{l}{M} T_0 \left\{ (1-\varepsilon) + \varepsilon \frac{1+2z_0^2}{\sqrt{1+z_0^2}} \right\}$$

in the case of G-model, and:

$$\gamma = \frac{l\bar{K}}{M} \left\{ [l(1-\varepsilon) - 2l_0\varepsilon] - l_0 \left[(1-\varepsilon) - \varepsilon \frac{l_0}{l} \right] \frac{1}{\left(\sqrt{1+z_0^2}\right)^3} + l\varepsilon \frac{1+2z_0^2}{\sqrt{1+z_0^2}} \right\}$$

for the M-model. In our numerical calculations we have always used values of Δt_k which verify condition (61) (⁶).

REMARK 4.4: *Following the papers* [15] *and* [17], *it could turn out that our solutions are discontinuous in some way. However, by the considerations made by Lax in* [18], *this circumstance has been eliminated.*

In fact, according Lax's work, if we consider the equation:

$$\frac{\partial^2 y}{\partial t^2} = Q^2 \left(\frac{\partial y}{\partial x} \right) \frac{\partial^2 y}{\partial x^2}$$

where, in our case,

$$Q(\xi) = \sqrt{A - \frac{B}{[\sqrt{1+\xi^2}]^3} + C \frac{1+2\xi^2}{\sqrt{1+\xi^2}}},$$

there can arise some discontinuities on $\frac{\partial y}{\partial x}$ at the time:

$$t_{crit} \approx \frac{1}{\frac{\partial Q}{\partial \xi}(0) \max \frac{d^2\alpha}{dx^2}}.$$

However, in our case, we have $\frac{\partial Q}{\partial \xi}(0) = 0$, so we obtain that in a finite time-interval such discontinuities do not arise.

5. - NUMERICAL RESULTS

We now present some numerical results about the two models considered. According to Remark 2.2, we made our calculations considering a realistic case of a circular

(⁶) Eventually, can be introduced a maximum value for Δt, in order to ensure sufficient accuracy.

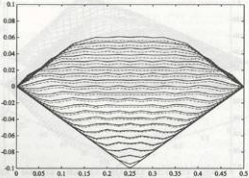

Fig. 1. – Behaviour of G-model with $\varepsilon = 0.9$ (solid) and $\varepsilon = 0.1$ (dashed) with triangular conditions, from 0 to 5 msec.: the increase in the frequency of oscillations is small, but quite evident.

rubber string with $l = 0, 5$ m, $l_0 = 0.25$ m, diameter of the normal section $= 2$ mm, density 1.1 g/cm^3, $T_0 =$ one third of the breakage tension, where the breakage stress is 20 N/mm^2.

Generally, G-model and M-model give very different results, but this is not surprising: in fact (see [11] and [2], even if in a different physical case), the introduction of a

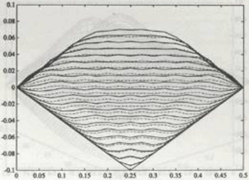

Fig. 2. – Behaviour of M-model with $\varepsilon = 0.9$ (solid) and $\varepsilon = 0.1$ (dashed) with triangular conditions, from 0 to 5 msec.

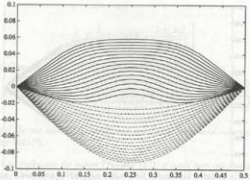

Fig. 3. – Behaviour of G-model with $\varepsilon = 0.5$ (solid) and D-model (dashed) with sinusoidal initial conditions, from 78.00 to 79.80 msec.

non-null value of l_0 causes substantial differences in the behaviour of the string, even if $\varepsilon = 0$, i.e. a linear stress-strain law it is considered and G-model becomes D'Alembert equation (D-model), while M-model becomes A-model. Actually, if the deformations are small, the differences between G-model and D-model, and between M-model and A-model are small too.

We present some graphics obtained starting from different initial conditions, by

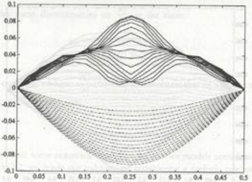

Fig. 4. – Behaviour of M-model with $\varepsilon = 0.5$ (solid) and D-model (dashed) with sinusoidal initial conditions conditions, from 78.00 to 79.80 msec.

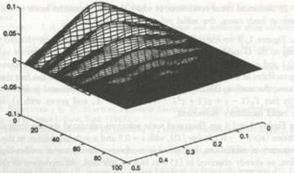

Fig. 5. – Behaviour of the damped G-model in 100 msec. ($\rho = 0.005$ Kg/sec; $\varepsilon = 0.5$).

using the *leap frog formulas* in the discrete equations (3) and (6): analogous results, according to the precision order of all the methods, i.e. the second, can be obtained employing the Method of Lines with spatial semidiscretization given by (3) and (6) and integration in time achieved either with a method proposed by Van der Houwen and Sommejer and adopted in [11], or with a Predictor-Corrector method.

Namely, we consider:

1) triangular initial conditions, i.e. the conditions of a plucked string: the displacement of the mid-point of the string is 0.1 m;

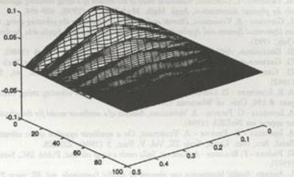

Fig. 6. – Behaviour of the damped M-model in 100 msec. ($\rho = 0.005$ Kg/sec; $\varepsilon = 0.5$).

2) sinusoidal initial conditions, in which the first sinusoidal mode of the string is excited; in both cases, the initial velocity is equal to zero.

In Figures 1, 2 we compare the behaviour of the solution of the plucked string ac cording to the G-model (Fig. 1) and the M-model (Fig. 2), with different values of ε ($\varepsilon = 0.1$ and $\varepsilon = 0.9$ (7)) and $\rho = 0$; it is possible to observe that when ε increases, the string oscillates at a higher frequency; in our opinion, this is due to the fact that when ε increases, the tension, that brings the string back to the horizontal position, increases too (in fact $T_0(1 - \varepsilon + \varepsilon(1+z^2)^{-3/2}(1+2z^2)) > T_0$ and grows with ε), causing a more rapid oscillatory movement.

In Figures 3 and 4 are illustrated some solutions calculated starting from sinusoidal conditions ($\alpha(x) = -0.1\sin(\pi x/l)$), with $\varepsilon = 0.5$ and $\rho = 0$: in addition to the increase of frequency of oscillations, it is possible to observe the arising of modes higher than the first, as already observed in [15]: in fact, in this case, superposition of the effects does not hold. The appearance of the higher modes is more evident in M-model, where the string loses very quickly the initial sine configuration.

In Figures 5 and 6 are illustrated the solutions calculated according to the G- and M-models in the damped case ($\rho = 0.005$ Kg/s, $\varepsilon = 0.5$): in the preceding figures we set $\rho = 0$ to better show the cited effects, but generally, it is clear that the choice $\rho \neq 0$ is more realistic.

(7) With the values adopted, condition (10) becomes $\varepsilon < 1$.

REFERENCES

[1] M. Frontini - L. Gotusso, *Numerical study of the motion of a string vibrating against an obstacle by physical discretization*, Appl. Math. Modelling, 14 (1990), 489-494.

[2] L. Gotusso - A. Veneziani, *Discrete and continuous models for the vibrating string*, to appear on *Dynamical Systems and Applications*, World Scientific Series in Applicable Analysis, Vol. 4 (July, 1995).

[3] D. Greenspan, *Arithmetic Applied Mathematics*, Pergamon, New York (1980).

[4] D. Greenspan, *Computer Oriented Mathematical Physic*, Pergamon, New York (1981).

[5] D. Greenspanc, *Quasimolecular Modelling*, World Scientific Series, 44, Singapore (1992).

[6] A. B. Schubert - D. Greenspan, *Numerical studies of discrete vibrating string*, Technical Report # 158, Univ. of Wisconsin (1972).

[7] A. Iannelli - G. Prouse - A. Veneziani, *Analysis of a nonlinear model for the vibrating string*, to appear on NoDEA (1994).

[8] A. Iannelli - G. Prouse - A. Veneziani, *On a nonlinear equation of the vibrating string*, Rend. Acc. Naz. Lincei, Serie IX, Vol. V, Fasc. 3 (1994), 223-228.

[9] G. Prouse - F. Rolandi - A. Zaretti, *Sulla corda e verga vibranti*, Pubbl. IAC, Serie III, 111 (1977).

[10] L. Amerio, *Analisi Matematica con elementi di Analisi Funzionale*, vol. III, parte 2ª, UTET, Torino (1982).

[11] A. Veneziani, *Numerical analysis of two nonlinear models of the vibrating string*, preprint.
[12] F. D. Auret - J. A. Snyman, *Numerical studies of linear and nonlinear string vibrations by means of physical discretization*, Appl. Math. Modelling, 2 (1978), 7-17.
[13] A. B. Davey - A. R. Payne, *Rubber in Engineering Practice*, McLaren, London (1964).
[14] G. D. Smith, *Numerical Solution of Partial Differential Equations: Finite Difference Method*, Oxford University Press, London (1978).
[15] E. Fermi - J. Pasta - S. Ulam, *Studies of Nonlinear Problems, I*, Los Alamos Report, 1940 (1955).
[16] R. D. Richtmyer - K. W. Morton, *Difference Methods for Initial-Value Problems*, 2nd ed., Wiley (Interscience), New York (1967).
[17] N. J. Zabusky, *Exact solution for the vibrations of a nonlinear continuous model string*, J. Math. Phys., 3, 5 (1962), 1028-1039.
[18] P. D. Lax, *Development of singularities of nonlinear hyperbolic partial differential equations*, J. Math. Phys., 5, 5 (1964), 611-613.
[19] P. J. Van der Houwen - B. Sommeijer, *Iterated θ-methods for hyperbolic equations*, Int. J. Numerical Analysis (1990).