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# On a Transmission Problem for Second Order Elliptic and Parabolic Equations in Planar Domains (**)

ABSTRACT. — We consider transmission problems for linear second order elliptic and parabolic equations in planar domains $\Omega$, $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$, where the interface $\Gamma$ meets the boundary of $\Omega$ in exactly two points $P$ and $Q$. The boundary is not necessarily smooth at these points. We assume the homogeneous boundary condition $u = (u_+, u_-) = 0$ on $\partial\Omega$, and $u_+ = u_-$ on $\Gamma$. The elliptic operator is formally self-adjoint and the first order condition on $u$ on the interface is the one we obtain from Green's formula. We show the existence of unique solutions to the considered problems in weighted Sobolev spaces $W_s^k(\Omega, \Sigma)$ and $L^2(0, T; W_s^k(\Omega, \Sigma))$, respectively ($\Sigma = \{P, Q\}$). Here $k$ denotes a non-negative integer, $0 \leq s \leq 1/2$, with $1 + k - s$ different from isolated exceptional numbers (which we determine), depending on the contact angles at $P$ and $Q$ as well as on the leading coefficients of the elliptic operator at $P$ and $Q$.

## Su un problema di trasmissione per equazioni del secondo ordine ellittiche e paraboliche in domini piani

RIASSUNTO. — Si considerano problemi di trasmissione per equazioni lineari del secondo ordine ellittiche e paraboliche in domini piani $\Omega$, $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$, ove la interfaccia $\Gamma$ incontra la frontiera di $\Omega$ in due soli punti $P$ e $Q$. La frontiera non è necessariamente regolare in questi punti. Si pone, sulla frontiera $\partial\Omega$, la condizione omogenea $u = (u_+, u_-) = 0$ e, su $\Gamma$, $u_+ = u_-$. L'operatore ellittico è formalmente autoaggiunto e la condizione del primo ordine sulla interfaccia è quella ottenuta dalla formula di Green. Viene dimostrata l'esistenza di soluzioni uniche per i problemi considerati negli spazi di Sobolev con peso $W_s^k(\Omega, \Sigma)$ e $L^2(0, T; W_s^k(\Omega, \Sigma))$, rispettivamente ($\Sigma = \{P, Q\}$). Con $k$ si indica un intero non negativo, è $0 \leq s \leq 1/2$, con $1 + k - s$ diverso da valori eccezionali (qui determinati) dipendenti dagli angoli di contatto in $P$ e $Q$, nonché dai coefficienti delle derivate seconde dell'operatore ellittico in $P$ e in $Q$.

## INTRODUCTION

The purpose of this paper is to study a class of transmission problems for linear second order elliptic and parabolic equations in planar domains.

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Let $\Omega$ be a bounded simply connected open subset of $R^2$ whose boundary consists of two $C^\infty$-curves $\partial\Omega^\pm$ with common end points $P$ and $Q$. Let $\Gamma$ be a $C^\infty$-curve dividing $\Omega$ into two simply connected sets $\Omega^\pm$ and intersecting $\partial\Omega$ at $P$ and $Q$.

Consider the elliptic transmission boundary-value problem:

$$(0.1)\qquad \begin{cases} A^\pm u_\pm = -\sum_{j,k=1}^{2} \frac{\partial}{\partial x_k}\left[a_{jk}^\pm(x)\frac{\partial}{\partial x_j}u_\pm\right] + a^\pm(x)u_\pm = f_\pm & \text{in } \Omega^\pm, \\ u_\pm = 0 \quad \text{on } \partial\Omega^\pm/\Gamma; \qquad B^+u_+ - B^-u_- = \phi, \quad u_+ = u_- \quad \text{on } \Gamma, \end{cases}$$

where

$$B^\pm u_\pm = \sum_{j,k=1}^{2} a_{jk}^\pm(x)\frac{\partial}{\partial x_j}u_\pm \cos(n, x_k).$$

The unit normal vector to $\Gamma$ oriented toward $\Omega^-$ is denoted by $n$.

Elliptic transmission and, related to them, mixed problems arise in numerous applications and thus far have been the subject of extensive studies [1], [2], [4], [5], [7], [9]-[15], [17] (see also the literature quoted in the above papers). It is known that for regular elliptic transmission problems of any order in $n$-dimensional space the Fredholm alternative holds in some weighted Sobolev spaces whose exponents are different from a set of numbers. The lack of information on the set of exceptional numbers is compensated by the generality of the problem.

In this paper by restricting to second order elliptic equations in planar domains we shall determine those exceptional numbers explicitly and show that they depend on the contact angles at $P$ and at $Q$ as well as on the values of $a_{jk}^\pm$ at $P$ and $Q$. Thus, when the contact angles are small and different from some exceptional ones (which are determined), the solution of the problem (0.1) is in the appropriate Sobolev space without loss of the top order derivative. The result seems new and besides being of interest in its own right may be needed in the study of free boundary problems. The proof is carried out in Section 3.

The canonical case of sectorial domains for $A^\pm = -\Delta$ is studied in Section 2. The asymptotics of the solutions near the origin is investigated.

In Section 4 we consider the parabolic transmission problem:

$$(0.2)\qquad \begin{cases} \frac{\partial}{\partial t}u_\pm + A^\pm u_\pm = f_\pm(x,t) \quad \text{in } \Omega^\pm \times (0,T); \\ u_\pm = 0 \quad \text{on } (\partial\Omega^\pm/\Gamma)\times(0,T), \\ u_+ = u_-, \quad B^+u_+ - B^-u_- = \phi(x) \quad \text{on } \Gamma\times(0,T), \\ u_\pm(x,0) = u_0^\pm(x) \quad \text{in } \Omega^\pm. \end{cases}$$

The existence of a unique weak solution of (0.2) is known, cf. [8]. In contrast to the elliptic case the literature on the regularity of the solution of (0.2) is almost non-existent. Using a discretisation of the time-variable and the results of Section 3 we shall prove the existence of a unique solution of (0.2)

in the appropriate weighted Sobolev spaces. When the contact angles are small, the solution belongs to the usual Sobolev spaces.

Notations are given in Section 1.

## Section 1

Let $\Omega$ be a bounded open subset of $R^2$ whose boundary consists of two $C^\infty$-curves $\partial\Omega^+$, $\partial\Omega^-$ with common end points $P$ and $Q$. Let $\Gamma$ be a $C^\infty$-curve dividing $\Omega$ into two simply connected open sets $\Omega^\pm$ and intersecting $\partial\Omega$ at $P$ and at $Q$. Throughout the paper we shall write: $\Sigma = \{P, Q\}$.

The interior angles made by $\partial\Omega^+$ with $\Gamma$ and with $\partial\Omega^-$ at $P$ are denoted by $\omega_1$ and $\gamma_1$ respectively. The corresponding angles at $Q$ are $\omega_2$ and $\gamma_2$. We have: $0 < \omega_j < \gamma_j \leq \leq \Pi$; $j = 1, 2$.

Let $k$ be a non-negative interger and $2 \leq p < \infty$. Then $W^{k,p}(\Omega)$ is the usual Sobolev space

$$W^{k,p}(\Omega) = \{u\colon D^\alpha u \text{ in } L^p(\Omega),\ |\alpha| \leq k\}$$

with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p}.$$

$W_0^{k,2}(\Omega)$ is the completion of $C_0^\infty$-functions with respect to the $W^{k,2}(\Omega)$-norm.

Let $\rho(x)$ be the distance from a point $x$ in $\Omega$ to the set $\Sigma = \{P, Q\}$. We introduce the weighted Sobolev spaces of Kondratiev [7]. By $H_\gamma^k(\Omega; \Sigma)$ where $k$ is a non-negative integer and $-\infty < \gamma < \infty$, we mean the Hilbert space with norm

$$\|u\|_{H_\gamma^k(\Omega;\Sigma)} = \left(\sum_{|\alpha| \leq k} \|\rho^{\gamma + |\alpha| - k} D^\alpha u\|^2_{L^2(\Omega)}\right)^{1/2}.$$

We define $H_\gamma^{k+1/2}(\Omega; \Sigma)$ to be the Hilbert space with norm

$$\|u\|^2_{H_\gamma^{k+1/2}(\Omega;\Sigma)} =$$

$$= \|u\|^2_{H_\gamma^k(\Omega;\Sigma)} + \sum_{|\alpha|=k} \int_\Omega \rho^{2\gamma}(x)\,dx \int_{|x-y| \leq (1/2)\rho(x)} |D^\alpha u(x) - D^\alpha u(y)|^2\,|x-y|^{-3}\,dy.$$

For functions prescribed on smooth manifolds these spaces can be defined in a standard way. The space of traces of functions in $H_\gamma^k(\Omega; \Sigma)$ coincides with $H_\gamma^{k-1/2}(\partial\Omega; \Sigma)$.

We shall write $H_\gamma^k(\Omega)$, $H_\gamma^{k-1/2}(\Gamma)$ for $H_\gamma^k(\Omega; \Sigma)$, $H_\gamma^{k-1/2}(\partial\Omega; \Sigma)$.

## Section 2

In this section we shall consider a transmission problem for the Laplace operator in sectorial domains. We shall determine the discriminant function associated with the

problem, study the asymptotics of the solutions near the origin and consider the case when the data are in some Sobolev spaces.

Let $S^{\pm}$ be the sectors

$$S^{+} = \{(r, \theta)\colon 0 < r < \infty,\ 0 < \theta < \alpha\},$$
$$S^{-} = \{(r, \theta)\colon 0 < r < \infty,\ \alpha < \theta < \beta \leq \pi\}$$

and let $\Gamma_0$, $\Gamma_\alpha$, $\Gamma_\beta$ be the rays $\theta = 0$, $\theta = \alpha$ and $\theta = \beta$ respectively.

We shall write $H_s^k(S^{\pm})$ for $H_s^k(S^{\pm}; 0)$.

Consider the transmission problem

$$(2.1)\qquad \begin{cases} -\Delta v_{\pm} = f_{\pm} \quad \text{in } S^{\pm}; & v_+ = 0 \quad \text{on } \Gamma_0, \quad v_- = 0 \quad \text{on } \Gamma_\beta, \\ v_+ - v_- = b_1 \quad \text{and} & \dfrac{\partial}{\partial n}(v_+ - v_-) = b_2 \quad \text{on } \Gamma_\alpha. \end{cases}$$

Set: $v_r^{\pm} = v_{\pm}\cos\theta$, $v_\theta^{\pm} = -v_{\pm}\sin\theta$.

In the new coordinates we have, cf. [14]

$$(2.2)\qquad \begin{cases} -r\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial}{\partial r}v_r^{\pm}\right\} - \dfrac{\partial^2}{\partial\theta^2}v_r^{\pm} + v_r^{\pm} + 2\dfrac{\partial}{\partial\theta}v_\theta^{\pm} = r^2 f_r^{\pm}, \\ -r\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial}{\partial r}v_\theta^{\pm}\right\} - \dfrac{\partial^2}{\partial\theta^2}v_\theta^{\pm} + v_\theta^{\pm} - 2\dfrac{\partial}{\partial\theta}v_r^{\pm} = r^2 f_\theta^{\pm} \quad \text{in } S^{\pm}, \end{cases}$$

where $f_r^{\pm} = f^{\pm}\cos\theta$, $f_\theta^{\pm} = -f^{\pm}\sin\theta$.

The boundary conditions are:

$$(2.3)\qquad \begin{cases} v_r^{+}(\cdot, 0) = v_\theta^{+}(\cdot, 0) = 0; & v_r^{-}(\cdot, \beta) = v_\theta^{-}(\cdot, \beta) = 0, \\ v_r^{+}(\cdot, \alpha) - v_r^{-}(\cdot, \alpha) = b_1\cos\alpha, & v_\theta^{+}(\cdot, \alpha) - v_\theta^{-}(\cdot, \alpha) = -b_1\sin\alpha, \\ \dfrac{\partial}{\partial\theta}(v_\theta^{+} - v_\theta^{-})\big|_{\theta=\alpha} = -b_1\cos\alpha - rb_2\sin\alpha = b_3. \end{cases}$$

Denote by

$$v^{\pm}(\sigma, \theta) = \int_0^\infty r^{-\sigma-1}v_r^{\pm}(r, \theta)\,dr, \qquad w^{\pm}(\sigma, \theta) = \int_0^\infty r^{-\sigma-1}v_\theta^{\pm}(r, \theta)\,dr$$

the Mellin transforms of $v_r^{\pm}$, $v_\theta^{\pm}$ and suppose that the functions $r^2 f_r^{\pm}$, $r^2 f_\theta^{\pm}$, $b_1$ and $b_2$ have Mellin transforms $g_1^{\pm}$, $g_2^{\pm}$, $\bar{b}_1$, $\bar{b}_2$ respectively.

The system (2.2) may be written as

$$(2.4)\qquad \begin{cases} -\dfrac{d^2}{d\theta^2}v^{\pm} + (1 - \sigma^2)v^{\pm} + 2\dfrac{d}{d\theta}w^{\pm} = g_1^{\pm}, \\ -\dfrac{d^2}{d\theta^2}w^{\pm} + (1 - \sigma^2)w^{\pm} - 2\dfrac{d}{d\theta}v^{\pm} = g_2^{\pm} \end{cases}$$

with

$$(2.5)\qquad \begin{cases} v^+(\sigma,0) = w^+(\sigma,0) = 0; \qquad v^-(\sigma,\beta) = w^-(\sigma,\beta) = 0, \\ v^+(\sigma,\alpha) - v^-(\sigma,\alpha) = \bar{b}_1 \cos\alpha, \qquad w^+(\sigma,\alpha) - w^-(\sigma,\alpha) = -\bar{b}_1 \sin\alpha, \\ \frac{\partial}{\partial\theta}(w^+ - w^-)|_{\theta=\alpha} = \bar{b}_3 = -\bar{b}_1 \cos\alpha - (\overline{\sigma b_2}) \sin\alpha. \end{cases}$$

Set $Y_\pm = (v^\pm, (v^\pm)', w^\pm, (w^\pm)')'$, then in matrix form the system (2.4) becomes

$$(2.6)\qquad \frac{d}{d\theta} Y_\pm = A Y_\pm - B_\pm$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1-\sigma^2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 1-\sigma^2 & 0 \end{pmatrix}$$

and

$$B_\pm = (0, g_1^\pm, 0, g_2^\pm)'.$$

For $\sigma \neq \pm 1$, the eigenvalues of $A$ are $\pm i(\sigma \pm 1)$ and the eigenvectors are $(1, i(\sigma + 1), i, -(\sigma+1))'$; $(1, i(\sigma - 1), -i, \sigma - 1)'$. The system (2.6) may be solved by the method of variation of parameters. Once $Y_\pm$ is obtained we consider the problem of matching the boundary conditions. The discriminant function arises in this context. We are led to the problem of finding all $\sigma$ for which the following system has a solution:

$$(2.7)\qquad \begin{cases} -\frac{d^2}{d\theta^2} v^\pm + (1-\sigma^2) v^\pm + 2\frac{d}{d\theta} w^\pm = 0, \\ -\frac{d^2}{d\theta^2} w^\pm + (1-\sigma^2) w^\pm - 2\frac{d}{d\theta} v^\pm = 0 \end{cases}$$

with

$$(2.8)\qquad \begin{cases} v^+(\sigma,0) = k_1^+(\sigma), \qquad w^+(\sigma,0) = k_2^+(\sigma), \\ v^-(\sigma,\beta) = k_1^-(\sigma), \qquad w^-(\sigma,\beta) = k_2^-(\sigma), \\ v^+(\sigma,\alpha) - v^-(\sigma,\alpha) = k_3(\sigma), \qquad w^+(\sigma,\alpha) - w^-(\sigma,\alpha) = k_4(\sigma), \\ \frac{d}{d\theta}(w^+ - w^-)|_{\theta=\alpha} = k_5(\sigma). \end{cases}$$

For $\sigma \neq \pm 1$, set

$$w^{\pm} = a_1^{\pm} \cos(\sigma - 1)\theta + a_2^{\pm} \sin(\sigma - 1)\theta + a_3^{\pm} \cos(\sigma + 1)\theta + a_4^{\pm} \sin(\sigma + 1)\theta,$$
$$v^{\pm} = -a_1^{\pm} \sin(\sigma - 1)\theta + a_2^{\pm} \cos(\sigma - 1)\theta + a_3^{\pm} \sin(\sigma + 1)\theta - a_4^{\pm} \cos(\sigma + 1)\theta.$$

Then $\{w^{\pm}, v^{\pm}\}$ satisfies (2.7).

For $\theta = 0$, we get: $a_1^+ + a_3^+ = k_1^+$, $a_2^+ - a_4^+ = k_2^+$. So

$$w^+ = a_1^+ \{\cos(\sigma - 1)\theta - \cos(\sigma + 1)\theta\} + a_2^+ \{\sin(\sigma - 1)\theta + \sin(\sigma + 1)\theta\} + \\ + k_2^+ \cos(\sigma + 1)\theta - k_1^+ \sin(\sigma + 1)\theta,$$

$$v^+ = -a_1^+ \{\sin(\sigma - 1)\theta + \sin(\sigma + 1)\theta\} + a_2^+ \{\cos(\sigma - 1)\theta - \cos(\sigma + 1)\theta\} + \\ + k_2^+ \sin(\sigma + 1)\theta + k_1^+ \cos(\sigma + 1)\theta.$$

On the ray $\Gamma_\beta$, we have:

$$(2.9) \qquad \begin{cases} a_1^- \cos(\sigma - 1)\beta + a_2^- \sin(\sigma - 1)\beta + \\ \qquad + a_3^- \cos(\sigma + 1)\beta + a_4^- \sin(\sigma + 1)\beta = k_2^-, \\ -a_1^- \sin(\sigma - 1)\beta + a_2^- \cos(\sigma - 1)\beta + \\ \qquad + a_3^- \sin(\sigma + 1)\beta - a_4^- \cos(\sigma + 1)\beta = k_1^-. \end{cases}$$

At the interface, the condition: $w^+ - w^- = k_4$ gives

$$(2.10) \qquad a_1^+ \{\cos(\sigma - 1)\alpha - \cos(\sigma + 1)\alpha\} + a_2^+ \{\sin(\sigma - 1)\alpha + \sin(\sigma + 1)\alpha\} - \\ - a_1^- \cos(\sigma - 1)\alpha - a_2^- \sin(\sigma - 1)\alpha - a_3^- \cos(\sigma + 1)\alpha - a_4^- \sin(\sigma + 1)\alpha = \\ = k_4 + k_1^+ \sin(\sigma + 1)\alpha - k_2^+ \cos(\sigma + 1)\alpha.$$

With $v^+ - v^- = k_5(\sigma)$ on $\Gamma_\alpha$, we obtain:

$$(2.11) \qquad -a_1^+ \{\sin(\sigma - 1)\alpha + \sin(\sigma + 1)\alpha\} + a_2^+ \{\cos(\sigma - 1)\alpha - \cos(\sigma + 1)\alpha\} + \\ + a_1^- \sin(\sigma - 1)\alpha - a_2^- \cos(\sigma - 1)\alpha - a_3^- \sin(\sigma + 1)\alpha + a_4^- \cos(\sigma + 1)\alpha = \\ = k_5 - k_2^+ \sin(\sigma + 1)\alpha - k_1^+ \cos(\sigma + 1)\alpha.$$

The interface condition $(d/d\theta)(w^+ - w^-)|_{\theta=\alpha} = k_5(\sigma)$ gives

$$(2.12) \qquad a_1^+ \{(\sigma + 1)\sin(\sigma + 1)\alpha - (\sigma - 1)\sin(\sigma - 1)\alpha\} + \\ + a_2^+ \{(\sigma - 1)\cos(\sigma - 1)\alpha + (\sigma + 1)\cos(\sigma + 1)\alpha\} + a_1^- (\sigma - 1)\sin(\sigma - 1)\alpha - \\ - a_2^- (\sigma - 1)\cos(\sigma - 1)\alpha + a_3^- (\sigma + 1)\sin(\sigma + 1)\alpha - a_4^- (\sigma + 1)\cos(\sigma + 1)\alpha = \\ = k_5 + k_2^+ (\sigma + 1)\sin(\sigma + 1)\alpha + k_1^+ (\sigma + 1)\cos(\sigma + 1)\alpha.$$

From (2.4), we have:

$$\frac{d^2}{d\theta^2}(v^+ - v^-)\big|_{\theta=\alpha} = 2k_3 + (1-\sigma^2)k_5 .$$

Thus,

$$(2.13)\qquad a_1^+\{(\sigma-1)^2\sin(\sigma-1)\alpha - (\sigma+1)^2\sin(\sigma+1)\alpha\} +$$

$$a_2^+\{(\sigma+1)^2\cos(\sigma+1)\alpha - (\sigma-1)^2\cos(\sigma-1)\alpha\} - a_3^-(\sigma-1)^2\sin(\sigma-1)\alpha +$$

$$+ a_3^-(\sigma-1)^2\cos(\sigma-1)\alpha + a_4^-(\sigma+1)^2\sin(\sigma+1)\alpha - a_4^-(\sigma+1)^2\cos(\sigma+1)\alpha =$$

$$= 2k_3 + (1-\sigma^2)k_5 + k_1^+(1+\sigma)^2\cos(\sigma+1)\alpha + k_2^+(1+\sigma)^2\sin(\sigma+1)\alpha .$$

We have a system of six equations with six unknowns which we write as $C\cdot(a_1^+, a_2^+, a_1^-, a_2^-, a_3^-, a_4^-)' = K^t$. The entries of the column matrix $K^t$ are the right hand sides of eq. (2.9)-(2.13). Row operations on the matrix $C$ show that the eq. (2.13) expresses the compatibility condition and thus we have a system of five equations with six unknowns. Further row operations give:

$$C_1\cdot(a_1^+, a_2^+, a_1^-, a_2^-, a_3^-, a_4^-)' = K^t$$

where $C_1^t$ is the matrix:

$$\begin{matrix}
0 & 0 & 2\sin\sigma\alpha\sin\alpha & -2\sigma\sin(\sigma-1)\alpha & 2\sigma\sin(\sigma+1)\alpha \\
0 & 0 & 2\sin\sigma\alpha\cos\alpha & 2\sigma\cos(\sigma-1)\alpha & 2\sigma\cos(\sigma+1)\alpha \\
\cos(\sigma-1)\beta & -\sin(\sigma-1)\beta & -\cos(\sigma-1)\alpha & 2\sigma\sin(\sigma-1)\alpha & 0 \\
\sin(\sigma-1)\beta & \cos(\sigma-1)\beta & -\sin(\sigma-1)\alpha & -2\sigma\cos(\sigma-1)\alpha & 0 \\
\cos(\sigma+1)\beta & \sin(\sigma+1)\beta & -\cos(\sigma+1)\alpha & 0 & 2\sigma\sin(\sigma+1)\alpha \\
\sin(\sigma+1)\beta & -\cos(\sigma+1)\beta & -\sin(\sigma+1)\alpha & 0 & -2\sigma\cos(\sigma+1)\alpha
\end{matrix}$$

The first three entries of $K_1^t$ and the fifth one are the same as those of $K^t$ but the fourth one is now:

$$k_5 + k_3 + k_2^+\sigma\sin(\sigma+1)\alpha + k_1^+\sigma\cos(\sigma+1)\alpha .$$

Since $\sin(\sigma+1)\beta$ and $\cos(\sigma+1)\beta$ cannot be both zero, the entries of the sixth column of $C_1$ are not all zeros for all $\sigma$.

Set $a_4^- = \lambda$ and consider the $5\times 5$ matrix $C_2$ obtained from $C_1$ by deleting the sixth column. A very lengthy computation yields:

$$\det C_2 = F(\sigma,\alpha,\beta) = 2^4\sigma^2\sin(\sigma\alpha)\sin(\sigma\beta)\sin(\sigma\beta+\alpha) .$$

Thus, for $\sigma$ with $F(\sigma,\alpha,\beta)\neq 0$ and for $a_4^- = \lambda$ we have $\{a_1^+, a_2^+, a_1^-, a_2^-, a_3^-\}$ as the unique solution of the equation

$$(2.14)\qquad C_2\cdot(a_1^+, a_2^+, a_1^-, a_2^-, a_3^-)' = K_2^t$$

where $K_1^*$ is the obvious column matrix arising from the above discussions. We have proved the following lemma:

LEMMA 2.1: *Suppose that $\sigma \neq \pm 1$, $n\pi/\alpha$, $m\pi/\beta$, $-\alpha/\beta + p\pi/\beta$ where $n$, $m$, $p$ are integers. Then the system (2.7)-(2.8) has a one-parameter family of solutions:*

$$w^+ = a_1^+ \{\cos(\sigma-1)\theta - \cos(\sigma+1)\theta\} + a_2^+ \{\sin(\sigma-1)\theta + \sin(\sigma+1)\theta\} + \\ + k_2^+ \cos(\sigma+1)\theta - k_1^+ \sin(\sigma+1)\theta,$$

$$v^+ = -a_1^+ \{\sin(\sigma-1)\theta + \sin(\sigma+1)\theta\} + a_2^+ \{\cos(\sigma-1)\theta - \cos(\sigma+1)\theta\} + \\ + k_2^+ \sin(\sigma+1)\theta + k_1^+ \cos(\sigma+1)\theta$$

*with*

$$w^- = a_1^- \cos(\sigma-1)\theta + a_2^- \sin(\sigma-1)\theta + a_3^- \cos(\sigma+1)\theta + \lambda \sin(\sigma+1)\theta,$$

$$v^- = -a_1^- \sin(\sigma-1)\theta + a_2^- \cos(\sigma-1)\theta + a_3^- \sin(\sigma+1)\theta - \lambda \cos(\sigma+1)\theta,$$

*where $\{a_1^+, a_2^+, a_1^-, a_2^-, a_3^-\}$ is the solution of (2.14).*

*Moreover:*

(i) *$\sigma = 0$ is a pole of order 4 for the solutions,*

(ii) *$\sigma = n\pi/\alpha$, $m\pi/\beta$, $-\alpha/\beta + p\pi/\beta$; $n$, $m$, $p$ positive integers, are simple poles for the solutions.*

One of the main results of the section is the following theorem.

THEOREM 2.1: *Let $k$ be a non-negative integer with $1 + k - s \neq \sigma$ where $\sigma = 1$ or is a root of the discriminant function*

$$F(\sigma, \alpha, \beta) = 2^4 \sigma^2 \sin(\sigma\alpha) \sin(\sigma\beta) \sin(\alpha + \sigma\beta).$$

*Then for $\{f_\pm, b_1, b_2\}$ in $H_r^k(S^+) \times H_r^{k+3/2}(\Gamma_\alpha) \times H_r^{k+1/2}(\Gamma_\alpha)$, there exists a unique solution $v_\pm$ in $H_r^{k+2}(S^\pm)$ of (2.1). Moreover:*

$$\|v_+\|_{H^{k+2}(S^+)} + \|v_-\|_{H^{k+2}(S^-)} \leq \\ \leq C\{\|f_+\|_{H^k(S^+)} + \|f_-\|_{H^k(S^-)} + \|b_1\|_{H^{k+3/2}(\Gamma_\alpha)} + \|b_2\|_{H^{k+1/2}(\Gamma_\alpha)}\}.$$

PROOF: With $f_\pm$, $b_1$ and $b_2$ as in the theorem the Mellin transforms of $r^2 f_r^\pm$, $r^2 f_\theta^\pm$, $b_1$ and $b_2$ are well-defined. The inverse Mellin transform of $Y_\pm$, solution of (2.6), belongs to the appropriate spaces. From Lemma 2.1 we know that the system (2.7)-(2.8) has a one-parameter family of solutions $\{v^\pm, w^\pm\}$. For the inverse Melling transforms of $\{v^\pm, w^\pm\}$ to be in $H_r^{k+2}(S^\pm)$, the parameter $\lambda$ of Lemma 2.1 has to be equal to zero. Now the estimates can be obtained as done in [7].

REMARK: When $\beta = \pi$, let $\sigma_0 = 0$, $\sigma_1 = \pi/\alpha$ and $\sigma_2 = -\pi/\alpha$. Then for $\operatorname{Re}\sigma > 0$, we should take $\sigma \neq \sigma_0 \pmod 1$, $\sigma \neq \sigma_2 \pmod 1$, $\sigma \neq \sigma_1 \pmod{\pi/\alpha}$ in Theorem 2.1. Those

numbers are the expectional ones and depend on the contact angle of the interface with the boundary.

Suppose that $f_\pm$ is in $H^{k_1}_{s_1}(S^+) \cap H^{k_2}_{s_2}(S^-)$ with $1 + k_1 - s_1 < 1 + k_2 - s_2$ and $1 + k_1 - s_1$, $1 + k_2 - s_2$ different from the roots of the discriminant function. According to Theorem 2.1 we have two solutions $u^{(1)}_\pm$, $u^{(2)}_\pm$. Our aim is to study the relationship between $u^{(1)}_\pm$ and $u^{(2)}_\pm$ and thus is led to consider the asymptotics of the solutions near the origin.

Suppose that $\sigma = \sigma_n = n\pi/\alpha$ and consider the homogeneous system arising from (2.7)-(2.8). We deduce from the matrix $C_1$ with $\sigma = \sigma_n$ that:

$$a_1^+ \sin\alpha + a_2^+ \cos\alpha = -a_3^- \sin\alpha + a_4^- \cos\alpha = a_3^- \sin\alpha + a_2^- \cos\alpha .$$

Let $D$ be the matrix obtained from $C_1$ be deleting the first two columns and the last row. Then $D$ is a $4 \times 4$ matrix and a simple calculation gives $\det D \neq 0$.

Thus, $a_1^- = a_2^- = a_3^- = a_4^- = 0$. We have proved the following lemma.

Lemma 2.2: *Let $\sigma = \sigma_n = n\pi/\alpha$, then the homogeneous system associated with (2.7)-(2.8) has a one-parameter family of solutions:*

$$v^-(\sigma_n, \theta) = w^-(\sigma_n, \theta) = 0 ,$$

$$w^+(\sigma_n, \theta) = a_1^+ \{\cos(\sigma_n - 1)\theta - \cos(\sigma_n + 1)\theta\} + a_2^+ \{\sin(\sigma_n - 1)\theta + \sin(\sigma_n + 1)\theta\} ,$$

$$v^+(\sigma_n, \theta) = -a_1^+ \{\sin(\sigma_n - 1)\theta + \sin(\sigma_n + 1)\theta\} + a_2^+ \{\cos(\sigma_n - 1)\theta - \cos(\sigma_n + 1)\theta\}$$

*with $a_1^+ \sin\alpha + a_2^+ \cos\alpha = 0$.*

*Moreover:*

$$v_r^-(r, \theta, \sigma_n) = 0 = v_\theta^-(r, \theta, \sigma_n) ,$$

$$v_r^+(r, \theta, \sigma_n) = r^{\sigma_n} v^+(\sigma_n, \theta) ,$$

$$v_\theta^+(r, \theta, \sigma_n) = r^{\sigma_n} w^+(\sigma_n, \theta) ,$$

*is a solution of the homogeneous system (2.2)-(2.3).*

Suppose that $\sigma = m\pi/\beta$ and consider the homogeneous system associated with (2.7)-(2.8). We study the matrix $C_1$ with $\sigma = m\pi/\beta$. After some simple but lengthy row operations, we obtain:

$$a_4^- = 0 = a_2^+ = a_2^- = a_4^+ ; \quad a_1^+ = -a_3^- = -a_3^+ ; \quad a_1^- = -a_3^- .$$

Thus, we have:

Lemma 2.4: *Let $\sigma = \bar{\sigma}_m = m\pi/\beta$, then the homogeneous system associated with (2.7)-(2.8) has a two-parameter family of solutions:*

$$v^-(m\pi/\beta, \theta) = -a_1^- \{\sin(\bar{\sigma}_m - 1)\theta + \sin(\bar{\sigma}_m + 1)\theta\} ,$$

$$w^-(m\pi/\beta, \theta) = a_1^- \{\cos(\bar{\sigma}_m - 1)\theta - \cos(\bar{\sigma}_m + 1)\theta\} ,$$

*with*

$$v^{+}(m\pi/\beta, \theta) = -a_1^{+}\{\sin(\bar{\sigma}_m - 1)\theta + \sin(\bar{\sigma}_m + 1)\theta\},$$

$$w^{+}(m\pi/\beta, \theta) = a_1^{+}\{\cos(\bar{\sigma}_m - 1)\theta - \cos(\bar{\sigma}_m + 1)\theta\}.$$

*Moreover:*

$$v_r^{+}(r, \theta, \bar{\sigma}_m) = r^{\bar{\sigma}_m} v^{+}(m\pi/\beta, \theta), \qquad v_\theta^{+}(r, \theta, \bar{\sigma}_m) = r^{\bar{\sigma}_m} w^{+}(m\pi/\beta, \theta),$$

*is a solution of the homogeneous system associated with* (2.2)-(2.3).

Suppose that $\sigma = \bar{\sigma}_p = -\alpha/\beta + p\pi/\beta$. As before we study the matrix $C_1$ with $\sigma = \bar{\sigma}_p$. Set $a_4^{-} = \lambda$ and the $5 \times 5$ matrix obtained by deleting from $C_1$ the fifth column has its determinant equal to zero. Thus, set $a_1^{-} = \mu$ and we obtain $a_1^{+}, a_2^{+}, a_2^{-}, a_3^{-}$ in terms of $\lambda$, $\mu$. We have:

Lemma 2.5: *Let $\sigma = \bar{\sigma}_p = -\alpha/\beta + p\pi/\beta$, then the homogeneous system associated with* (2.7)-(2.8) *has a two-parameter family of solutions* $\{v^{\pm}(\bar{\sigma}_p, \theta), w^{\pm}(\bar{\sigma}_p, \theta)\}$. *Moreover*

$$v_r^{\pm}(r, \theta, \bar{\sigma}_p) = r^{\bar{\sigma}_p} v^{\pm}(\bar{\sigma}_p, \theta), \qquad v_\theta^{\pm}(r, \theta, \bar{\sigma}_p) = r^{\bar{\sigma}_p} w^{\pm}(\bar{\sigma}_p, \theta)$$

*is a solution of the homogeneous problem* (2.2)-(2.3).

Finally we consider the case when $\sigma = 0$. It is a zero multiplicity 4 of the discriminant function of Lemma 2.1. Setting $\sigma = 0$ in the matrix $C_1$, we get after some row operations: $a_1^{+} + a_3^{+} = 0$, $a_2^{+} - a_4^{+} = 0$, $a_1^{-} + a_3^{-} = 0$, $a_2^{-} - a_4^{-} = 0$. We have:

$$w^{\pm} = a_1^{+}\{\cos(\sigma - 1)\theta - \cos(\sigma + 1)\theta\} + \\ + a_2^{\pm}\{\sin(\sigma - 1)\theta + \sin(\sigma + 1)\theta\}|_{\sigma=0} = 0,$$

$$v^{\pm} = -a_1^{+}\{\sin(\sigma - 1)\theta + \sin(\sigma + 1)\theta\} + \\ + a_2^{\pm}\{\cos(\sigma - 1)\theta - \cos(\sigma + 1)\theta\}|_{\sigma=0} = 0.$$

Theorem 2.2: *Let $\{f_\pm, h_1, h_2\}$ be in*

$$\{H_{s_1}^{k_1}(S^{\pm}) \cap H_{s_2}^{k_2}(S^{\pm})\} \times \\ \times \{H_{s_1}^{k_1+1/2}(\Gamma_a) \cap H_{s_2}^{k_2+1/2}(\Gamma_a)\} \times \{H_{s_1}^{k_1+1/2}(\Gamma_a) \cap H_{s_2}^{k_2+1/2}(\Gamma_a)\}.$$

*Suppose that $1 + k_1 - s_1 < 1 + k_2 - s_2$ and that $1 + k_1 - s_1$, $1 + k_2 - s_2 \neq \sigma$ with $\sigma = 1$ or to a root of the discriminant function of Lemma 2.1. Let $v_\pm^1$, $v_\pm^2$ be the solutions of* (2.1) *in $H_{s_1}^{2+k_1}(S^{\pm})$, $H_{s_2}^{2+k_2}(S^{\pm})$ given by Theorem 2.1. Then:*

$$v_\pm^1 = v_\pm^2 + \sum c_j^{\pm} v^{\pm}(\sigma_j, x)$$

*where the summation is extended over all the zeros of the discriminant function in the strip* $1 + k_1 - s_1 < \operatorname{Re}\sigma < 1 + k_2 - s_2$; $v^{\pm}(\sigma_j, x)$ *are defined by Lemmas 2.2-2.5.*

*The coefficients* $c_j^{\pm}$ *depend on* $f_{\pm}$, $b_1$, $b_2$.

PROOF: The theorem is an immediate consequence of Theorem 2.1 and of Lemmas 2.2-2.5.

Since there is no injection mapping of $W^{k,p}(S^{\pm})$ into $H_v^k(S^{\pm})$ we now consider the case when the data are in the usual Sobolev spaces and not in the weighted ones. Suppose $f_{\pm}$ is in $W^{k,p}(S^{\pm})$ with $\operatorname{supp} f_{\pm} \subset B_R$, where $B_R$ is a ball of radius $R$ centered at the origin and $2 < p < \infty$, $1 \leq k < \infty$. By the Sobolev imbedding theorem, $\partial^j f_{\pm}/\partial r^j$ is continuous in $\operatorname{cl}(S^{\pm}) \cap \overline{B}_R$ for $j \leq k - 1$ and

$$\int_0^{\infty} r \left| \frac{\partial^j f_{\pm}}{\partial r^j} \right|^p dr < \infty .$$

Let

$$P_f^{\pm}(x) = \sum_{j \leq k-1} \frac{1}{j!} \frac{\partial^j f_{\pm}}{\partial r^j} \bigg|_{r=0}^{r^j} = \sum_{j=0}^{k-1} b_j^{\pm}(\theta) r^j . \tag{2.15}$$

The polynomial $P_f^{\pm}$ has a meaning. Set:

$$g_{\pm} = f_{\pm} - P_f^{\pm} . \tag{2.16}$$

Then as in [7], we have by an application of the Hardy inequality

$$\|g_{\pm}\|_{H_v^k(S^{\pm} \cap B_R)} < C \|f_{\pm}\|_{W^{k,p}(S^{\pm} \cap B_R)} .$$

For simplicity suppose that $b_1 = 0$ and that $b_2$ is in $W^{k+1/p,p}(\Gamma_{\pm})$ with $\operatorname{supp} b_2 \subset \subset \Gamma_{\pm} \cap B_R$. As above

$$b(r) = b_2(r) - \sum_{j \leq k-1} \frac{1}{j!} \frac{\partial^j f_{\pm}}{\partial r^j} \bigg|_{r=0}^{r^j}$$

is in $H_0^{k+1/2}(\Gamma_{\pm} \cap B_R)$. Also:

$$\|b\|_{H_0^{k+1/2}(\Gamma_{\pm} \cap B_R)} \leq C \|b_2\|_{W^{k+1/p,p}(\Gamma_{\pm} \cap B_R)} . \tag{2.17}$$

Instead of the problem (2.1), we now have:

$$\begin{cases} -\Delta u_{\pm} = g_{\pm} & \text{in } S^{\pm} \cap B_R ; \\ u_+ = 0 \quad \text{on } \Gamma_0 \cap B_R , & u_- = 0 \quad \text{on } \Gamma_3 \cap B_R , \\ u_+ = u_- \quad \text{and} \quad \dfrac{\partial}{\partial n}(u_+ - u_-) = b & \text{on } \Gamma_{\pm} \cap B_R . \end{cases} \tag{2.18}$$

with

$$(2.19)\quad \begin{cases} -\Delta w_\pm = \sum_{j=0}^{k-1} b_j^\pm(\theta) r^j \ \text{ in } S^\pm; \qquad w_+ = 0 \ \text{ on } \Gamma_0, \qquad w_- = 0 \ \text{ on } \Gamma_\beta, \\ w_+ = w_- \qquad \text{and} \qquad \dfrac{\partial}{\partial\theta}(w_+ - w_-) = \sum_{j \le k-1} \dfrac{1}{j!} \dfrac{\partial^j f_\pm}{\partial r^j}\Big|_{r=0} \ \text{ on } \Gamma_\alpha. \end{cases}$$

An application of Theorem 2.1 gives the following result.

LEMMA 2.6: *Let $\{f_\pm, b_2\}$ be in $W^{k,p}(S^\pm) \times W^{k+1/p,p}(\Gamma_\alpha)$ with $2 < p < \infty$ and $1 \le k < \infty$. Suppose that:*

(i) *$\operatorname{supp} f_\pm$, $\operatorname{supp} b_2$ are contained in $B_\delta \cap S^\pm$ and in $B_\delta \cap \Gamma_\alpha$ respectively,*

(ii) *$\alpha \ne n\pi/k$, $\beta \ne m\pi/k$ and $\alpha + \beta \ne -\beta k + j\pi$ where $m$, $n$, $j$ are non-negative integers.*

*Then there exists a unique $u_\pm$ in $H_0^{k+2}(S^\pm)$, solution of (2.18). Moreover:*

$$\|u_+\|_{H^{k+2}(S^+)} + \|u_-\|_{H^{k+2}(S^-)} \le C\{\|f_+\|_{W^{k,p}(S^+)} + \|f_-\|_{W^{k,p}(S^-)} + \|b_2\|_{W^{k+1/p,p}(\Gamma_\alpha)}\}.$$

PROOF: Let $\xi(x)$ be a $C_0^\infty(B_\delta)$ with $\xi(x) = 1$ on $\operatorname{supp} f_\pm \cup \operatorname{supp} b_2$. In (2.18), we consider $\xi g$, $\xi b$ instead of $g$ and of $b$.

With our hypotheses on $\alpha$, $\beta$ and on $k$, we have: $1 + k \ne \sigma$ where $\sigma$ is a root of the discriminant function of Lemma 2.1. Taking (2.16)-(2.17) into account and applying Theorem 2.1 we obtain the Lemma.

We now consider the problem (2.19). Set:

$$w_\pm = \sum_{j=0}^{k-1} a_j^\pm(\theta) r^{j+2}.$$

The equation (2.19) becomes:

$$\sum_{j=0}^{k-1} \left\{ \frac{d^2}{d\theta^2}(a_j^\pm) + (j+2)^2 a_j^\pm \right\} r^j = \sum_{j=0}^{k-1} b_j^\pm(\theta) r^j.$$

So:

$$\frac{d^2}{d\theta^2}(a_j^\pm) + (j+2)^2 a_j^\pm = b_j^\pm(\theta); \qquad 0 \le j \le k-1.$$

Thus,

$$a_j^\pm(\theta) = c_{\xi}^\pm \cos(j+2)\theta + c_{\xi}^\pm \sin(j+2)\theta - (j+2)^{-1} \int_0^\theta b_j^\pm(\eta) \sin(j+2)(\eta - \theta)\, d\eta.$$

The condition: $w_+(r, 0) = 0$ gives: $c_{1j}^+ = 0$. Similarly $w_-(r, \beta) = 0$ gives:

$$(2.20) \qquad 0 = c_{1j}^- \cos(j+2)\beta + c_{2j}^- \sin(j+2)\beta - \\ - (j+2)^{-1} \int_0^\beta b_j^-(\eta) \sin(j+2)(\eta - \beta)\, d\eta .$$

Since $w_+(r, \alpha) = w_-(r, \alpha)$, we have:

$$(2.21) \qquad c_{2j}^+ \sin(j+2)\alpha - c_{1j}^- \cos(j+2)\alpha - c_{2j}^- \sin(j+2)\alpha = \\ = (j+2)^{-1} \int_0^\alpha (b_j^+(\eta) - b_j^-(\eta)) \sin(j+2)(\eta - \alpha)\, d\eta .$$

The interface condition

$$\frac{d}{d\theta}(w_+ - w_-)\Big|_{\theta=\alpha} = \sum_{j \le k-1} \frac{\partial^j b_2}{\partial r^j}\Big|_{r=0}^{r^j}$$

gives:

$$(2.22) \qquad c_{2j}^+ \cos(j+2)\alpha + c_{1j}^- \sin(j+2)\alpha - c_{2j}^- \cos(j+2)\alpha = \\ = H_j + (j+2)^{-1} \int_0^\alpha \{b_j^-(\eta) - b_j^+(\eta)\} \cos(j+2)(\eta - \alpha)\, d\eta$$

where $H_j = (\partial^j b_2 / \partial r^j)|_{r=0}$.

From (2.20)-(2.22) we have a system of three equations with three unknowns $\{c_{2j}^+, c_{1j}^-, c_{2j}^-\}$. An easy computation shows that the system is uniquely solvable and thus $w_\pm$ is determined.

THEOREM 2.3: *Let $\{f_\pm, b_2\}$ and $\alpha, \beta$ be as in Lemma 2.6. Then there exists a solution $v_\pm$ of the problem* (2.1) *with $b_1 = 0$. Moreover:*

$$\|v_+\|_{H^{k+2}(U^+ \cap B_R)} + \|v_-\|_{H^{k+2}(U^- \cap B_R)} \le C\{\|f_+\|_{W^{k,p}(U^+)} + \|f_-\|_{W^{k,p}(U^-)} + \|b_2\|_{W^{k+1/p,p}(\Gamma_R)}\} .$$

PROOF: Let $u_\pm$ be as in Lemma 2.6 and let $w_\pm$ be as above. Then: $v_\pm = u_\pm - w_\pm$. The estimate is now trivial to establish.

## SECTION 3

In this section we shall study a transmission problem for second order linear elliptic equations in bounded planar regions. Let

$$(3.1) \qquad A^\pm u_\pm = \sum_{j,k=1}^{2} \frac{\partial}{\partial x_k}\left[a_{jk}^\pm(x) \frac{\partial}{\partial x_j} u_\pm\right] + a^\pm(x) u_\pm .$$

We assume that:

ASSUMPTION (I): (i) $a_{jk}^{\pm}(x)$, $a^{\pm}(x)$ *are real-valued* $C^{\infty}$*-functions in* $\mathrm{cl}(\Omega^{\pm})$.

(ii) $a^{\pm}(x) \geq c > 0$ *in* $\mathrm{cl}(\Omega^{\pm})$ *and* $a_{jk}^{\pm}(x) = a_{kj}^{\pm}(x)$.

(iii) $\sum_{j,k=1}^{2} a_{jk}^{\pm}(x)\xi_j\xi_k \geq c_1 |\xi|^2$ *for all* $\xi$ *in* $R^2$ *and all* $x$ *in* $\mathrm{cl}(\Omega^{\pm})$.

(iv) $a_{jk}^{+}(P) = a_{jk}^{-}(P)$, $a_{jk}^{+}(Q) = a_{jk}^{-}(Q)$.

REMARK 3.1: The hypothesis (iv) is only needed in order to simplify the algebraic calculations involved in the determination of the discriminant function associated with a transmission problem for (3.1).

Consider the transmission problem

$$(3.2)\qquad \begin{cases} A^{\pm}u_{\pm} = f^{\pm} & \text{in } \Omega^{\pm}, \qquad u_{\pm} = 0 \quad \text{on } \partial\Omega^{\pm}/\Gamma, \\ u_{+} = u_{-}, & B^{+}u_{+} - B^{-}u_{-} = \phi \quad \text{on } \Gamma, \end{cases}$$

where

$$B^{\pm}u_{\pm} = \sum_{j,k=1}^{2} a_{jk}^{\pm} \frac{\partial}{\partial x_j} u_{\pm} \cdot \cos(n, x_k).$$

The unit normal vector to $\Gamma$ oriented toward $\Omega^{-}$ is denoted by $n$ and the interior angles made by $\partial\Omega^{+}/\Gamma$, $\partial\Omega^{-}/\Gamma$ with $\Gamma$ at $P$ and at $Q$ are $(\alpha, \beta_1 - \alpha_1)$ and $(\alpha_2, \beta_2 - \alpha_2)$ respectively.

We shall first establish an a priori estimate for solutions of (3.2) and clearly the main difficulties are the estimates near $P$ and $Q$. Without loss of generality we may assume that $P$ is the origin and that $\partial\Omega^{+}/\Gamma$, $\partial\Omega^{-}/\Gamma$ and $\Gamma$ are given by:

$$x_2 = g_1(x_1), \qquad x_2 = g_3(x_1), \qquad x_1 = g_2(x_1),$$

with $g_i$ in $C^{\infty}$ and $g_1(0) = g_2(0) = g_3(0) = g_1'(0) = 0$, $g_2'(0) = \operatorname{cotan}\alpha$, $g_3'(0) = \tan\beta$.

Set:

$$(3.3)\qquad A_0 u_{\pm} = \sum_{j,k=1}^{2} a_{jk}(0) \frac{\partial^2}{\partial x_j \partial x_k} u_{\pm}$$

with $a_{jk}(0) = a_{jk}^{+}(0) = a_{jk}^{-}(0)$.

We transform $A_0$ into its normal form. It is known that for elliptic operators of second order in two independent variables such a transformation $\psi$ exists. Since $A_0$ has constant coefficients the transformation $\psi$ may be written explicitly:

$$\psi(x_1, x_2) = x_2 + mx_1 = U(x_1, x_2) + iV(x_1, x_2)$$

where $m = \{-a_{12}(0) + i[a_{11}(0)a_{22}(0) - (a_{12}(0))^2]^{1/2}\}/a_{11}(0)$. E.g. cf. [3] p. 88.

In terms of the new coordinates $\{y_1 = U(x_1, x_2), y_2 = V(x_1, x_2)\}$, the equation (3.3) becomes

$$b\,\Delta v_\pm = F^\pm \quad \text{in } N^\pm(R) = \bar{\Omega}^\pm \cap B(R). \tag{3.4}$$

The constant $b$ is given by the expression

$$b = a_{11}(0)\left(\frac{\partial U}{\partial x_1}\right)^2 + 2a_{12}(0)\frac{\partial U}{\partial x_1}\frac{\partial U}{\partial x_2} + a_{22}(0)\left(\frac{\partial U}{\partial x_2}\right)^2.$$

The transformation maps $\partial\Omega^\pm/\Gamma$ and $\Gamma$ into $\Gamma^\pm$ and $\Gamma^*$. The jacobian of the transformation is a non-zero constant. Let $\omega$ be the angle made by $\Gamma^+$ with $\Gamma^*$ then by setting $x_1 = g_2(x_2)$ we obtain by an elementary argument:

$$\tan\omega = \{a_{11}(0)a_{22}(0) - (a_{12}(0))^2\}^{1/2}\{a_{11}(0)\operatorname{cotan}\alpha + a_{12}(0)\}. \tag{3.5}$$

So if $A_0^+ = \Delta$, we have $\omega = \alpha$.

Let $\gamma$ be the angle made by $\Gamma^*$ with $\Gamma^-$. Setting $x_2 = g_3(x_1)$ and computing $\gamma$, we get:

$$\tan\gamma = \{a_{11}(0)a_{22}(0) - (a_{12}(0))^2\}^{1/2}\{a_{11}(0)\operatorname{cotan}\beta + a_{12}(0)\} \tag{3.6}$$

with $\gamma = \pi$ if $\beta = \pi$.

Again if $A_0^- = \Delta$, then $\gamma = \beta$.

Consider the transmission problem:

$$\begin{cases} A_0 u_\pm = F^\pm & \text{in } \Omega^\pm \cap B_R, \qquad \operatorname{supp} u_\pm \subset B_R, \\ u_\pm = 0 & \text{on } (\partial\Omega^\pm/\Gamma) \cap B_R, \\ u_+ = u_-, & B_0^+ u_+ - B_0^- u_- = \phi \quad \text{on } \Gamma \cap B_R. \end{cases} \tag{3.7}$$

In the new coordinates $\{y_1, y_2\}$ the problem (3.7) becomes:

$$\begin{cases} b\,\Delta v_\pm = F^\pm & \text{in } N^\pm(R), \qquad \operatorname{supp} v_\pm \subset N^\pm(R), \\ v_\pm = 0 & \text{on } \Gamma^\pm \cap B(R); \qquad v_+ - v_- = 0, \\ b(\nabla v_+ - \nabla v_-)\cdot n = \phi & \text{on } \Gamma^* \cap B(R). \end{cases} \tag{3.8}$$

It is known (e.g. cf. [16]) that there exists a conformal mapping taking $N^\pm(R)$ into $S^\pm \cap B_R$ where

$$S^+ = \{(r, \theta):\ 0 < r < \infty,\ 0 < \theta < \omega\},$$

$$S^- = \{(r, \theta):\ 0 < r < \infty,\ \omega < \theta < \gamma\}.$$

The problem (3.8) becomes:

$$(3.9)\qquad \begin{cases} \Delta w_\pm = f^\pm & \text{in } S^\pm \cap B_R, \qquad \operatorname{supp} w_\pm \subset B_R, \\ w_+ = 0 & \text{on } \Gamma_0 \cap B_R, \qquad w_- = 0 \quad \text{on } \Gamma_\gamma \cap B_R, \\ w_+ = w_-, & \dfrac{\partial}{\partial\theta}(w_+ - w_-) = \phi/b \quad \text{on } \Gamma_\omega \cap B_R. \end{cases}$$

$\Gamma_0$, $\Gamma_\omega$ and $\Gamma_\gamma$ are the rays $\theta = 0$, $\theta = \omega$ and $\theta = \gamma$ respectively.

Remark 3.2: The condition (iv) of Assumption (I) is needed so that in (3.4) we have $b = b^+ = b^-$. It then allows us to get in (3.8), the simple condition $\nabla(w_+ - w_-)\cdot n = \phi/b$ on $\Gamma^\omega \cap B(R)$. If the condition (iv) is replaced by the weaker one:

$$(\text{iv})' \qquad a_{12}^+(0)/a_{11}^+(0) = a_{12}^-(0)/a_{11}^-(0); \qquad a_{22}^+(0)/a_{11}^+(0) = a_{22}^-(0)/a_{11}^-(0)$$

then we still have $m = m_+ = m_-$ in the definition of the transformation $\phi$ but $b_+ \neq b_-$. Thus in (3.9) we are led to the boundary condition $k^+(\partial/\partial\theta)w_+ - k^-(\partial/\partial\theta)w_- = b$ on $\Gamma_\omega \cap B_R$. The computation of the discriminant function in Section 2 becomes too complex although there is no technical difficulty. We assume (iv) only in order to make the paper more readable.

Lemma 3.1: *Let $A^\pm$ be two linear elliptic operators on $\Omega^\pm$ satisfying Assumption* (I). *Let $\omega, \gamma$ be as in* (3.5)-(3.6) *and let $k$ be a non-negative integer with $0 \leq s \leq 1/2$. Suppose that $1 + k - s \neq \sigma$ where $\sigma = 0, 1, n\pi/\omega, m\pi/\gamma, -\omega/\gamma + p\pi/\gamma$ with $n, m, p$ non-negative integers. Then there exists a constant $C$ independent of $\delta$ such that:*

$$\|u_+\|_{H_s^{k+2}(\Omega^+ \cap B_\delta)} + \|u_-\|_{H_s^{k+2}(\Omega^- \cap B_\delta)} \leq$$

$$\leq C\{\|A^+ u_+\|_{H_s^k(\Omega^+ \cap B_\delta)} + \|A^- u_-\|_{H_s^k(\Omega^- \cap B_\delta)} + \|B^+ u_+ - B^- u_-\|_{H_s^{k+1/2}(\Gamma \cap B_\delta)}\}$$

*for all $u_\pm$ in $H_s^{k+2}(\Omega^\pm \cap B_\delta)$, $\operatorname{supp} u_\pm \subset B_\delta$ and $u_\pm = 0$ on $(\partial\Omega^\pm/\Gamma) \cap B_\delta$ with $u_+ = u_-$ on $\Gamma \cap B_\delta$.*

*$B_\delta$ is the ball centred at $P$ with radius $\delta$.*

Proof: Set

$$A_0 u_\pm = -\sum_{j,k=1}^{2} a_{jk}(0)\frac{\partial^2}{\partial x_j \partial x_k} u_\pm .$$

As above we first transform $A_0$ into its normal form and then use a conformal mapping to reduce to the problem (3.9). The transformations are all of class $C^\infty$ and are 1-1. The function $u_\pm$ becomes $w_\pm$ and from Theorem 2.1 we get:

$$\|w_+\|_{H_s^{k+2}(G^+ \cap B_\delta)} + \|w_-\|_{H_s^{k+2}(G^- \cap B_\delta)} \leq$$

$$\leq C\{\|f^+ u_+\|_{H_s^k(G^+ \cap B_\delta)} + \|f^-\|_{H_s^k(G^- \cap B_\delta)} + \|\phi\|_{H_s^{k+1/2}(\Gamma_\omega \cap B_\delta)}\}.$$

Returning to the original variables we obtain

$$\|u_+\|_{H^{s+2}(\Omega^+\cap B_\delta)} + \|u_-\|_{H^{s+2}(\Omega^-\cap B_\delta)} \leq$$

$$\leq C\{\|A_0 u_+\|_{H^s(\Omega^+\cap B_\delta)} + \|A_0 u_-\|_{H^{s+2}(\Omega^-\cap B_\delta)} + \|B_0^+ u_+ - B_0^- u_-\|_{H^{s+1/2}(\Gamma\cap B_\delta)}\}.$$

Since the coefficients of $A^\pm$, $B^\pm$ are in $C^\infty$, we have by a standard argument (cf. [7]):

$$(1-\varepsilon c_1)\|u_+\|_{H^{s+2}(\Omega^+\cap B_\delta)} + (1-\varepsilon c_2)\|u_-\|_{H^{s+2}(\Omega^-\cap B_\delta)} \leq$$

$$\leq C\{\|A^+ u_+\|_{H^s(\Omega^+\cap B_\delta)} + \|A^- u_-\|_{H^s(\Omega^-\cap B_\delta)} + \|B^+ u_+ - B^- u_-\|_{H^{s+1/2}(\Gamma\cap B_\delta)}\},$$

for $0 < \delta \leq \delta_0(\varepsilon)$.

Take $\varepsilon > 0$ small and the lemma is proved.

THEOREM 3.1: *Let $A^\pm$ be two linear elliptic operators satisfying Assumption* (I). *Let $k$ be a non-negative integer such that $1 + k - s \neq \sigma_j$ where $\sigma_j = 0, 1, n\pi/\omega_j, m\pi/\gamma_j, -\omega_j/\gamma_j + p\pi/\gamma_j$ with $\omega_1 = \omega(P)$, $\omega_2 = \omega(Q)$, $\gamma_1 = \gamma(P)$, $\gamma_2 = \gamma(Q)$ and $\omega, \gamma$ defined by* (3.5)-(3.6). *Suppose that $0 \leq s \leq 1/2$. Then there exists $C$ such that:*

$$\|u_+\|_{H^{s+2}(\Omega^+)} + \|u_-\|_{H^{s+2}(\Omega^-)} \leq$$

$$\leq C\{\|A^+ u_+\|_{H^s(\Omega^+)} + \|A^- u_-\|_{H^s(\Omega^-)} + \|B^+ u_+ - B^- u_-\|_{H^{s+1/2}(\Gamma)}\}$$

*for all $u_\pm$ in $H_c^{s+2}(\Omega^\pm)$, $u_\pm = 0$ on $\partial\Omega^\pm/\Gamma$ and $u_+ = u_-$ on $\Gamma$.*

PROOF: Let $\{\phi_j\}$ be a finite partition of unity corresponding to a covering of $\Omega^+ \cup \cup \Omega^-$ by balls of radius $\delta$. Denote by $N(P)$ a neighbourhood of $P$.

1) For all $\phi_j$ with $\operatorname{supp}\phi_j \cap \{\Gamma \cup N(P) \cup N(Q)\} = \emptyset$, we have the known estimate:

$$\|\phi_j u_\pm\|_{H^{s+2}(\Omega^\pm)} \leq$$

$$\leq C\{\|A^+(\phi_j u_+)\|_{H^s(\Omega^+)} + \|A^-(\phi_j u_-)\|_{H^s(\Omega^-)} + \|\phi_j u_+\|_{H^s(\Omega^+)} + \|\phi_j u_-\|_{H^s(\Omega^-)}\}.$$

$C$ is independent of $j$.

2) For all $\phi_j$ with $\operatorname{supp}\phi_j \cap \{N(P) \cup N(Q)\} \neq \emptyset$, then Lemma 3.1 gives:

$$\|\phi_j u_\pm\|_{H^{s+2}(\Omega^\pm)} + \|\phi_j u_-\|_{H^{s+2}(\Omega^-)} \leq$$

$$\leq C\{\|A^+(\phi_j u_+)\|_{H^s(\Omega^+)} + \|A^-(\phi_j u_-)\|_{H^s(\Omega^-)} + \|B^+(\phi_j u_+) - B^-(\phi_j u_-)\|_{H^{s+1/2}(\Gamma)}\}.$$

3) We now consider the case when $\operatorname{supp}\phi_j \cap \Gamma \neq \emptyset$ and $\operatorname{supp}\phi_j \cap \{\partial\Omega^\pm/\Gamma \cup \cup N(P) \cup N(Q)\} = \emptyset$, set:

$$A = \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix}, \quad M = \begin{pmatrix} B^+ & -B^- \\ I & -I \end{pmatrix}, \quad u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}.$$

We have the elliptic systems:

$$A(\phi_j u) = F_j, \quad M(\phi_j u) = G_j \quad \text{on } \Gamma.$$

Thus by the usual theory of elliptic systems we get:

$$\|\phi_j u_+\|_{C^{k+2}(\Omega^+)} + \|\phi_j u_-\|_{C^{k+2}(\Omega^-)} \leq$$

$$\leq C\{\|A^+(\phi_j u_+)\|_{C^k(\Omega^+)} + \|A^-(\phi_j u_-)\|_{C^k(\Omega^-)} + \|B^+(\phi_j u_+) - B^-(\phi_j u_-)\|_{C^{k+1}(\Gamma)} +$$

$$+ \|\phi_j u_+\|_{C^k(\Omega^+)} + \|\phi_j u_-\|_{C^k(\Omega^-)}\}.$$

Combining the estimates and taking the summation with respect to $j$, we obtain by a standard argument:

$$(3.10) \qquad \|u_+\|_{C^{k+2}(\Omega^+)} + \|u_-\|_{C^{k+2}(\Omega^-)} \leq$$

$$\leq C\{\|A^+ u_+\|_{C^k(\Omega^+)} + \|A^- u_-\|_{C^k(\Omega^-)} + \|B^+ u_+ - B^- u_-\|_{C^{k+1}(\Gamma)} +$$

$$+ \|u_+\|_{C^k(\Omega^+)} + \|u_-\|_{C^k(\Omega^-)}\},$$

for all $u_\pm$ as stated in the theorem.

4) We now show that

$$\|u_+\|_{C^k(\Omega^+)} + \|u_-\|_{C^k(\Omega^-)} \leq$$

$$\leq C\{\|A^+ u_+\|_{C^k(\Omega^+)} + \|A^- u_-\|_{C^k(\Omega^-)} + \|B^+ u_+ - B^- u_-\|_{C^{k+1}(\Gamma)}\}.$$

Set: $\bar{u} = u_+$ on $\Omega^+ \cup \Gamma$, $\bar{u} = u_-$ on $\Omega^-$. Since $u_\pm$ is in $H_s^2(\Omega^\pm)$ and $u_\pm = 0$ on $\partial\Omega^\pm/\Gamma$ with $u_+ = u_-$ on $\Gamma$, we have $\bar{u}$ in $W_0^{1,2}(\Omega)$. Let $\rho(x)$ be the distance from a point $x$ of $\Omega$ to $\Sigma = \{P, Q\}$. It follows from Hardy's inequality that $\rho^{-1}\bar{u}_-$ is in $L^2(\Omega)$ and:

$$\|\rho^{-1}u_+\|_{L^2(\Omega^+)} + \|\rho^{-1}u_-\|_{L^2(\Omega^-)} \leq C\|\bar{u}\|_{W_0^{1,2}(\Omega)} \leq C_1\{\|u_+\|_{W^{1,2}(\Omega^+)} + \|u_-\|_{W^{1,2}(\Omega^-)}\}.$$

Let $\psi$ be an arbitrary element of $W_0^{1,2}(\Omega)$ and $u_\pm$ be as in the theorem. An integration by parts gives:

$$\sum_{j,k=1}^{2}\left\{\int_{\Omega^+} a_{jk}^+ \frac{\partial}{\partial x_j} u_+ \frac{\partial}{\partial x_k}\psi\,dx + \int_{\Omega^-} a_{jk}^- \frac{\partial}{\partial x_j} u_- \frac{\partial}{\partial x_k}\psi\,dx\right\} +$$

$$+ \int_{\Omega^+} a^+ u_+ \psi\,dx + \int_{\Omega^-} a^- u_- \psi\,dx + \int_{\Gamma} (B^- u_- - B^+ u_+)\psi\,d\sigma =$$

$$+ \int_{\Omega^+} A^+ u_+ \psi\,dx + \int_{\Omega^-} A^- u_- \psi\,dx.$$

From the above argument we may take $\psi = \bar{u}$. Then:

$$C_1 \{\|u_+\|^2_{H^{s+1}(\Omega^+)} + \|u_-\|^2_{H^{s+1}(\Omega^-)}\} \leq c \int_{\Omega^+} \rho^s |A^+ u_+| \rho^{-1} |u_+| dx +$$

$$+ c \int_{\Omega^-} \rho^s |A^- u_-| \rho^{-1} |u_-| dx + \int_{\Gamma} |B^+ u_+ - B^- u_-| \rho^{s-1/2} |u_+| \rho^{1/2-s} d\sigma.$$

Since we assume that $0 \leq s \leq 1/2$, we obtain:

$$\|u_+\|^2_{H^{s+1}(\Omega^+)} + \|u_-\|^2_{H^{s+1}(\Omega^-)} \leq$$

$$\leq c\{\|A^+ u_+\|^2_{L^2(\Omega^+)} + \|A^- u_-\|^2_{L^2(\Omega^-)} + \|B^+ u_+ - B^- u_-\|^2_{L^2(\Omega)} + \varepsilon \|u_+\|^2_{H^{s+1}(\Omega^+)}\}.$$

We have used the Hardy inequality in the above estimate.
So:

$$(3.11) \qquad \|u_+\|^2_{H^{s+1}(\Omega^+)} + \|u_-\|^2_{H^{s+1}(\Omega^-)} \leq$$

$$\leq C_1 \{\|A^+ u_+\|^2_{L^2(\Omega^+)} + \|A^- u_-\|^2_{L^2(\Omega^-)} + \|B^+ u_+ - B^- u_-\|^2_{L^2(\Omega)}\}.$$

The estimate of the theorem follows from (3.10)-(3.11).

THEOREM 3.2: *Suppose all the hypotheses of Theorem 3.1 are satisfied. Let* $\{f_\pm, \phi\}$ *be in* $H_s^k(\Omega^\pm) \times H_s^{k+1/2}(\Gamma)$, *then there exists a unique solution* $u_\pm$ *of* (3.2) *in* $H_s^{k+2}(\Omega^\pm)$. *Moreover:*

$$\|u_+\|_{H^{k+2}(\Omega^+)} + \|u_-\|_{H^{k+2}(\Omega^-)} \leq c\{\|f_+\|_{H^k(\Omega^+)} + \|f_-\|_{H^k(\Omega^-)} + \|\phi\|_{H^{k+1/2}(\Gamma)}\}.$$

PROOF: 1) Let $X$ be the Hilbert space

$$X = \{u: u = (u_+, u_-),\ u_\pm \text{ in } H_s^{k+2}(\Omega^\pm),\ u_\pm = 0 \text{ on } \partial\Omega^\pm / \Gamma,\ u_+ = u_- \text{ on } \Gamma\}$$

with the obvious norm. Set:

$$Y = H_s^k(\Omega^+) \times H_s^k(\Omega^-) \times H_s^{k+1/2}(\Gamma).$$

We define the operator $\mathfrak{A}$ from $X$ into $Y$ by

$$\mathfrak{A}u = \{A^+ u_+|_{\Omega^+},\ A^- u_-|_{\Omega^-},\ (B^+ u_+ - B^- u_-)|_\Gamma\}.$$

From the estimates of Theorem 3.1 we deduce that $\mathfrak{A}$ is 1-1 and the range $R(\mathfrak{A})$ is closed in $Y$.

Suppose $R(\mathfrak{A})$ is strictly contained in $Y$. Then there exists $\{g_+, g_-, g\}$ in $Y$ with $\{g_+, g_-, g\}$ not in $R(\mathfrak{A})$.

By the Hahn-Banach theorem there exists $\{b_+, b_-, b\}$ in $Y$ such that:

$$(3.12) \qquad (b_+, g_+)_{H^k(\Omega^+)} + (b_-, g_-)_{H^k(\Omega^-)} + (b, g)_{H^{k+1/2}(\Gamma)} \neq 0$$

with

$$(3.13)\qquad (A^+u_+, b_+)_{H^0(\Omega^+)} + (A^-u_-, b_-)_{H^0(\Omega^-)} + (B^+u_+ - B^-u_-, b)_{H^{1/2}(\Gamma)} = 0$$

for all $u = (u_+, u_-)$ in $X$.

2) Let

$$Z = \{v: v = (v_+, v_-),\ v_\pm \text{ in } W^{1,2}(\Omega^\pm),\ v_\pm = 0 \text{ on } \partial\Omega^\pm/\Gamma,$$

$$v_+ = v_- \text{ on } \Gamma,\ A^\pm v_\pm \text{ in } H_t^k(\Omega^\pm),\ B^+v_+ - B^-v_- \text{ in } H_t^{k+1/2}(\Gamma)\}.$$

By regularity theorem (see Appendix)

$$(3.14)\qquad (A^+u_+, b_+)_{H^0(\Omega^+)} + (A^-u_-, b_-)_{H^0(\Omega^-)} + (B^+u_+ - B^-u_-, b)_{H^{1/2}(\Gamma)} = 0$$

for all $u = (u_+, u_-)$ in $Z$.

3) Let $S$ be the Hilbert space

$$S = \{u: u = (u_+, u_-),\ u_\pm \text{ in } W^{1,2}(\Omega^\pm),\ u_\pm = 0 \text{ on } \partial\Omega^\pm/\Gamma,\ u_+ = u_- \text{ on } \Gamma\}$$

with the usual norm. With $v$ in $S$, set $\bar{v} = v_+$ on $\Omega^+ \cup \Gamma$ and $\bar{v} = v_-$ on $\Omega^-$. Then $\bar{v}$ is in $W_0^{1,2}(\Omega)$ and the Hardy inequality give:

$$\|\rho^{-1}v_\pm\|_{L^2(\Omega^\pm)} \leq c\|\rho^{-1}\bar{v}\|_{L^2(\Omega)} \leq c\|\bar{v}\|_{W_0^{1,2}(\Omega)}.$$

Thus, the linear form:

$$L(b_\pm, b; v) = \int_{\Omega^+} b_+ v_+\, dx + \int_{\Omega^-} b_- v_-\, dx + \int_\Gamma b \cdot v_+\, d\sigma$$

is well-defined for $\{b_+, b_-, b\}$ in $Y$ and $v$ in $S$. We have: $L(b_\pm, b; v) = \langle G, v\rangle$ for all $v$ in $S$. The pairing between $S$ and its dual $S^*$ is denoted by $\langle\cdot,\cdot\rangle$.

Let $u$, $v$ be in $S$ and consider the bilinear form

$$\mathcal{L}(u, v) = \sum_{j,k=1}^{2}\left\{\int_{\Omega^+} a_{jk}^+ \frac{\partial}{\partial x_j}u_+ \frac{\partial}{\partial x_k}v_+\, dx + \int_{\Omega^-} a_{jk}^- \frac{\partial}{\partial x_j}u_- \frac{\partial}{\partial x_k}v_-\, dx\right\} +$$

$$+ \int_{\Omega^+} a^+ u_+ \cdot v_+\, dx + \int_{\Omega^-} a^- u_- \cdot v_-\, dx.$$

Then: $|\mathcal{L}(u, v)| \leq M\|u\|_S\|v\|_S$ for all $u$, $v$ in $S$ and thus, $\mathcal{L}(u, v) = \langle Au, v\rangle$.

With our assumptions on $A^\pm$, we get:

$$c\|u\|_S^2 \leq \mathcal{L}(u, u) = \langle Au, u\rangle = \langle u, Au\rangle.$$

By a well-known argument we have a unique $u$ in $S$ such that:

$$\mathcal{L}(u, v) = L(b_+, b_-, b; v) \quad \text{for all } v \text{ in } S.$$

Taking $v = (\phi_+, 0)$ with $\phi_+$ in $C_0^\infty(\Omega^+)$, we obtain:

$$A^+ u_+ = b_+ .$$

Similarly: $A^- u_- = b_-$.

Since $u_\pm$ is in $W^{1,2}(\Omega^\pm)$ and $A^\pm u_\pm$ is in $H_0^k(\Omega^\pm)$, $B^\pm u_\pm$ is in $W^{-1/2,2}(\Gamma)$. Let $\phi$ be in $C_0^\infty(\Omega)$, then:

$$\int_{\Omega^+} A^+ u_+ \cdot \phi \, dx + \int_{\Omega^-} A^- u_- \cdot \phi \, dx = ((B^- u_- - B^+ u_+, \phi)) + \mathcal{E}(u; \phi).$$

The pairing between $W^{1/2,2}(\Gamma)$ and its dual $W^{-1/2,2}(\Gamma)$ is denoted by $((\cdot,\cdot))$. It then follows that:

$$((B^- u_- - B^+ u_+ + b, \phi)) = 0$$

for all $\phi$ in $C_0^\infty(\Omega)$. Therefore: $B^+ u_+ - B^- u_- = b$ on $\Gamma$.

Replacing $\{b_+, b_-, b\}$ by $\{A^+ u_+, A^- u_-, B^+ u_+ - B^- u_-\}$ in (3.14) and we deduce that

$$A^+ u_+ = b_+ = 0, \quad A^- u_- = b_- = 0, \quad B^+ u_+ - B^- u_- = b = 0.$$

Comparing with (3.12) and we have a contradiction. Therefore $R(\mathcal{A}) = Y$ and the theorem is proved.

Corollary 3.1: *Suppose all the hypotheses of Theorems 3.1-3.2 are satisfied. Suppose further that:*

1) $0 < \omega_j < \pi/(1+k)$; $j = 1, 2$ *with* $k = 1, 2, \ldots$,

2) $\pi/(1+k) < \gamma_j < \pi$; $\gamma_j \neq m\pi/(1+k)$, $m = 2, 3, \ldots$.

*Then for* $\{f^\pm, \phi\}$ *in* $H_0^k(\Omega^\pm) \times H_0^{k+1/2}(\Gamma)$, *the unique solution* $u_\pm$ *of* (3.2) *given by Theorem 3.2 is also in* $W^{k+2,2}(\Omega^\pm)$.

Proof: One of the hypotheses of Theorem 3.1 is that

$$1 + k - s \neq \sigma_j \quad \text{where} \quad \sigma_j = 0, \quad 1, \quad n\pi/\omega_j, \quad m\pi/\gamma_j, \quad -\omega_j/\gamma_j + p\pi/\gamma_j.$$

Taking $s = 0$, we can take $\omega_j < \pi/(1+k)$ and $\pi/(1+k) < \gamma_j < \pi$, $\gamma_j \neq m\pi/(1+k)$ with $k = 1, 2, 3, \ldots$ and $m = 2, 3, \ldots$. Then $u_\pm$ is in $H_0^{k+2}(\Omega^\pm)$ and from the definition of the space together with the boundedness of $\Omega^\pm$, we get $u_\pm$ in $W^{k+2,2}(\Omega^\pm)$.

It is clear that when $\Omega$ has a smooth boundary, $\gamma_1 = \gamma_2 = \pi$ and thus $s \neq 0$. Consequently $u_\pm$ will only be in $W^{k+1,2}(\Omega^\pm)$.

## Section 4

In this section we shall study a transmission problem for linear parabolic equations

$$
(4.1)\qquad \begin{cases} \dfrac{\partial}{\partial t} u_\pm + A^\pm u_\pm = f_\pm \ \text{ in } \Omega^\pm \times (0,T); \qquad u_\pm = 0 \ \text{ on } (\partial\Omega^\pm / \Gamma) \times (0,T), \\ u_+ = u_-, \qquad B^+ u_+ - B^- u_- = \phi(x) \ \text{ on } \Gamma \times (0,T), \\ u_\pm(x,0) = u_0^\pm(x) \ \text{ in } \Omega^\pm. \end{cases}
$$

Set $b = T/N$ where $N$ is a large positive integer and denote by:

$$
f_\pm^k(x) = b^{-1} \int_{kb}^{(k+1)b} f_\pm(x,t)\,dt, \qquad 0 \le k \le N-1.
$$

We consider the discrete version of (4.1):

$$
(4.2)\qquad \begin{cases} u_\pm^k - u_\pm^{k-1} + bA^\pm u_\pm^k = bf_\pm^k \quad \text{in } \Omega^\pm; \qquad u_\pm^k = 0 \quad \text{on } \partial\Omega^\pm / \Gamma, \\ u_+^k = u_-^k, \qquad B^+ u_+^k - B^- u_-^k = \phi(x) \quad \text{on } \Gamma, \\ u_\pm^0(x) = u_0^\pm(x) \quad \text{in } \Omega, \ 1 \le k \le N-1. \end{cases}
$$

Proposition 4.1: *Let $u_\pm, v_\pm$ be in $W^{1,2}(\Omega^\pm)$ and let $A^\pm$ be as in Assumption (I). Then:*

$$
|a_\pm(u_\pm, v_\pm)| \le \{a_\pm(u_\pm, u_\pm) a_\pm(v_\pm, v_\pm)\}^{1/2}
$$

*where*

$$
a_\pm(u_\pm, v_\pm) = \int_{\Omega^\pm} a^\pm(x) u_\pm v_\pm\,dx + \sum_{j,k=1}^{2} \int_{\Omega^\pm} a_{jk}^\pm \frac{\partial}{\partial x_j} u_\pm \frac{\partial}{\partial x_k} v_\pm\,dx.
$$

Proof: It is an immediate consequence of Theorem 29, p. 33 in [6].

Lemma 4.1: *Suppose all the hypotheses of Theorem 3.1 are satisfied. Let $\{f_\pm^k, \phi(x), u_0^\pm\}$ be in $L^2(\Omega^\pm) \times H_s^{1/2}(\Gamma) \times W^{1,2}(\Omega^\pm)$. Suppose further that $u_0^\pm = 0$ on $\partial\Omega^\pm / \Gamma$, $u_0^+ = u_0^-$ on $\Gamma$. Then for each $k$ there exists a solution $u_\pm^k$ of (4.2) in $H_s^2(\Omega^\pm)$. Moreover:*

$$
\|u_\pm^k\|_{W^{1,2}(\Omega^\pm)} + \sum_{j=1}^{k} b \|A^\pm u_\pm^j\|_{L^2(\Omega^\pm)} \le
$$

$$
\le C \left\{ \|u_0^+\|_{W^{1,2}(\Omega^+)} + \|u_0^-\|_{W^{1,2}(\Omega^-)} + \|\phi\|_{C^{1+1/2}(\Gamma)} + b \sum_{j=1}^{k} (\|f_+^j\|_{L^2(\Omega^+)} + \|f_-^j\|_{L^2(\Omega^-)}) \right\}.
$$

*$C$ is independent of $b$, $k$.*

PROOF: The existence of a unique solution $u_\pm^k$ of (4.2) in $H_\nu^2(\Omega^\pm)$ with $A^\pm u_\pm^k$ in $L^2(\Omega^\pm)$ follows from Theorem 3.2 since $f_\pm$ is in $L^2(\Omega^\pm)$ and hence in $H_\nu^0(\Omega^\pm)$. We now establish the estimates of the lemma.

We multiply (4.2) by $A^\pm u_\pm^k$ and integrate by parts. With $a_\pm(v, w)$ as in Proposition 4.1, we get:

$$a_+(u_+^k, u_+^k) + b\|A^+ u_+^k\|^2_{L^2(\Omega^+)} \leq b\|f_+^k\|_{L^2(\Omega^+)}\|A^+ u_+^k\|_{L^2(\Omega^+)} +$$

$$+ a_+(u_+^k, u_+^{k-1}) + \int_\Gamma B^+ u_+^k (u_+^k - u_+^{k-1}) d\sigma.$$

It follows from Proposition 4.1 that:

$$(4.3) \quad a_+(u_+^k, u_+^k) + b\|A^+ u_+^k\|^2_{L^2(\Omega^+)} \leq$$

$$\leq b\|f_+^k\|_{L^2(\Omega^+)} + a_+(u_+^{k-1}, u_+^{k-1}) + 2\int_\Gamma B^+ u_+^k (u_+^k - u_+^{k-1}) d\sigma.$$

Similarly

$$(4.4) \quad a_-(u_-^k, u_-^k) + b\|A^- u_-^k\|^2_{L^2(\Omega^-)} \leq$$

$$\leq b\|f_-^k\|_{L^2(\Omega^-)} + a_-(u_-^{k-1}, u_-^{k-1}) - 2\int_\Gamma B^- u_-^k (u_-^k - u_-^{k-1}) d\sigma.$$

The minus sign on the right hand side of (4.4) is due to the orientation of the normal to $\Gamma$. Since $u_+^k = u_-^k$ on $\Gamma$, we obtain by adding (4.3), (4.4):

$$a_+(u_+^k, u_+^k) + a_-(u_-^k, u_-^k) + b\|A^+ u_+^k\|^2_{L^2(\Omega^+)} + b\|A^- u_-^k\|^2_{L^2(\Omega^-)} \leq$$

$$\leq b(\|f_+^k\|^2_{L^2(\Omega^+)} + \|f_-^k\|^2_{L^2(\Omega^-)}) + a_+(u_+^{k-1}, u_+^{k-1}) +$$

$$+ a_-(u_-^{k-1}, u_-^{k-1}) + 2\int_\Gamma \phi(u_+^k - u_-^{k-1}) d\sigma.$$

Adding from $k = 1$ to $j$ and we have:

$$(4.5) \quad a_+(u_+^j, u_+^j) + a_-(u_-^j, u_-^j) + b\sum_{k=1}^{j} (\|A^+ u_+^k\|^2_{L^2(\Omega^+)} + \|A^- u_-^k\|^2_{L^2(\Omega^-)}) \leq$$

$$\leq a_+(u_+^0, u_+^0) + a_-(u_-^0, u_-^0) + b\sum_{k=1}^{j} (\|f_+^k\|^2_{L^2(\Omega^+)} + \|f_-^k\|^2_{L^2(\Omega^-)}) + 2\int_\Gamma \phi(u_+^j - u_-^0) d\sigma.$$

Since $0 < s \leq 1/2$, it is clear that:

$$\left| \int_\Gamma \rho^{s-1/2} \phi \cdot \rho^{1/2-s} u_+^j \, d\sigma \right| \leq \|\phi\|_{L^\infty(\Gamma)} \|u_+^j\|_{L^2(\Gamma)} \leq C\|\phi\|_{L^\infty(\Gamma)} \|u_+^j\|_{H^{1/2}(\Omega^+)}.$$

The estimate of the lemma then is an immediate consequence of (4.5) and of Assumption (I).

LEMMA 4.2: *Suppose all the hypotheses of Lemma 4.1 are satisfied and that $f_\pm$ is in $L^2(0, T; L^2(\Omega^\pm))$. Then:*

$$\sum_{j=1}^{k} b \|(u_\pm^j - u_\pm^{j-1})/b\|^2_{L^2(\Omega_\pm)} \leq C.$$

*$C$ is independent of $k$, $b$.*

PROOF: Let $u_\pm^k$ be as in Lemma 4.1, then:

$$\|(u_\pm^k - u_\pm^{k-1})/b\|^2_{L^2(\Omega^\pm)} \leq 2(\|A^\pm u_\pm^k\|^2_{L^2(\Omega^\pm)} + \|f_\pm^k\|^2_{L^2(\Omega^\pm)}).$$

The lemma is then a consequence of the estimate of Lemma 4.1.

THEOREM 4.1: *Suppose all the hypotheses of Theorems 3.1-3.2 are satisfied. Let $\{f_\pm, \phi, u_\pm^0\}$ be in $L^2(0, T; L^2(\Omega^\pm)) \times H_t^{1/2}(\Gamma) \times W^{1,2}(\Omega^\pm)$ with $0 \leq s \leq 1/2$ and $u_\pm^0 = 0$ on $\partial\Omega^\pm/\Gamma$, $u_+^0 = u_-^0$ on $\Gamma$. Then there exists a unique solution $u_\pm$ of (4.1) in $L^2(0, T; H_t^2(\Omega^\pm))$. Moreover:*

$$\left\| \frac{\partial}{\partial t} u_\pm \right\|_{L^2(0, T; L^2(\Omega^\pm))} + \|u_\pm\|_{L^\infty(0, T; W^{1,2}(\Omega^\pm))} + \|u_\pm\|_{L^2(0, T; H_t^2(\Omega^\pm))} \leq$$

$$\leq C\{\|u_+^0\|_{W^{1,2}(\Omega^+)} + \|u_-^0\|_{W^{1,2}(\Omega^-)} + \|\phi\|_{H_t^{1/2}(\Gamma)} + \|f_+\|_{L^2(0, T; L^2(\Omega^+))} + \|f_-\|_{L^2(0, T; L^2(\Omega^-))}\}.$$

PROOF: 1) Let $u_\pm^k$ be as in Lemma 4.1 and set:

$$u_b^\pm(x, t) = u_\pm^k(x) \quad \text{for } kb < t \leq (k+1)b.$$

From the estimate of Lemma 4.1, we have:

$$\|u_b^\pm\|_{L^\infty(0, T; W^{1,2}(\Omega^\pm))} + \|A^\pm u_b^\pm\|_{L^2(0, T; L^2(\Omega^\pm))} \leq C.$$

By taking subsequences, we get: $u_b^\pm \to u_\pm$ in the weak*-topology of $L^\infty(0, T; W^{1,2}(\Omega^\pm))$, $A^\pm u_b^\pm \to A^\pm u_\pm$ weakly in $L^2(0, T; L^2(\Omega^\pm))$ as $b \to 0^+$.

Let $\bar{u}_b^\pm$ be a piecewise linear function; continuous in $[0, T]$ and such that $\bar{u}_b^\pm(x, jk) = u_\pm^{j-1}$, $1 \leq j \leq N-1$; $\bar{u}_b^\pm(x, 0) = u_\pm^0$.

From the estimate of Lemma 4.2, we have:

$$\frac{d}{dt}\bar{u}_b^\pm \to \frac{\partial u_\pm}{\partial t} \quad \text{weakly in } L^2(0, T; L^2(\Omega^\pm)) \text{ as } b \to 0^+.$$

A standard argument gives

$$(4.6)\qquad \begin{cases} \dfrac{\partial}{\partial t}u_\pm + A^\pm u_\pm = f_\pm \ \text{ in } \Omega^\pm \times (0,T); \qquad u_\pm = 0 \ \text{ on } (\partial\Omega^\pm/\Gamma)\times(0,T), \\ u_+ = u_-, \qquad B^+u_+ - B^-u_- = \phi(x) \ \text{ on } \Gamma\times(0,T), \\ u_\pm(x,0) = u_\pm^0(x) \ \text{ in } \Omega^\pm. \end{cases}$$

2) The solution obtained is unique. Indeed suppose that $v_\pm$ is another solution of (4.1) with the properties stated in the theorem. Then:

$$\frac{\partial}{\partial t}w_\pm + A^\pm w_\pm = 0 \ \text{ in } \Omega^\pm\times(0,T); \qquad w_\pm = 0 \ \text{ in } (\partial\Omega^\pm/\Gamma)\times(0,T),$$

$$w_+ = w_-, \qquad B^+w_+ - B^-w_- = 0 \ \text{ on } \Gamma\times(0,T),$$

$$w_\pm(x,0) = 0 \ \text{ in } \Omega^\pm \ \text{ with } w_\pm = u_\pm - v_\pm.$$

Multiplying the equation by $w_\pm$ and integrating we obtain:

$$\frac{d}{dt}(\|w_+\|^2_{L^2(\Omega^+)} + \|w_-\|^2_{L^2(\Omega^-)}) + c(\|w_+\|^2_{W^{1,2}(\Omega^+)} + \|w_-\|^2_{W^{1,2}(\Omega^-)}) \leq 0.$$

So $w_+ = w_- = 0$.

3) Writing (4.6) as:

$$(4.7)\qquad \begin{cases} A^\pm u_\pm = F^\pm = f_\pm - \dfrac{\partial}{\partial t}u_\pm \ \text{ in } \Omega^\pm, \qquad u_\pm = 0 \ \text{ on } \partial\Omega^\pm/\Gamma, \\ u_+ = u_-, \qquad B^+u_+ - B^-u_- = \phi \ \text{ on } \Gamma, \end{cases}$$

for almost all $t$ in $(0,T)$, we have an elliptic problem. Since $F^\pm$ is in $L^2(\Omega^\pm)$ and hence in $H_t^0(\Omega^\pm)$ for almost all $t$, it follows from Theorem 3.2 that

$$\|u_+(\cdot,t)\|_{H^2(\Omega^+)} + \|u_-(\cdot,t)\|_{H^2(\Omega^-)} \leq$$

$$\leq C\left\{\|f_+(\cdot,t)\|_{L^2(\Omega^+)} + \|f_-(\cdot,t)\|_{L^2(\Omega^-)} + \left\|\frac{\partial}{\partial t}u_+(\cdot,t)\right\|_{L^2(\Omega^+)} + \left\|\frac{\partial}{\partial t}u_-(\cdot,t)\right\|_{L^2(\Omega^-)}\right\}.$$

Hence: $u_\pm$ is in $L^2(0,T;H_t^2(\Omega^\pm))$.

We shall now proceed to establish some global regularity results.

LEMMA 4.3: *Suppose all the hypotheses of Theorem 4.1 are satisfied. Let $\{f_\pm, (\partial/\partial t)f_\pm\}$ be in $L^2(0,T;H_t^1(\Omega^\pm))\times L^2(0,T;L^2(\Omega^\pm))$ and let $\{\phi, u_\pm^0\}$ be in $H_t^{3/2}(\Gamma)\times\{W^{2,2}(\Omega^\pm)\cap H_t^2(\Omega^\pm)\}$ with $u_\pm^0 = 0$ on $\partial\Omega^\pm/\Gamma$, $u_+^0 = u_-^0$ on $\Gamma$. Then the solution $u_\pm$ of (4.1) given by Theorem 4.1 is in $L^2(0,T;H_t^3(\Omega^\pm))$ with $\{(\partial/\partial t)u_\pm, (\partial^2/\partial t^2)u_\pm\}$ in $L^2(0,T;H_t^s(\Omega^\pm))\times L^2(0,T;L^2(\Omega^\pm))$ for $0\leq s\leq 1/2$ with $2-s\neq\pi/\omega_j$, $\pi/\gamma_j$ where $\omega_j$, $\gamma_j$ are as in Theorem 3.1.*

*Furthermore if $0<\omega_1$, $\omega_2<\pi/2<\gamma_1$, $\gamma_2<\pi$ then we may take $s=0$ and $\{u_\pm, (\partial/\partial t)u_\pm\}$ is in $L^2(0,T;W^{3,2}(\Omega^\pm))\times L^2(0,T;W^{2,2}(\Omega^\pm))$.*

PROOF: Set: $v_\pm = u_\pm - u_\pm^0$ and we have the initial boundary-value problem:

$$\frac{\partial}{\partial t} v_\pm + A^\pm v_\pm = f_\pm - A^\pm u_\pm^0 \quad \text{in } \Omega^\pm \times (0, T); \qquad v_\pm = 0 \quad \text{in } (\partial\Omega^\pm / \Gamma) \times (0, T),$$

$$v_+ = v_-, \qquad B^+ v_+ - B^- v_- = \frac{\phi}{\rho} - (B^+ u_+^0 - B^- u_-^0) \quad \text{on } \Gamma \times (0, T),$$

$$v_\pm (x, 0) = 0 \quad \text{in } \Omega^\pm .$$

With our hypotheses on $u_\pm^0$, $A^\pm u_\pm^0$ is in $L^2(\Omega^\pm)$ and $B^+ u_+^0 - B^- u_-^0$ is in $H_s^{3/2}(\Gamma)$. Let $d_b w(x, t) = \{w(x, t + b) - w(x, t)\} b^{-1}$ be the difference quotient with respect to $t$. The estimates of Theorem 4.1 give:

$$\left\| \frac{\partial}{\partial t} (d_b v_\pm) \right\|_{L^2(0, T_1; L^2(\Omega^\pm))} + \| A^\pm (d_b v_\pm) \|_{L^2(0, T_1; L^2(\Omega^\pm))} \leq$$

$$\leq C \{ \| d_b f_+ \|_{L^2(0, T_1; L^2(\Omega^+))} + \| d_b f_- \|_{L^2(0, T_1; L^2(\Omega^-))} \}$$

for $T_1 < T$. The constant $C$ is independent of $b$.

Let $b \to 0^+$ and we have $\{(\partial^2/\partial t^2) v_\pm, (\partial/\partial t)(A^\pm v_\pm)\}$ in $L^2(0, T; L^2(\Omega^\pm))$. Consider the problem:

$$A^\pm \left( \frac{\partial}{\partial t} v_\pm \right) = \frac{\partial}{\partial t} f_\pm - \frac{\partial^2}{\partial t^2} (v_\pm) \quad \text{in } \Omega^\pm; \qquad \frac{\partial}{\partial t} v_\pm = 0 \quad \text{on } \Omega^\pm / \Gamma,$$

$$\frac{\partial}{\partial t} v_+ = \frac{\partial}{\partial t} v_-, \qquad B^+ \left( \frac{\partial v_+}{\partial t} \right) - B^- \left( \frac{\partial v_-}{\partial t} \right) = 0 \quad \text{on } \Gamma,$$

for almost all $t$.

From Theorem 3.2 we have $(\partial/\partial t) v_\pm$ in $L^2(0, T; H_s^2(\Omega^\pm))$. So now with $(\partial/\partial t) u_\pm$ in $L^2(0, T; H_s^2(\Omega^\pm))$ and $f_\pm$ in $L^2(0, T; H_s^1(\Omega^\pm))$, by applying Theorem 3.2 to (4.7) we obtain $u_\pm$ in $L^2(0, T; H_s^3(\Omega^\pm))$ for $2 - s \neq \pi/\gamma_j$, $(p\pi - \omega_j)/\gamma_j$.

It is clear that if $0 < \omega_1, \omega_2 < \pi/2 < \gamma_1, \gamma_2 < \pi$ and $\gamma_j \neq (p\pi - \omega_j)/2$, then we may take $s = 0$ and $u_\pm$ is in $L^2(0, T; W^{2,3}(\Omega^\pm))$.

LEMMA 4.4: *Suppose all the hypotheses of Lemma 4.3 are satisfied and suppose further that* $\{f_\pm, \phi, u_\pm^0\}$ *is in* $L^2(0, T; H_s^2(\Omega^\pm)) \times H_s^{3/2}(\Gamma) \times H_s^3(\Omega^\pm)$ *with* $3 - s \neq \pi/\omega_j$, $\pi/\gamma_j$, $(p\pi - \omega_j)/\gamma_j$ *where* $p = 1, 2, \ldots$ *and* $1 \leq j \leq 2$. *Then the solution* $u_\pm$ *of* (4.1) *is in* $L^2(0, T; H_s^4(\Omega^\pm))$.

*If* $0 < \omega_1, \omega_2 < \pi/3 < \gamma_1, \gamma_2 < \pi$ *and* $\gamma_j \neq (p\pi - \omega_j)/3$ *then* $u_\pm$ *is in* $L^2(0, T; H_s^4(\Omega^\pm))$.

PROOF: With our additional regularity hypotheses of $f_\pm$, $\phi$ and $u_\pm^0$, the stated result follows from Theorem 3.2.

## APPENDIX

Here we would like to outline two different methods of completing the proof of Theorem 3.2. The first method consists of showing the existence of a solution

of a perturbed problem, the other one is based on local estimates near singular points.

1) Let $A_0 = (A_0^+, A_0^-)$ be an elliptic operator (satisfying all stated assumptions):

$$A_0^{\pm} u_{\pm} = -\sum_{j,l=1}^{2} \frac{\partial}{\partial x_j}\left(\overset{\circ}{a}{}_{jl}^{\pm}(x)\frac{\partial u_{\pm}}{\partial x_l}\right)$$

where now $\overset{\circ}{a}{}_{jl}^{+}(x) = a_{jl}(P)$ near $P$ and $\overset{\circ}{a}{}_{jl}^{-}(x) = a_{jl}(Q)$ near $Q$, $1 \leq j, l \leq 2$.

From Theorems 2.1 and 3.1 we deduce that our transmission problem for the operator $A_0$ (and corresponding to it boundary operator $B_0$) possesses a unique strong solution, as in Theorem 3.2. Let $A = (A^+, A^-)$ be our original operator and define operators:

$$L_0, L_1 : X \to Y$$

by

$$L_0 u = [A_0 u_+, A_0 u_-, B_0^+ u_+ - B_0^- u_-]$$

and

$$L_1 u = [(A_0^+ - A^+) u_+, (A_0^- - A^-) u_-, (B_0^+ - B^+) u_+ - (B_0^- - B^-) u_-].$$

$L_0$ is invertible and $L_0^{-1}$ is bounded. (Theorems 2.1, 3.1 and existence of a weak solution). We write our original transmission problem as a perturbation of the problem for $A_0$, in the form:

$$L_0 u = [f^+, f^-, \Phi] + L_1 u, \qquad u \in X, \tag{1}$$

and show that equation (1) admits a unique solution in $X$, under the assumptions of Theorem 3.2. We write (1) as

$$(I - L_0^{-1} L_1) u = [f^+, f^-, \Phi]$$

and then $u$ is given by $u = (E - L_0^{-1} L_1)^{-1} L_0^{-1} [f^+, f^-, \Phi]$, if only $\|L_0^{-1} L_1\|_{L(X,X)} < 1$. It is easy to see that due to Assumption (I) (cf. [7]) we have

$$\|L_0^{-1} L_1 u\|_X \leq \|L_0^{-1}\| \cdot \|L_1 u\|_Y \leq \|L_0^{-1}\| \delta_0 \|u\|_X < \|u\|_X$$

and thus $I - L_0^{-1} L_1$ is invertible in $X$. This ends the proof.

2) To be able to show the higher regularity of a weak solution by local estimates near singular points we need somewhat stronger assumptions, namely

$$|a_{jl}^{\pm}(x) - a_{jl}(P)| \leq C\rho^{2+k}$$

and

$$|D^{\alpha}(a_{jl}^{\pm}(x) - a_{jl}(P))| \leq C\rho^{2+|\alpha|}$$

near $P$ (and similar assumptions near $Q$). These assumptions allow us to keep $L_1 u$ in $Y$, where now $u$ is a weak solution of $(L_0 + L_1)u = [f^+, f^1, \Phi]$. First we proceed as in [14] to get $u_\pm$ in $H^{k+1}_{\varepsilon,\delta}{}^{2}_{\varepsilon+2}(\Omega^\pm)$, then show that $L_1 u$ is in $Y$ and in the end use Theorem 2.1 to prove the demanded regularity.

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