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Connectedness and Disconnectedness in Pseudocompact Groups (**) (***).

SUMMARY. — We discuss connectedness and disconnectedness in pseudocompact groups. In particular we answer negatively a question of Comfort and van Mill [CvM] whether a precompact hereditarily disconnected group is totally disconnected. We strenghten a result of [CvM] showing that each precompact connected abelian group is the connected component of a pseudocompact abelian group.

Connessione e sconnessione nei gruppi pseudocompatti

RIASSUNTO. — Si studia connessione e sconnessione nei gruppi pseudocompatti. In particolare, si risponde negativamente alla questione aperta di Comfort e van Mill [CvM] se un gruppo precompatto e ereditariamente sconnesso sia anche totalmente sconnesso. Si generalizza un altro risultato di [CvM] dimostrando che ogni gruppo precompatto abeliano connesso è la componente connessa di un gruppo pseudocompatto abeliano.

0. - INTRODUCTION

Throughout this paper all topological groups are assumed to be Hausdorff. For a topological group G the quasi component $q(G)$ of the neutral element 1 of G is the intersection of all clopen (= closed and open) sets containing 1. We denote by $c(G)$ the connected component of 1 in G by $o(G)$ the intersection of all open normal subgroups of G and by $\tilde{G}$ the Weil completion of G. The group G is *hereditarily disconnected* if $c(G) = \{1\}$, *totally disconnected* if $q(G) = \{1\}$ and *zero-dimensional* if the topology of G has a base of clopen sets ([E]). Total disconnectedness yields hereditary disconnectedness and zero-dimensionality yields total disconnectedness for arbitrary topological groups. Therefore zero-dimensionality can be considered as stronger version of total disconnectedness. The aim of this paper is to investigate the various degrees of disconnectedness in this sense.

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The following well known fact is our starting point.

0.1. Fact. ([HR, (7.7), (7.8)]): *For every locally compact group G $c(G) = q(G) = o(G)$ and $G/c(G)$ is zero-dimensional. In particular, every hereditarily disconnected locally compact group is zero-dimensional.*

This shows that compact-like properties of the group G may play an important role in our investigation. We consider the following generalizations of compactness for G: *precompact* (i.e. $\tilde{G}$ is compact), *pseudocompact* (every continuous function $G \to \mathbf{R}$ is bounded), *countably compact* (each open countable cover of G admits a finite subcover), *minimal* (each continuous isomorphism $G \to H$ is open), *totally minimal* (each Hausdorff quotient of G is minimal).

Comfort and van Mill [CvM, Corollary 7.7] showed that a pseudocompact totally disconnected group need not be zero-dimensional. They left open the following question regarding hereditary and total disconnectedness in the precompact case.

0.2. Question ([CvM, Remark 7.8]): *Is every precompact hereditarily disconnected group totally disconnected?*

Here we give counter-examples to this question following three different ways. The first two constructions, given in sect. 2, are based on the additional set-theoretic assumption $2^{\omega_1} = 2^{\omega}$, known as Lusin's hypothesis. They provide groups with the additional property of being totally minimal. The third construction, given in sect. 3, makes no recourse to Lusin's hypothesis.

The first construction in sect. 2 is based on the following.

0.3. Lemma [$2^{\omega_1} = 2^{\omega}$]: *Let G be a compact Abelian group of weight ω_1 having no closed, torsion G_δ-subgroups and let C be a torsion-free subgroup of G with $|C| < \mathfrak{c}$. Then G contains a proper, dense, pseudocompact and totally minimal subgroup H such that $H \cap C = 0$.*

This lemma is essentially contained in the proof of Lemma 5.2 of [DS1]. It provides a family of 2^{ω} pairwise non-isomorphic one-dimensional totally minimal hereditarily disconnected pseudocompact groups which are not totally disconnected (Theorem 2.1 and Remark 2.2*b*)).

The second construction in sect. 2 provides totally minimal, hereditarily disconnected, pseudocompact groups of arbitrary dimension which are not totally disconnected (Theorem 2.3). It makes use of the following.

0.4. Lemma [$2^{\omega_1} = 2^{\omega}$]: *Let p be a prime number. Then there exists a cyclic subgroup C of $Z_p^{\omega_1}$ and a dense pseudocompact subgroup H of $Z_p^{\omega_1}$ such that H^{ω} is totally minimal and $H \cap C = 0$.*

This lemma can be obtained from the proof of [DS2, Theorem 1.16] by setting $\sigma = \omega_1$.

In both lemmas the subgroup H has stronger properties than what we really needed to answer Question 0.2. In sect. 3 we relax the condition of total minimality on H

and in this way the assumption $[2^{\omega_1} = 2^{\omega}]$ is not needed any more. We stress the fact that in both 0.3 and 0.4 there exists a dense pseudocompact subgroup H of a compact group G which avoids some (fixed) subgroup C of G, *i.e.* $G \cap C = 0$ holds. This suggested the following general question which is one of the main topics of sect. 3.

0.5. Question: *Let G be a compact group of weight $\sigma > \omega$, and C be a closed subgroup of G. Under which conditions there exists a dense pseudocompact subgroup H of G «avoiding» C, i.e. $H \cap C = 0$?*

Wilcox [W, Example 2.5] showed that without some reasonable restriction on the groups in Question 0.5 the answer is strongly negative (see Example 3.9 below). It should be mentioned that whenever $C \neq 0$ in 0.5 the subgroup H cannot be totally minimal according to the total minimality criterion 1.6.

In Lemma 3.1 we carry out a general construction by transfinite recursion, following (to some extent) and strengthening the idea from [CvM, Theorem 7.6]. This construction is applied then in various cases. In Theorem 3.2 we show that 0.5 has a positive solution whenewer $r(C) \leq \sigma^{\omega} < r(G)$, in particular when $r(G) = |G|$ and $r(C) \leq \sigma = \sigma^{\omega}$. In Corollary 3.4 we give the following important case: $r(G) > \sigma^{\omega}$ and C is metrizable. For example, groups G with $r(G) > 2^{\omega} \geq \sigma$ have this property. In particular, 0.5 has a positive solution for groups G of weight ω_1, $r(G) > 2^{\omega}$ and C metrizable. It is shown in Example 3.8 that the condition $r(G) > \sigma^{\omega}$ cannot be omitted in Corollary 3.4. As an application we get in Corollary 3.7 a negative answer to Question 0.2 in ZFC. As another application we construct in Corollary 3.6 pseudocompact Abelian groups with prescribed quasi-component and connected component generalizing substantially Theorem 7.6 of [CvM].

A problem in a somewhat opposite direction with respect to 0.5 has been treated by Wilcox [W] where the following positive result can be found when the subgroup *C is not fixed in advance*: if G is abelian and $\sigma^{\omega} < 2^{\sigma}$ then for every dense pseudocompact subgroup H of G with $|H| \leq \sigma^{\omega}$ *there exists* an infinite compact subgroup C of G such that $H \cap C = 0$ holds ([W, Theorem 2.2]).

In Section 1 we collect some useful properties of the quasi-component, which are not easily found in the existing literature on topological groups. They give some immediate relations between zero-dimensionality and total (hereditary) disconnectedness of pseudocompact groups in the presence of further compact-like properties (Corollary 1.5, Theorem 1.7, Corollary 1.8).

The notation follows [HR], [E] and [DPS]. In particular, $\boldsymbol{Z}$ denotes the integers, $\boldsymbol{Q}$—the rationals, $\boldsymbol{R}$—the reals, $\boldsymbol{T}^n = (\boldsymbol{T}/\boldsymbol{Z})^n$—the n-dimensional torus, $\boldsymbol{Z}_p$—the group of p-adic integers. We fix $|X|$ for denoting the cardinality of a set X. The symbols ω and ω_1 denote the first infinite cardinal and the first uncountable cardinal respectively. For groups which are not necessarily abelian multiplicative notation is used, while for abelian groups—always additive one; in particular, the neutral element is always denoted by 1 or 0 resp. For an abelian group G we denote by $r(G)$ the free-rank of G. If X is a subset of a topological group G, then $\langle X \rangle$ is the smallest subgroup of G that contains X and $\overline{X}$ is the closure of X.

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1. - The quasi-component

Throughout this paper we consider only the covering Čech-Lebesgue dimension dim. For pseudocompact groups the coincidence of all three dimensions was proved by Tkačenko [T]: $\dim G = \operatorname{ind} G = \operatorname{Ind} \beta G$ for such a group G. Shakhmatov [S1] showed that in the larger class of precompact groups, these dimensions need not coincide. Nevertheless, zero-dimensionality does, *i.e.* $\dim G = 0$ and $\operatorname{ind} G = 0$ are equivalent for a precompact group G ([S2]). Since all groups considered in this paper are precompact, the reader should not worry that zero-dimensionality was defined above with respect to the small inductive dimension ind ([E]).

The following notion is needed to characterize the pseudocompact groups: a subset Y of topological space X is G_δ*-dense* if Y meets every non-empty G_δ-set of X.

1.1. Fact ([CR, Theorem 1.2, Theorem 4.1]): *Every pseudocompact group is precompact. Let G be a precompact group. Then the following are equivalent:* 1) *G is pseudocompact;* 2) *G is G_δ-dense in $\tilde{G}$;* 3) $\tilde{G} = \beta G$.

1.2. Fact: *Let G be a pseudocompact group. Then:*

a) $\dim G = \dim \tilde{G}$.

b) *G is connected iff $\tilde{G}$ is connected.*

Let G be a topological group, clearly $o(G)$ is a closed normal subgroup of G and $q(G) \subseteq o(G)$.

1.3. Lemma: *Let G be a topological group and H be a dense subgroup of G. Then* $o(H) = o(G) \cap H$.

Proof: Clearly $o(H) \subseteq o(G) \cap H$. Let O be an open subgroup of H, then its closure $\overline{O}$ in G is open and $\overline{O} \cap H = O$ since O is also closed in H. Hence $o(H) \supseteq o(G) \cap H$. Q.E.D.

1.4. Lemma: *Let G be a pseudocompact group. Then $q(G) = o(G) = G \cap c(\tilde{G})$, in particular $q(G)$ is a closed normal subgroup of G.*

Proof: Let O be a clopen subset of G and $1 \in O$. By fact 1.1 $\tilde{G}$ is compact and $\tilde{G} = \beta G$, so the closure of O in $\tilde{G}$ is clopen. On the other hand, for every clopen subset W of $\tilde{G}$ the intersection $W \cap G$ is a clopen subset of G. This shows that $q(G) = q(\tilde{G}) \cap G$. Since $q(\tilde{G}) = o(\tilde{G})$ by Fact 0.1, we get $q(G) = o(\tilde{G}) \cap G = o(G)$ by Lemma 1.3. Q.E.D.

Pseudocompactness is essential in the above lemma, in fact for the subgroup $G = \boldsymbol{Q}/\boldsymbol{Z}$ of $\boldsymbol{T}$ $q(G) = 0$, while $o(G) = G$.

A topological group G is said to have *linear topology* if the open normal subgroups of G form a base of open neighbourhoods of 1. Clearly, every topological group G admits a finest linear group topology λ_G coarser than the given topology of G, namely the family of all open normal subgroups of G form a base of open neighbourhoods of 1 in λ_G. Clearly, λ_G is Hausdorff iff $o(G) = 1$. Hence a group G admits a coarser linear group topology iff $o(G) = 1$. Every linear group topology is zero-dimensional since every open subgroup is closed, so a simple application of Lemma 1.4 gives the following

1.5. COROLLARY ([S3]): *Let G be a pseudocompact totally disconnected group. Then G admits a coarser zero-dimensional group topology. In particular, a minimal pseudocompact Abelian group G is totally disconnected iff* $\dim G = 0$.

The following criterion will often be used in the sequel (see [DP, Theorem 3.3] and also [DPS, Theorem 4.3.3]). We remind that a subgroup H of a topological group G is *totally dense* if $N = \overline{N \cap H}$ for every closed normal subgroup N of G.

1.6. TOTAL MINIMALITY CRITERION: *A precompact group G is totally minimal iff G is totally dense in $\tilde{G}$.*

1.7. THEOREM: *Let G be a totally minimal Abelian group such that every closed normal subgroup of G is pseudocompact. Then $q(G) = c(G) = o(G)$. In particular, the following conditions are equivalent for G:*

a) $c(G) = 0$;

b) $q(G) = 0$;

c) $o(G) = 0$;

d) G has linear topology.

PROOF: Since the group G itself is pseudocompact, the completion $\tilde{G}$ is compact. Let $C = c(\tilde{G})$. Then $G \cap C$ is a dense subgroup of C by the total minimality criterion 1.6. On the other hand, $G \cap C$ is pseudocompact as a closed normal subgroup of G. Thus C is the Čech-Stone compactification of $C \cap G$ by Fact 1.1. According to Fact 1.2 *b*), the connectedness of C implies that $G \cap C$ is connected. This proves that $G \cap \cap C \subseteq c(G)$. On the other hand, $C = o(\tilde{G})$, thus $o(G) \subseteq C \cap G \subseteq c(G)$. The rest is obvious. Q.E.D.

Now we turn to the countably compact case.

1.8. COROLLARY: *Let G be a countably compact group which is either totally minimal or minimal and Abelian. Then $q(G) = c(G) = o(G)$, in particular, if G is hereditarily disconnected, then G is 0-dimensional.*

PROOF: Apply the above theorem in the case G is totally minimal. Now assume that G is minimal and Abelian. Then G contains $c(\tilde{G})$ according to [D2]. Thus $q(G) = c(G) = o(G)$ by Lemma 1.4. Q.E.D.

It will be shown in the next section under Lusin's hypothesis that «countably compact» cannot be substituted by «pseudocompact» in the above corollary even in the case of Abelian groups (compare with Corollary 1.5).

2. - Counterexamples under Lusin's hypothesis

In this section we work under Lusin's hypothesis $2^{\omega_1} = 2^{\omega}$.

2.1. Theorem: *Assume Lusin's hypothesis. There exist a pseudocompact and totally minimal hereditarily disconnected Abelian group G such that* $\dim G = 1$.

Proof: To construct G fix an infinite cyclic subgroup Z of the torus T and a prime number p. The compact abelian group $G = T \times Z_p^{\omega_1}$ has no closed, torsion G_δ-subgroups, since the subgroup $N = \{0\} \times Z_p^{\omega_1}$ is torsion-free and G_δ. Hence we can apply Lemma 0.3 to obtain a dense, pseudocompact and totally minimal subgroup H of the compact group G with

$$(1) \qquad H \cap (Z \times \{0\}) = 0 .$$

It follows from Lemma 1.4 that $q(H) = H \cap (T \times \{0\})$, since $T \times \{0\}$ is the connected component of G. Hence the connected component $c(H)$ of H is contained in $H \cap \cap (T \times \{0\})$. According to (1) $q(H) \subseteq ((T \setminus Z) \times \{0\}) \cup \{(0, 0)\}$ and the latter subspace of G is zero-dimensional (hence totally disconnected), since Z is dense in T. Thus $c(H) \subseteq q(q(H)) = 0$. Therefore the group H is hereditarily disconnected. Now $\dim H = \dim G = 1$ by Fact 1.2 *a*). Finally, to see that H is not totally disconnected it suffices to apply Corollary 1.5 (for an alternative argument concerning the total disconnectedness see Remark 2.2*a*)). Q.E.D.

2.2. Remark: *a*) By the total minimality Criterion 1.6 the group H, contains the torsion part $Q/Z \times \{0\}$ of its completion G (see also [DP] or [DPS, Corollary 4.3.4]). So $Q/Z \times \{0\} \subseteq q(H)$, hence H is not totally disconnected.

b) Note that the group H obtained in the above proof depends strongly on the choice of the subgroup Z. In fact, if we denote by H_Z the group H obtained by means of Z, then for $Z \neq Z'$ the groups H_Z and $H_{Z'}$ are not topologically isomorphic. In fact, even $q(H_Z)$ and $q(H_{Z'})$ are not isomorphic, since every topological isomorphism $\varphi\colon q(H_Z) \to q(H_{Z'})$ extends to a topological automorphism $\bar{\varphi}$ of the common completion $T \times \{0\}$ (by a)), thus $\bar{\varphi} = \pm \mathrm{id}_{T \times \{0\}}$. This implies $Z = Z'$—a contradiction. Since T has a family of cardinality 2^{ω} of distinct infinite cyclic subgroups, this means that there is a family of cardinality 2^{ω} of pairwise non-isomorphic one-dimensional pseudocompact, totally minimal and hereditarily disconnected Abelian groups G of weight ω_1 which are not totally disconnected. Note that there are at most 2^{ω_1} pairwise non- isomorphic compact abelian groups of weight ω_1. Hence, in view of the blanket assumption $2^{\omega_1} = 2^{\omega}$, the family we get has the maximal possible cardinality.

We can obtain totally disconnected pseudocompact groups of higher dimensions if we lean on another idea.

2.3. THEOREM: *Assume Lusin's hypothesis. For every $n \in \mathbf{N}$ there exists a pseudocompact, totally minimal, hereditarily disconnected but not totally disconnected Abelian group H_n such that* $\dim H_n = n$.

PROOF: To construct H_n fix a prime number p and apply Lemma 0.4 to obtain an infinite cyclic subgroup C of the compact group $K = \mathbf{Z}_p^{\omega}$ and a totally dense pseudocompact subgroup H of K with H^{ω} totally minimal and $H \cap C = 0$. Note that by Comfort-Ross' theorem [CRs] pseudocompactness is preserved by products, thus H^{ω} is pseudocompact. Hence by Fact 1.1 H^{ω} is G_δ-dense in K^{ω}.

Note that $r(C^{\omega}) = |T^n| = 2^c$. Let F be a free subgroup of C^{ω} of rank 2^c and let $\phi: F \to T^n$ be a surjective homomorphism. Using the divisibility of T^n we extend ϕ to a homomorphism $\phi': C^{\omega} \to T^n$. Since T^n is divisible and $H^{\omega} \cap C^{\omega} = 0$, we can extend ϕ' to a homomorphism $\phi: K^{\omega} \to T^n$ such that $\phi(H^{\omega}) = 0$. Set $M = K^{\omega} \times T^n$ and consider the subgroup $G = \{(x, \phi(x)) \in M : x \in K^{\omega}\}$ of the compact group M. This is the graph of ϕ, so that the equality

$$(2) \qquad G \cap (\{0\} \times T^n) = 0,$$

is easy to check.

Let us check that G is G_δ-dense in M. Let $O \neq \emptyset$ be a G_δ set in M. We can assume wlog that $O = U \times V$, where $\emptyset \neq U \subseteq K^{\omega}$ and $\emptyset \neq V \subseteq T^n$ are G_δ sets. Take any $y \in V$. Since ϕ is surjective there exists $x \in K^{\omega}$ such that $y = \phi(x)$. Now by the G_δ-density of H^{ω} in K^{ω} there exists $b \in H^{\omega} \cap (U - x)$. Then the definition of ϕ yields $\phi(b) = 0$, so that $\phi(x + b) = y$. Hence $(x + b, y) \in G \cap (U \times V)$.

Now set $H_n = G + (\{0\} \times (\mathbf{Q}/\mathbf{Z})^n)$. Then also H_n is G_δ-dense in M, so by Fact 1.1 H_n is pseudocompact. By (2) and the modular low for subgroups

$$(3) \qquad H_n \cap (\{0\} \times T^n) = \{0\} \times (\mathbf{Q}/\mathbf{Z})^n.$$

Now Lemma 1.4 and (3) yield $q(H_n) = \{0\} \times (\mathbf{Q}/\mathbf{Z})^n$. Since the latter group is totally disconnected (even zero-dimensional), it follows that $c(H_n) \subseteq q(q(H_n)) = 0$. Thus the group H_n is hereditarily disconnected.

We show next that the group H_n is totally minimal. By the obvious inclusion $H^{\omega} \times \{0\} \subseteq G$ and by the definition of H_n it follows that H_n contains the subgroup $B = H^{\omega} \times (\mathbf{Q}/\mathbf{Z})^n$ of M. So by the total minimality Criterion 1.6, it suffices to see that B is totally minimal. This follows from the total minimality of H^{ω} and the perfect total minimality of $(\mathbf{Q}/\mathbf{Z})^n$ (see [D1], or [DPS, Corollary 6.1.18]).

On the other hand $\dim T^n = n$, so that Fact 1.2*a*) gives $\dim H_n = n$. To finish the proof we note that by (3) H_n is not totally disconnected (this follows also from Corollary 1.5 and $\dim H_n > 0$). Q.E.D.

In both constructions we got a pseudocompact group H such that $0 = q(q(H)) \neq q(H)$. This shows that the quasi-component is not idempotent considered as a functorial subgroup.

3. - Dense pseudocompact subgroups avoiding some compact subgroup: counterexamples in ZFC

The following general lemma covers all cases we are interested in.

3.1. Lemma: *Let C be a non-zero subgroup of an abelian group G and let $\{L_\gamma\}_{\lambda<\alpha}$ be a collection of subgroups of G such that*

(4) $$r(L_\gamma) > \alpha \geq r(C),$$

holds for each $\gamma < \alpha$. Then for each collection $\{x_\gamma\}_{\gamma<\alpha}$ of elements of G there exists a subgroup H of G such that $H \cap C = 0$ and $H \cap (x_\gamma + L_\gamma) \neq \emptyset$ holds for every $\gamma < \alpha$.

Proof: We will construct by transfinite recursion an increasing chain $\{H_\gamma : \gamma < \alpha\}$ of subgroups of G such that for all $\gamma < \alpha$ the following conditions will be satisfied:

(i_γ) $r(H_\gamma) \leq \max\{\omega, \gamma\}$,

(ii_γ) $H_\gamma \cap (x_\gamma + L_\gamma) \neq \emptyset$,

(iii_γ) $H_\gamma \cap C = \{0\}$.

Then the subgroup $H = \bigcup_{\gamma<\alpha} H_\gamma$ will clearly have the desired properties.

To start the recursion set $H_{-1} = \{0\}$ for convenience. Then suppose that $\gamma < \alpha$ and that H_λ satisfying (i_λ)-(iii_λ) have already been defined for $\lambda < \gamma$. Let us define H_γ.

Set $H_\gamma^* = \bigcup\{H_\lambda : \lambda < \gamma\}$. If γ is a non-limit cardinal, then simply $H_\gamma^* = H_{\gamma-1}$. To check that H_γ^* satisfies (i_γ) in the case of limit γ consider an independent subset S of H_γ^*. We have to show that $|S| \leq \max\{\omega, \gamma\}$. For each $\lambda < \gamma$ the subset $S \cap H_\lambda$ of H_λ is independent, so that (i_λ) yields $|S \cap H_\lambda| \leq \max\{\omega, \lambda\} \leq \max\{\omega, \gamma\}$. This ensures ($\mathrm{i}_\gamma$) for H_γ^* since $S = \bigcup_{\lambda<\gamma}(S \cap H_\lambda)$. Clearly H_γ^* satisfies also (iii_γ), so we can set $H_\gamma = H_\gamma^*$ if (ii_γ) holds as well. Otherwise, as the remaining part of the proof shows, we can enlarge slightly the subgroup H_γ^* to get also (ii_γ) by keeping (i_γ) and (iii_γ) satisfied.

For each $x \in L_\gamma$ consider the subgroup $K_x = H_\gamma^* + \langle x + x_\gamma\rangle$ of G. If

(5_x) $$K_x \cap C \neq 0,$$

then there exists $k_x \in Z$, $c \in C$ and $b \in H_\gamma^*$ such that $k_x(x + x_\gamma) + b = c \neq 0$. If $k_x(x + x_\gamma) = 0$, then $b = c \in H_\gamma \cap C = 0$—a contradiction. Thus $k_x(x + x_\gamma) \neq 0$. Hence for each $x \in L_\gamma$ such that (5_x) holds there exists a non-zero integer k_x such that $k_x x \in S = \langle x_\gamma, H_\gamma^* + C\rangle$. Since

$$r(S) \leq r(C) + r(H_\gamma^*) + 1 \leq r(C) + \max\{\omega, \gamma\} \leq \alpha < r(L_\gamma),$$

according to (4), we get $r(L_\gamma) > r(S)$. Hence there exists $x \in L_\gamma$ such that (5_x) fails, *i.e.*

(6) $$K_x \cap C = 0,$$

holds. Now $H_\gamma = K_\gamma$ satisfies (iii$_\gamma$) by (6). Moreover, $x + x_\gamma \in H_\gamma \cap (L_\gamma + x_\gamma)$, so also (ii$_\gamma$) is satisfied. Finally, (i$_\gamma$) follows from the trivial inequality $r(H_\gamma) \leq \leq \max\{r(H_\gamma^*), \omega\}$. Q.E.D.

3.2. Theorem: *For every compact Abelian group G with $r(G) > w(G)^\omega$ and compact subgroup C of G such that $w(G)^\omega \geq r(C)$ there exists a dense pseudocompact subgroup H of G such that $C \cap H = 0$.*

Proof: Set $\alpha = w(G)^\omega$. Let $\{L_\gamma\}_{\lambda < \alpha}$ be the collection of closed normal G_δ-subgroups of G each one taken 2^ω times (see the proof of Theorem 4.2 [CvM] for the possibility of such an enumeration). For each L_γ, $\lambda < \alpha$, the quotient G/L_γ is metrizable, thus

$$(7) \qquad r(G/L_\gamma) \leq |G/L_\gamma| \leq 2^\omega.$$

By hypothesis $r(G) > w(G)^\omega \geq 2^\omega$, thus $r(G) > 2^\omega$. Hence (7) yields

$$r(G) = \max\{r(G/L_\gamma), r(L_\gamma)\} = r(L_\gamma) = r(G) > \alpha \geq r(C).$$

Thus (4) holds. By (7) one can enumerate by $\{x_\gamma + L_\gamma\}_{\gamma < \alpha}$ all cosets $\{G/L_\gamma\}_{\gamma < \alpha}$ for an appropriate collection $\{x_\gamma\}_{\gamma < \alpha}$ of elements of G. Now Lemma 3.1 provides a subgroup H of G which meets all cosets $\{x_\gamma + L_\gamma\}_{\gamma < \alpha}$. This means that the subgroup H of G is G_δ-dense, since each G_δ-subset of G containing 0 contains a G_δ-subgroup of G according to [CR, Lemma 1.6 (*b*)]. Then H is pseudocompact by Fact 1.1. Q.E.D.

3.3. Remark: Example 3.8 below shows that the condition $r(G) > w(G)^\omega$ cannot be removed, although we are not certain it is also necessary. This condition is fulfilled for every compact Abelian group G with $r(G) = |G|$ (in particular, torsion-free) and $w(G) = w(G)^\omega$, since $|G| = 2^{w(G)}$ (see [C] for this relation).

3.4. Corollary: *Let G be an infinite compact Abelian group, and C be a metrizable closed subgroup of G. Under the condition $r(G) > w(G)^\omega$ there exists a dense pseudocompact subgroup H of G «avoiding» C, i.e. $H \cap C = 0$.*

Proof: Since $r(C) \leq 2^\omega \leq w(G)^\omega$ Theorem 3.2 can be applied. Q.E.D.

The condition $r(G) > w(G)^\omega$ implies that the group G is not metrizable.

3.5. Corollary: *Let $0 < n \leq \omega$ and C be a compact connected Abelian group of dimension n. Then for every subgroup L of C there exists a pseudocompact group H such that $\dim H = n$ and $q(H) \cong L$.*

Proof: We follow the proof of [CvM, Theorem 7.6], in particular, set $\beta = |C|$ and $G = C \times M^{\beta}$, where M is a torsion-free zero-dimensional and metrizable compact abelian group (for example Z_p for some prime p). Then $\beta^{\omega} = \beta = 2^{w(C)}$ and $r(G) = r(M^{\beta}) = 2^{\beta}$, so to the group G and its subgroup $C \times \{0\}$ the above corollary can be applied to produce a dense pseudocompact subgroup $\tilde{H}$ of G with

$$\tilde{H} \cap (C \times \{0\}) = 0. \tag{8}$$

Set $H = \tilde{H} + (L \times \{0\})$. Then H is pseudocompact by Fact 1.1. By Fact 1.2 *a*) $\dim H = \dim G$, so we get $\dim H = n$. On the other hand, (8) and the modular low for subgroups yield that $(C \times \{0\}) \cap H = L \times \{0\}$. Now it suffices to note that $C \times \{0\} = c(G)$ and apply Lemma 1.4 to get $q(H) = (C \times \{0\}) \cap H = L \times \{0\}$. Q.E.D.

3.6. Corollary: *Let L be a precompact (connected) Abelian group. Then there exists a pseudocompact group H such that $q(H) \cong L$ (resp. $c(H) \cong L$).*

Proof: Consider first the case when L is a precompact Abelian group. Then by Peter-Weyl's theorem there exists a topological group embedding of L into a power T^{τ}. Now for $C = T^{\tau}$ apply Corollary 3.5 to get a pseudocompact group H such that $q(H) \cong L$. This proves the first part of the corollary. It remains to observe that if L is also connected, then obviously $c(H) = q(H)$ since $q(H)$ is connected. Q.E.D.

The case when L is connected and $\hat{L}$ is torsion-free coincides with Theorem 7.6 in [CvM]. Note that the second condition is rather strong, since $\hat{L}$ may have torsion element even if H is torsion-free.

The next corollary gives a negative answer to Question 0.2. It should be stressed that this example makes no recourse to Lusin's hypothesis.

3.7. Corollary: *For every $0 < n \leq \omega$ there exists a hereditarily disconnected and non-totally disconnected pseudocompact group of dimension n.*

Proof: Take L to be a non-zero hereditarily disconnected subgroup of C in Corollary 3.5, for example $C = T^{n}$ and $L = (Q/Z)^{n}$. Q.E.D.

The following example shows that it is not possible to omit the condition $r(G) > w(G)^{\omega}$ in Theorem 3.2.

3.8. Example: Let K be a compact metrizable connected non-trivial group, say $K = T$. Let p be a prime number and $G = Z(p)^{\omega} \times K$, where $Z(p) = Z/pZ$. Then every dense subgroup H of G meets non-trivially the subgroup $C = \{0\} \times K$. In fact, assume $H \cap C = 0$, then H is algebraically isomorphic to a subgroup of $G/C \cong Z(p)^{\omega}$, hence $pH = 0$ and consequently $pG = 0$ by the density of H in G. This contradicts the choice of K. Note, that $r(G) = w(G)^{\omega} = r(C) = 2^{\omega}$. It can be shown that every dense pseudocompact subgroup of G contains the subgroup C.

The following example shows that for a torsion group G there is no hope to resolve 0.5 even for very small subgroups C.

3.9. EXAMPLE ([W, Example 2.5]): Let $G = Z(2)^{\beta} \times Z(4)$, where $\beta > \omega$. Then every dense pseudocompact subgroup of G contains the non-trivial subgroup $C = \{0\} \times Z(2)$ of G. It is easy to see that pseudocompactness is not necessary here.

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